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TIGHTNESS IN σ-COMPACT SPACES

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Abstract. In 1993 Arhangelskii and Stavrova defined the notion of the k-tightness number of a space and its hereditary version. They proved that the hereditary k-tightness of a compact space is equal to the standard notion of tightness. Out of this grew the notion of what one might call σ-compact tightness: the closure of a set is the union of the closures of all its σ-compact subsets. We contribute to the question of whether σ-compact tightness is equivalent to countable tightness.

1. Introduction

From the paper [1], the cardinal invariant $t_k(X)$ is defined for a space X and is called the k-tightness of the space. The subscript k is a common method of referring to the notion of compactness. A subset B of a space X is called τ-compact, for a cardinal τ, if B can be written as a union of a family of cardinality at most τ consisting of compact subsets of X (or B). The k-tightness, $t_k(X)$, does not exceed τ if for every set $M \subset X$ which is not closed in X, there is a τ-compact set $B \subset X$ such that $M \cap B$ is also not closed in X. This is a natural generalization of a k-space, since in a k-space X a set $M \subset X$ is closed if $M \cap B$ is closed for every compact set $B \subset X$. Now, still following [1], let $ht_k(X)$ denote the hereditary version of $t_k(X)$. Thus, $ht_k(X) \leq \tau$ if for each $A \subset X$, we have that $t_k(A) \leq \tau$.

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It is shown in [1] that $ht_k(X)$ is equal to $t(X)$ for compact spaces. Recall that the tightness, $t(X)$, of a space X is the least infinite cardinal τ such that whenever a point x is in the closure of $A \subset X$, then there is a set $B \subset A$ of cardinality at most τ, such that x is in the closure of B. Of course it follows that every subspace of X has tightness no larger than the tightness of X.

Proposition 1.1. [1] *If X is a compact space, then $ht_k(X) = t(X)$.*

A similar, perhaps more natural, generalization of $t(X)$ is also formulated in [1].

Definition 1.2. Define $t^*_k(X) \leq \tau$ iff for every $A \subset X$ and $x \in \overline{A}$, there is a τ-compact $B \subset A$ such that $x \in \overline{B}$.

Furthermore, it is established that

Proposition 1.3 ([1])**.** *For every X we have that $ht_k(X) \leq t^*_k(X) \leq t(X)$.*

In answer to one of the questions from [1, Problem 19], we show that there are examples of a hereditarily Lindelöf and ccc σ-compact spaces for which $\omega = ht_k(X) < t(X)$. These examples also (partially) answer [1, Problem 25].

The very interesting question of whether $t^*_k(X) = \omega$ implies, in general, that $t(X) = \omega$, remains open, but we are able to establish that the answer is affirmative for spaces with weight at most ω_1. We present examples which we feel illustrate the complexity of the problem, and we establish the following result about locally compact spaces of countable tightness.

Theorem 1.4. *Each locally compact space of countable tightness has a dense subset with the property that every σ-compact subset of it is contained in the closure of a countable subset of X.*

2. Hereditary k-tightness

In [1, Problem 19], it is asked if each σ-compact (or Lindelöf) space X with $ht_k(X) \leq \omega$ will have countable tightness.

Theorem 2.1. *There is a hereditarily Lindelöf space X with $ht_k(X) = \omega$ and uncountable tightness.*

Proof. It is well-known that the second author proved that there is a hereditary Lindelöf space L which has countable tightness, weight ω_1, and is nowhere separable [9]. Such a space is ccc, and it is shown in [5], that $\omega \times L$ has a remote point p. That is, p is a point in the Stone-Cech remainder of $\omega \times L$ which is not in the closure of any nowhere dense subset of $\omega \times L$. Our example is $X = (\omega \times L) \cup \{p\}$. Since X is nowhere separable and p is a remote point, it follows that p is not a limit point of any countable subset of X. Therefore X has uncountable tightness. Since $\omega \times L$ has countable tightness, it follows immediately that $ht_k(\omega \times L) \leq \omega$. To show that X itself satisfies that $ht_k(X) \leq \omega$ we assume that $Y \subset X$ and that $A \subset Y$ is not closed in Y. We must find a σ-compact set $B \subset Y$ such that $B \cap A$ is not closed in Y. If p is not a limit of A, then the fact that $t(L) = \omega$, implies there is a countable subset of A which accumulates to a point of $Y \setminus A$; and so B can be taken to be such a countable set. Otherwise, we may assume that p is a limit point of A. Since p is a remote point of $\omega \times L$, and L has no isolated points, there is an n such that $A \cap (\{n\} \times L)$ is not discrete. Choose B to be any countable subset of $A \cap (\{n\} \times L)$ which is not closed in Y. Again, such a B is a witness to $t_k(Y) = \omega$. □

Now we turn our attention to this question for σ-compact spaces. Let us also mention Problem 25 from [1] which asks if $ht_k(X) \leq \omega$ implies $t(X) = \omega$ for spaces X that are equal to a countable union of compact subspaces of countable tightness. More generally they ask what additional conditions are sufficient to ensure countable tightness. We consider spaces of the form $\{p\} \cup \bigcup_n X_n$ where $X = \bigcup_n X_n$ is σ-compact and of countable tightness. Surprisingly, it turns out that the answer is generally no, even with additional properties imposed on the subspace X. We first obtain a condition that ensures that $ht_k(X \cup \{p\}) \leq \omega$ holds for such spaces.

Lemma 2.2. *If X is σ-compact and has countable tightness, then for any extension $X \cup \{p\}$ of X, $ht_k(X \cup \{p\}) \leq \omega$ providing p has countable tightness with respect to discrete subsets of X.*

Proof. Suppose that $Y \subset X \cup \{p\}$ and that $A \subset Y$ is not closed in Y. We must produce a σ-compact $B \subset Y$ such that $A \cap B$ is not closed in Y. Since X has countable tightness, it follows that we may do so long as A has any limit points in $X \cap Y$. Otherwise we have that $A \cap X$ is discrete and $p \in Y$ is the only limit point in Y. □

Rather than simply getting examples with $ht_k(X \cup \{p\}) = \omega < t(X \cup \{p\})$, we seek examples where p is not in the closure of any countable subset of X. In view of Lemma 2.2 and [3], such a space $X \cup \{p\}$ may be called a discrete-remote weak-P-extension of X, that is, the point p is a non-isolated weak P-point in the space $X \cup \{p\}$ which is also not in the closure of any discrete subset of X.

For spaces which are not ccc one has the following result of van Mill for finding weak P-points, but little is known, in ZFC, for finding discrete-remote points.

Proposition 2.3 ([8]). *If $X = \Sigma X_n$ is nowhere ccc, and if D is a nowhere dense subset of X, then there is a free filter $\mathcal{F}$ of closed sets with the property that for each ccc $E \subset X$, there is an $F \in \mathcal{F}$ such that $F \cap \overline{E \cup D}$ is empty. In particular, X has a weak P-point extension.*

In fact, for locally compact spaces of countable tightness, weak P-points exist so long as the space is nowhere separable.

Theorem 2.4. *If X is a locally compact, nowhere separable, σ-compact, non-compact space of countable tightness, then X has a weak P-point extension.*

For the sake of continuity, we defer the proof of Theorem 2.4 until the end of this section. In a positive direction, in terms of getting $ht_k(X \cup \{p\}) = \omega$ to imply countable tightness, we have the following two Fréchet non-ccc examples. The main idea to these results is that in these examples we can show that a discrete-remote point must be a remote point, and the fact from [4] that such products of cellularity larger than ω_1 do not have remote points.

Proposition 2.5. *If $A(\omega_2)$ is the one-point compactification of the discrete space ω_2, then $X = \omega \times (A(\omega_2))^\omega$ has no discrete-remote weak-P-extension.*

Proof. It is shown in [4] that this space X has no remote points. The result will follow once we show that each nowhere dense subset of $(A(\omega_2))^\omega$ is contained in the closure of a discrete set. Let $K \subset (A(\omega_2))^\omega$ be non-empty and nowhere dense. For $t \in \omega_2^{<\omega}$, let $[t]$ denote the basic clopen set $\{x \in (A(\omega_2))^\omega : t \subset x\}$. Let T denote the set of $t \in \omega_2^{<\omega}$ which are minimal with respect to having $[t]$ disjoint from K. Since K is nowhere dense and the family $\{[t] : t \in \omega_2^{<\omega}\}$ is a π-base for $(A(\omega_2))^\omega$, we have that $\bigcup\{[t] : t \in T\}$ is dense. Let T_1 denote the set of one-point extensions in $\omega_2^{<\omega}$ of members of T. For each $t \in T_1$ we will choose a single point $x_t \in [t]$; it is evident that $D = \{x_t : t \in T_1\}$ will be a discrete subset of $(A(\omega_2))^\omega$. The selection can be inductively defined so as to ensure that for each basic clopen subset of $(A(\omega_2))^\omega$ which meets K will also meet D. Since $(A(\omega_2))^\omega$ has weight ω_2, this selection can be made so long as we can show that for each clopen set W meeting K, the set of $t \in T_1$ satisfying that $W \cap [t]$ is non-empty has cardinality ω_2.

A basic clopen set W will have the form $\Pi_{i<n} W_i$ where for each $i < n$, either W_i is a singleton set from ω_2, or $W_i = A(\omega_2) \setminus F_i$ for some finite $F_i \subset \omega_2$. Assume that W meets K. If each W_i is a singleton, then $W = [s]$ for $s = \langle \beta_i : i < n \rangle$ and $[s] \cap K \neq \emptyset$. Therefore there is an extension $t_s \in T$ of s and we have that W contains $[t]$ for each $t \in T_1$ which extends t_s. Otherwise, let i be the minimal element of n such that

W_i is not a singleton. Then, for each $\beta \in \omega_2 \setminus F_i$, there is some $t_\beta \in T_1$ satisfying that $[t_\beta] \cap W \neq \emptyset$ and $t_\beta(i) = \beta$. This completes the proof. □

By similar reasoning we have the following result for a first-countable non-ccc space.

Proposition 2.6. *If K is the lexicographically ordered square, then $X = \omega \times K^\omega$ has a discrete-remote weak-P-extension if and only if the continuum hypothesis holds.*

Proof. The same argument as used in the proof of Lemma 2.5 shows that a discrete-remote point of X is also a remote point. Also, if $\mathfrak{c} > \omega_1$, then it follows from [4] that X has no remote points. On the other hand, it is shown in [7, 1.3] that X does have remote points if the continuum hypothesis holds. □

Now we turn to considering ccc examples and the existence of discrete-remote weak P-point extensions; hence the failure of σ-compact tightness implying countable tightness. We first use a space constructed by Bell [2] to provide a ccc first-countable example. We will also show that it is independent of the usual axioms as to whether there is a locally compact ccc example.

We begin by recalling the ingenious example from [2].

Definition 2.7. The Pixley-Roy space $F[2^\omega]$ over the Cantor set 2^ω is a topology on the family of non-empty finite subsets of 2^ω in which a base for the topology is given by the collection of sets $[H, O] = \{G \in F[2^\omega] : H \subseteq G \subset O\}$ where O is a clopen subset of 2^ω.

Proposition 2.8 ([2], 7.1 [13])**.** *There is a first-countable σ-compact extension B of the Pixley-Roy space $F[2^\omega]$. That is, the Pixley-Roy space is a dense subspace of B; hence B is ccc and nowhere-separable.*

Definition 2.9. A point p of $\beta X \setminus X$ is a remote point if p is not in the closure of any nowhere dense subset of X. A collection $\mathcal{L}$ of subsets of a space X is a remote collection if for each nowhere dense set $D \subset X$, there is an $L \in \mathcal{L}$ so that L and D have disjoint closures.

Proposition 2.10 ([3])**.** *If a space X has, for each $n \in \omega$, a remote n-linked collection, then the free sum $\Sigma_n X$ has remote points.*

The ideas for this next proof are taken from [6].

Lemma 2.11. *If $F[2^\omega]$ is dense in a space K then K has remote n-linked collections for each $n \in \omega$.*

Proof. Let $\{O_j : j \in \omega\}$ be an enumeration of the clopen subsets of 2^ω. For each $m \in \omega$, let $\mathcal{P}_m$ denote the collection $\{[H, O_j] : j < m, |H| \leq$

$m, H \subset O_j\}$. Let $\mathcal{P}$ be the union of the increasing chain $\{\mathcal{P}_j : j \in \omega\}$. Choose any increasing sequence $\{m_k : k \in \omega\} \subset \omega$ with the property that $(m_k)^2 < m_{k+1}$ and each non-empty finite intersection of members of $\{O_j : j < m_k\}$ is a member of $\{O_j : j < m_{k+1}\}$. It follows then that each non-empty intersection of at most m_k members of $\mathcal{P}_{m_k}$ is an element of $\mathcal{P}_{m_{k+1}}$.

For a finite subset L of $\mathcal{P}$, let us say that L is in $\mathcal{P}_j^+$ if $\bigcup L$ meets every member of $\mathcal{P}_j$.

Claim 1. If $\{A_i : i \in \omega\} \subset \mathcal{P}$ has dense union, then, for each $j \in \omega$, there is an $n_j \in \omega$ such that $\{A_i : i < n_j\} \in \mathcal{P}_j^+$.

We prove the Claim by contradiction. For each $n \in \omega$, assume there is an $[H_n, O_{\ell_n}] \in \mathcal{P}_j$ which is disjoint from $\bigcup_{i<n} A_i$. By passing to a subcollection, we may assume that there is a pair $m, \ell \leq j$ such that each H_n has cardinality m and each ℓ_n is equal to ℓ. Identifying each H_n with a member of the compact product space O_ℓ^m, we may assume that the sequence $\{H_n : n \in \omega\}$ converges to some $H \in O_\ell^m$. Choose any $k \in \omega$ so that A_k meets $[H, O_\ell]$. Let $A_k = [H', O']$ and notice that $H \cup H' \subset O' \cap O_\ell$. Since $\{H_n : n \in \omega\}$ converges to H, there is an $n > k$ so that $H_n \subset O'$. It follows that $A_k \cap [H_n, O_{\ell_n}]$ is not empty, which is a contradiction.

We define the desired n-linked remote collections by induction. Let $\mathcal{L}_0$ be equal to the entire collection $\mathcal{P}$ (i.e. of singleton elements of $\mathcal{P}$).

For each $n \in \omega$, we have that for a finite $L \subset \mathcal{P}$, $\bigcup L$ is a member of $\mathcal{L}_{n+1}$, if there is k such that $\bigcup(L \cap \mathcal{P}_{m_k})$ is a member of $\mathcal{L}_n$ and L is a member of $\mathcal{P}_{m_{k+1}}^+$. We prove that each $\mathcal{L}_n$ is n-linked.

Claim 2. For each $n \in \omega$ and $\{L_i : i < n\} \subset \mathcal{L}_n$, there is a selection $P_i \in L_i$ $(i < n)$ such that $\bigcap P_i$ is non-empty.

We prove this claim by induction. Suppose that it holds for n and let $\{L_i : i < n+1\}$ be a family of finite subsets of $\mathcal{P}$ so that $\bigcup L_i \in \mathcal{L}_{n+1}$ for each $i < n+1$. Choose the indexing so that there is m_k witnessing that $L_n \in \mathcal{L}_{n+1}$, as in L_n is a member of $\mathcal{P}_{m_{k+1}}^+$ and, so that for all $i < n+1$, $\bigcup(L_i \cap \mathcal{P}_{m_k})$ is a member of $\mathcal{L}_n$. By the inductive assumption, there is a selection $\{P_i : i < n\}$ with non-empty intersection such that $P_i \in L_i \cap \mathcal{P}_{m_k}$ for each $i < n$. Recall that $\bigcap_{i<n} P_i \in \mathcal{P}_{m_{k+1}}$, and therefore there is a $P_n \in L_n$ meeting this intersection.

We finish the proof by showing each $\mathcal{L}_n$ is a remote collection. Let D be any closed nowhere dense subset of K. Let $\mathcal{A} \subset \mathcal{P}$ be all those members of $\mathcal{P}$ whose closure in K is disjoint from D. Since $F[2^\omega]$ is dense in K, $\bigcup \mathcal{A}$ is dense in K. Since $F[2^\omega]$ is ccc, there is a countable subcollection $\{A_i : i \in \omega\} \subset \mathcal{A}$ which also has dense union. By induction on n, assume there is a value i_n such that $\bigcup\{A_i : i < i_n\} \in \mathcal{L}_n$. Fix k so that $\{A_i : i < i_n\} \subset \mathcal{P}_{m_k}$, and then, by Claim 1, choose $i_{n+1} > i_n$ so that $\{A_i : i < i_{n+1}\} \in \mathcal{P}^+_{m_{k+1}}$. □

Corollary 2.12. *There is a σ-compact ccc first-countable space X with a discrete-remote weak P-extension.*

Proof. Let $X = \Sigma_n B$ with B as in Proposition 2.8. Since $\Sigma_n F[2^\omega]$ is a dense subset of X it follows from Lemma 2.11 and Proposition 2.10 that X has a remote point p. Since X is nowhere separable we have that p is not in the closure of any countable subset of X. □

We now turn our attention to locally compact spaces of countable tightness. This next result is offered to highlight the fact that Bell's space can not be made to be locally compact.

Proposition 2.13. *The space B (and $F[2^\omega]$) does not have a compactification with countable tightness.*

We obtain the proof of the following based on Sapirovskii's more general result given below in 2.18.

Proof. Suppose that K is a compactification of $F[2^\omega]$. Let $\{x_\alpha : \alpha \in \omega_1\} \subset 2^\omega$ be distinct points. For each $\alpha \in \omega_1$, there is a clopen set $C_\alpha \ni x_\alpha$ such that $[\{x_\alpha\}, C_\alpha]$ and $F[2^\omega] \setminus [\{x_\alpha\}, 2^\omega]$ have disjoint closures in K. By shrinking the family, we may assume that there is a single clopen set C such that $C_\alpha = C$ for all $\alpha \in \omega_1$.

For each α, the family

$$\mathcal{F}_\alpha = \{[\{x_\beta\}, C] : \beta < \alpha\} \cup \{F[2^\omega] \setminus [\{x_\gamma\}, 2^\omega] : \alpha \le \gamma\}$$

has the finite intersection property. If $H \in [\omega_1]^{<\omega}$ and $H_0 = H \cap \alpha$, then choose any clopen $O \subset C$ so that $O \cap H = H_0$. Then it is easily checked that $[H_0, O]$ is contained in

$$\bigcap_{\beta \in H_0} [\{x_\beta\}, C] \setminus \bigcup_{\gamma \in H \setminus H_0} [\{x_\gamma\}, 2^\omega]$$

We may choose $k_\alpha \in K$ so that k_α is in the closure of each F from the filter generated by $\mathcal{F}_\alpha$. Since $\{k_\beta : \beta < \alpha\}$ is contained in the closure of $[\{x_\alpha\}, C]$ and $\{k_\gamma : \gamma \ge \alpha\}$ is contained in the closure of $F[2^\omega] \setminus [\{x_\alpha\}, 2^\omega]$, it follows that $\{k_\alpha : \alpha \in \omega_1\}$ is a free sequence in K. □

In our construction of weak P-points and of discrete-remote points, we will need this next result.

Proposition 2.14 ([5]). *If $X = \Sigma X_n$ is ccc and has π-weight at most ω_1, then X has remote points.*

We are now able to utilize some powerful results of Todorcevic and Sapirovskii to establish this next independence result.

Theorem 2.15. *The existence of a locally compact ccc countably tight [first-countable] space with a discrete-remote weak P-point extension is equivalent to the failure of $MA(\omega_1)$*

Before giving the proof, let us recall the needed results of Todorcevic and Sapirovskii.

Proposition 2.16 ([13, 3.4])**.** *If $MA(\omega_1)$ fails, there is a compact ccc first-countable space of weight ω_1 which is nowhere separable.*

Proposition 2.17 ([11])**.** *If $MA(\omega_1)$ holds, then each locally compact ccc space of countable tightness is separable.*

Proof of Theorem 2.15. If $\mathrm{MA}(\omega_1)$ fails, then let K be the space provided by Proposition 2.16. By Proposition 2.14, $X = \omega \times K$ has a remote point $p \in \beta X \setminus X$. Since K is nowhere separable, p is also a weak P-point of X. On the other hand, suppose that $\mathrm{MA}(\omega_1)$ holds and that X is a locally compact ccc countably tight space. By Proposition 2.17, X is separable. Therefore X does not have a weak P-point extension. □

In preparation for the proof of Theorem 2.4, we will also need the following easy consequence of [13, 3.1].

Proposition 2.18 (Sapirovskii, Todorcevic)**.** *If X is compact, nowhere separable, and has countable tightness, then there is a continuous map f onto a space Y which has weight ω_1 and is also nowhere separable.*

Proof. The actual statement from Todorcevic [13, 3.1] is that compact countably tight spaces have a point-countable π-base. Let κ be the π-weight of X and fix any standard embedding e of X into $I^{w(X)}$. Let $\mathcal{B} = \{b_\alpha : \alpha \in \kappa\}$ be a point-countable π-base of $e[X]$ so that each b_α is the intersection with X of a cozero subset of $I^{w(X)}$. Let M be an elementary submodel of $H(\theta)$ for suitably large θ with $e, X, \mathcal{B} \in M$. Assume also that $\omega_1 \subset M$ and that M has cardinality ω_1. Simple elementary submodel properties implies that $\{b_\alpha \restriction M : \alpha \in \kappa \cap M\}$ is a π-base for $e[X] \restriction M$ (under suitable identifications). We check that $Y = e[X] \restriction M$ is nowhere separable. Assume that $\alpha_0 \in \kappa \cap M$ is such that $b_{\alpha_0} \restriction M$ is separable. Let $J = \{\alpha \in \kappa : b_\alpha \cap b_{\alpha_0} \neq \emptyset\}$. Since J is uncountable, it follows that $J \cap M$ is uncountable, and that there is some $y \in Y$ such that $J_y = \{\alpha \in M \cap J : y \in e[b_\alpha] \restriction M\}$ is uncountable. Choose any $x \in X$ such that $e(x) \restriction M = y$. It follows that $x \in b_\alpha$ for all $\alpha \in J_y$, which is clearly a contradiction. □

Theorem 2.19. *If X is a locally compact σ-compact non-compact nowhere separable space of countable tightness, then X has a weak P-point extension.*

Proof. Choose any unbounded positive real-valued function f on X. Assume that $\{r_n : n \in \omega\}$ is contained in the range of f and that $r_{n+1} > r_n + 2$ for each n. For each n, choose a cozero set U_n of X so that the closure, X_n, is compact, $f[X_n]$ is a subset of $(r_n - 1, r_n + 1)$ and so that U_n is either ccc or nowhere ccc. It is well-known that the closure in βX of the subspace $\Sigma_n X_n = \bigcup_n X_n$ is homeomorphic to $\beta(\Sigma_n X_n)$. Let $D = \bigcup_n X_n \setminus U_n$ and note that D is a nowhere dense subset of $\Sigma_n X_n$. Since X is normal, it follows that if $F \subset \bigcup_n U_n$ is a closed subset of X, then F and $X \setminus \Sigma_n X_n$ have disjoint closures in βX.

We proceed by cases.

Case 1. There are infinitely many n such that U_n is nowhere ccc.

Let I be the set of n such that U_n is nowhere ccc. Apply Proposition 2.3, to $\Sigma_{n\in I} X_n$ and $D \cap \Sigma_{n\in I} X_n$ to obtain the described filter $\mathcal{F}$. If $p \in \beta(\Sigma_n X_n)$ is any point which is the closure of each member of $\mathcal{F}$, then clearly p is not a limit point of any countable subset of $\Sigma_{n\in I} X_n$. In addition, p is not in the closure of $X \setminus \bigcup_{n\in I} U_n$ because X is normal, and there are members F of $\mathcal{F}$ which are disjoint from D.

Case 2. There is an infinite $I \subset \omega$ such that U_n is ccc for each $n \in I$.

For each $n \in I$, fix a space Y_n as in Proposition 2.18 with a mapping f_n from X_n onto Y_n. In the proof of Proposition 2.18, choose each member of the π-base $\mathcal{B}$ to be a subset of U_n. In this way we obtain that $f[X_n \setminus U_n]$ is nowhere dense in Y_n. By Proposition 2.14, there is a point q in $\beta(\Sigma_{n\in I} Y_n)$ which is a remote point. The mapping $\Sigma_{n\in I} f_n$ maps $\Sigma_{n\in I} X_n$ onto $\Sigma_{n\in I} Y_n$ and extends to a mapping f between the Stone-Cech compactifications. Choose $p \in \beta(\Sigma_{n\in I} X_n)$ so that $f(p) = q$. Since each countable subset A of $\Sigma_{n\in I} X_n$ maps to a nowhere dense subset of $\Sigma_{n\in I} Y_n$, it follows that p is not in the closure of such a set A. In addition, let us note that p is not in the closure of $\bigcup_{n\in I} X_n \setminus U_n$ since q is remote. Again, it follows from the normality of X and basic properties of the Stone-Cech compactification, that p is not in the closure of $X \setminus \bigcup_{n\in I} X_n$. It should now be clear that p is not in the closure of any countable subset of X. □

3. On σ-compact tightness

We interpret the property $t_k^*(X) \leq \omega$ as that the space X has σ-compact tightness. As mentioned above, it is shown in [1] that each compact space having σ-compact tightness actually has countable tightness.

In fact, this statement follows easily from Arhangelskii's earlier result that a compact space with uncountable tightness must contain an uncountable free sequence. It is immediate that no complete accumulation point of the uncountable free sequence is in the closure of a σ-compact subset of the same free sequence.

Our main new idea in the investigation of σ-compact tightness is the following observation about left-separated families of G_δ's. Say that a family $\{Z_\alpha : \alpha \in \mu\}$ is left-separated if, for each $\alpha < \mu$, Z_α is disjoint from the closure of $\bigcup_{\beta<\alpha} Z_\beta$.

Theorem 3.1. *If $\{Z_\alpha : \alpha \in \mu\}$ is a left-separated family of G_δ subsets of a space X, then each compact subset of the union is covered by a countable subcollection.*

Proof. Let $B \subset \bigcup\{Z_\alpha : \alpha \in \mu\}$ be compact. Assume that $I_0 = \{\alpha \in \mu : B \cap Z_\alpha \neq \emptyset\}$ is uncountable. For each $\alpha \in I_0$, choose any $b_\alpha \in B \cap Z_\alpha$. Since B is compact, we may choose $\beta_0 \in I_0$ minimal such that the family $\{b_\alpha : \alpha \in I_0\}$ has a complete accumulation point z_0 in Z_{β_0}. Since Z_{β_0} is a G_δ, it has an open neighborhood U_0 such that $I_1 = \{\alpha \in I_0 : b_\alpha \notin U_0\}$ is uncountable. Let β_1 be minimal such that $\{b_\alpha : \alpha \in I_1\}$ has a complete accumulation z_1 point in Z_{β_1}. Notice that $\beta_0 < \beta_1$. We may continue this process and thereby choose an increasing sequence β_i $(i \in \omega)$ with points $z_i \in B \cap Z_{\beta_i}$, together with a decreasing sequence $\{I_i : i \in \omega\}$ of uncountable subsets of μ, so that for each i, z_i is a complete accumulation point of $\{b_\alpha : \alpha \in I_i\}$, and no point of $\bigcup_{\gamma<\beta_i} Z_\gamma$ is a complete accumulation point of $\{b_\alpha : \alpha \in I_i\}$.

It follows from these assumptions that the set $\{z_i : i \in \omega\}$ has no accumulation point in the compact set B – which is the contradiction we seek. To see this, notice that any accumulation point z of $\{z_i : i \in \omega\}$ is a complete accumulation point of $\{b_\alpha : \alpha \in I_i\}$ for each $i \in \omega$. By the minimality of β_i, $z \notin Z_\gamma$ for any $\gamma < \beta_i$. On the other hand, since the family is left-separated, it is also the case that $z \notin Z_\gamma$ for all $\gamma \geq \sup\{\beta_i : i \in \omega\}$. □

A set $Z \subset X$ is called a G_κ-set, for a cardinal κ, if Z is equal to the intersection of a family of at most κ-many open sets. The following result is extracted from Arhangelskii's investigation of free sequences.

Lemma 3.2. *If $t(X) \leq \tau$ for a compact space X, then for each non-empty closed $Z \subset X$ which is a G_τ set, there is a set $S \subset X$ of cardinality at most τ such that $\overline{S}$ contains a non-empty G_τ subset of Z.*

Proof. Let Z be any non-empty closed set which is a G_τ-set in X. Assume we recursively choose points $\{z_\beta : \beta < \alpha < \tau^+\}$ from Z together with a descending sequence $\{Z_\beta : \beta < \alpha < \tau^+\}$ of non-empty closed G_τ subsets of Z so that $z_\beta \in Z_\beta$ and Z_β is disjoint from $\overline{\{z_\xi : \xi < \beta\}}$. This recursion must stop at some $\alpha < \tau^+$, since otherwise, we will have constructed a free sequence of length τ^+, which would contradict $t(X) \leq \tau$. Evidently, the failure implies that for some α, the non-empty G_τ-set, $\bigcap\{Z_\beta : \beta < \alpha\}$ is contained in the closure of the set $\{z_\beta : \beta < \alpha\}$. □

Corollary 3.3. *If $t_k^*(X) \leq \omega$ and a point $p \in X$ is in the closure of $\bigcup_n B_n$ where $\{B_n : n \in \omega\}$ is a pairwise disjoint family of compact subsets of X, then p is in the closure of a countable subset of $\bigcup_n B_n$.*

Proof. Since $t_k^*(X) \leq \omega$, each B_n has countable tightness. For each n, let $\mathcal{Z}_n$ be the collection of closed relative G_δ subsets of B_n which are contained in the closure of a countable subset of B_n. Fix any maximal left-separated subfamily $\{Z(n,\alpha) : \alpha < \mu_n\}$ of $\mathcal{Z}_n$. By Lemma 3.2, the union of this collection is a dense subset of B_n. By the assumption that $t_k^*(X) \leq \omega$ and Theorem 3.1, there is a countable subcollection $\mathcal{Z}_p$ of the collection $\{Z(n,\alpha) : n \in \omega, \alpha \in \mu_n\}$ whose union has p in its closure. Since each $Z \in \mathcal{Z}_p$ is contained in the closure of a countable subset of $\bigcup_n B_n$, we have that p also is contained in the closure of a countable subset of $\bigcup_n B_n$. □

If X is a space with $t_k^*(X) \leq \omega$ which does not have countable tightness, then it is evident that there is a point $p \in X$ and a countable increasing sequence $\mathcal{S} = \{X_n : n \in \omega\}$ of compact subsets of X, such that p is in the closure of $\bigcup_n X_n$, but is not in the closure of any countable subset of $\bigcup_n X_n$. By passing to a subspace, we can assume that $X = \{p\} \cup \bigcup_n X_n$.

Definition 3.4. Say that a space X is an $\mathcal{S}$-example if $t_k^*(X) \leq \omega$, $\mathcal{S}$ is a countable increasing sequence of compact subsets, X is equal to $\{p\} \cup \bigcup \mathcal{S}$, and a set $F \subset \bigcup \mathcal{S}$ is closed in $X \setminus \{p\}$ if, for each $S \in \mathcal{S}$, $F \cap S$ is compact.

We next show that if there is such an example, we can assume that the subspace $\bigcup_n X_n$ is equipped with the finest topology in which each X_n is compact. The sequence $\mathcal{S} = \{X_n : n \in \omega\}$ will be a parameter in discussing the example. Notice that if a countable increasing union of compact Hausdorff spaces are endowed with the fine topology, then this topology is completely regular. We should also remark that a different selection of the sequence $\mathcal{S}$ can result in a different topology. Observe also that if X is an $\mathcal{S}$-example then the subspace $\bigcup \mathcal{S}$ has countable tightness.

Lemma 3.5. *If there is a space X which has σ-compact tightness and uncountable tightness, then there is a nowhere locally compact $\mathcal{S}$-example with uncountable tightness.*

Proof. Assume that $p \in X$ is in the closure of $A \subset X$ but is not in the closure of any countable subset of A. Since $t_k^*(X) \leq \omega$, there is a family

$\{X_n : n \in \omega\}$ of compact subsets of A such that p is in the closure of $\bigcup_n X_n$. Consider the space $\{p\} \cup \bigcup_n X_n$ with the possibly finer topology in which the neighborhood base for p is from the usual subspace topology, but each $F \subset \bigcup_n X_n$ is declared closed if $F \cap X_n$ is closed for each n. This is a finer topology, hence it still has uncountable tightness at p with respect to $\bigcup_n X_n$. In addition, since we are not changing the topology at p, it is easy to check that if p is in the closure of some $Y \subset \bigcup_n X_n$, then Y will contain a σ-compact set (in the old topology) which has p in the closure. It is easily checked that such a σ-compact set is also σ-compact in the finer topology. Also, $\bigcup_n X_n$ with the fine topology has countable tightness since each X_n has countable tightness.

Let U be the open set of points of X that have a compact neighborhood, and let $\mathcal{U}$ be any maximal pairwise disjoint family of regular-closed compact subsets of U. The point p can not be in the closure of U since by Corollary 3.3, p would not be a limit point of any countable subfamily of $\mathcal{U}$, which contradicts that $t_k^*(X) \leq \omega$. By passing to the subspace $X \setminus \overline{U}$, we may thereby assume that X is nowhere locally compact and each X_n is a nowhere dense subset. □

If there were a Tychonoff example (in fact regular) example of such a space with uncountable tightness, then this raises the obvious question of whether there is a bound on the value of $t(p, X)$ for $p \in \beta X$ with X being a σ-compact space of countable tightness. We discuss this in example 3.10 below.

We define a special notion which is an $\mathcal{S}$ version of a nowhere dense zero-set. A set is a zero-set if it is the preimage of 0 under a real-valued continuous function.

Definition 3.6. For a zero-set $Z \subset X$ and an increasing sequence $\mathcal{S}$ of compact subsets of X, the $\mathcal{S}$-acccessible points of Z, $\mathcal{S}\text{-}acc(Z)$, will be defined as all those points of Z which are in the closure of $S \setminus Z$ for some $S \in \mathcal{S}$. In other words, $Z \cap \bigcup_{S\in\mathcal{S}} \overline{S \setminus Z}$ is the set of $\mathcal{S}$-accessible points. Analogously, the $\mathcal{S}$-interior of Z, denoted $\mathcal{S}\text{-}int(Z)$, will be the points of Z which are not in the closure of the $\mathcal{S}$-accessible points, namely $\mathcal{S}\text{-}int(Z) = Z \setminus \overline{Z \cap \bigcup_{S\in\mathcal{S}} \overline{S \setminus Z}}$.

Using our method of constructing left-separated families of G_δ's, we are able to exclude a large family of $\mathcal{S}$-spaces as potential counterexamples.

Theorem 3.7. *If X is an $\mathcal{S}$-example with uncountable tightness, then p is not in the closure of the union of all zero-sets which have empty $\mathcal{S}$-interior.*

Proof. Let $\mathcal{Z}$ be the family of all zero-sets of $\bigcup \mathcal{S}$ which have empty $\mathcal{S}$-interior. Notice that any zero-set which is contained in a member of $\mathcal{Z}$ is itself a member of $\mathcal{Z}$. Let $Y = \bigcup \mathcal{Z}$ and assume that $p \in \overline{Y}$. Recursively choose a maximal left-separated family $\{Z_\alpha : \alpha < \mu\} \subset \mathcal{Z}$. That is, having chosen $\{Z_\beta : \beta < \alpha\} \subset \mathcal{Z}$, if $Y \setminus \overline{\bigcup\{Z_\beta : \beta < \alpha\}}$ is non-empty, we may choose $Z_\alpha \in \mathcal{Z}$ to be disjoint from $\overline{\bigcup\{Z_\beta : \beta < \alpha\}}$. Clearly, there will be some μ so that $\bigcup\{Z_\alpha : \alpha \in \mu\}$ is dense in Y. This of course means that p is in the closure of the union of this left-separated collection of G_δ's.

We apply the hypothesis that X is a t^*_k-space and find that, by Theorem 3.1, there is a countable subfamily of $\{Z_\beta : \beta < \mu\}$ which has p in the closure of its union. By re-indexing, we may assume that p is in the closure of $\bigcup\{Z_j : j \in \omega\}$.

Fix an enumeration, $\{X_n : n \in \omega\}$ of $\mathcal{S}$. Let K denote the closure of $\bigcup_j Z_j$ and let $\mathcal{S}_K = \{X_n \cap K : n \in \omega\}$. We may view $\{p\} \cup K$ as an $\mathcal{S}_K$-example. With this change the $\mathcal{S}_K$-interior of some of the Z_j's may be non-empty, and $\mathcal{S}_K\text{-}acc(Z_j)$ may be strictly smaller than $\mathcal{S}\text{-}acc(Z_j)$ for some j.

We break into two cases.

Case 1. p is in the closure of $\bigcup_j \mathcal{S}_K\text{-}acc(Z_j)$.

Select any increasing sequence $\{L_k : k \in \omega\}$ of compact sets such that $L_k \subset \bigcup_j \mathcal{S}_K\text{-}acc(Z_j)$ and p is in the closure of $\bigcup_k L_k$. For each j, n, k, let $L_{j,k,n} = L_k \cap Z_j \cap \overline{X_n \cap K \setminus Z_j}$. It follows that $L_k \subset \bigcup_{j,n} L_{j,k,n}$. It is evident that $L_{j,k,n}$ is contained in the closure of $\bigcup\{Z_\ell \cap X_n : \ell > n\}$, and so we have that p is in the closure of the union of the collection $\{Z_\ell \cap X_\ell : \ell \in \omega\}$. By Corollary 3.3, we have that $t(p, X) = \omega$.

Case 2. p is in the closure of $\bigcup_j \mathcal{S}\text{-}acc(Z_j)$ but not in the closure of $\bigcup_j \mathcal{S}_K\text{-}acc(Z_j)$.

For each j, let $\{U(j,\ell) : \ell \in \omega\}$ be a sequence of open neighborhoods of Z_j so that $\overline{U(j,\ell+1)} \subset U(j,\ell)$ and $\bigcap_\ell U(j,\ell) = Z_j$. We may also assume that for each $i < j \leq \ell$, $U(j,\ell)$ and $U(i,\ell)$ have disjoint closures.

Again choose a sequence $\{L_{j,k,n} : j,k \in \omega\}$ of compact sets so that, for each j,k,n, $L_{j,k,n} \subset \overline{(X_n \setminus Z_j)} \cap Z_j \setminus \overline{\mathcal{S}_K\text{-}acc(Z_j)}$ and so that p is in the closure of the union of the sequence $\{L_{j,k,n} : j,k,n \in \omega\}$. We may assume that for each j,k,n, $L_{j,k,n} \subset L_{j,k,n+1} \subset L_{j,k+1,n+1}$.

Now we use the fact that each $L_{j,k,n}$ is contained in the $\mathcal{S}_K$-interior of Z_j. For each n, choose a sequence $\{W_{j,n} : j < n\}$ of open sets with disjoint closures so that, for each $j < n$, $\bigcup_{k<n} L_{j,k,n} \subset W_{j,n}$ and $\overline{W}_{j,n} \cap \left(X_n \cap \bigcup_{\ell \neq j} Z_\ell\right)$ is empty. By induction on n, we may then also ensure that $\overline{W}_{j,n}$ is disjoint from $\overline{W}_{i,m}$ for all $j \neq i \leq m \leq n$.

For each $j \leq n \in \omega$, now set $B(j,n)$ to be the closure of

$$X_n \cap \bigcup_{m \leq n} W_{j,m} \cap (U(j,n) \setminus U(j,n+1))$$

Note that $W_{j,m} \cap X_m \cap U(j,m) \subset \bigcup_n B(j,n)$ and that $L_{j,m,m}$ is contained in the closure of $W_{j,m} \cap X_m \setminus Z_j$. Therefore it follows that $\bigcup_{k,m} L_{j,k,m}$ is contained in the closure of the union of the family $\{B(j,n) : n \in \omega\}$.

Let us again note that for $i \neq j$, the set $\bigcup_n \overline{W}_{j,n}$ is disjoint from $\bigcup_n \overline{W}_{i,n}$, and therefore $\bigcup_n B(j,n)$ is disjoint from $\bigcup_n B(i,n)$. Furthermore, for $n - m > 1$, we have that $B(j,n)$ and $B(j,m)$ are disjoint since $B(j,m) \cap U(j,m+1)$ is empty and $B(j,n) \subset U(j,m+1)$. Since p is in the closure of the union of one of the two collections $\{B(j,2n) : n \in \omega\}$, $\{B(j,2n+1) : n \in \omega\}$, we are again done by invoking Corollary 3.3. □

A well-known class of spaces are the almost P-spaces. These are the spaces in which every non-empty G_δ has non-empty interior. The natural generalization (weakening) to $\mathcal{S}$-spaces is relevant.

Definition 3.8. An $\mathcal{S}$-example X is an almost $\mathcal{S}$-P-space if every non-empty zero-set has non-empty $\mathcal{S}$-interior.

It follows immediately from Theorem 3.7 that if there is a space with σ-compact tightness and uncountable tightness, then there is an increasing countable sequence $\mathcal{S}$ of compact subsets whose closure is of uncountable tightness and which is an almost $\mathcal{S}$-P-space.

Lemma 3.9. *If an $\mathcal{S}$-example X is an almost $\mathcal{S}$-P-space then for each $x \in \bigcup \mathcal{S}$, there is an $S \in \mathcal{S}$ such that for each zero-set Z of X with $x \in Z$, x in the closure of the set of points of S which are in the $\mathcal{S}$-interior of Z.*

Proof. If the lemma fails for some $x \in \mathcal{S}$, then, for each $S \in \mathcal{S}$ there is a zero-set Z_S such that $x \in Z_S$ and there is a zero-set neighborhood W_S of x such that $S \cap W_S$ is disjoint from $\mathcal{S}\text{-}int(Z_S)$. Since $\mathcal{S}$ is countable, the set $Z = \bigcap\{Z_S \cap W_S : S \in \mathcal{S}\}$ is also a zero-set with $x \in Z$. For each $S \in \mathcal{S}$, the set $S \cap (\mathcal{S}\text{-}int(Z))$ is a subset of $W_S \cap \mathcal{S}\text{-}int(Z_S)$. Clearly then Z has empty $\mathcal{S}$-interior, contradicting that X is an almost $\mathcal{S}$-P-space. □

We next present an instructive example of an almost P-space which also shows that there is no bound on the tightness for Tychonoff one-point extensions of σ-compact spaces. The verification that no one-point extension of this space will have σ-compact tightness seems to require new ideas. These ideas allow us to at least rule out examples of weight ω_1. However we first present a similar example (first discovered by Okunev [10]) to illustrate the tightness behavior for points in the Stone-Cech extension.

Example 3.10. *For each uncountable cardinal κ there is a σ-compact Fréchet-Urysohn space $X = \bigcup_n X_n$ for which there are points $z \in \beta X$ such that $t(z, X) \geq \kappa$.*

Proof. Fix any uncountable cardinal κ. For each n, let $X_n = [\kappa]^{\leq n}$ be the subsets of κ with cardinality at most n. The set X_0 has the single element $\emptyset$. As usual, $[\kappa]^{<\omega}$ denotes the family of all finite subsets of κ and $X = [\kappa]^{<\omega}$.

For each disjoint pair $t, F \in [\kappa]^{<\omega}$, let

$$[t; F] = \{s \in [\kappa]^{<\omega} : t \subset s \subset \kappa \setminus F\}$$

and topologize X by using this family as an open base. Since $[x; t \setminus x] \cap [t; F]$ is empty for $x \in [\kappa]^{<\omega} \setminus [t; F]$, it follows that these sets are clopen.

For each n, X_n with the subspace topology is easily seen to be compact. It is also Fréchet-Urysohn, since a point t is a limit of a set $A \subset [t; \emptyset]$ if and only if $\{\alpha : A \cap [t \cup \{\alpha\}; \emptyset] \neq \emptyset\}$ is infinite. The space X is not an almost P-space since, for each infinite set $S \subset \kappa$ and $t \in X$, the set $\bigcap\{[t; \{\alpha\}] : \alpha \in S \setminus t\}$ is nowhere dense.

Now consider the filter base $\mathcal{F} = \{[t; \emptyset] : t \in [\kappa]^{<\omega}\}$. Choose any point $z \in \beta X$ such that $z \in \mathrm{cl}_{\beta X}[t; \emptyset]$ for all $t \in [\kappa]^{<\omega}$. If $Y \subset X$ has cardinality less than κ, then $\bigcup Y$ (the union of this family of finite sets) has cardinality less than κ. Then for any $\alpha \in \kappa \setminus \bigcup Y$, we have that $\mathrm{cl}_{\beta X}[\{\alpha\}; \emptyset]$ is a clopen neighborhood of z which misses Y. □

The space in Example 3.10 is not an almost P-space, but the following slight modification is. We do not know if there is an almost P-space as in Example 3.10.

Example 3.11. *For each uncountable cardinal κ, the set $\kappa^{<\omega}$ (of ordered finite sequences) has the natural σ-compact Fréchet-Urysohn topology in which each set $[t] = \{s \in \kappa^{<\omega} : t \subseteq s\}$ is clopen. This space is an almost P-space. Each one-point extension $\kappa^{<\omega} \cup \{p\}$ which has σ-compact tightness, also has countable tightness. Each $z \in \beta(\kappa^{<\omega})$ is in the closure of some subset of $\kappa^{<\omega}$ of size at most $\mathfrak{c}$.*

Proof. We content ourselves with proving that if $\kappa^{<\omega} \cup \{p\}$ has σ-compact tightness, then p is in the closure of some countable subset of $\kappa^{<\omega}$ because this is the feature that is important in our investigation. The proof is easily adapted to show that $\kappa^{<\omega} \cup \{p\}$ has countable tightness.

Of course we assume that p is not isolated. We first prove that p is in the closure of a nowhere dense set. We next show that each nowhere dense set is contained in the closure of a discrete set, which of course completes the verification.

The set $E = \bigcup\{\kappa^{2n} : n \in \omega\}$ is dense and so p is in the closure. By the assumption of σ-compact tightness, we choose a sequence $\{L_k : k \in \omega\}$ of compact subsets of E so that p is in the closure of $\bigcup_k L_k$. We check that $\bigcup_k L_k$ is nowhere dense.

To do so, we observe that for each $t \in \kappa^{<\omega} \setminus E$, there is an α such that $[t \cup \{\alpha\}]$ is disjoint from $\bigcup_k L_k$. If there were no such α, then there would be a k such that $\{\alpha : [t \cup \{\alpha\}] \cap L_k \neq \emptyset\}$ is uncountable. This is impossible since this would imply that t was in the closure of L_k.

Now that we know that $\bigcup_k L_k$ is nowhere dense, we may simply work with any closed nowhere dense set L which has p in its closure. We set T to be the collection of all minimal $t \in \kappa^{<\omega}$ with the property that $[t] \cap \bigcup_k L_k$ is empty. A special property of this space ensures that L is contained in the closure of the set T. To see this assume that $s \in L$ – hence no initial segment of s is in T. For each $\alpha \in \kappa$, there is a $t_\alpha \in T$ which extends s and satisfies that $t(|s|) = \alpha$. Now note that s is in the closure of each infinite subset of $\{t_\alpha : \alpha \in \kappa\}$.

It is immediate that the set T is a discrete subset of $\kappa^{<\omega}$, and so the only σ-compact subsets of T are countable. □

Now we use the ideas developed the analysis of Example 3.11 to establish our final result.

Theorem 3.12. *If a space with σ-compact tightness has the property that each compact subset has weight at most ω_1, then the space has countable tightness.*

Proof. Let X be a space with σ-compact tightness and assume that each compact subset of X has weight at most ω_1. Suppose that a point p is in the closure of the union of an increasing sequence $\mathcal{S} = \{X_n : n \in \omega\}$ of compact subsets of X. We must show that p is in the closure of a countable subset of $\bigcup_n X_n$. As we showed in the proof of Lemma 3.5, we may assume that X is an $\mathcal{S}$-example and that X is nowhere locally compact. Therefore, by Definition 3.4, we have that for each m, n, $X_n \cap \bigcup_{k>m} \overline{X_k \setminus X_m}$ is dense in X_n. Also, by Theorem 3.7, we may assume that $\bigcup_n X_n$ is an almost $\mathcal{S}$-P-space.

We will build a tree $\{Z_t : t \in \omega_1^{<\omega}\}$ of zero-sets of $\bigcup_n X_n$ in an effort to mimick the approach in Example 3.11. We let $Z_\emptyset = \bigcup_n X_n$, and one of our inductive assumptions is that for $s \subset t$, Z_t is contained in the $\mathcal{S}$-interior of Z_s. We will also arrange that Z_t is disjoint from $X_{|t|}$. The family $\{Z_{t^\frown\alpha} : \alpha \in \omega_1\}$ will be $\mathcal{S}$-left-separated and the union will be dense in Z_t. By $\mathcal{S}$-left-separated we mean that, for each $\alpha < \omega_1$ and each $n \in \omega$, $Z_{t^\frown\alpha}$ will be disjoint from the closure of $X_n \cap \bigcup_{\beta<\alpha} Z_{t^\frown\beta}$. In other words, for each n, the family $\{Z_{t^\frown\alpha} \cap X_n : \alpha \in \omega_1\}$ will be left-separated.

Here is an informal description of the construction. We will be developing a listing $\{x(t,\gamma) : \gamma \in \omega_1\}$ of points which will be a dense subset of $Z_t \setminus \bigcup\{Z_{t^\frown\alpha} : \alpha \in \omega_1\}$. This is easily done, modulo standard enumeration methods, by using the fact that the weight of each X_n is at most ω_1, and, so choosing, for each $\alpha \in \omega_1$, a dense subset of the set of limit points of the family $\{Z_{t^\frown\beta} : \beta < \alpha\}$.

For each $n \in \omega$, let $\{W(n,\gamma) : \gamma \in \omega_1\} \subset \mathcal{P}(X_n)$ enumerate an open base for X_n. For convenience, we may assume that each element is listed cofinally many times.

Fix a one-to-one function g from $\omega \times \omega_1$ onto ω_1 so that $g(n,\alpha) \geq \alpha$ for all $(n,\alpha) \in \omega \times \omega_1$. Also let $\prec$ be any well-ordering of X. We may suppose that $g(m,0) = m$ for each $m \in \omega$. For each α, we will choose a point $x(t,\alpha) \in Z_t$. To start, if $W(m,0)$ meets Z_t, choose $x(t,m)$ to be the $\prec$-minimum point in $Z_t \cap W(m,0)$, otherwise let $x(t,m)$ be the $\prec$-minimum point of Z_t. Whenever we specify a choice of an $x(t,\gamma)$ we will assume without further mention that we make the $\prec$-least possible choice. Notice that for all ζ such that $W(m,\zeta) = W(m,\xi)$, our choice convention ensures that $x(t,g(m,\zeta))$ is also chosen to be the same point. In this way, we have a natural method of ensuring that each point is listed uncountably many times.

For each m, let $\Xi(m,t,0)$ denote the set of $\xi \in \omega_1$ such that $W(m,\xi)$ contains an $\mathcal{S}$-accessible point of Z_t. We begin by choosing $x(t,g(m,\xi))$ for all (m,ξ) such that either $m \leq |t|$ or $\xi \in \Xi(m,t,0)$. If $m \leq |t|$ then we note that $W(m,\xi) \subset X_{|t|}$ is disjoint from Z_t and we define, for all $\xi \in \omega_1$, $x(t,g(m,\xi))$ be the $\prec$-minimum point of Z_t. If $m > |t|$ and $\xi \in \Xi(m,t,0)$, then $W(m,\xi)$ contains an $\mathcal{S}$-accessible point of Z_t and we set $x(t,\gamma)$ to be the $\prec$-least such point.

Observe that each $Z_{t^\frown\alpha}$ is required to avoid all the points that have been so selected. We continue by induction on $\alpha \in \omega_1$. We choose each $Z_{t^\frown\alpha}$ as well as ensuring that $x(t,\alpha)$ has been chosen, along with possibly many more choices for points $x(t,\gamma)$. Assume that $\{Z_{t^\frown\beta} : \beta < \alpha\}$ have been chosen.

At stage α, we first add to the list of selected points. For each $\ell \in \omega$, let $L(\ell,t,\alpha)$ denote the set of limit points of the collection $\{X_\ell \cap Z_{t^\frown\beta} : \beta < \alpha\}$, more precisely

$$L(\ell,t,\alpha) = \overline{\bigcup_{\beta<\alpha} X_\ell \cap Z_{t^\frown\beta}} \setminus \bigcup_{\beta<\alpha} X_\ell \cap Z_{t^\frown\beta} \ .$$

If $\alpha < \omega$ or $m \leq |t|$, then $L(\ell,t,\alpha)$ is empty. For each m and $0 < \alpha$, let $\Xi(t,m,\alpha) = \{\xi : W(m,\xi) \cap \bigcup_\ell L(\ell,t,\alpha) \neq \emptyset\}$. Of course $\Xi(t,m,j)$ is empty for all $0 < j \in \omega$. We note that for $\xi \in \Xi(m,t,0)$, $x(t,g(m,\xi))$ has been defined above. We inductively assume that, at stage α, $x(t,\gamma)$ has been defined for precisely all γ in the set

$$\alpha \cup \{g(m,\xi) : m \leq |t| \text{ or } \xi \in \bigcup\{\Xi(m,t,\beta) : \omega \leq \beta < \alpha\}\} \ .$$

Now we choose, if necessary $x(t,\alpha)$, and points $x(t,g(m,\xi))$ for each $m \geq |t|$ and each $\xi \in \Xi(m,t,\alpha) \setminus \bigcup_{\beta<\alpha} \Xi(m,t,\beta)$. If $|t| \leq m$ and $\xi \in \Xi(m,t,\alpha) \setminus \bigcup_{\beta<\alpha} \Xi(m,t,\beta)$, then $W(m,\xi)$ meets $\bigcup_\ell L(\ell,t,\alpha)$; choose $x(t,g(m,\xi))$ to be any point in the intersection. If $x(t,\alpha)$ has not yet been chosen, then set $x(t,\alpha) = x(t,0)$.

We must choose $Z_{t^\frown\alpha}$ to be contained in $Z_t \setminus \bigcup_\ell \overline{\bigcup_{\beta<\alpha} X_\ell \cap Z_{t^\frown\beta}}$ (i.e. the $\mathcal{S}$-interior). First identify the unique (m,ξ) so that $g(m,\xi) = \alpha$. If $m = 2k$ for some integer k, then the only additional demand on $Z_{t^\frown\alpha}$ is that if $W(k,\xi)$ meets $Z_t \setminus \bigcup_\ell \overline{\bigcup_{\beta<\alpha} X_\ell \cap Z_{t^\frown\beta}}$ then $Z_{t^\frown\alpha}$ must also meet $W(k,\xi)$. If $m = 2k+1$, then we go about getting $Z_{t^\frown\alpha}$ to be contained in the first α many neighborhoods of $x(t,\xi)$. That is, first choose a zero-set $Z(t,\alpha)$ with $x(t,\xi) \in Z(t,\alpha)$ and ensure that $X_j \cap Z(t,\alpha) \subset W(j,\zeta)$ for each $\zeta < \alpha$ and each j for which $x(t,\xi) \in W(j,\zeta)$. Since for some j, $x(t,\xi) \in X_j$ is not isolated we may ensure that $x(t,\xi)$ is in the $\mathcal{S}$-interior of $Z(t,\alpha)$. Additionally, we arrange that $Z(t,\alpha) \cap Z_{t^\frown\beta}$ is empty for all $\beta < \alpha$. Then, since X is an almost $\mathcal{S}$-P-space, we may choose $Z_{t^\frown\alpha}$ to be contained in the $\mathcal{S}$-interior of $Z(t,\alpha)$ (which means that $x(t,\xi) \notin Z_{t^\frown\alpha}$).

This completes the construction of the family of zero-sets $\{Z_t : t \in \omega_1^{<\omega}\}$, and the associated points $\{x(t,\alpha) : t \in \omega_1^{<\omega}, \alpha \in \omega_1\}$.

Claim 1. For each $t \in \omega_1^{<\omega}$, the set $\{x(t,\gamma) : \gamma \in \omega_1\}$ is dense in $Z_t \setminus \bigcup\{Z_{t^\frown\alpha} : \alpha \in \omega_1\}$.

Proof. Suppose that $x \in Z_t \setminus \bigcup\{Z_{t^\frown\alpha} : \alpha \in \omega_1\}$ and choose any $k > |t|$ so that $x \in X_k$, and by Lemma 3.9, so that for every zero set Z containing x, x is in the closure of the points of X_k which are in the $\mathcal{S}$-interior of Z. The first step of the construction of the collection $\{x(t,\gamma) : \gamma \in \omega_1\}$ was to choose a dense subset of the collection of the $\mathcal{S}$-accessible points of Z_t, so we may as well assume that x is not an $\mathcal{S}$-accessible point of Z_t. Choose any $\xi, \zeta \in \omega_1$ so that $x \in W(k,\xi)$ and $\overline{W(k,\xi)} \subset W(k,\zeta)$ and, towards a contradiction, suppose that $W(k,\zeta)$ is disjoint from the closure of the set $\{x(t,\gamma) : \gamma \in \omega_1\}$. Let $\alpha = g(2k,\xi)$ and consider the stage α in the construction. Since X_k is compact, it follows from the selection of $x(t,g(k,\zeta))$ that $W(k,\xi)$ must meet only finitely many of the sets in $\{Z_{t^\frown\alpha} : \alpha \in \omega_1\}$ (otherwise $W(k,\zeta)$ would include a limit point). Since $W(k,\xi)$ is listed cofinally often, we can assume that ξ is so large that $W(k,\xi)$ is disjoint from $Z_{t^\frown\alpha}$. However, this now contradicts our choice of $Z_{t^\frown\alpha}$, since $W(k,\xi)$ does, by the assumption from Lemma 3.9, meet $Z_t \setminus \bigcup_\ell \overline{\bigcup_{\beta<\alpha} X_\ell \cap Z_{t^\frown\beta}}$. □

Claim 2. For each $(t,\gamma) \in \omega_1^{<\omega} \times \omega_1$ and each countable sequence $\{W_m : m \in \omega\}$ of neighborhoods of $x(t,\gamma)$, there are uncountably many α with $Z_{t^\frown\alpha} \subset \bigcap_m W_m$.

Proof. Fix any (k,ξ) so that $g(k,\xi)=\gamma$. Let n be minimal such that $x(t,\gamma)\in X_n$. For each $\ell\geq n$ and $m\in\omega$, choose $\beta_{m,\ell}$ so that $x(t,\gamma)\in W(\ell,\beta_{m,\ell})\subset W_m$. Remember that there are uncountably many ζ so that $x(t,g(k,\zeta))=x(t,\gamma)$. For each $\zeta\geq\sup_{n\leq\ell}\beta_{m,\ell}$ and $\alpha=g(2k+1,\zeta)$, $Z(t,\alpha)$ was chosen so that $Z(t,\alpha)\cap X_\ell$ is contained in $W(\ell,\beta_{m,\ell})$ for each $\ell\in\omega\setminus n$. Therefore $Z(t,\alpha)$ is contained in W, and so is $Z_{t^\frown\alpha}$. □

It follows from Claim 1 that the set $\{x(t,\gamma): t\in\omega_1^{<\omega}\ ,\ \gamma\in\omega_1\}$ is dense in X. By symmetry, we will now assume that p is in the closure of $Y=\{x(t,\gamma): t\in\bigcup_n\omega_1^{2n}\ ,\gamma\in\omega_1\}$. By the σ-compact tightness assumption, we choose a sequence $\{L_n:n\in\omega\}$ of compact sets so that $L_n\subset X_n\cap Y$ for each n, and so that p is in the closure of $\bigcup_n L_n$.

Let T be the set of minimal elements of $\{t\in\omega_1^{<\omega}: Z_t\cap\bigcup_n L_n=\emptyset\}$. Evidently, T is an antichain in $\omega_1^{<\omega}$.

Claim 3. For every n, the family $\{X_n\cap Z_t: t\in T\}$ is left-separated.

We use the lexicographic ordering on members of T: specifically $s<_\ell t$ providing there is a $k\in dom(s)\cap dom(t)$ such that $s(k)<t(k)$. Let $t\in T$ and for each $j\in dom(t)$, let $L(T,t,j)=\{s\in T\cap\omega^j: s\restriction j\subset t\ \ ,\ \ s(j)<t(j)\}$. It suffices to show that, for each $j\in dom(t)$, Z_t is disjoint from the closure of the union of the family $\{Z_s: s\in L(T,t,j)\}$. Notice that for $s\in L(T,t,j)$, $Z_s\subseteq Z_{s\restriction j+1}$. Also, $Z_t\subset Z_{t\restriction j}$ and by construction, $Z_{t\restriction j}$ is disjoint from the closure of the union of the family $\{Z_{s\restriction j+1}: s\in L(T,t,j)\}$. Therefore the union of the family $\{Z_s : s\in L(T,t,j)\}$ is contained in the union of the family $\{Z_{s\restriction j+1}: s\in L(T,t,j)\}$, and we have established the Claim.

Claim 4. p is in the closure of the set $\{x(t,0): t\in T\}$.

To establish this claim, we simply show that $\bigcup_n L_n$ is contained in the closure of the set $\{x(t,0): t\in T\}$. Choose any point $y\in\bigcup_n L_n$ and fix $(s,\gamma)\in\omega_1^{<\omega}\times\omega_1$ so that $y=x(s,\gamma)$. Since $x(s,\gamma)\in\bigcup_n L_n$, no initial segment of s is a member of T. Let W be any open set containing $x(s,\gamma)$, we show there is a $t\in T$ such that $x(t,0)\in W$. Applying Claim 2, there is an extension t of s with $dom(t)=dom(s)+1$, such that $Z_t\subset W$. Since $dom(t)$ is an odd integer, $x(t,\xi)\notin\bigcup_n L_n$ for all ξ. If Z_t is disjoint from $\bigcup_n L_n$, then $t\in T$ and we have found our desired $x(t,0)\in W$. Otherwise we now check that there is α such that $t^\frown\alpha\in T$. Since $x(t^\frown\alpha,0)\in Z_{t^\frown\alpha}\subset Z_t\subset W$ this will complete the proof of the claim. For each m, choose an open W_m containing $x(t,0)$ wich is disjoint from L_m. Again we use Claim 2 to choose an α so that $Z_{t^\frown\alpha}\subset\bigcap_m W_m$, and this completes the proof of the Claim.

Claim 5. The point p is in the closure of a countable subset of X.

Putting Theorem 3.1 and Claim 3 together, we have that each σ-compact subset of $\{x(t,0) : t \in T\}$ is countable. Since we are assuming that $\{p\} \cup X$ has σ-compact tightness, we are done by Claim 4. □

We must end with a question.

Question 1. Is there a bound on the tightness of regular spaces with σ-compact tightness?

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