In this session, we'll learn how to solve problems related to **place value**, also known as positional notation. This is one of the fundamental concepts in arithmetic, something every elementary and middle school mathematics teacher should understand profoundly. You already know that the digits used to represent a positive integer have different meanings depending on their position. This is commonly called *place value.* [1]

Example 1. Pick a three digit number. Multiply it by 7. Then multiply your answer by 11, and finally multiply by 13. Explain why you got that answer.

Example 2. Next, consider the following problem. Find a four-digit number $\underline{abcd}$ which is reversed when multiplied by 9. In other words, find digits a, b, c, and d such that

$$9 \cdot \underline{abcd} = \underline{dcba}.$$

Solution. Our solution method is to reason digit by digit. First note that $a = 1$ since otherwise $9 \cdot \underline{abcd} \geq 9 \cdot 2000 = 18000$, which is a five-digit number. Since $a = 1$, it follows that $d = 9$. Now the equation take the following form:

$$9 \cdot \underline{1bc9} = \underline{9cb1}.$$

We can express this in decimal notation (in contrast to the underline notation we have been using) as follows:

$$9 \cdot (1009 + 100b + 10c) = 9001 + 100c + 10b.$$

Distributing the 9 across the $\cdot$ yields

$$9081 + 900b + 90c = 9001 + 100c + 10b$$

from which it follows that

$$80 + 890b = 10c.$$

Since the right side $10c$ is at most 90 (c is a digit), we can conclude that $b = 0$, and hence $c = 8$. Therefore $\underline{abcd} = 1089$ is the only such number.

[1]Unauthorized reproduction/photocopying prohibited by law' ©

In the exercises, you will be asked to investigate larger numbers that reverse when multiplied by 9.

Thinking more about the number 1089 enables us to put this phenomenon in perspective. The base b number that looks somewhat like 1089 is $N = \underline{10(b-2)(b-1)}$. We use underline so you know we don't mean multiplication. The value of N is $b^3 + (b-2)b + b - 1$. If we multiply this by $b-1$, we get $b^4 - 2b^2 + 1$. On the other hand the base b number that looks like 9801 would be written $\underline{(b-1)(b-2)01}$ and its value is $(b-1)b^3 + (b-2)b^2 + 1$, and this too has value $b^4 - 2b^2 + 1$. Thus, multiplication by $b - 1$ turns the base b numeral $\underline{10(b-2)(b-1)}$ around. Alternatively, $b^4 - 2b^2 + 1 = b^4 - b^3 + b^3 - 2b^2 + 1 = (b-1)b^3 + (b-2)b^2 + 1$, which is the base b number we would write as $(b-1)(b-2)01$. As a challenge, see if you can recognize the four digit number 2178 as a tip of an iceberg in the same way that 1089 is. Did you notice that $2178 = 2 \cdot 1089$?

In the next example we deal with several digits at once.

Example 3. Find a 6-digit number $\underline{abcdef}$ that becomes 5 times as large when the units digit f is moved to the left end of the number. In other words, solve $5 \cdot \underline{abcdef} = \underline{fabcde}$.

Solution. Before we solve this, let's consider how a six-digit number changes when the rightmost digit is moved to the left end. Take 123456 as an example. Note that $612345 = 600000 + 12345$, whereas $123456 = 123450 + 6 = 12345 \cdot 10 + 6$. If we give the name x to 12345, a common technique in algebra, we can write $123456 = 10x + 6$ and $612345 = 6 \cdot 10^5 + x$. What the hypothesis tells us is that $5(10x + 6) = 6 \cdot 10^5 + x$. Of course 123456 does not satisfy the equation, but replacing *abcde* with x reduces the six variables to just two. Of course we don't know that $f = 6$ works so we need to solve

$$5 \cdot (10x + f) = f \cdot 10^5 + x.$$

Distributing the 5 and migrating we get $50x + 5f = 10^5 f + x$ which is equivalent to $49x = (10^5 - 5)f = 99995f$. Both sides are multiples of 7 so we can write $7x = 14285 \cdot f$. Now the left side is a multiple of 7, so the right side must also be a multiple of 7. Since 14285 is not a multiple of 7, it follows (from the Fundamental Theorem of Arithmetic, which we have yet to prove) that f must be a multiple of 7. Since f is a digit, it must **be** 7. And x must be 14285. This technique is one that you will use repeatedly in the next few weeks.

Example 4. The amazing number 1089. Start with your favorite two-digit number, reverse it and then subtract the smaller from the larger. What can you say about the result? Now pick a three-digit number like 742. Reverse it to get 247 and subtract the smaller of the two from the larger. Here we get $742 - 247 = 495$. Now take the answer and reverse it to get 594, then add these two to get $495 + 594 = 1089$. What did you get? Finally, compute the product of 1089 and 9. You get $1089 \cdot 9 = 9801$. Isn't that odd, multiplying by 9 had the effect of reversing the number. Is there a connection between these two properties?

Here's a rationale for always getting 1089. First note, assuming $a > c$ that $\underline{abc} - \underline{cba} = 100(a - c) + c - a = 99(a - c) = 99d$. Now suppose $99d = \underline{u9v}$. Why is the middle digit of $99d$ always a 9? Is there a connection between u and v? For $d = 1$, we have $\underline{u9v} = 099$, so $u = 0$ and $v = 9$. For $d = 2$, $\underline{u9v} = 198$. Again we see that $u + v = 9$. Check out what we get for each of $d = 3, \ldots, 9$. In each case $u + v = 9$. Thus $\underline{u9v} + \underline{v9u} = 100(u + v) + 18 + v + u = 900 + 18 + 9 = 1089$.

Example 5. Suppose a, b, c and d are digits (ie, in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$), and the sum of the two four-digit numbers $\underline{abcd}$ and $\underline{dabc}$ is 6017. Find all four-digit numbers $\underline{abcd}$ with this property. Note that $\underline{abcd}$ is a four-digit number only if $a \neq 0$.

Recall the two functions *floor* ($\lfloor\rfloor$) and *fractional part* ($\langle\rangle$), defined by $\lfloor x\rfloor$ is the largest integer that is less than or equal to x, and $\langle x\rangle = x - \lfloor x\rfloor$. For example, $\lfloor\pi\rfloor = 3$ and $\langle\pi\rangle = \pi - 3$, $\lfloor-\pi\rfloor = -4$ and $\langle-\pi\rangle = -\pi - (-4) = 4 - \pi$. Also, notice that for any real number x, $x = \lfloor x\rfloor + \langle x\rangle$. For example, $\lfloor 3.15\rfloor + \langle 3.15\rangle = 3 + 0.15 = 3.15$.

Example 6. Let N be a four-digit number and let $N' = \langle N/10\rangle \cdot 10^4 + \lfloor N/10\rfloor$. Suppose $N - N' = 3105$. Find all possible values of N. Recall the two functions *floor* ($\lfloor\rfloor$) and *fractional part* ($\langle\rangle$), are defined by $\lfloor x\rfloor$ is the largest integer that is less than or equal to x, and $\langle x\rangle = x - \lfloor x\rfloor$. **Solution.**

Let $N = \underline{abcd}$. Then $\langle N/10\rangle = \langle \underline{abc.d}\rangle = 0.d$, so $10000\langle N/10\rangle$ could be written as $1000d$. On the other hand $\lfloor N/10\rfloor = \lfloor \underline{abc.d}\rfloor = \underline{abc}$. In other words, $N' = \underline{dabc}$, just as in the problem above. The equation $10x + d - (1000d + x) = 3105$ is equivalent to $9x - 999d = 3105$, and dividing both sides by 9 yields $x - 111d = 345$. There are six solutions for $x \geq 100$:

$$\begin{aligned}
d = 0 &\;\Rightarrow\; x = 345, N = 3450\\
d = 1 &\;\Rightarrow\; x = 456, N = 4561\\
d = 2 &\;\Rightarrow\; x = 567, N = 5672\\
d = 3 &\;\Rightarrow\; x = 678, N = 6783\\
d = 4 &\;\Rightarrow\; x = 789, N = 7894\\
d = 5 &\;\Rightarrow\; x = 900, N = 9005
\end{aligned}$$

Example 7. Consider the equations

$$1 \times 8 + 1 = 9$$

$$12 \times 8 + 2 = 98$$

$$123 \times 8 + 3 = 987$$

$$1234 \times 8 + 4 = 9876$$

Find the next few equations in the list and prove that each one follows from the one above it.

$$\begin{aligned}
\underline{12\ldots d} \times 8 + d &= (\underline{12\ldots(d-1)0} + d) \times 8 + d \\
&= \underline{12\ldots(d-1)0} \times 8 + 8d + d \\
&= 10(\underline{12\ldots(d-1)} \times 8) + 9d \\
&= 10(\underline{12\ldots(d-1)} \times 8) + 10(d-1) + (10-d) \\
&= 10(\underline{12\ldots(d-1)} \times 8 + d - 1) + (10-d) \\
&= 10(\underline{987\ldots(10-d+1)}) + (10-d) \\
&= \underline{987\ldots(10-d+1)0} + (10-d) \\
&= \underline{987\ldots(10-d+1)(10-d)}
\end{aligned}$$

Yes, you can go further. The next equation is

$$(1234567890 + 10) \times 8 + 10 = 9876543210.$$

1 Problems

During the rest of the session, please work on the following problems. You cannot finish them, but perhaps you will have time to work on them later.

1. **X'ing digits.** Consider the number

   $$N = 123456789101112\ldots5960,$$

   obtained by writing the numbers from 1 to 60 in order next to one another. What is the largest number that can be produced by crossing out 100 digits of N? What is the smallest number that can be produced by crossing out 100 digits of N? Here the number is allowed to start with some zero digits.

2. The number 123456789101112...484950 is obtained by writing the integers from 1 to 50 inclusive one after the other in their natural order. Some digits are removed so that the remaining ones have sum 200. Find the first ten digits from the left of the largest possible number obtained this way.

3. Suppose a, b, c and d are digits satisfying $abcd - cdab = 1485$. Find a relationship between the two digit numbers ab and cd. If the four digits are not required to be distinct, how many solutions are there? In case the digits are required to be distinct, how many solutions are there?

4. The digits a, b, c, d, e, f need not be distinct, and $\underline{abcdef} + \underline{efabcd} - \underline{cdefab} = 123456$. Find $\underline{abcdef}$.

5. Let $\underline{uvw}$ be a selected three-digit number. Find digits a, b, c, not necessarily distinct, for which

$$abc + cab - bca = uvw.$$

   (a) $uvw = 608$
   (b) $uvw = 708$
   (c) $uvw = 717$
   (d) $uvw = 707$

6. Suppose $\underline{ab3}/\underline{cd1} = \underline{ab}/\underline{cd}$ where a, b, c, d are distinct digits. Find the digits.

7. For each of the following problems, let $S(n) = n$ in case n is a single digit integer. If $n \geq 10$ is an integer, $S(n)$ is the sum of the digits of n. Similarly $P(n)$ is n if n is a positive single digit integer and the product of the digits of n otherwise. If there is no solution, prove it.

   (a) What is the smallest solution to $S(n) = 2005$. Express your answer in exponential notation.
   (b) How many five-digit numbers n satisfy $S(S(n)) + S(n) = 50$. For a very hard challenge, try finding the number of six-digit numbers for which $S(S(n)) + S(n) = 50$.
   (c) Find all solutions to $S(S(S(n))) + S(S(n)) + S(n) = 100$.
   (d) Find all solutions to $S(S(n)) + S(n) + n = 2007$.
   (e) (2019 AIME) Consider the integer

$$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots99}_{\text{321 digits}}.$$

   Find the sum of the digits of N.
   (f) (2007 North Carolina High School Math Contest) Find the sum $S(1) + S(2) + \cdots + S(2007)$.

(g) Can both $S(a)$ and $S(a+1)$ be multiples of 49? If so, find the smallest such a. If we change 49 to 100, how many digits are in a?

(h) Find a number n such that $S(n) = 2S(n+1)$. For what values of k does there exist n such that $S(n) = kS(n+1)$?

(i) Find, with proof, the largest n for which $n = 7S(n)$.

(j) Find the smallest positive integer n satisfying $S(S(n)) \geq 10$.

(k) Find the smallest positive integer n satisfying $S(S(n)) \geq 100$.

(l) Find the smallest positive integer n such that $S(n^2) = 27$.

(m) Find all three digit numbers n such that $n = 3(S(n))^2$.

(n) Find all positive integers that are 34 times the sum of their digits.

(o) How many four digit numbers N satisfy $S(N) = P(N)$?

(p) Let N denote the smallest positive integer such that $N + S(N) + S(S(N)) = 99$. What is $S(N)$?

(q) Solve the equation $n = 37S(n)$.

(r) Find the sum of the digits of $10^{2017} - 2017$.

(s) Solve: (a) $n + S(n) + S(S(n)) = 2019$. and
(b) $n + S(n) + S(S(n)) + S(S(S(n))) = 2019$.

8. In this problem we'll explore the sum of $S(n)$ and $P(n)$.

(a) Find a number n that is the sum of its digit sum and its digit product. That is find n such that $n = P(n) + S(n)$.

(b) Find n such that $P(n) + S(n) = 100$.

(c) Find n such that $P(n) + S(n) = 1000$.

(d) Find the two three-digit numbers n that satisfy $n = S(n) \cdot P(n)$.

(e) Find three 3-digit numbers n such that

$$P(P(P(P(n)))) < P(P(P(n))) < P(P(n)) < P(n) < n.$$

(f) Find an integer N such that

- $N = 3P(N) + 19S(N)$
- $N = 5P(N) + 3S(N)$.

(g) Find the smallest integer n such that $P(n) \cdot S(P(n)) = 1000$.

(h) Find the smallest integer n such that $P(n)7!$.

(i) Find an integer n that is exactly 9 times the product of its digits.

(j) Find an integer n that is exactly 5 times the product of its digits.

(k) An n-digit number N satisfies

- $P(N) = 300$
- $S(N) = 18$
- $Q(N) = 76$,
  where $Q(N)$ is the sum of the squares of the digits.

What is n?

(l) Let $R_i = 111\ldots1$ with i 1's. This is called the i^{th} repunit. Let $N = \sum_{i=1}^{i=100} R_i$. Find $S(N)$.

9. For each value of $k \in \{4, 5, 6, 7, 8\}$ find the smallest integer n for which $P(n) = k!$. Then find the number of such numbers have the same number of digits as n. For example, for $k = 4$, $k! = 24$, the smallest n with $P(n) = 24$ is $n = 38$. There are four two-digit numbers that satisfy $P(n) = 24$.

10. Let T denote the set of all 100-digit (decimal) integers. As usual, $S(n)$ and $P(n)$ denote the sum and the product of the digits of integer n.

(a) How many 100 digit numbers are there? In other words, find the cardinality of T. Here and in the problems below, you can use exponential notation. For example, if the question was how many 4 digit numbers there are, you could answer $9 \cdot 10^3$. It will be helpful to note that the largest member of T is $10^{100} - 1$ and the smallest is 10^{99}.

(b) What is the largest multiple of 72 in the set T?

(c) What is the smallest multiple of 72 in the set T?

(d) Among the multiples n of 72 in T, what is the largest $S(n)$?

(e) Among the multiples n of 72 in T, what is the largest $P(n)$?

(f) Does T contain a number multiple of 72 n such that both $S(n)$ and $P(n)$ are also multiples of 72?

(g) For each of the following values of k, find the largest and smallest member of T which is a multiple of k.

i. $k = 77$

ii. $k = 91$

iii. $k = 75$

iv. $k = 37$

v. $k = 143$

vi. $k = 45$

vii. $k = 27$.

11. Find four digits `a,b,c,d` which satisfy 3 $\underline{\texttt{abcd}}=\underline{\texttt{cbad}}$. Can you prove that $\underline{\texttt{cbad}}$ is a multiple of 27?

12. Find four digits `a,b,c,d` which satisfy 3 $\underline{\texttt{abcd}}=\underline{\texttt{cabd}}$. Can you prove that $\underline{\texttt{cabd}}$ is a multiple of 27?

13. Let N be the smallest four digit number such that the three digit number obtained by removing the leftmost digit is one ninth of the original number. What is the sum of the digits of N?

14. Use each of the five digits $1, 3, 5, 7$ and 9 exactly once to build two numbers A and B such that $A \cdot B$ is as large as possible. Then build two numbers C and D such that $C \cdot D$ is as small as possible.

15. Find all 3-digit numbers m which are equal to the arithmetic mean of the six numbers one obtains by rearranging the digits of m in all possible ways.

16. Notice that $4^2 = 16, 34^2 = 1156$ and $334^2 = 111556$. Find the pattern and prove it always works.

17. Recall the two functions *floor* ($\lfloor\rfloor$) and *fractional part* ($\langle\rangle$), defined by $\lfloor x \rfloor$ is the largest integer that is less than or equal to x, and $\langle x \rangle = x - \lfloor x \rfloor$. Also, the *ceiling* function, which rounds upward to the next integer, is denoted $\lceil x \rceil$.

    (a) For each member x of the set S, evaluate $\langle x \rangle$ and $\lfloor x \rfloor$ and $\lceil x \rceil$. $S = \{\pi, 1.234, -1.234, 12.34, -12.34, 123.4, \frac{7}{3}, -\frac{7}{3}\}$

    (b) Define another function f by $f(x) = x - 10\lfloor \frac{x}{10} \rfloor$. Find $f(x)$ and $f(\lfloor x \rfloor)$ for each x in S.

    (c) Let $g(x) = \lfloor \frac{\lfloor x \rfloor}{10} \rfloor - 10\lfloor \frac{\lfloor x \rfloor}{100} \rfloor$. Evaluate g at each of the members of S.

    (d) Prove that for any positive real number $x = 100a + 10b + c + f$, where a is a positive integer, b is a digit, c is a digit, and $0 \leq f < 1$, $g(x) = b$. In other words, $g(x)$ is the tens digit of x. What about $g(-x)$?

    (e) Prove that $\lceil x \rceil = -\lfloor -x \rfloor$.

18. The numbers $1, 2, 3, 6, 7, 8$ are arranged in a multiplication table, with three along the top and the other three down the column. The multiplication table is completed and the sum of the nine entries is tabulated. What is the largest possible sum obtainable.

$\times$	a	b	c
d			
e			
f			

19. A *special-8* number is one whose decimal representation consists entirely of 0's and 8's. For example 0.8808 and $0.\overline{08}$ are special numbers. What is the fewest special numbers whose sum is 1.

20. A *special-3* number is one whose decimal representation consists entirely of 0's and 3's. For example 0.3033 and $0.\overline{03}$ are special numbers. What is the fewest special numbers whose sum is 1.

21. A *special-7* number is one whose decimal representation consists entirely of 0's and 7's. For example 0.7707 and $0.\overline{07}$ are special numbers. What is the fewest special numbers whose sum is 1.

22. What is the largest 5-digit multiple of 11 that has exactly 3 different digits? Suppose we require that there be four different digits? What if all five digits are required to be different?

23. How many positive odd integers n less than 1000 have the property that the product of the digits of n is 252?

24. Given that $29a031 \cdot 342 = 100900b02$ where a and b denote two missing digits, what is the value of $a + b$?

25. A two-digit integer N that is not a multiple of 10 is k times the sum of its digits. The number formed by interchanging the digits is m times the sum of the digits. What is the relationship between m and k?

26. A check is written for x dollars and y cents, both x and y two-digit numbers. In error it is cashed for y dollars and x cents, the incorrect amount exceeding the correct amount by \$17.82. Find a possible value for x and y.

27. Solve the alpha-numeric problem $\underline{abcd} \times 4 = \underline{dcba}$, where a, b, c and d are decimal digits.

28. When the decimal point of a certain positive decimal number is moved four places to the right, the new number is four times as big as the reciprocal of the original number. What is the number?

29. The rightmost digit of a six-digit number N is moved to the left end. The new number obtained is five times N. What is N?

30. Repeat the same problem with the 5 changed to a 4. That is $4(\underline{abcdef}) = \underline{fabcde}$

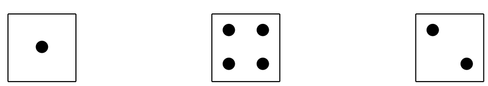
31. Let $N = 1234567891011\ldots19992000$ be the integer obtained by appending the decimal representations of the numbers from 1 to 2000 together in order. What is the remainder when N is divided by 9?

32. Ashley, John and Peter have a large supply of identical cubical dice. Each one builds a pile of them directly upon one another. When they are done, they notice that:

    (a) When they read the tops of the three columns from left to right as one number, the number is exactly the total number of dots showing on all the dice. And

    (b) When the top numbers are multiplied together, the product is exactly the number of dice used to build the three columns.

    For example, if the top three dice looked like

    we would be interpreted as the number 142. How many dice are there among the three piles?

33. Find all three-digit numbers whose squares end with the digits 444.

34. A six-digit number $N = abcdef$ has the property that $7(\underline{abcdef}) = 6(\underline{defabc})$, where both digits a and d are non-zero. What is N?

35. Let a, b, c, d, and e be digits satisfying $4 \cdot \underline{abcde4} = \underline{4abcde}$. Find all five of the digits.

36. Let a, b, c, d, e be digits satisfying $\underline{abc} \cdot a = \underline{bda}$ and $\underline{bda} \cdot a = \underline{cdde}$. What is $\underline{cdde} \cdot a$? A reminder about notation: the string of digits abc can be interpreted two ways, first as $a \cdot b \cdot c$ and secondly as $100a + 10b + c$. To distinguish these two interpretations, we use the underline notation $\underline{abc}$ for the latter of these.

37. Let $N = \underline{abcdef}$ be a six-digit number such that $\underline{defabc}$ is six times the value of $\underline{abcdef}$. What is the sum of the digits of N?

38. **Maximizing Products**

(a) Using all nonzero digits each once, build two numbers A and B so that $A \cdot B$ is as large as possible.

(b) Using all nonzero digits each once, build three numbers A, B and C so that $A \cdot B \cdot C$ is as large as possible.

(c) Using all nonzero digits each once, build four numbers A, B, C and D so that $A \cdot B \cdot C \cdot D$ is as large as possible.

(d) If we build five two-digit numbers using each of the digits 0 through 9 exactly once, and the product of the five numbers is maximized, find the greatest number among them.

(e) Suppose we want to use each non-zero digit once to create three integers whose product is as large as possible but unlike part (b) or numbers are required to have 1, 3 and 5 digits.

39. **Calling All Digits**

(a) Using each nonzero digit exactly once, create three 3-digit numbers A, B, and C, such that $A + B = C$.

(b) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $1 : 3 : 5$.

(c) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $1 : 2 : 3$.

(d) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $4 : 5 : 6$.

(e) Again using each nonzero digit exactly once, create three 3-digit numbers A, B, and C that are in the ratio $3 : 7 : 8$.

(f) Are there any more single digit ratios $a : b : c$ for which the nine nonzero digits can be used to build three numbers A, B, and C in the ratio $a : b : c$.

(g) Using the ten digits each exactly once, create three numbers A, B, and C, such that $A + B = C$.

(h) Use each of the nine nonzero digits exactly once to construct three prime numbers A, B and C such that the sum $A + B + C$ is as small as possible.

(i) Now use each of the ten digits exactly once to construct three prime numbers A, B and C such that the sum $A + B + C$ is as small as possible.

40. The number $N = 123456789101112\ldots999$ is formed stringing together all the numbers from 1 to 999. What is the product of the 2007^{th} and 2008^{th} digits of N?

41. The number $N = 37! = 1 \cdot 2 \cdot 3 \cdots \cdot 37$ is a 44-digit number. The first 33 digits are $K = 137637530912263450463159795815809$. In fact, $N = K \cdot 10^{11} + L \cdot 10^{8}$, where L is less than 1000. Find the number L.

42. Find the smallest integer multiple of 84 and whose decimal representation uses just the two digits 6 and 7.

43. (Mathcounts 2009) Find a six-digit number $\underline{abcdef}$ such that $4 \cdot \underline{abcdef} = 3 \cdot \underline{defabc}$.

44. Find the greatest 9-digit number whose digits' product is 9!.

45. Find values for each of the digits A, B and C.

$$\begin{array}{r} BA \\ AB \\ \underline{+AB} \\ CAA \end{array}$$

46. Notice that
$$\frac{19}{95} = \frac{1\cancel{9}}{\cancel{9}5} = \frac{1}{5}.$$
Can you find more pairs of two-digit numbers, with the smaller one on top, so that cancellation of this type works? Do you have them all?

47. The digits of $S = 2^{2008}$ are written from left to right followed by the digits of $T = 5^{2008}$. How many digits are written altogether? We can build some notation to simplify the solution and to help us think about the problem. Let $x||y$ denote the concatenation of integers x and y. Thus $2^4||5^2 = 16||25 = 1625$.

48. The number 2^{29} contains nine digits, all of them distinct. Which one is missing?

49. A positive integer c bigger than 1 can be *split* into two positive integer summands a and b. The *value* of the split is ab. For example the value of the split of 7 into 2 and 5 is $2 \cdot 5 = 10$. The number 20 is written on a board. The splitting process is repeated 19 times until the only unsplit numbers are 1's. What is the greatest possible sum of the values of the 19 splits?

50. The numbers $1, 2, 3, \ldots, 100$ are written on a board. A *move* consists of choosing a pair of numbers x and y, and replacing them with $x \oplus y = xy + x + y$. So each move reduces the number of numbers on the board by 1. After 99 moves just one huge number is left.

    (a) Give a name to the operation. Let's call it $\oplus$. What properties does it have?

    (b) Finally, define a new operation $\ominus$ as follows:

    $$x \ominus y = xy - x - y.$$

    Answer the same questions about $\ominus$ as you did about $\oplus$.

51. The number 20! is a nineteen digit number $243290200817664d000$ where d is a digit. What is d?

52. Let S be the set of integers $n > 1$ for which $\frac{1}{n} = 0.d_1d_2d_3d_4\ldots$, an infinite decimal that has the property that $d_i = d_{i+12}$ for all positive integers i. Given that 9901 is prime, how many positive integers are in S?

53. A *palindrome* is a positive integer that is the same when read forwards or backwards. For example 151 is a palindrome. What is the largest palindrome less than 200 that is the sum of three consecutive integers?

54. All the coefficients of a polynomial $p(x)$ are positive integers, $p(1) = 7$, and $p(17) = 84103$. What is $p(16)$?

55. All the coefficients of a polynomial $p(x)$ are positive integers, $p(1) = 9$, and $p(3/2) = 100$. What is $p(2)$?

56. The sum of the following seven numbers is exactly 19: $a_1 = 2.56$, $a_2 = 2.61$, $a_3 = 2.65$, $a_4 = 2.71$, $a_5 = 2.79$, $a_6 = 2.81$, $a_7 = 2.86$. It is desired to replace each a_i by an integer approximation A_i, $1 \le i \le 7$, so that the sum of the A_i's is also 19 and so that M, the maximum of the 'errors' $\|A_i - a_i\|$, the maximum absolute value of the difference, is as small as possible. For this minimum M, what is $100M$?

57. Let $T = \{9^k : k \text{ is an integer}, 0 \le k \le 4000\}$. Given that 9^{4000} has 3817 digits and that its first (leftmost) digit is 9, how many elements of T have 9 as their leftmost digit?

58. Let S be the set of all rational numbers r, $0 < r < 1$, that have a repeating decimal expansion in the form $0.abcabcabc\ldots = 0.\overline{abc}$, where the digits a, b, and c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

59. The equation $232_r = (2)(114_r)$ holds in base r. What is r?

60. Nick is 17 years older than Katherine. If his age were written after hers, the result would be a 4-digit perfect square. The same statement could be made 13 years from now. What is Katherine's present age?

61. Replace the letters with digits to maximize the expressions

$$NO + MORE + MATH.$$

62. Adding a zero to the end of a number makes the new number greater by 252. What is the original number?

63. If you insert a zero in the middle of a two-digit number, you get a three digit number that is 9 times as big as the original.

64. Find the biggest 6-digit number such that each of its digits, except for the last two, is equal to the sum of its two right neighbors.

65. Replace the letters with digits to maximize the expressions

$$SEND + MORE + MONEY.$$

As before, different letters stand for different digits.

66. A three digit multiple of 10 is 351 larger than the two-digit number obtained by removing the zero. What is the number?

67. A three-digit number has a hundreds digit of 9. If the 9 is moved to the right end of the number, the new number is 216 less than the old number. What is the number?

68. A three-digit number starts with the digit 4. If the 4 is relocated to the right end of the number, the new number is three-fourths of the old number. What is the old number?

69. At this stage of 'the course,' we have seen three four-digit numbers which can be permuted by multiplication: $9 \cdot 1089 = 9801, 4 \cdot 2178 = 8712$, and $3 \cdot 2475 = 7425$. What do the three numbers have in common? Can you find other numbers $\underline{abcd}$ with the property that there exists a digit t such that $t \cdot abcd$ is a four digit number whose digits are a, b, c and d?

70. Note that $\sqrt{2 \cdot 3 \cdot 4 \cdot 5 + 1} = 11$. Without a calculator, find $\sqrt{28 \cdot 29 \cdot 30 \cdot 31 + 1}$.

71. **Flipable Numbers.** A flipable number n is a base-b number for which there is a base-b digit f, such that $f \cdot n = \overline{n}$, where $\overline{n}$ is the

reverse of n. For example $9 \cdot 1089 = 9801$, so 1089 is a decimal flipable number with flip digit 9. In this problem we are finding four-digit flipable numbers and attempting to understand their structure. In each part, find the four digit number(s), then compute their decimal equivalent and find the factorization into primes. For example, solve in base 3 $abcd \cdot 2 = dcba$. Of course the base 3 digits a, b, c, d need not be distinct. Answer $1012_3 = 27 + 3 + 2 = 32 = 2^5$. Of course $dcba_3 = 2^6$. Notice that base 11 has five flip digits for four digit numbers.

(a) In base 11 solve $abcd \cdot 2 = dcba$.

(b) In base 11 solve $abcd \cdot 3 = dcba$.

(c) In base 11 solve $abcd \cdot 5 = dcba$.

(d) In base 11 solve $abcd \cdot 7 = dcba$.

(e) In base 11 solve $abcd \cdot 10 = dcba$.