The Euclidean Algorithm

The Decanting Problem is a liquid measuring problem that begins with two unmarked decanters with capacities a and b.[1] Usually a and b are integers. The problem is to determine the smallest amount of liquid that can be measured and how such amount can be obtained, by a process of filling, pouring, and dumping. Specifically, there are three actions we can take:

1. fill an empty decanter,
2. dump out a full decanter, and
3. pour from one decanter to the other until either the receiving decanter is full or the poured decanter is empty.

Let's look at an easy one first. Let $a = 3$ and $b = 5$. We can fill the 3 unit decanter twice, and dump the 5 unit decanter once to get 1 unit of liquid. Algebraically, $2 \cdot 3 - 1 \cdot 5 = 1$.

Next, suppose the decanters have capacities 5 units and 7 units. A little experimentation leads to the conclusion that 1 unit of water can be obtained by filling the 5 unit decanter 3 times, pouring repeatedly from the 5 unit to the 7 unit decanter and dumping out the 7 unit decanter twice. A finite state diagram is helpful to follow the procedure:

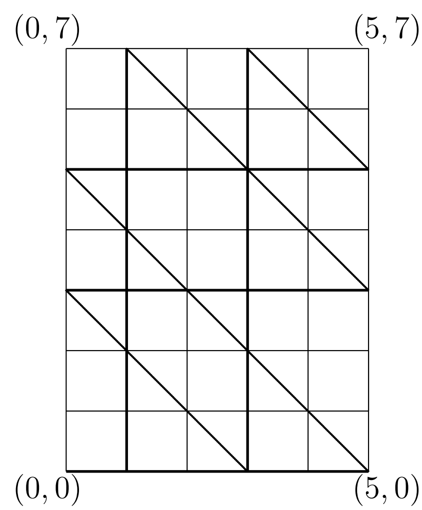
$$(0,0) \Longrightarrow (5,0) \Longrightarrow (0,5) \Longrightarrow (5,5) \Longrightarrow (3,7) \Longrightarrow (3,0)$$

$$\Longrightarrow (0,3) \Longrightarrow (5,3) \Longrightarrow (1,7) \Longrightarrow (1,0),$$

where the notation (x, y) means the 5-unit container has x units of liquid and the 7-unit container has y units. Notice that the procedure includes 3 fills and 2 dumps, with fills and dumps alternating and separated by 4 pours. An arithmetic equation representing this is

$$3 \cdot 5 - 2 \cdot 7 = 1.$$

Notice that not only does the arithmetic equation follow from the state diagram, the reverse is also true. That is, given the arithmetic equation, it is an easy matter to construct the state diagram, shown below:



In the next example, the least amount that can be measured is not 1. Let the decanters have sizes 15 and 99. Before reading on, can you see why it is impossible to obtain exactly one unit of water? An equation can be obtained for any sequence of moves. Such an equation is of the form

$$15x + 99y = z$$

where exactly one of the integers x and y is negative, and z is the amount obtained. Now notice that the left side is a multiple of 3, so the right side must be also. Thus the least positive amount that can be measured is a multiple of 3. One can also reason this as follows: each fill adds a multiple of 3 units of water to the total amount on hand, each pour leaves the total number unchanged, and each dump removes a multiple of three units from the total, so the amount on hand at each stage is a multiple of 3.

In general, when a and b are integers, the least amount that can be measured is the greatest common divisor of the two decanter sizes, and the Euclidean algorithm, as explained below, tells us how to proceed. Suppose $c = GCD(a, b)$. The Euclidean algorithm yields a solution to

$$c = ax + by$$

where x and y are integers exactly one of which is positive and, except in trivial cases, the other is negative. For convenience, we assume x is positive. Then the solution to the decanting problem is to fill the a capacity decanter x times, repeatedly pouring its contents into the b unit decanter. The b unit decanter will be dumped y times, so the total liquid on hand at the end is the difference $ax - by = c$.

Let's look at another specific example. Again we use the Euclidean Algorithm to solve the decanting problem. There are two stages. The first stage is a sequence of divisions. The second is a sequence of 'unwindings'. For this example, let the decanter sizes be $a = 257$ and $b = 341$. Use the division algorithm to get $341 = 1 \cdot 257 + 84$. Then divide 257 by 84 to get $q = 3$ and $r = 5$. That is, $257 = 3 \cdot 84 + 5$. Continue dividing until the dividend is less than the divisor. Thus 84 divided by 5 yields $84 = 16 \cdot 5 + 4$. Finally, divide 5 by 4 to get $5 = 1 \cdot 4 + 1$. This completes the first stage. Now to unwind, start with the final division scheme, writing $1 = 5 - 1 \cdot 4$. Then replace the 4 with $84 - 16 \cdot 5$ to get $1 = 5 - 1(84 - 16 \cdot 5)$. This is equivalent to $1 = 17 \cdot 5 - 1 \cdot 84$. Check this to be sure. Then replace 5 with $257 - 3 \cdot 84$ to get

$$1 = 17 \cdot (257 - 3 \cdot 84) - 1 \cdot 84,$$

i.e., $1 = 17 \cdot 257 - 52 \cdot 84$. Finally, replace 84 with $341 - 257$ to get $1 = 17 \cdot 257 - 52(341 - 257)$, which we can rewrite as

$$1 = 69 \cdot 257 - 52 \cdot 341.$$

Thus, the solution to the decanting problem is to measure out 1 unit of liquid by filling the 257 unit decanter 69 times, repeatedly pouring its contents into the 341 unit decanter, and, in the process, dumping out the 341 unit decanter 52 times.

There is a related problem sometimes called the postage stamp problem. Here we are given an unlimited supply of two denominations of postage, a and b. If a and b are not relatively prime, and say $d > 1$ is the gcd of a and b, then there is no hope of specifying n cents in postage using as and bs unless n is a multiple of d. If a and b are relatively prime, then there is a largest number k that cannot be made. Here we are repeatedly solving the problem $n = ax + by, x, y \geq 0$.

Theorem Suppose m and n are relatively prime positive integers such that $m > n \geq 2$. Then

1. The equation $mx + ny = mn - m - n$ has no solution with $x \geq 0$ and $y \geq 0$.
2. For all $t \geq 1$, the equation $mx + ny = mn - m - n + t$ has a solution with $x \geq 0$ and $y \geq 0$.

Proof

1. Suppose that $mx + ny = mn - m - n$ with $x \geq 0$ and $y \geq 0$. Since $mn - m - n$ is a positive number that is not a multiple of either m or n, both x and y are positive. Next note that $xm = mn - m - n - yn = mn - m - (y + 1)n$ which implies that $m \mid (y + 1)n$. But since m and n are relatively prime, it follows that $m \mid y + 1$. Therefore $y + 1 = cm$ for some $c \geq 1$. It follows that $y = cm - 1$. Likewise there is a $d \geq 1$ such that $x = dn - 1$. Putting all this together, we have $(dn - 1)m + (cm - 1)n = mn - m - n$, and from this it follows that $(c + d)mn = mn$. But this can happen only when $c + d = 1$, a contradiction.
2. We show that for all $t \geq 1$, there exists $x \geq 0$ and $y \geq 0$ such that $mx + ny = mn - m - n + t$. By the Euclidean algorithm, there exists integers $\overline{x}, \overline{y}$ satisfying $m\overline{x} + n\overline{y} = mn - m - n + t$. Therefore, , for all

integers c, $x = \overline{x} + cn, y = \overline{y} - cn$ also satisfy $mx + ny = mn - m - n + t$. Since $n < m$ we can choose c so that $0 \leq x \leq n - 1$, and $mx + ny = mn - m - n + t$. If $y \geq 0$, we are done. Suppose $y \leq -1$. Then $mx + ny \leq (n-1)m - n = mn - m - n$ which contradicts the fact that $mx + ny = mn - m - n + t$, where $t \geq 1$. This complete the proof of the theorem.

Problems

1. Is it possible to express the fraction 1/360 as a sum of three fractions whose denominators are pairwise relatively prime.

2. For each algebraic fraction below, find simple fractions whose sum is the given fraction. In calculus, problems of this sort are called partial fraction problems.

   (a) $8/(x-1)^2(x^2+3)$

   (b) $1/(x-1)(x-2)(x-3)$.

3. For each pair of values s and t below, use repeated division to find $gcd(s,t)$ the greatest common divisor of s and t and then use the Euclidean Algorithm to solve the equation $gcd(s,t) = xs + yt$, where x and y are integers. In other words, solve the decanting problems for decanters of sizes s and t.

   (a) $s = 22$ and $t = 37$

   (b) $s = 105$ and $t = 95$

   (c) $s = 89$ and $t = 144$

   (d) Suppose you have decanters of sizes 99 and 105. Find the least amount of liquid that can be measured, show how to measure that amount, and explain in English why you cannot do better.

4. **Dinner Bill Splitting.** Years ago, my neighbors agreed to celebrate our wedding anniversary with my wife and me. The four of us went to a lovely restaurant, enjoyed a fine dinner, and asked for the bill. When it came, we asked that it be split in half. Realizing the waiter's discomfort, we all set to work on the problem. The bill was for an odd amount, so it could not be split perfectly. However, we realized that, except for the penny problem, we could take half the bill by simply reversing the dollars and the cents. In other words, if we double t dollars and s cents, the result differs by 1 cent from s dollars and t cents. We told the waiter about this. He was astounded: "I never knew you could do it that way." Later, over another dinner with mathematical friends, the question of uniqueness came up, and pretty soon we realized that this number is the only one with this surprising splitting property. What was the amount of the original bill?

5. Chicken McNuggets can be purchased in quantities of 6, 9, and 20 pieces. You can buy exactly 15 pieces by purchasing a 6 and a 9, but you can't buy exactly 10 McNuggets. What is the largest number of McNuggets that can NOT be purchased?

   See `http://www.mathnerds.com/mathnerds/best/mcnuggets/solution.aspx`

   Other McNugget problems.

   (a) Krish Korrapati, a fifth grader at Metrolina Scholars (2020) proposed $21x + 28y + 50z$. What is the largest unachievable value, for $x, y, z \geq 0$?

   (b) Suppose packages come in sizes $12, 16$ and 45. What is the largest non-achievable number?

   (c) For this problem you're given that there are packages of sizes 33 and 44, and the largest unachievable value is 505. What could the size of the third package be?

   (d) Make up your own McNuggets problem, solve it and submit it to me for addition to this paper.

6. Does the equation $399x + 703y = 114$ have an integer solution in x and y?

7. Postage Stamps Revisited. We'‘ve discussed three types of integer problems, decanting, McNuggets, and postage stamps. All three of these involve integer solutions to problems like $ax + by + cz = N$ where a, b, c, and N are positive integers. In case of decanters, we usually have $c = 0$ except on rare occasions like problem the last three problems in this paper. In decanting one of x, y is negative. In McNuggets, we usually have $gcd(a, b, c) = 1$ and the question is What is the smallest non-achievable number, where x, y, and z are nonnegative? In case of postage stamps, we usually have two or three parameters a, b, c and a large N. Here the question is how many solutions, $x, y, z \geq 0$? For example, we have 7 and 9 unit postage stamps, and we need to frank the package with 1000 units. How many ways can we do this?

8. Does the equation $399x + 703y = 115$ have an integer solution in x and y?

9. For m and n integers, characterize those integers k for which the equation $mx + ny = k$ has integer solutions in x and y.

10. **The Subtraction Game** In the Subtraction Game, two players start with some positive integers written on a board. The first player must find a pair of numbers whose positive difference is not already written on the board. Then he writes this new number on the board. At each stage, the next player finds a positive difference between two numbers on the board that is not already written on the board and writes it on the board. The first player who cannot find a new positive difference loses. For each of the sets of numbers listed below, decide how many numbers will be on the board at the end of the game. Use this information to state whether you would like to move first or not (in order to win).

    (a) $101, 102, 103$

    (b) 3105 and 4104

(c) $21, 24, 81, 87$

11. You have three decanters, a 12 cup size, a 7 cup size and a 6 cup size. To begin, there and 12 cups of water in the largest decanter, and the other two decanters are empty. You have to somehow measure out 9 cups of water into the largest decanter. You can't pour any water on the ground and you can't add any water. There are no markings on the decanters, so you can't 'judge' a partial amount. You must pour from one cup into another until either the target cup is full or the cup you are pouring from is empty. So you need to find a sequence of these pourings that will end up with nine cups of water in the large cup.

12. You have three decanters of sizes 7, 13, and 19. The first two are full and the third empty. Your job is to measure out exactly 10 units of liquid.

13. Three baskets contain 6, 7, and 11 marbles. A move consists of moving marbles from one basket to another, but the number of marbles in the basket receiving the marbles must double. For example, the move $(6, 7, 11) \rightarrowtail (12, 1, 11)$ is allowed. Is it possible to arrange exactly eight marbles in each basket?

14. There is a lovely application of the Euclidean algorithm which which every mathematician is familiar. It is in the proof of the Fundamental Theorem of Arithmetic. The FTA asserts that every positive integer bigger than 1 has a unique factorization into prime numbers. This set of exercises is designed to help you understand how the Euclidean algorithm is helpful to this cause. We've seen in this paper that for any two positive relatively prime integers a and b, we can find integers x and y so that $ax+by=1$. This is the decanting problem with decanters of sizes a and b whose gcd is 1. One important application of this result is the following. If p is a prime and $p|bc$ (read p divides bc), then either $p|b$ or $p|c$.

    (a) Prove that if bc is even, then either b is even or c is even.

(b) Prove that if bc is a multiple of 3, the either b or c is a multiple of 3

(c) Prove the result for 5

(d) Show that the results is not true for 6

(e) Prove that any prime p which divides bc must divide b or c.