Polynomials are to algebra what numbers are to arithmetic. In fact a deep understanding of arithmetic and facility with arithmetic operations paves the way to understanding algebra. Not surprisingly, polynomials problems are quite popular in mathematics competitions.

Just as the sets Z of integers, Q of rational numbers and R is real numbers are the underlying sets for algebraic structures, so the polynomials can also be used to build algebraic structures. And just a integers can be represented in many ways so can polynomials. An integer, like 144 can be written in *standard* form (the ordinary place value form), *factored* form $2^4 \cdot 3^2$, or in some *hybrid* form, like $(3+9)(24-12)$. Likewise a polynomial like x^2+4x+4 has standard form x^2+4x+4, factored form $(x+2)(x+2)$, and hybrid form $(x+2)x+2(x+2)$. The polynomials admit factorizations very much like the factorization of integers into primes, but these factorizations are a bit more complicated. Every real polynomial (one whose coefficients are real numbers) has a **complete** factorization if we allow complex numbers. For example $x^2+1 = (x-i)(x+i)$ where i is the imaginary unit. If we don't allow complex numbers, we must accept parts like x^2+1 as **primes**. In this case all real polynomials have prime factorizations that include linear factors (of the form $ax+b$) and quadratic factors (with negative discriminant $D = b^2-4ac$).

There are just a few theorems that you need to know before attacking the problems below. By far, the most popular theorem about polynomials is Vieta's Theorem.

1 Vieta's Theorem

The following is copied with thanks from The Art of Problem Solving website. Vieta's Formulas were discovered by the French mathematician Franois Vite. Vieta's Formulas can be used to relate the sum and product of the roots of a polynomial to its coefficients. The simplest application of this is with quadratics. If we have a quadratic $x^2+ax+b=0$ with solutions p and q, then we know that we can factor it as: $x^2+ax+b=(x-p)(x-q)$

(Note that the first term is x^2, not ax^2.) Using the distributive property to expand the right side we now have $x^2+ax+b = x^2-(p+q)x+pq$

Vieta's Formulas are often used when finding the sum and products of the roots of a quadratic in the form $ax^2 + bx + c$ with roots r_1 and r_2. They state that:

$$r_1 + r_2 = -\frac{b}{a}$$

and

$$r_1 \cdot r_2 = \frac{c}{a}.$$

We know that two polynomials are equal if and only if their coefficients are equal, so $x^2 + ax + b = x^2 - (p+q)x + pq$ means that $a = -(p+q)$ and $b = pq$. In other words, the product of the roots is equal to the constant term, and the sum of the roots is the opposite of the coefficient of the x term.

A similar set of relations for cubics can be found by expanding $x^3 + ax^2 + bx + c = (x-p)(x-q)(x-r)$.

We can state Vieta's formulas more rigorously and generally. Let $P(x)$ be a polynomial of degree n, so $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the coefficient of x^i is a_i and $a_n \neq 0$. As a consequence of the Fundamental Theorem of Algebra, we can also write $P(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$, where r_i are the roots of $P(x)$. We thus have that $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$.

Expanding out the right-hand side gives us $a_n x^n - a_n(r_1 + r_2 + \cdots + r_n)x^{n-1} + a_n(r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n)x^{n-2} + \cdots + (-1)^n a_n r_1 r_2 \cdots r_n$.

The coefficient of x^k in this expression will be the $(n-k)$-th elementary symmetric sum of the r_i.

We now have two different expressions for $P(x)$. These must be equal. However, the only way for two polynomials to be equal for all values of x is for each of their corresponding coefficients to be equal. So, starting with the coefficient of x^n, we see that $a_n = a_n$ $a_{n-1} = -a_n(r_1 + r_2 + \cdots + r_n)$ $a_{n-2} = a_n(r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n)$ $\vdots$ $a_0 = (-1)^n a_n r_1 r_2 \cdots r_n$

More commonly, these are written with the roots on one side and the a_i on the other (this can be arrived at by dividing both sides of all the equations by a_n).

If we denote σ_k as the k-th elementary symmetric sum, then we can write those formulas more compactly as $\sigma_k = (-1)^k \cdot \frac{a_{n-k}}{a_n}$, for $1 \leq k \leq n$. Also, $-b/a = p + q, c/a = p \cdot q$. Proving Vieta's Formula

Basic proof: This has already been proved earlier, but I will explain it more. If we have $x^2 + ax + b = (x - p)(x - q)$, the roots are p and q. Now expanding the left side, we get: $x^2 + ax + b = x^2 - qx - px + pq$. Factor out an x on the right hand side and we get: $x^2 + ax + b = x^2 - x(p + q) + pq$ Looking at the two sides, we can quickly see that the coefficient a is equal to $-(p + q)$. $p + q$ is the actual sum of roots, however. Therefore, it makes sense that $p + q = \frac{-b}{a}$. The same proof can be given for $pq = \frac{c}{a}$.

Note: If you do not understand why we must divide by a, try rewriting the original equation as $ax^2 + bx + c = (x - p)(x - q)$

General Form

For a polynomial of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ with roots $r_1, r_2, r_3, ...r_n$, Vieta's formulas state that:

$$\begin{aligned}
s_1 = &\quad r_1 + r_2 + r_3 + \cdots + r_n &= -\frac{a_{n-1}}{a_n} \\
s_2 = &\quad r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-2}r_{n-1} &= \frac{a_{n-2}}{a_n} \\
s_3 = &\quad r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n &= -\frac{a_{n-3}}{a_n} \\
&\quad \vdots & \\
s_n = &\quad r_1r_2r_3 \cdots r_n &= (-1)^n \frac{a_0}{a_n}.
\end{aligned}$$

These formulas are widely used in competitions, and it is best to remember that when the n roots are taken in groups of k (i.e. $r_1 + r_2 + r_3... + r_n$ is taken in groups of 1 and $r_1r_2r_3...r_n$ is taken in groups of n), this is equivalent to $(-1)^k \frac{a_{n-k}}{a_n}$.

2 Remainder and Factor Theorems

To be done.

3 Problems with Polynomials

1. Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

2. Let f be a function satisfying $f(xy) = \frac{f(x)}{y}$ for all positive real numbers x and y. If $f(500) = 3$, what is the value of $f(600)$?

3. Compute the sum of all the roots of $(2x+3)(x-4)+(2x+3)(x-6) = 0$

4. Let d and e denote the solutions of $2x^2 + 3x - 5 = 0$. What is the value of $(d-1)(e-1)$?

5. There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

6. Let $A = x^2 - 6x - 6$ and $B = x^2 + 4x - 60$. Find all possible values of x such that $A^B = 1$.

7. For how many integers x is the number $x^4 - 51x^2 + 50$ negative?

8. The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

9. The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

10. Polynomial p has non-negative integer coefficients, and satisfies $p(1) = 21$ and $p(11) = 2021$. What is $p(10)$?

11. Two roots of the polynomial $3x^3 + \alpha x^2 - 5x - 10$ are r and $-r$ for some real number r. What is the value of α?

12. For what positive integer b does the number whose base-b expansion is 265 equal the number whose base-9 expansion is $1b1$?

13. List all prime numbers which are of the form $x^3 - 11x^2 - 107x + 1177$ for some integer x.

14. For how many integers x between 1 and 91 inclusive is it true that $x^2 - 3x + 2$ is a multiple of 91?

15. What is the smallest positive value of $(4x^2 + 8x + 5)/(6x + 6)$ for all real numbers x?

16. Express the solution set of the inequality $\frac{1}{x} + 2x \geq 3$ as a union of intervals.

17. There are two values of m so that the equation

$$p(x) = x^4 - (3m + 2)x^2 + m^2 = 0$$

has four real roots in arithmetic progression. What is the smaller of these two values of m?

18. Suppose $p > 0$ and $x^2 + px + 1 = 0$ has solutions which differ by 1. What is the value of p?

19. Let r_1, r_2, r_3, and r_4 denote the (possibly complex) roots of the polynomial $p(x) = 20x^4 + 13x^3 + 11x^2 + 16$. What is the numerical value of $(r_1 + 1)(r_2 + 1)(r_3 + 1)(r_4 + 1)$? What is the sum $\Sigma_{i=1}^{4}(r_i + 1)$?

20. What is the least possible value of $(xy - 1)^2 + (x + y)^2$ for real numbers x and y?

21. All the roots of the polynomial $p(z) = z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. Find the values of A, B, C, D.

22. What is the value of b if a, b, and c satisfy the following equations?

$$1 + a + b + c = 16 + 8a + 4b + 2c = 81 + 27a + 9b + 3c = 256 + 64a + 16b + 4c$$

23. Find an equation for a line parallel to the line $5x - 3y = 7$ and which goes through the point $(-3, 9)$. Express your answer in slope-intercept form.

24. Find the exact value of the expression $|5\pi - 17| + |5\pi - 12| + |3\pi - 8|$. Express your answer the form $a\pi + b$.

25. How many points (x, y) in the plane satisfy both $x^2 + y^2 = 900$ and $x^2 - 10x + y^2 - 24y = -88$? The correct answer is worth 2 points, the correct explanation is worth 10 points.

26. Suppose f, g and h are polynomials of degrees $3, 4$ and 7 respectively. What is the degree of the product $(f \circ g) \cdot (f + g + h)$, where $\circ$ means composition?

4 Other Problems

27. Let $h(x) = \frac{x(2x+11)(2x+7)}{(x-1)^2(3x-12)}$.

    (a) Find the asymptotes and the zeros of h.

    (b) Build the sign chart for $h(x)$.

    (c) Sketch the graph of $h(x)$ USING the information in (a) and (b).

28. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

29. What is the distance from the center of the circle $x^2 + y^2 + 4y = 21$ to the point $(1, 3)$? Is the point $(1, 3)$ **inside**, **outside**, or **on** the circle?

30. The points $(1, 0), (-3, -1), (u, v)$, and $(0, 4)$ are the vertices of a square. Find u and v.

31. Consider the three circles in the plane $A : \{(x, y)|x^2 + (y - 10)^2 = 1\}$, $B : \{(x, y)|(x - 8)^2 + y^2 = 1\}$, $C : \{(x, y)|(x + 8)^2 + y^2 = 1\}$. How many circles in the plane have exactly one point in each of the three circles. Find an equation for one such circle.

32. Find the domain of the function

$$g(x) = \frac{\sqrt{x^2 - 2x - 3}}{x + 9}.$$

Express your answer as a union of intervals. That is, use interval notation.