These problems are harder than those in Place Value and Fusing Dots, but they require the same kind of ideas. [1]

1. **Emphatic Countdown.** Find the decimal representation of $(10/9!) \cdot 8! \cdot 7! - 6! \cdot 5!/4! + 3! \cdot 2! + 1!$.

2. Evaluate the expression $\sqrt{\frac{-1}{64}}$. It describes my behavior in late November.

3. Cross out 10 digits from the number $N = 1234512345123451234512345$ so that what remains is as large as possible.

4. Find all integers n for which the sum of the digits is 11 and the product of the digits is 24.

5. For each positive integer n, let $G(n)$ denote the product of the five consecutive integers beginning at n. For example $G(1) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. Let $R(n)$ denote the rightmost nonzero digit of n when n is written in decimal notation. For example $R(G(1)) = R(120) = 2$.
   (a) Find the smallest n for which $R(G(n))$ is an odd number.
   (b) Find the smallest n for which $R(G(n))$ is 5.
   (c) Find the smallest n for which $R(G(n))$ is the digit 9.
   (d) Find the smallest n for which $R(G(n))$ is the digit 7.

6. What are the largest and the smallest five-digit numbers the product of whose digits is 2520?

7. What is the value of the base 3 repeating decimal $0.\overline{12}_3$ expressed as a base-ten common fraction?

8. The product of two three-digit numbers $\underline{abc}$ and $\underline{cba}$ is 396396, where $c < a$. Find the value of $\underline{abc}$.

9. Multiplicative Palindromes Notice that $69 \times 64 = 46 \times 96$. Its also true that $3516 \times 8274 = 4728 \times 6153$, and $992 \cdot 483 \cdot 156 = 651 \cdot 384 \cdot 299$. What we want to explore here is does the first equation hold for any other pairs of numbers. See the problem on multiplicative palindromes.

10. Suppose a, b and c are three different decimal digits. Then there are six different three digit numbers that can be built using all of the three digits. Suppose the difference between the largest such number and the smallest such number is one of the six numbers. What are the three digits?

11. The product of the number 7 and the six-digit number $\overline{abcdef}$ is the six-digit number $6 \cdot \overline{defabc}$. What is the number $\overline{abcdef}$?

12. **Great Numbers.** Call a nine-digit number *great* if it consists of nine different nonzero digits. It is possible that two great numbers can have a great sum. In fact, it is even possible that two great numbers can both have a great sum and also be identical in seven places. Find two such numbers.

13. **Finding the unknown digit.**
Let $N = \underline{abcde}$ denote the five digit number with digits a, b, c, d, e and $a \neq 0$. Let $N' = \underline{edcba}$ denote the reverse of N. Suppose that $N > N'$ and that $N - N' = 670x3$ where x is a digit. What is x?

14. Recall also that the remainder when a decimal integer is divided by 9 is the same as the remainder when the sum of its digits is divided by 9. For example $1234 = 9k + 1$ since $1 + 2 + 3 + 4 = 10 = 9 + 1$.

    Let $a_1 = 1$ and for each $n \geq 1$, define a_{n+1} as follows:

$$a_{n+1} = n + 1 + a_n \cdot 10^{1+\lfloor \log(n+1) \rfloor}.$$

Let's work out the first few values of the sequence. For example,

$$\begin{aligned} a_2 &= 2 + a_1 \cdot 10^{1+\lfloor \log(n+1) \rfloor} \\ &= 2 + 1 \cdot 10^{1+\lfloor \log(2) \rfloor} \\ &= 2 + 1 \cdot 10^{1+0} \\ &= 2 + 1 \cdot 10 = 12 \end{aligned}$$

and a_3 is

$$\begin{aligned} a_3 &= 3 + a_2 \cdot 10^{1+\lfloor \log(2+1) \rfloor} \\ &= 3 + 12 \cdot 10^{1+\lfloor \log(3) \rfloor} \\ &= 3 + 12 \cdot 10^{1+0} \\ &= 3 + 12 \cdot 10 = 3 + 120 = 123. \end{aligned}$$

Using the definition, we can show that $a_4 = 1234$, etc. This definition is said to be *recursive*. Recursion is a popular topic in discrete mathematics. We'll see later that the Principle of Mathematical Induction is an important tool in reasoning about recursively defined sequences.

(a) How many digits does a_{2008} have?

(b) Find the smallest n such that a_n has at least 2008 digits.

(c) What is $S(a_{2008})$?

(d) How many 1's are in the representation of a_{2008}?

(e) How many 0's are in the representation of a_{2008}?

(f) Find the first 5 multiples of 9 among the a_n.

(g) Find the remainder when a_{2008} is divided by 9.

(h) What is the smallest n such that a_n is a multiple of 99?

15. Let $b_1 = 1$ and for each $n \geq 1$, define b_{n+1} as follows:

$$b_{n+1} = (n+1)^2 + b_n \cdot 10^{1+\lfloor \log(n+1)^2 \rfloor}.$$

Let's work out the first few values of the sequence. The function log refers to the common logarithm, ie. $\log_{10}$. For example,

$$\begin{aligned} b_2 &= 2^2 + b_1 \cdot 10^{1+\lfloor \log(n+1)^2 \rfloor} \\ &= 4 + 1 \cdot 10^{1+\lfloor \log(4) \rfloor} \\ &= 4 + 1 \cdot 10^{1+0} \\ &= 4 + 1 \cdot 10 = 14 \end{aligned}$$

and b_3 is

$$\begin{aligned} b_3 &= 3^2 + b_2 \cdot 10^{1+\lfloor \log(2+1)^2 \rfloor} \\ &= 9 + 14 \cdot 10^{1+\lfloor \log(9) \rfloor} \\ &= 9 + 14 \cdot 10^{1+0} \\ &= 9 + 14 \cdot 10 = 9 + 140 = 149. \end{aligned}$$

(a) Using the definition and the fact that $b_3 = 149$, show that $b_4 = 14916$.

(b) How many digits does b_9 have?

(c) How many digits does b_{200} have?

(d) Find the smallest n such that b_n has at least 2000 digits.

(e) Find the first four multiples of 9 among the b_n.

(f) Find the remainder when b_{200} is divided by 9.

16. Define another sequence t_n as follows:

$$t_n = a_n/10^{1+\lfloor \log a_n \rfloor}.$$

Thus $t_1 = 0.1$, $t_2 = 0.12$ and $t_3 = 123/10^{1+\lfloor \log 123 \rfloor} = 123/1000 = 0.123$. Let $a = \lim_{n\to\infty} t_n$. In other words, $a = 0.1234567891011\ldots$. Prove that a is irrational by showing that there is no repeating block of digits.

17. The other number is defined in the same way, except that we use the sequence b_n. Thus, $b = 0.149162536496481100121\ldots$. This number is also irrational.

18. The third sequence of interest is c_n. The first few members are $c_1 = 1$, $c_2 = 13$, and $c_3 = 135$. This sequence is obtained by appending to each c_n the decimal representation of the $n + 1^{\text{st}}$ odd positive integer. Thus, for example, $c_{12} = 13579111315171921 23$.

    (a) Use the ideas in the first two problems to define the sequence without using the word 'append'. IE, multiply the number by enough to create some zeros at the right end and then add to this an integer.

    (b) How many digits does each of the following numbers have?

    i. c_{10}
    ii. c_{100}
    iii. c_{1000}

    (c) How many times does the digit 1 appear in each of the numbers below? look at the three answers below. Can you generalize?

    i. c_{10}
    ii. c_{100}
    iii. c_{1000}

    (d) Which of the following numbers are multiples of 9? How many of the first 100 c_n's are multiples of 9?

    i. c_9
    ii. c_{99}
    iii. c_{999}

19. The fourth sequence of interest is d_n. The first few members are $d_1 = 2$, $d_2 = 24$, and $d_3 = 246$. This sequence is obtained by appending to each d_n the decimal representation of the $n+1^{\text{st}}$ even positive integer. Thus, for example, $d_{12} = 24681012141618202224$.

(a) Use the ideas in the first three problems to define the sequence without using the word 'append'. IE, multiply the number by just enough to create some zeros at the right end and then add to this an integer. $d_n = d_{n-1} \cdot 10^{\lfloor \log(2n) \rfloor + 1} + 2n$.

(b) How many digits does each of the following numbers have?

   i. d_{10}
   ii. d_{100}
   iii. d_{1000}

(c) How many times does the digit 1 appear in each of the numbers below? look at the three answers below. Can you generalize?

   i. d_{10}
   ii. d_{100}
   iii. d_{1000}

(d) How many of the first 1000 d_n's are multiples of k where

   i. $k = 3$
   ii. $k = 6$
   iii. $k = 8$
   iv. $k = 9$
   v. $k = 11$
   vi. $k = 37$

20. What is the mean of the list L of nine numbers?

$$L = (9, 99, 999, 9999, \ldots, 999999999)$$

21. Consider the set of single digit prime numbers, $S = \{2, 3, 5, 7\}$. For each nonempty subset D of S, let $P(D)$ denote the product of the members of D. There are 15 such nonempty subsets, $A_1, A_2, A_3, \ldots, A_{15}$. What is the sum of the 15 numbers $P(A_1) + P(A_2) + \cdots + P(A_{15})$?

22. What is the smallest n such that $1/n$ has a decimal that repeats in blocks of 5.

23. A positive integer equals 11 times the sum of its digits. What is the this number?

24. It is surprising that each of the following infinite decimals is actually a repeating decimal. Find the pair of integers whose ratio is the given decimal.

(a) $0.0010020040080160320 64\ldots$

(b) $0.0101020305081321\ldots$

25. Find a five digit positive integer in which all the digits are different and are in increasing order from left to right. The number is a multiple of four and its square is a ten-digit number.

26. Find a three digit positive integer in which all the digits are different and are in increasing order from left to right which has a square satisfying the same increasing digit condition.

27. Is it possible to write the numbers $1, 2, \ldots, 10$ in a list so that each pair of adjacent numbers differ by either 3 or 5? (As an example, the list: $1, 4, 7, 10, 5, 8, 3, 6, 9, 2$ would not suffice since 9 and 2 differ by 7.)

28. Consider the equation
$$\left\lfloor \frac{N}{10} \right\rfloor + \left(N - 10 \left\lfloor \frac{N}{10} \right\rfloor \right) 10^{\lfloor \log_{10} N \rfloor} = \frac{2N}{3}.$$
Show that $N = 5294117647058823$ is a solution and find all the other positive integer solutions.

29. The integers from 1 to 999 are written on the board. What is the smallest number of these integers that can be wiped off so that the product of the remaining integers ends in 8?

30. Find the sum of all 5-digit numbers that use only the digits $1, 2, 8, 9$ when repetition is allowed. Also, what is the sum of all four-digit numbers that can be built if repetition is not allowed.

31. For positive integer n let a_n be the integer consisting of n digits of 9 followed by the digits 488. For example, $a_3 = 999, 488$ and $a_7 = 9, 999, 999, 488$. Find the value of n so that a_n is divisible by the highest power of 2.

32. Find the least three-digit number that is equal to the sum of its digits plus twice the product of its digits.

33. Define the function f by
$$f(x) = \langle x/10 \rangle \cdot 100 + \lfloor x/10 \rfloor.$$
Compute
$$\sum_{k=10}^{99} f(k).$$

34. The digits of $S = 2^{2008}$ are written from left to right followed by the digits of $T = 5^{2008}$. How many digits are written altogether? We can build some notation to simplify the solution and to help us think about the problem. Let $x||y$ denote the concatenation of integers x and y. Thus $2^4||5^2 = 16||25 = 1625$.

35. Let $N = 7 + 77 + 707 + 7007 \cdots + (7 \cdot 10^{30} + 7)$. When N is written in decimal (base 10) notation, what is the sum of the digits of N?

36. The number 2^{29} contains nine digits, all of them distinct. Which one is missing?

37. A faulty car odometer proceeds from digit 3 to digit 5 , always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 000050. If the odometer now reads 002005, how many miles has the car actually traveled?

38. You're allowed to perform two operations with numbers. You can either add 1 to the number you have or you can double it. Starting with 0, what is the fewest operations you can perform to get 100?

39. You're allowed to perform three operations with numbers. You can either add 1 or 2 to the number you have or you can triple it. Starting with 0, what is the fewest operations you can perform to get 100?

40. A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 000050. If the odometer now reads 002005, how many miles has the car actually traveled?

41. Let a, b, c, d, e, f, g, and h be distinct elements in the set

$$\{-7, -5, -3, -2, 2, 4, 6, 13\}.$$

What is the minimum possible value of

$$(a + b + c + d)^2 + (e + f + g + h)^2?$$

42. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?

43. Find all three digit numbers $\underline{abc}$ such that $\underline{abc}^2$ ends in the three digits abc. Of course $\underline{abc}^2$ means $(100a + 10b + c)^2$.

44. Joy writes down a nine-digit positive integer by first selecting a three-digit positive integer N, then writing $N, N+1$, and $N+2$ in that order. What is the smallest such number that is a multiple of 37? What is the largest such number that is a multiple of 37?

45. Let x and b be positive integers. Suppose that x is represented as 324 in base b, and x is represented as 155 in base $b+2$. What is b?

46. The product of the digits of Ashley's age is the same nonzero number as it was six years ago. In how many years will it be the same again?

47. Define the sequence $a_1, a_2, a_3, \ldots$ by

$$a_i = \left\lfloor 10^i \times \frac{1}{13} \right\rfloor - 10 \times \left\lfloor 10^{i-1} \times 1/13 \right\rfloor, \text{ for } i = 1, 2, \ldots.$$

What is the largest value of any a_i?

48. The product of the digits of a four-digit number is 810. If none of the digits are repeated, what is the sum of the digits?

49. Prove that any number of the form $111\ldots1222\ldots2$ with the same number of 1's and 2's is the product of two consecutive integers.

50. What is the tenth digit to the right of the point in the decimal representation of $1/98$?

51. Each power of 11 $(11, 121, 1331, 14641, \ldots$ has one more digit than the previous power until we get to 26. There turns out to be no power of eleven that is 26 digits long. Which is the next failing count of digits?

52. In the list of numbers $1, 2, \ldots, 9999$, the digits 0 through 9 are replaced with the letters A through J, respectively. For example, the number 501 is replaced by the string 'FAB' and 8243 is replaced by the string 'ICED'. The resulting list of 9999 strings is sorted alphabetically. How many strings appear before 'CHAI' in this list?

53. The integer n is the smallest positive multiple of 15 such that every digit of n is either 8 or 0. Compute $\frac{n}{15}$.

54. What is the largest even integer that cannot be written as the sum of two odd composite numbers?

55. The increasing sequence $1, 3, 4, 9, 10, 12, 13 \cdots$ consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100^{th} term of this sequence.

56. In a parlor game, the magician asks one of the participants to think of a three digit number (abc) where a, b, and c represent digits in base 10 in the order indicated. The magician then asks this person to form the numbers (acb), (bca), (bac), (cab), and (cba), to add these five numbers, and to reveal their sum, N. If told the value of N, the magician can identify the original number, (abc). Play the role of the magician and determine the (abc) if $N = 3194$.

57. An ordered pair (m, n) of non-negative integers is called "simple" if the addition $m + n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to 1492.

58. Find the largest possible value of k for which 3^{11} is expressible as the sum of k consecutive positive integers.

59. Find the smallest positive integer whose cube ends in 888.

60. Suppose n is a positive integer and d is a single digit in base 10. Find n if $\frac{n}{810} = 0.d25d25d25\ldots$

61. For how many pairs of consecutive integers in $\{1000, 1001, 1002, \ldots, 2000\}$ is no carrying required when the two integers are added?

62. Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one non-zero digit, then $p(n)$ is equal to that digit.) Let $S = p(1) + p(2) + p(3) + \cdots + p(999)$. What is the largest prime factor of S?

63. It is possible to place the nine digits on the boxes so that the arithmetic is satisfied. The operations are assumed to take place left to right and top to bottom. For example, $5 + 4 \div 3 = 3$.

$$\begin{array}{ccccccc}
\square & \times & \square & - & \square & = & 3 \\
\times & & - & & - & & \\
\square & + & \square & \div & \square & = & 3 \\
- & & \times & & + & & \\
\square & - & \square & + & \square & = & 3 \\
= & & = & & = & & \\
3 & & 3 & & 3 & &
\end{array}$$

64. Determine the positive integer n such that each of the digits $0, 1, 2, \ldots, 9$ occurs exactly once in either n^3 or n^4, but not in both, i.e. if a digit is used in n^3, then it is not used in n^4 and vice versa.

65. Consider the integer $N = 142857$. If you multiply N by 5, you get 714285, which is N with all but he rightmost digit shifted right and the rightmost digit switched to the left end. Let's call this a **right shift**. If instead, you multiply N be 3, you get 428571, which is N with all digits except the leftmost shifted to the left and the leftmost digit moved to the right end. Call this a **left shift**.

    (a) Find all positive integers for which multiplying by 2 is the same as a right shift.

    (b) Find all positive integers for which multiplying by 2 is the same as a left shift.

    (c) Find all positive integers for which multiplying by 3 is the same as a right shift.

    (d) Find all positive integers for which multiplying by 3 is the same as a left shift.

66. If 135^k divides $2007!$ and 135^{k+1} does not, what is the value of k?

67. Let a, b, c, d be positive real numbers with $a < b < c < d$. Given that a, b, c, d are the first four terms in an arithmetic sequence, and a, b, d are the first three terms in a geometric sequence, what is the value of $\frac{ad}{bc}$?

68. Let S denote the list of positive integers whose digit sum $S(N)$ is 9, written in order of value. Thus $S = \{9, 18, 27, 36, \ldots, \}$. What number is the 2020^{th} in the list.

69. Several sets of prime numbers, such as $\{7, 83, 421, 659\}$ use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?