# MA AND THREE FRÉCHET SPACES

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ABSTRACT. M. Scheepers introduced the notion of selectively separable as one of many interesting selection principles. Fréchet spaces are known to be selectively separable but neither of these properties is well-behaved in products, even for countable spaces. We prove that the open coloring axiom, OCA, implies that the product of two countable Fréchet spaces is selectively separable. We prove that Martin's Axiom implies there are three countable Fréchet spaces whose product is not selectively separable.

## 1. INTRODUCTION

A space is Fréchet (often called Fréchet-Urysohn) if a point $x$ of $X$ is an accumulation point of a subset $A$ of $X$ if and only if there is a sequence from $A$ converging to $x$. While the property of being Fréchet is a local property, it has been discovered to have an effect on some selection principles connected to density. The first of these was introduced by M. Scheepers [10] and called selectively separable. It is often now referred to as M-separable so as to fit a pattern.

**Definition 1.** Let $\{D_n : n \in \omega\}$ be dense subsets of a space $X$. Then, $X$ is said to be

1. M-separable if there is a selection $H_n \subset D_n$ of finite sets such that $\bigcup\{H_n : n \in \omega\}$ is dense,
2. R-separable if there is a selection of points $d_n \in D_n$ such that $\{d_n : n \in \omega\}$ is dense,
3. H-separable if there is sequence of finite sets $H_n \subset D_n$ so that $\bigcup\{H_n : n \in H\}$ is dense for every infinite set $H \subset \omega$,
4. mH-separable if there is a sequence of finite sets $H_n \subset \bigcup\{D_m : m \geq n\}$ so that $\bigcup\{H_n : n \in H\}$ is dense for every infinite $H \subset \omega$ (equivalently, the H-separable property holds for descending sequences of dense sets).

The above notions were introduced and studied in [6]. It is evident that an M-separable space is separable, that every R-separable space is M-separable, and every mH-separable space is M-separable.

In this paper our interest will be on countable zero-dimensional Hausdorff Fréchet spaces. It was shown in [4] that such spaces are M-separable. It is also known that such spaces are R-separable and mH-separable [7]. In the remainder of the paper, when we say that a space is a countable Fréchet space it will also mean that the space is Hausdorff and has a basis consisting of clopen sets. It was proven in [5] that PFA implies that the product of two countable Fréchet spaces is M-separable. It was noted in [3] that the proof yields that such products are R-separable. It was

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shown in [3, Theorem 1.2] that it is consistent with Martin's Axiom (and follows from CH) that there is a product of two countable zero-dimensional Fréchet spaces that is not M-separable.

Two questions, among many, remain (as noted in [3]): does the PFA result extend to products of three or more spaces as (inadvertently) claimed in [5] and can the conclusion for the product of two be strengthened to mH-separable. We answer with the following three theorems.

**Theorem 1.** *Martin's Axiom implies the existence of 3 Fréchet countable regular spaces with non-M-separable product.*

**Theorem 2.** *Martin's Axiom implies the existence of 2 Fréchet countable regular spaces with non-mH-separable product.*

**Theorem 3.** *The Open Graph Axiom implies that the product of 2 Fréchet countable regular spaces is M-separable.*

Theorem 1 is proven via the constructions in sections 2 through section 6. Section 7 completes the discussion of the constructed examples to prove Theorem 2. Theorem 3 is restated (and the Open Graph Axiom is recalled) and proven in the final section.

## 2. Ground 0 inductive assumptions

Choose a dense subset $D$ of $\mathbb{Q}^{\{1,2,3\}} = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ such that, for each $i = 1, 2, 3$, the projection map $\pi_i \restriction D$ is 1-to-1 and let $Q_i = \pi_i[D]$. For convenience also assume (arrange) that $\{Q_1, Q_2, Q_3\}$ is a partition of $\mathbb{Q}$. It will also be convenient to assume that $\mathbb{Q}$ denotes the rational points in $[0, 1]$, and when we speak of the product space $[0, 1]^{\{1,2,3\}}$ we mean with respect to the usual topology. We also let $\pi_{1,2}$ denote the projection map from $D$ to $Q_1 \times Q_2$.

For the remainder of the paper, until stated otherwise at the beginning of section 8, we assume that Martin's Axiom holds. In fact we only need Martin's Axiom for posets that are $\sigma$-2-linked. The main idea of the proof relies on [1]

Fix a countable base $\tau_0^i$ for each $Q_i$ consisting of non-empty clopen sets and closed under complements.

For each $q \in Q_1 \cup Q_2 \cup Q_3$, let $i_q \in \{1, 2, 3\}$ indicate that $q \in Q_{i_q}$.

For each $i \in \{1, 2, 3\}$ and subset $A \neq \emptyset$ of $Q_i$, let $\tilde{A} = \pi_i^{-1}[A] \cap D$.

Fix a partition, $\{D_n : n \in \omega\}$, of $D$ into dense subsets. These will provide the family of dense sets that will witness the failure of the product being M-separable. We will simultaneously ensure that the sequence $\{\pi_{1,2}[D_n] : n \in \omega\}$ will witness that the product $Q_1 \times Q_2$ is not mH-separable.

**Proposition 4.** *For each $q \in Q_1$, $n \in \omega$, $U_2 \in \tau_0^2$ and $U_3 \in \tau_0^3$, there is a sequence $I \subset \pi_1[D_n \cap \tilde{U}_2 \cap \tilde{U}_3]$ such that*

(1) *$I$ converges to $q$,*
(2) *for $j \in \{2, 3\}$, $\pi_j[\tilde{I}]$ is closed and discrete in $Q_j$ (wrt $\tau_0^j$).*

*Proof.* By the assumption that $D_n$ is dense, it follows that $\pi_1[D_n \cap \pi_2^{-1}[U_2] \cap \pi_3^{-1}[U_3]]$ is dense in $Q_1$. Since $D_n \cap \pi_2^{-1}[U_2] \cap \pi_3^{-1}[U_3] = D_n \cap \tilde{U}_2 \cap \tilde{U}_3$, $\pi_1[D_n \cap \tilde{U}_2 \cap \tilde{U}_3]$ is dense in $Q_1$. Choose a pair of $\{U_{2,\ell} : \ell \in \omega\} \subset \tau_0^2$ and $\{U_{3,\ell} : \ell \in \omega\} \subset \tau_0^3$ that are partitions of $Q_2$ and $Q_3$ respectively. Since $Q_1$ is Fréchet (in fact first-countable) there is a sequence $I = \{i_\ell : \ell \in \omega\} \subset Q_1$ converging to $q$ such that $i_\ell \in \pi[D_n \cap \tilde{U}_{2,\ell} \cap \tilde{U}_{3,\ell}]$ for each $\ell$. It follows that $\pi_j[\tilde{I}] \cap U_{j,\ell}$ is a singleton for each $\ell \in \omega$. This certainly ensures that $\pi_j[\tilde{I}]$ is closed and discrete. $\square$

For each $q \in \mathbb{Q}$, choose a countable family $\mathcal{I}_0^q$ of subsets of $Q_{i_q}$ satisfying that:

(1) each $I \in \mathcal{I}_0^q$ converges to $q$,
(2) for each $n \in \omega$, and each $(U_2, U_3) \in \tau_0^{j_2} \times \tau_0^{j_3}$ (where $\{j_2, j_3\} = \{1,2,3\} \setminus \{i_q\}$), there is an $I \in \mathcal{I}_0^q$ such that $I \subset \pi_{i_q}[D_n \cap \tilde{U_{j_2}} \cap \tilde{U_{j_3}}]$,
(3) for each $i_q \neq j \in \{1,2,3\}$, $\pi_j[\tilde{I}]$ is closed discrete in $Q_j$ in the topology $\tau_0^j$,
(4) for each $I \in \mathcal{I}_0^q$, $\tilde{I}$ converges in $[0,1]^{\{1,2,3\}}$.

## 3. Inductive assumptions

For each $\beta \leq \gamma < \alpha < \mathfrak{c}$, each $q \in \mathbb{Q}$, and each $i \in \{1,2,3\}$:

(1) $\tau_\beta^i \subset \tau_\gamma^i$ are bases of clopen (closed under complements) non-empty subsets of $Q_i$
(2) $\tau_\gamma^i$ has cardinality equal to $\aleph_0 + |\gamma|$,
(3) $\mathcal{I}_\beta^q$ is a set of at most $\aleph_0 + |\beta|$ many subsets of $Q_{i_q}$ that $\tau_\beta^i$-converge to $q$,
(4) $\mathcal{I}_\beta^q \subset \mathcal{I}_\gamma^q$,
(5) for $i_q \neq j \in \{1,2,3\}$ and $I \in \mathcal{I}_\beta^q$, $\tilde{I}$ converges in $[0,1]^{\{1,2,3\}}$, and $\pi_j[\tilde{I}]$ is closed and discrete in $\tau_\beta^j$,
(6) for $n \in \omega$, $U_2 \in \tau_\beta^{j_2}$, $U_3 \in \tau_\beta^{j_3}$ (where $\{i_q, j_2, j_3\} = \{1,2,3\}$), there is an $I \in \mathcal{I}_\beta^q$ that meets $\pi_{i_q}[D_n \cap \tilde{U}_2 \cap \tilde{U}_3]$ in an infinite set.

Assume also that $\{A_\beta : \beta < \mathfrak{c}\}$ enumerates the family $[Q_1]^\omega \cup [Q_2]^\omega \cup [Q_3]^\omega$ and that $\{H_\beta : \beta < \mathfrak{c}\}$ enumerates the family of subsets $H$ of $D$ that satisfy that $H \cap D_n$ is finite for each $n \in \omega$.

For convenience, $\mathcal{I}_\alpha^{\mathbb{Q}} = \bigcup\{\mathcal{I}_\alpha^q : q \in \mathbb{Q}\}$ and, for each $i \in \{1,2,3\}$, let $\mathcal{I}_\alpha^{Q_i} = \bigcup\{\mathcal{I}_\alpha^q : q \in Q_i\}$. Since the partition $Q_1, Q_2, Q_3$ of $\mathbb{Q}$ is fixed, we can let $\tau_\alpha^{\mathbb{Q}}$ denote the union $\tau_\alpha^1 \cup \tau_\alpha^2 \cup \tau_\alpha^3$ and understand that each $\tau_\alpha^i$ is reconstructible from $\tau_\alpha^{\mathbb{Q}}$. With this convention, it is easy to describe that $(\tau_\gamma^{\mathbb{Q}}, \mathcal{I}_\gamma^{\mathbb{Q}})$ is an extension of $(\tau_\beta^{\mathbb{Q}}, \mathcal{I}_\beta^{\mathbb{Q}})$ simply to mean that $\tau_\beta^{\mathbb{Q}} \subset \tau_\gamma^{\mathbb{Q}}$, $\mathcal{I}_\beta^{\mathbb{Q}} \subset \mathcal{I}_\gamma^{\mathbb{Q}}$ and that induction hypotheses (1)-(6) are satisfied for each.

Now add the next three inductive assumptions for $\beta + \omega < \alpha < \mathfrak{c}$:

(7) if $A_\beta \subset Q_i$ and $q \in Q_i$ is a limit point of $A_\beta$ with respect to the topology $\tau_{\beta+\omega}^i$ , then there is an $I \in \mathcal{I}_{\beta+\omega}^q$ such that $I \cap A_\beta$ is infinite,
(8) The set $H_\beta$ has no limit points in $Q_1 \times Q_2 \times Q_3$ with respect to the product topology $\tau_{\beta+\omega}^1 \times \tau_{\beta+\omega}^2 \times \tau_{\beta+\omega}^3$.
(9) If $\{k : H_\beta \cap D_k = \emptyset\}$ is infinite, there is an infinite sequence $\{k_\ell : \ell \in \omega\} \subset \{k : H_\beta \cap D_k = \emptyset\}$ such that the set

$$\pi_{1,2}\left[\bigcup\{H_\beta \cap D_n : \ (\exists \ell \in \omega)\ (k_{2\ell} \leq n < k_{2\ell+1})\}\right]$$

has no limit points in $Q_1 \times Q_2$ with respect to the topology $\tau^1_{\beta+\omega} \times \tau^2_{\beta+\omega}$.

Let $IH(\alpha, 1{-}6)$ denote the statement that the induction hypotheses 1-6 are satisfied for all $\beta < \alpha$. Similarly define $IH(\alpha, 1{-}e)$ $(e \in \{8, 9\})$.

If we succeed in extending the induction out to $\mathfrak{c}$, then let $\tau^i_{\mathfrak{c}} = \bigcup\{\tau^i_\alpha : \alpha < \mathfrak{c}\}$ and we have

(1) Induction hypotheses (3), (7) ensure that each $(Q_i, \tau^i_{\mathfrak{c}})$ is Fréchet.
(2) Induction hypothesis (6) ensures that $\{D_n : n \in \omega\}$ is a family of dense subsets of $Q_1 \times Q_2 \times Q_3$.
(3) Induction hypothesis (8), combined with the previous item, ensures that this product is not $M$-separable.

We also have that the product $Q_1 \times Q_2$ is not mH-separable as we now check. Suppose that $\langle F_n : n \in \omega\rangle$ is a sequence of finite sets satisfying that $F_n \subset \bigcup\{D_m : n \leq m \in \omega\}$ for all $n$. Assume also that $F_n \cap D_n$ is non-empty for all $n$. Recursively choose an increasing sequence $\{m_\ell : \ell \in \omega\}$ so that, for each $n < m_\ell$, $F_n \subset \bigcup\{D_m : m < m_{\ell+1}\}$. Choose $\beta < \mathfrak{c}$ so that $H_\beta = \bigcup_n F_n \setminus \bigcup_\ell D_{m_{2\ell+1}}$. Clearly the set of $k$ such that $H_\beta \cap D_k = \emptyset$ is equal to $\{m_{2\ell+1} : \ell \in \omega\}$. By Induction hypothesis (9) we have the infinite sequence $\{k_\ell : \ell \in \omega\}$ contained in $\{m_{2\ell+1} : \ell \in \omega\}$. For each $k_{2\ell} = m_{2\ell'+1}$, the interval $[m_{2\ell'+2}, m_{2\ell'+3})$ is contained in $[k_{2\ell}, k_{2\ell+1})$. Therefore $\bigcup\{H_\beta \cap D_n : (\exists \ell \in \omega)\ (k_{2\ell} \leq n < k_{2\ell+1})\}$ contains $F_n$ for infinitely many $n$. Similarly $\pi_{1,2}\left[\bigcup\{H_\beta \cap D_n : (\exists \ell \in \omega)\ (k_{2\ell} \leq n < k_{2\ell+1})\}\right]$ contains $\pi_{1,2}[F_n]$ for infinitely many $n$. In other words the selection $\{\pi_{1,2}[F_n] \subset \pi_{1,2}[\bigcup\{D_m : n \leq n \in \omega\}]$ fails to satisfy the requirement for the mH-separable property.

Now we make two trivial assumptions for convenience that in no way affect the outcome. For each successor $\beta < \mathfrak{c}$, $A_\beta = Q_1$ and $H_\beta$ is empty. For each limit $\beta$, one of $A_\beta$ and $H_\beta$ is empty. With these innocuous assumptions, we have, for each limit $\beta$, one task that must be completed by stage $\beta + \omega$.

We handle each of the three types, (7), (8), and (9), of inductive steps in their own section.

We note that for limit $\alpha \leq \mathfrak{c}$, the induction hypotheses are preserved if we simply set $\mathcal{I}^q_\alpha = \bigcup\{\mathcal{I}^q_\xi : \xi < \alpha\}$ for all $q \in \mathbb{Q}$, and $\tau^i_\alpha = \bigcup \tau^i_\xi : \xi < \alpha\}$ for $i \in \{1, 2, 3\}$.

## 4. Safely adding converging sequences and clopen sets

In this section we introduce our method for adding new members to any $\tau^i_\beta$. The procedure will be used again with milder assumptions on the set $B$.

The following observation, a strengthening of Arhangelskii's $\alpha_1$-property for first countable spaces, has proven useful.

**Proposition 5.** *Let $\mathcal{I}$ be a family of sequences in a countable space $X$ all converging to a single point $x$ that has countable character. If $\mathcal{I}$ has cardinality less than $\mathfrak{b}$, then there is a single sequence $S$ converging to $x$ that mod finite contains every member of $\mathcal{I}$.*

*Proof.* Fix a descending neighborhood basis, $\{U_n : n \in \omega\}$, for $x$ with $U_0 = X$. For each $n \in \omega$, let $X_n = U_n \setminus U_{n+1}$. There is nothing to prove if $x$ has a neighborhood that is simply a converging sequence, so we may assume that each $X_n$ is infinite. For each $n \in \omega$, choose an enumeration, $\{x(n, m) : m \in \omega\}$ of $X_n$. For each $I \in \mathcal{I}$, there is a function $f_I \in \omega^\omega$ satisfying that $I \subset \bigcup_n \{x(n,m) : m < f_I(n)\}$. Therefore, if $|\mathcal{I}| < \mathfrak{b}$, we may choose a function $f \in \omega^\omega$ so that $f$ is eventually larger than each $f_I$. It is easy to check that $S = \bigcup_n \{x(n,m) : m < f(n)\}$ is a sequence that converges to $x$ and which satisfies that $I \setminus S$ is finite for all $I \in \mathcal{I}$. $\square$

For an ideal $\mathcal{I}$ of subsets of a set $X$ (the value of $X$ should be clear from the context), we let $\mathcal{I}^\perp$ denote the ideal of all subsets of $X$ that are almost disjoint from every member of $\mathcal{I}$.

**Lemma 6.** *Assume $IH(\beta+1, 1\text{–}6)$. Suppose that $B \subset Q_i$ ($i \in \{1,2,3\}$) satisfies that $B \in \left(\mathcal{I}_\beta^{Q_i}\right)^\perp$ and $\tilde{B} \cap \tilde{J} \cap D_n$ is finite for all $n \in \omega$ and $J \in \mathcal{I}_\beta^{\mathbb{Q}}$.*

*Then there is a function $\varphi_\beta : Q_i \to \omega$ such that*

1. *$\varphi_\beta \restriction B$ is 1-to-1,*
2. *for all $q \in Q_i$ and $I \in \mathcal{I}_\beta^q$, $\varphi_\beta[I]$ "converges to" $\varphi_\beta(q)$ (with respect to the discrete topology on $\omega$).*
3. *$IH(\beta+2, 1\text{–}6)$ holds after we set*
   (a) *$\mathcal{I}_{\beta+1}^q = \mathcal{I}_\beta^q$ for all $q \in \mathbb{Q}$,*
   (b) *$\tau_{\beta+1}^j = \tau_\beta^j$ for $i \neq j \in \{1,2,3\}$,*
   (c) *$\tau_{\beta+1}^i$ is the topology generated by $\tau_\beta^i \cup \{\varphi_\beta^{-1}(m) : m \in \omega\}$.*

*Proof.* It will be more convenient notationally to assume that $i = 1$. Begin by applying Proposition 5 to choose, for each $r \in Q_1$ a sequence $I_\beta^r \subset Q_1$ converging to $r$ with respect to $\tau_0^1$ satisfying that $I \subset^* I_\beta^r$ for all $I \in \mathcal{I}_\beta^r$. Remove a finite set from each such $I_\beta^r$ so as to ensure that the family $\{I_\beta^r : r \in Q_1\}$ is a pairwise disjoint family. Also ensure that $I_\beta^r$ is disjoint from $B$, which we may do because of the assumption that $B \in \left(\mathcal{I}_\beta^r\right)^\perp$.

Next we will choose $L_\beta^r \subset I_\beta^r$ so that

1. $L_\beta^r \cap I$ is finite for all $I \in \mathcal{I}_\beta^r$,
2. for all $J \in \mathcal{I}_\beta^{Q_2} \cup \mathcal{I}_\beta^{Q_3}$, if $\pi_1[\tilde{J}] \cap I_\beta^r$ is infinite, then $L_\beta^r \cap \pi_1[\tilde{J}]$ is also infinite.

This is a simple application of the fact that $\mathcal{I}_\beta^{\mathbb{Q}}$ has cardinality less than $\mathfrak{p}$ together with the fact, by induction hypothesis (5), that $\pi_1[\tilde{J}] \cap I$ is finite for all $J \in \mathcal{I}_\beta^{Q_2} \cup \mathcal{I}_\beta^{Q_3}$ and $I \in \mathcal{I}_\beta^r$. There is no loss to assuming that each $L_\beta^r$ is infinite.

Fix an enumeration $\{r_\ell : \ell \in \omega\}$ of $Q_1$. Let $\varphi_\beta^0$ be any 1-to-1 function from $B \cup \{r_0\}$ into $\omega$. We "Cohen generically" perform a recursion to define an increasing sequence of functions $\varphi_\beta^k$ ($k < \omega$) from subsets of $Q_1$ to $\omega$. To make the Cohen genericity precise, choose any elementary submodel $M_\beta$ of cardinality less than $\mathfrak{c}$ that contains $\{Q_1, Q_2, Q_3, \} \cup \{D_n : n \in \omega\}$ and every element of $\mathcal{I}_\beta^{\mathbb{Q}}$ and $\tau_\beta^{\mathbb{Q}}$. Also ensure that $\{I_\beta^r : r \in Q_1\} \cup \{L_\beta^r : r \in Q_1\}$ is an element of $M_\beta$. Let $g_\beta$ be any function from $\omega$ to $\omega$ that is generic over $M_\beta$ with respect to the poset $\omega^{<\omega}$.

The domain of $\varphi_\beta^k$ will equal $B \cup \{r_\ell : \ell \leq k\} \cup \bigcup \{I_\beta^{r_\ell} \setminus L_\beta^{r_\ell} : \ell < k\}$ union some finite subset of $Q_1$. The inductive hypothesis is that, similar to condition (2) of the statement of the Lemma, $\varphi_\beta^k[I^*] = \varphi_\beta^k(r)$ for some cofinite $I^* \subset I_\beta^r \setminus L_\beta^r$, holds for each $r \in \{r_\ell : \ell < k\}$. We may note here that $L_\beta^r \cap dom(\varphi_\beta^k)$ will be finite for all $r \in Q_1$ and $k \in \omega$.

Now we describe how to define $\varphi_\beta^{k+1}$. If $r_{k+1} \notin dom(\varphi_\beta^k)$, then set $\varphi_\beta^{k+1}(r_{k+1}) = g_\beta(k+1)$. Next, let $\{r_{\ell_i} : i < g_\beta(k)\}$ denote the first $g_\beta(k)$ many elements of of

$Q_1 \setminus (\{r_{k+1} \cup dom(\varphi_\beta^k))$. Define $\varphi_\beta^{k+1}(r_{\ell_i}) = g_\beta(\ell_i)$ for each $i < g_\beta(k)$. Finally, have $\varphi_\beta^{k+1}$ send the remainder of $I_\beta^{r_k} \setminus L_\beta^{r_k}$ to the value $\varphi_\beta^{k+1}(r_k)$.

The purpose of the Cohen generic choices (the details of which we skip) together with the choices of $L_\beta^r$ ($r \in Q_1$) is simply this: Suppose that $J \in \mathcal{I}_\beta^{Q_2} \cup \mathcal{I}_\beta^{Q_3}$ satisfies that $\tilde{J} \cap D_n$ is infinite for some $n \in \omega$. Then $\pi_1[\tilde{J} \cap D_n] \cap B$ is finite and so, for any $m \in \omega$, some infinite subset of $\pi_1[\tilde{J} \cap D_n]$ will be sent to $m$ by $\varphi_\beta = \bigcup_k \varphi_\beta^k$. The reason this is true is that $\pi_1[\tilde{J} \cap D_n] \setminus dom(\varphi_\beta^k)$ is infinite for every $k$ because either $\pi_1[\tilde{J} \cap D_n]$ meets $I_\beta^{r_k}$ in a finite set, or $\pi_1[\tilde{J}]$ meets $L_\beta^{r_k}$ in an infinite set. It is easily seen that this same genericity will ensure that $\varphi^{-1}(m)$ is a $\tau_\beta^1$-dense subset of $Q_1$.

The only induction hypothesis that needs checking for $IH(\beta+2, 1\text{–}6)$ is number (6). However this is routine. Let $U_1 \in \tau_\beta^1$, $U_2 \in \tau_\beta^2$ and $U_3 \in \tau_\beta^3$. We must check the property in (6) with respect to $U = U_1 \cap \varphi_\beta^{-1}(m)$ for any $m$. Referring to the statement in (6), there are two cases $i_q = 1$ and $i_q = 2$ (because $i_q = 3$ follows the same proof as for $i_q = 2$). In the case that $i_q = 1$ there is nothing new that needs proving (because $\mathcal{I}_\beta^q \subset \mathcal{I}_{\beta+1}^q$). Now let $i_q = 2$ and choose a $J \in \mathcal{I}_\beta^q$ such that $J$ meets $\pi_2[D_n \cap \tilde{U}_1 \cap \tilde{U}_3]$ in an infinite set. Equivalently $\pi_1[\tilde{J} \cap D_n \cap \tilde{U}_1 \cap \tilde{U}_3]$ is an infinite subset of $\pi_1[\tilde{J} \cap D_n]$. Again, by Cohen genericity, we have that $\varphi^{-1}(m)$ will contain an infinite subset of $\pi_1[\tilde{J} \cap D_n \cap \tilde{U}_1 \cap \tilde{U}_3]$. This is equivalent to the statement that $J$ meets the set $\pi_2[D_n \cap \tilde{U} \cap \tilde{U}_3]$ as required. □

**Corollary 7.** *Assume $IH(\beta+1, 1\text{–}6)$. Suppose that $B \subset Q_i$ satisfies that*
(1) *$\tilde{B}$ converges in the product space $[0,1]^{\{1,2,3\}}$,*
(2) *$B$ converges to $q \in Q_i$ with respect to $\tau_\beta^i$,*
(3) *$\pi_j[\tilde{B}]$ is in $\left(\mathcal{I}^{Q_j}\right)^\perp$ for each $j = 1, 2, 3$.*

*Then there are $\varphi_\beta : Q_{j_2} \to \omega$ and $\psi_\beta : Q_{j_3} \to \omega$ so that, with $\{i, j_2, j_3\} = \{1, 2, 3\}$, $IH(\beta+2, 1\text{–}6)$ holds after we set*
(1) *$\mathcal{I}_{\beta+1}^r = \mathcal{I}_\beta^r$ for all $q \neq r \in \mathbb{Q}$,*
(2) *$\mathcal{I}_{\beta+1}^q = \{B\} \cup \mathcal{I}_\beta^q$,*
(3) *$\tau_{\beta+1}^i = \tau_\beta^i$*
(4) *$\tau_{\beta+1}^{j_2}$ is the topology generated by $\tau_\beta^{j_2} \cup \{\varphi_\beta^{-1}(m) : m \in \omega\}$,*
(5) *$\tau_{\beta+1}^{j_3}$ is the topology generated by $\tau_\beta^{j_2} \cup \{\varphi_\beta^{-1}(m) : m \in \omega\}$*

*Proof.* First apply Lemma 6 in two consecutive steps to obtain $\varphi_\beta$ and then $\psi_\beta$. This gives the new topologies $\tau_{\beta+1}^{j_2}$ and $\tau_{\beta+1}^{j_3}$ so that $\pi_j[\tilde{B}]$ is closed and discrete for $j = j_2, j_3$. Finally, the assumptions on $B$ now ensure that $B$ can be added to $\mathcal{I}_{\beta+1}^q$. □

## 5. Ensure Fréchet in each coordinate

In this section we assume that the induction hypotheses hold at stage $\alpha$ and that $A_\alpha$ is a subset of $Q_1$. The case when $A_\alpha$ is a subset of any other $Q_i$ is completely symmetric and it is notationally clearer to be specific about the value of $i$.

Fix an enumeration $\{r_\ell : \ell \in \omega\}$ of $Q_1$. We will take $\omega$ many steps (from $\alpha$ up to $\alpha + \omega$) so as to ensure that either $r_\ell$ is not in the closure of $A_\alpha$ (with respect to $\tau_\alpha^1$ in fact) or there is an $I \subset A_\alpha$ which is an element of $\mathcal{I}_{\beta+\ell+1}^{r_\ell}$. This will require that we enlarge each of $\tau^2_{\beta+\ell}$ and $\tau^3_{\beta+\ell}$ so as to ensure induction hypothesis (5) remains valid. If we complete this process, we obtain that induction hypothesis (7) is extended. No changes are needed for induction hypotheses (8,9). Induction hypothesis (6) will be retained in the process because we will be applying Corollary 7 at each step.

**Lemma 8.** *Let $\ell < \omega$ and assume $IH(\alpha+\ell+1, 1\text{–}6)$. Assume that $A_\alpha \in \left(\mathcal{I}^{r_\ell}_{\alpha+\ell}\right)^\perp$ and $r_\ell$ is a $\tau^1_{\alpha+\ell}$-limit point of $A_\alpha$. Then there is an extension $(\tau^{\mathbb{Q}}_{\alpha+\ell+1}, \mathcal{I}^{\mathbb{Q}}_{\alpha+\ell+1})$ of $(\tau^{\mathbb{Q}}_{\alpha+\ell}, \mathcal{I}^{\mathbb{Q}}_{\alpha+\ell})$ so that $IH(\alpha+\ell+2, 1\text{–}6)$ holds and there is an $I \in \mathcal{I}^{r_\ell}_{\alpha+\ell+1}$ that is a subset of $A_\alpha$.*

*Proof.* First choose any infinite $B \subset Q_1$ that is a pseudointersection of the filter base $\{U \cap A_\alpha : r_\ell \in U \in \tau^1_{\alpha+\ell}\}$. By passing to a subsequence we can assume that $\tilde{B}$ converges in $[0,1]^{\{1,2,3\}}$. Since $B$ is a pseudointersection of the neighborhood base for $r_\ell$, it is clear that $B$ converges to $r_\ell \in Q_1$ with respect to $\tau^1_{\alpha+\ell}$. We check that $\pi_j[\tilde{B}]$ is in $\left(\mathcal{I}^{Q_j}_{\alpha+\ell}\right)^\perp$ for each $j = 1,2,3$. The first case is that $\pi_1[\tilde{B}] = B$ is in $\left(\mathcal{I}^{r_\ell}_{\alpha+\ell}\right)^\perp$ because we assume that $A_\alpha \in \left(\mathcal{I}^{r_\ell}_{\alpha+\ell}\right)^\perp$. For any $j \in \{1,2,3\}$ and $J \in \mathcal{I}^r_{\alpha+\ell}$ with $r \neq r_\ell$, we have that $r_\ell$ is not a $\tau^1_{\alpha+\ell}$-limit of $\pi_1[\tilde{J}]$. Therefore there is some $r_\ell \in U \in \tau^1_{\alpha+\ell}$ that is almost disjoint from $\pi_1[\tilde{J}]$. This implies that $B$ is almost disjoint from $\pi_1[\tilde{J}]$, which, by symmetry, implies that $\pi_j[\tilde{B}]$ is almost disjoint from $J$.

We finish the proof of the Lemma by applying Corollary 7. □

## 6. Ensure M-separable fails for the product

In this section we assume that, for a limit ordinal $\alpha < \mathfrak{c}$, $\langle(\tau^{\mathbb{Q}}_\beta, \mathcal{I}^{\mathbb{Q}}_\beta) : \beta \leq \alpha\rangle$ is a system that satisfies $IH(\alpha, 1\text{–}6)$ and that $H_\alpha$ is an infinite subset of $D$ and we recall that $H_\alpha \cap D_n$ is finite for each $n \in \omega$.

Two ideals $\mathcal{I}_1, \mathcal{I}_2$ on $\mathcal{P}(\omega)$ are orthogonal if $I_1 \cap I_2$ is finite for all $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. We will use the following notion from [1].

**Definition 2.** A finite family $\{\mathcal{I}_i : i \in n\}$ of mutually orthogonal ideals of $\mathcal{P}(\omega)$ is an *$n$-gap* if for every collection $\{C_i : i \in n\} \subset \mathcal{P}(\omega)$ satisfying that $I \subset^* C_i$ for all $i \in n$ and $I \in \mathcal{I}_i$, we have that $C_0 \cap C_2 \cap \cdots \cap C_{n-1}$ is infinite.

**Proposition 9** ([1, Corollary 21])**.** *The hypothesis $MA_\theta$($\sigma$-$k$-linked) implies that for $n > k$ there exist no $n$-gaps of $\theta$-generated ideals in $\mathcal{P}(\omega)/fin$.*

Of course Martin's Axiom implies $\mathrm{MA}_\theta$($\sigma$-$k$-linked) for all $k \in \omega$ and $\theta < \mathfrak{c}$.

**Lemma 10.** *The set of ideals $\{\mathcal{I}_i : i \in \{1,2,3\}\}$ is mutually orthogonal, where, for each $i \in \{1,2,3\}$, $\mathcal{I}_i$ is the ideal of subsets of $H_\alpha$ generated by the family $\{\tilde{I} : I \in \mathcal{I}^{Q_i}_\alpha\}$.*

*Proof.* Let $1 \leq j_1 < j_2 \leq 3$ and suppose that $r_1 \in Q_{j_1}$ and $r_2 \in Q_{j_2}$. Furthermore assume that $I \in \mathcal{I}^{r_1}_\alpha$ and $J \in \mathcal{I}^{r_2}_\alpha$. By induction hypothesis (4), $\pi_{j_1}[\tilde{J}]$ is closed and discrete with respect to $\tau^{j_1}_\alpha$, while, by (3), $I$ is a converging sequence. Therefore $I \cap \pi_{j_1}[\tilde{J}]$ is finite. Since $\pi_{j_1}$ is 1-to-1 and $\pi_{j_1}[\tilde{I}] = I$, it follows that $\tilde{I} \cap \tilde{J}$ is finite. Since these are arbitrary generators of the ideals $\mathcal{I}_{j_1}$ and $\mathcal{I}_{j_2}$, this proves the Lemma. □

Of course it follows from Proposition 9 that $\{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$ of Lemma 10 is not a 3-gap. So therefore, we may choose sets $C_1, C_2, C_3$ satisfying that $C_1 \cap C_2 \cap C_3 = \emptyset$, and, for each $I \in \mathcal{I}_\alpha^{Q_i}$ and $i \in \{1,2,3\}$, $H_\alpha \cap \tilde{I} \subset^* C_i$, and (with no loss of generality) $C_1 \cup C_2 \cup C_3 = H_\alpha$.

**Proposition 11.** *For $i \in \{1,2,3\}$ and $B_i = \pi_i[H_\alpha \setminus C_i]$, we have that $B_i \in \left(\mathcal{I}_\alpha^{Q_i}\right)^\perp$ and $\tilde{B}_i \cap \tilde{J} \cap D_n$ is finite for all $n \in \omega$ and $J \in \mathcal{I}_\alpha^{\mathbb{Q}}$.*

*Proof.* First observe that $\tilde{I} \cap H_\alpha \subset^* C_i$ for all $I \in \mathcal{I}_\alpha^{Q_i}$. This is equivalent to the desired conclusion that $B_i \in \left(\mathcal{I}_\alpha^{Q_i}\right)^\perp$. In fact, let us highlight the fact that $\tilde{B}_i$ is equal to $H_\alpha \setminus C_i$ and recall that $H_\alpha \cap D_n$ is finite for all $n \in \omega$. Now suppose that $J \in \mathcal{I}_\alpha^{\mathbb{Q}}$. It then follows trivially that $\tilde{B}_i \cap (\tilde{J} \cap D_n)$ is finite for all $n \in \omega$. □

**Lemma 12.** *There is a sequence of extension $\langle(\tau_{\alpha+i}^{\mathbb{Q}}, \mathcal{I}_{\alpha+i}^{\mathbb{Q}}) : i = 1,2,3\rangle$ of $(\tau_\alpha^{\mathbb{Q}}, \mathcal{I}_\alpha^{\mathbb{Q}})$ such that there are functions $\varphi_\alpha^i : Q_i \to \omega$, for $i = 1,2,3$ satisfying*

1. *for all $q \in \mathbb{Q}$, $\mathcal{I}_{\alpha+3}^q = \mathcal{I}_\alpha^q$,*
2. *$\varphi_\alpha^i \restriction B_i$ is 1-to-1 where $B_i = \pi_i[H_\alpha \setminus C_i]$,*
3. *$H_\alpha \cap \tilde{I} \subset^* C_i$ for all $I \in \mathcal{I}_\alpha^{\mathbb{Q}}$,*
4. *$C_1 \cap C_2 \cap C_3$ is empty and $H_\alpha \subset C_1 \cup C_2 \cup C_3$,*
5. *$\varphi_\alpha^i$ is continuous with respect to $\tau_{\alpha+3}^i$,*
6. *$IH(\alpha+4, 1\text{–}6)$ holds.*

*Proof.* By induction on $i = \{1,2,3\}$, apply Lemma 6 with $B = B_i$ and $\beta = \alpha + (i{-}1)$, to obtain $\varphi_\alpha^i$ to be $\varphi_\beta$ in the statement of the Lemma. Define $(\tau_{\alpha+i}^{\mathbb{Q}}, \mathcal{I}_{\alpha+i}^{\mathbb{Q}})$ as described in Lemma 6. □

**Corollary 13.** *For limit $\alpha < \mathfrak{c}$ for which $H_\alpha$ is an infinite subset of $D$ and the system $\langle(\tau_\beta^{\mathbb{Q}}, \mathcal{I}_\beta^{\mathbb{Q}}) : \beta \le \alpha\rangle$ satisfies $IH(\alpha+1, 1\text{–}9)$, there is an extension $\langle(\tau_\beta^{\mathbb{Q}}, \mathcal{I}_\beta^{\mathbb{Q}}) : \beta \le \alpha+3\rangle$ satisfying $IH(\alpha+4, 1\text{–}8)$.*

*Proof.* First choose $(\tau_{\alpha+3}^{\mathbb{Q}}, \mathcal{I}_{\alpha+3}^{\mathbb{Q}})$ together with the functions $\varphi_\alpha^i$ $(i = 1,2,3)$ as in Lemma 12. We have to show that induction hypothesis (8) holds, namely that, that $H_\alpha$ is closed and discrete in $Q_1 \times Q_2 \times Q_3$ with respect to the topology $\tau_{\alpha+3}^1 \times \tau_{\alpha+3}^2 \times \tau_{\alpha+3}^1$. For each $i = 1,2,3$, the function $\varphi_\alpha^i \circ \pi_i$ is a continuous function from $Q_1 \times Q_2 \times Q_3$ into $\omega$. For each $i = 1,2,3$, $\varphi_\alpha^i \circ \pi_i \restriction (H_\alpha \setminus C_i)$ is 1-to-1. Therefore, for $i = 1,2,3$, $H_\alpha \setminus C_i$ is closed and discrete in $Q_1 \times Q_2 \times Q_3$. Of course this implies that $H_\alpha$ is closed and discrete since $H_\alpha = H_\alpha \setminus (C_1 \cap C_2 \cap C_3)$ and $H_\alpha \setminus (C_1 \cap C_2 \cap C_3) = (H_\alpha \setminus C_1) \cup (H_\alpha \setminus C_2) \cup (H_\alpha \setminus C_3)$. □

## 7. Ensure the failure of mH-separable for the product of two

In this section, we continue from Section 6, and we assume that $H_\alpha$ is not empty and that $\langle(\tau_{\alpha+i}^{\mathbb{Q}}, \mathcal{I}_{\alpha+i}^{\mathbb{Q}}) : i = 1,2,3\rangle$ have been defined as in Lemma 12 so that $IH(\alpha, 1{-}9)$ and $IH(\alpha+4, 1-8)$ hold. Suppose further that $\{k \in \omega : H_\alpha \cap D_k = \emptyset\}$ is infinite and let $\{m_\ell : \ell \in \omega\}$ enumerate this set. For each $\ell$, let $h_\ell$ be the finite set $H_\alpha \cap \bigcup\{D_n : m_\ell \le n < m_{\ell+1}\}$.

**Lemma 14.** *There is an infinite set $L \subset \omega$ and a partition $C_1, C_2$ of $H_L = \bigcup\{h_\ell : \ell \in L\}$ satisfying that $H_L \cap \tilde{I} \subset^* C_1$ for all $I \in \mathcal{I}_{\alpha+3}^{Q_1}$, and $H_L \cap \tilde{I} \subset^* C_2$ for all $I \in \mathcal{I}_{\alpha+3}^{Q_2}$.*

*Proof.* Given an infinite set $L$, let $\mathcal{Q}_L$ denote the poset consisting of the set of triples $(j, c, \mathcal{I})$ where $j \in \omega$, $c \subset \bigcup\{h_\ell : \ell \in L \cap j\}$, and $\mathcal{I}$ is a finite subset of $\mathcal{I}_\alpha^{Q_1} \cup \mathcal{I}_\alpha^{Q_2}$ satisfying that $\tilde{I}_1 \cap \tilde{I}_2 \cap h_\ell$ is empty for all $\ell \in \omega \setminus j$. A condition $(j_2, c_2, \mathcal{I}_2)$ extends $(j, c, \mathcal{I})$ providing

(1) $j \leq j_2$, $c_2 \cap \bigcup\{H_\ell : \ell \in L \cap j\} = c$,
(2) $\mathcal{I}_2 \supset \mathcal{I}$,
(3) $\tilde{I}_1 \cap h_\ell \subset c_2$ for all $I_1 \in \mathcal{I}_\alpha^{Q_1} \cap \mathcal{I}$ and $\ell \in L \cap j_2 \setminus j$,
(4) $\tilde{I}_2 \cap h_\ell$ is disjoint from $c_2$ for all $I_2 \in \mathcal{I}_\alpha^{Q_2} \cap \mathcal{I}$ and $\ell \in L \cap j_2 \setminus j$.

The poset $\mathcal{Q}_L$ is a standard poset for adding a separation of two orthogonal ideals. However it is not always ccc.

If we let $\dot{L}$ denote the set $\dot{g}^{-1}(1)$ where $\dot{g}$ is a generic for the standard poset $2^{<\omega}$, then we check that $2^{<\omega} * \mathcal{Q}_{\dot{L}}$ is ccc. It should be clear that this will then prove the Lemma. The elements of $\mathcal{Q}_{\dot{L}}$ are simply elements of $\mathcal{Q}_\omega$ but the ordering on $\mathcal{Q}_{\dot{L}}$ depends on the generic. To prove that $2^{<\omega} * \mathcal{Q}_{\dot{L}}$ is ccc, we may pass to the dense set of conditions that have the form $(\sigma, (j, c, \mathcal{I}))$ where $\sigma \in 2^k$ for some $k \geq j$. Let $\{(\sigma_\xi, (j_\xi, c_\xi, \mathcal{I}_\xi)) : 2 < \xi < \omega_1\}$ be any family of such conditions. For each $\xi \in \omega_1$ and $i = 1, 2$, let $I_{\xi,i} = \bigcup\left(\mathcal{I}_\xi \cap \mathcal{I}_\alpha^{Q_i}\right)$. By passing to an uncountable subset we may assume that there are $\sigma, j, c$ so that, for all $2 < \xi < \omega_1$, $\sigma_\xi = \sigma$, $j_\xi = j$, and $c_\xi = c$. Let $k$ be the domain of $\sigma$. Pass to a further uncountable subset so that for any $\xi, \eta$ and any $\ell < k$, $\tilde{I}_{\xi,1} \cap h_\ell = \tilde{I}_{\eta,1} \cap h_\ell$ and $\tilde{I}_{\xi,2} \cap h_\ell = \tilde{I}_{\eta,2} \cap h_\ell$. Set

$$c_2 = c \cup \left(\tilde{I}_{\xi,1} \cap \bigcup\{h_\ell : j \leq \ell \text{ and } \sigma(\ell) = 1\}\right) .$$

Now fix such a distinct pair $\xi, \eta$. Choose $j_2 > k$ large enough so that for each $I_1 \in \mathcal{I}_\alpha^{Q_1} \cap (\mathcal{I}_\xi \cup \mathcal{I}_\eta)$ and $I_2 \in \mathcal{I}_\alpha^{Q_2} \cap (\mathcal{I}_\xi \cup \mathcal{I}_\eta)$, $\tilde{I}_1 \cap \tilde{I}_2 \cap h_\ell$ is empty for all $\ell \geq j_2$. Define $\sigma \subset \tau \in 2^{j_2}$ so that $\tau(\ell) = 0$ for all $k \leq \ell < j_2$. Routine checking shows that $(\tau, (j_2, c_2, \mathcal{I}_\xi \cup \mathcal{I}_\eta))$ is a common extension of each of $(\sigma_\xi, (j_\xi, c_\xi, \mathcal{I}_\xi))$ and $(\sigma_\eta, (j_\eta, c_\eta, \mathcal{I}_\eta))$ in $2^{<\omega} * \mathcal{Q}_{\dot{L}}$.

□

Now we have, completely analogous to Proposition 11, the following result.

**Proposition 15.** *For $i \in \{1, 2\}$ and $B_i = \pi_i[H_L \setminus C_i]$, we have that $B_i \in \left(\mathcal{I}_{\alpha+3}^{Q_i}\right)^\perp$ and $\tilde{B}_i \cap \tilde{J} \cap D_n$ is finite for all $n \in \omega$ and $J \in \mathcal{I}_{\alpha+3}^{\mathbb{Q}}$.*

**Lemma 16.** *There are extensions $(\tau_{\alpha+4}^{\mathbb{Q}}, \mathcal{I}_{\alpha+4}^{\mathbb{Q}})$ and $(\tau_{\alpha+5}^{\mathbb{Q}}, \mathcal{I}_{\alpha+5}^{\mathbb{Q}})$ of $(\tau_{\alpha+3}^{\mathbb{Q}}, \mathcal{I}_{\alpha+3}^{\mathbb{Q}})$ such that there are functions $\varphi_{\alpha+3}^i : Q_i \to \omega$, for $i = 1, 2$ satisfying*

(1) *for all $q \in \mathbb{Q}$, $\mathcal{I}_{\alpha+5}^q = \mathcal{I}_{\alpha+3}^q$,*
(2) *$\varphi_{\alpha+3}^i \restriction B_i$ is 1-to-1 where $B_i = \pi_i[H_L \setminus C_i]$,*
(3) *$H_L \cap \tilde{I} \subset^* C_i$ for all $I \in \mathcal{I}_\alpha^{Q_i}$,*
(4) *$C_1 \cap C_2$ is empty and $H_L \subset C_1 \cup C_2$,*
(5) *$\varphi_{\alpha+3}^i$ is continuous with respect to $\tau_{\alpha+5}^i$,*
(6) *$IH(\alpha + 6, 1–6)$ holds for the system $\langle(\tau_\beta^{\mathbb{Q}}, \mathcal{I}_\beta^{\mathbb{Q}}) : \beta < \alpha + 5\rangle$.*

*Proof.* By induction on $i = \{1, 2\}$, apply Lemma 6 with $B = B_i$ and $\beta = \alpha + (i-1)$, to obtain $\varphi_\alpha^i$ to be $\varphi_\beta$ in the statement of the Lemma. Define $(\tau_{\alpha+i}^{\mathbb{Q}}, \mathcal{I}_{\alpha+i}^{\mathbb{Q}})$ as described in Lemma 6. □

**Corollary 17.** *For limit $\alpha < \mathfrak{c}$ for which $H_\alpha$ is an infinite subset of $D$ and $\langle(\tau^{\mathbb{Q}}_\beta, \mathcal{I}^{\mathbb{Q}}_\beta) : \beta \leq \alpha\rangle$ satisfies $IH(\alpha, 1\text{–}9)$, there is an extension $\langle(\tau^{\mathbb{Q}}_\beta, \mathcal{I}^{\mathbb{Q}}_\beta) : \beta \leq \alpha+\omega\rangle$ satisfying $IH(\alpha+\omega, 1\text{–}9)$.*

*Proof.* First choose $(\tau^{\mathbb{Q}}_{\alpha+i}, \mathcal{I}^{\mathbb{Q}}_{\alpha+i})$ together with the functions $\varphi^i_\alpha$ $(i = 1, 2, 3)$ following the proof of Lemma 12. Following that, we have chosen $H_L$ as in Lemma 14, and apply Lemma 16 to choose $(\tau^{\mathbb{Q}}_{\alpha+i}, \mathcal{I}^{\mathbb{Q}}_{\alpha+i})$ $(i = 4, 5)$. Finally, for all $\alpha+5 \leq \beta \leq \alpha+\omega$, we set $(\tau^{\mathbb{Q}}_\beta, \mathcal{I}^{\mathbb{Q}}_\beta)$ equal to $(\tau^{\mathbb{Q}}_{\alpha+5}, \mathcal{I}^{\mathbb{Q}}_{\alpha+5})$.

We have to show that induction hypothesis (9) holds. It will be sufficient to show that $\pi_{1,2}[H_L]$ is closed and discrete in $Q_1 \times Q_2$ with respect to the topology $\tau^1_{\alpha+\omega} \times \tau^2_{\alpha+\omega}$. For each $i = 1, 2$, the function $\varphi^i_{\alpha+3}$ is a continuous function from $Q_i$ into $\omega$. The sequence $\{\left(\varphi^1_{\alpha+3}\right)^{-1}(m) \times Q_2 : m \in \omega\}$ is a clopen partition of $Q_1 \times Q_2$. Since $\varphi^1_{\alpha+3} \circ \pi_1$ is 1-to-1 on $H_L \setminus C_1$ (as in Lemma 16), it follows that the set $\pi_{1,2}(H_L \setminus C_1)$ has at most one point in common with each element of this partition. Similarly, $\pi_{1,2}(H_L \setminus C_2)$ has at most one point in common with each member of the clopen partition $\{Q_1 \times \left(\varphi^1_{\alpha+3}\right)^{-1}(m) : m \in \omega\}$. Since $C_1, C_2$ is a partition of $H_L$, it follows that $\pi_{1,2}(H_L)$ is the union of two closed discrete subsets of $Q_1 \times Q_2$. □

## 8. OGA and the product of two Fréchet spaces

In this section we are no longer assuming any form of Martin's Axiom and we prove that OGA implies that the product of two countable Fréchet spaces is M-separable. This improves the result in [5] where it was proven under the assumption of PFA. Our current proof is similar to the proof in [5] but is easier to read. See also [11] for a similar application to products of Fréchet spaces. It is noted in [3] that PFA yields the stronger result that the product of two countable Fréchet spaces is R-separable. We do not know if OCA implies the same.

The version of the open graph axiom, referred to as OCA in the initial reference [12], is due to Todorčević. The original OCA, referred to as $\mathrm{OCA}_{ARS}$ in [8], was introduced in [2].

OGA is the assertion that every open graph on a separable metric space is either countably chromatic or else has an uncountable complete subgraph. Here a graph is open if the adjacency relation on the vertex set is topologically open in the product topology.

**Theorem 3** (OGA). *The product of two countable Fréchet spaces is M-separable.*

*Proof.* Let $\tau_1, \tau_2$ be two Fréchet topologies on the underlying space $\omega$. Let $\{E_n : n \in \omega\}$ be dense subsets of the product space $(\omega, \tau_1) \times (\omega, \tau_2)$. It is well-known (and easily checked) that if we let $(x, y) \in \omega^2$ be an arbitrary point and we are able to find a sequence $H_n \subset E_n$ $(n \in \omega)$ of finite sets satisfying that $(x, y)$ is a limit point of the union $\bigcup\{H_n : n \in \omega\}$, then the product is M-separable. So we fix such a pair $(x, y)$.

Next we choose a pair of converging 1-to-1 sequences $\langle x_n : n \in \omega\rangle$ and $\langle y_n : n \in \omega\rangle$ such that the first sequence $\tau_1$-converges to $x$ and the second $\tau_2$-converges to $y$. Now let $\{U^0_n : n \in \omega\} \subset \tau_1$ be pairwise disjoint with $x_n \in U^0_n$ for each $n$. Similarly choose pairwise disjoint $\{W^0_n : n \in \omega\} \subset \tau_2$ such that $y_n \in W^0_n$ for each $n$. Let $D$ be the union of the family $\{E_n \cap (U^0_n \times W^0_n) : n \in \omega\}$.

The fact that the sequence $\langle (x_n, y_n) : n \in \omega \rangle$ converges to $(x, y)$ will play a key role in the proof. Observe the trivial fact that, for each $n \in \omega$, $(x_n, y_n)$ is a limit point of the interior of the closure of the set $D \cap (U_n^0 \times W_n^0)$.

Let $\mathcal{I}_x$ be the set of all infinite sequences that $\tau_1$-converge to $x$. Similarly let $\mathcal{J}_y$ be the set of all infinite sequences that $\tau_2$-converge to $y$. For each $I \in \mathcal{I}_x$, let $\tilde{I} = D \cap (I \times \omega)$. For each $J \in \mathcal{J}_y$, let $\tilde{J} = D \cap (\omega \times J)$.

Suppose there is some $I \in \mathcal{I}_x$ and $J \in \mathcal{J}_y$ such that $(x, y)$ is a limit point of $\tilde{I} \cap \tilde{J}$. Since $\{(x, y)\} \cup \left(\tilde{I} \cap \tilde{J}\right)$ is metrizable, this would imply there is a sequence $S$ from $D$ converging to $(x, y)$. Since $S_n = S \cap (U_n^0 \times W_n^0)$ is finite for each $n$, it follows that $\{S_n : n \in \omega\}$ is the sequence needed for the verification of M-separability.

Therefore we now assume there is no such $I \in \mathcal{I}_x$ and $J \in \mathcal{J}_y$. Since, for such a pair $(I, J)$, the family $\{\{(x, y)\} \cup \left(\widetilde{(I \setminus k)} \cap \widetilde{(J \setminus k)}\right) : k \in \omega\}$ is a neighborhood base for $(x, y)$ in the above mentioned subspace, it follows that for each such $(I, J) \in \mathcal{I}_x \times \mathcal{J}_y$, there is an integer $k$ satisfying that $\widetilde{(I \setminus k)} \cap \widetilde{(J \setminus k)}$ is empty.

Let $\mathcal{X}$ be the family of pairs $(I, J) \in \mathcal{I}_x \times \mathcal{J}_y$ such that $\tilde{I}$ and $\tilde{J}$ are disjoint. The separable metric topology that we place on $\mathcal{X}$ is the standard one where for each integer $n \in \omega$ and subsets $s, t$ of $n$, $[s, t; n] = \{(I, J) \in \mathcal{X} : I \cap n = s \text{ and } J \cap n = t\}$.

We define the graph $G$ to consist of all pairs $\langle (I, J), (I', J') \rangle$ from $\mathcal{X}^2$ that satisfy that at least one of $\tilde{I} \cap \tilde{J}'$ and $\tilde{J} \cap \tilde{I}'$ is non-empty. This graph is open since being an edge depends only on a single element of $D$.

We first show that $G$ is not countably chromatic. Suppose that $\{\mathcal{X}_k : k \in \omega\}$ is a family of induced subgraphs, each containing no edges. Set $S_k = \bigcup\{I : (\exists J)\ (I, J) \in \mathcal{X}_k\}$ and $T_k = \bigcup\{J : (\exists I)\ (I, J) \in \mathcal{X}_k\}$. Note that $D \cap (S_k \times \omega) \cap (\omega \times T_k)$ is empty for all $k$. Let $L_0 = \omega$ and $\{U_n^0, W_n^0 : n \in L_0\}$ be as above. We perform a recursion in which we choose $L_{k+1}$, an infinite subset of $L_k$, and sets $U_n^{k+1}, W_n^{k+1}$ $(n \in L_{k+1})$ where, for all $n \in L_{k+1}$, either

(Case 1) $U_n^{k+1} = U_n^k$ and $W_n^{k+1} = W_n^k \setminus T_k$, or
(Case 2) $U_n^{k+1} = U_n^k \setminus S_k$ and $W_n^{k+1} = W_n^k$.

The inductive assumption is that, for all $n \in L_k$, $(x_n, y_n)$ is a limit point of the interior of the closure of $D \cap \left(U_n^k \times W_n^k\right)$.

Suppose we have so chosen $L_k, \{U_n^k, W_n^k : n \in L_k\}$. Since $D \cap (U_n^k \times W_n^k)$ is equal to the union of $D \cap (U_n^k \times (W_n^k \setminus T_k))$ and $D \cap ((U_n^k \setminus S_k) \times W_n^k)$, $(x_n, y_n)$ is a limit point of the interior of the closure of one of these sets. If the set of $n \in L_k$ such that $(x_n, y_n)$ is a limit point of the interior of the closure of the first of these sets is infinite, then we are in (Case 1) and this is the choice for $L_{k+1}$ and the sequence $\{(U_n^{k+1}, W_n^{k+1}) : n \in L_{k+1}\}$. Otherwise $L_{k+1}$ is the subset of $L_k$ in which (Case 1) condition fails, and we have (Case 2) holding for all $n \in L_{k+1}$.

Having completed the induction, choose a strictly increasing sequence $\{n_k : k \in \omega\}$ so that $n_k \in L_{k+1}$. For each $k$ choose a pair of sequence $I_k, J_k$ so that $I_k \subset U_{n_k}^{k+1}$ converges to $x_{n_k}$ and $J_k \subset W_{n_k}^{k+1}$ converges to $y_{n_k}$.

Since $x$ is a limit point of the set $\bigcup\{I_k : k \in \omega\}$, there is a sequence $I \in \mathcal{I}_x$ that is contained in this union. Let $L' = \{k : I \cap I_k \neq \emptyset\}$ and, by the same argument, there is a sequence $J \subset \bigcup\{J_k : k \in L'\}$ that converges to $y$. By possibly removing a finite set from each of $I$ and $J$, we have that $(I, J) \in \mathcal{X}$. The proof that $\mathcal{X}$ is not countably chromatic is finished if we prove that $(I, J) \notin \mathcal{X}_k$ for any $k \in \omega$. The reason this is true is that, for each $k$, either (Case 1) held at step $k$

and $J \subset^* \bigcup\{W_n^{k+1} : n \in L_{k+1}\}$ is almost disjoint from $T_k$ or (Case 2) held at step $k$ and $I \subset^* \bigcup\{U_n^{k+1} : n \in L_{k+1}\}$ is almost disjoint from $S_k$.

Therefore, from OGA, we can conclude there is an uncountable family $\{(I_\alpha, J_\alpha) : \alpha < \omega_1\} \subset \mathcal{X}$ such that $\langle (I_\alpha, J_\alpha), (I_\beta, J_\beta)\rangle \in G$ for all $\alpha < \beta \in \omega_1$. Now we check that the family $\{(\tilde{I}_\alpha, \tilde{J}_\alpha) : \alpha \in \omega_1\}$ forms a Luzin-type gap in the following sense. For all $\alpha < \omega_1$, $\tilde{I}_\alpha$ and $\tilde{J}_\alpha$ are disjoint (as per the definition of $\mathcal{X}$), while for $\alpha < \beta$, one of $\tilde{I}_\alpha \cap \tilde{J}_\beta$ or $\tilde{I}_\beta \cap \tilde{J}_\alpha$ is not empty (as per the definition of $G$). Every uncountable subset of pairs from a Luzin-type gap is also a Luzin-type gap.

Say that a set $S$ separates a family $\mathcal{A}$ from a family $\mathcal{B}$ if every member of $\mathcal{A}$ is mod finite contained in $S$ and every member of $\mathcal{B}$ is almost disjoint from $S$. It is well-known that the pairs from a Luzin-type gap can not be separated.

We are just a couple of steps away from choosing our sequence of finite sets $\{H_n \subset E_n : n \in \omega\}$. First partition $D$ into $D_0 = \bigcup\{D \cap (\{n\} \times n) : n \in \omega\}$ and $D_1 = D \setminus D_0 = \bigcup\{D \cap (n{+}1 \times \{n\}) : n \in \omega\}$. So $D_0$ is finite in every column (vertical fiber) and $D_1$ is finite in every row (horizontal fiber).

Assume there are $S_i \subset D_i$ $(i = 0, 1)$ that separates the family $\{\tilde{I}_\alpha \cap D_i : \alpha \in \omega_1\}$ and the family $\{\tilde{J}_\alpha \cap D_i : \alpha \in \omega_1\}$. Then clearly $S_0 \cup S_1$ would separate the family $\{\tilde{I}_\alpha : \alpha < \omega_1\}$ and the family $\{\tilde{J}_\alpha : \alpha < \omega_1\}$. Therefore, by symmetry, we may as well assume that the families $\{\tilde{I}_\alpha \cap D_0 : \alpha \in \omega_1\}$ and $\{\tilde{J}_\alpha \cap D_0 : \alpha \in \omega_1\}$ can not be separated. Notice now that $\tilde{I}_\alpha \cap D_0$ is finite in every column. Also, $I_\alpha \cap U_n^0$ is finite for every $n$. Therefore, using that OGA implies that $\mathfrak{b} > \omega_1$, there is a sequence of finite sets $H_n \subset (U_n^0 \times W_n^0) \cap D$, such that $\tilde{I}_\alpha \cap D_0$ is mod finite contained in $\bigcup\{H_n : n \in \omega\}$ for all $\alpha < \omega_1$. Supplying more details, we can let $\{h(n,m) : m \in \omega\}$ be any enumeration of $E_n \supset E_{n,0} = D_0 \cap (U_n^0 \times W_n^0)$ for all $n$. For each $\alpha < \omega_1$, there is a function $f_\alpha \in \omega^\omega$ so that $\tilde{I}_\alpha \cap E_{n,0} \subset \{h(n,m) : m < f_\alpha(n)\}$. We simply choose $f \in \omega^\omega$ so that $f_\alpha <^* f$ for all $\alpha$ and set $H_n = \{h(n,m) : m < f(n)\}$ for all $n \in \omega$. Again we note that $\bigcup_n H_n$ contains, mod finite, every $\tilde{I}_\alpha \cap D_0$.

Now let $x \in U \in \tau_1$ and $y \in W \in \tau_2$. Clearly $U \times \omega$ contains, mod finite, $\tilde{I}_\alpha$ for every $\alpha \in \omega_1$. Choose any $\beta < \omega$ such that $(U \times \omega) \cap \bigcup_n H_n$ meets $\tilde{J}_\beta \cap D_0$ in an infinite set. By removing a finite set from $J_\beta$ we can assume that $J_\beta \subset W$. Pick any $h = (u, w) \in \bigcup_n H_n$ such that $(u, w) \in (U \times \omega) \cap \tilde{J}_\beta$. It follows that $(u, w) \in U \times W$. $\square$

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