

AUTOMORPHISMS OF $\mathcal{P}(\omega)/\text{fin}$ AND LARGE CONTINUUM

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ABSTRACT. We prove that it is consistent with $\mathfrak{c} > \aleph_2$ that all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial.

1. INTRODUCTION

The study of automorphisms of $\mathcal{P}(\omega)/\text{fin}$ has, by now, an extensive and fascinating history. Naturally $\mathcal{P}(\omega)/\text{fin}$ is the quotient Boolean algebra of the complete Boolean power set algebra $\mathcal{P}(\omega)$ by the ideal fin of finite sets. Every bijection between cofinite subsets of ω induces an automorphism of $\mathcal{P}(\omega)/\text{fin}$ and such automorphisms are said to be trivial. W. Rudin [6] established that the continuum hypothesis implied that there were non-trivial automorphisms. S. Shelah [7] established that it was consistent that all automorphisms were trivial and Shelah and Steprans [8] proved that this was a consequence of PFA. Our results follow the basic approach of both [7, 8] but also benefit from the considerable contributions in [2, 11, 12] and others. Shelah and Steprans [9] have shown that it is consistent with Martin's Axiom plus $\mathfrak{c} > \aleph_2$ that there are non-trivial, even nowhere trivial, automorphisms. In this paper we establish that it is consistent with Martin's Axiom plus $\mathfrak{c} > \aleph_2$ that all automorphisms are trivial.

Our original submission of this paper established the result primarily as a forcing construction and argument but it also featured the identification of some basic, and familiar, combinatorial principles such as ω^ω -cohere and P -cohere (see Definitions 2.3 and 2.6) that help constrain the existence of non-trivial automorphisms and proving that they held in the constructed forcing model. At the *instruction* of the referee we have worked much harder to present the proof of the main result in the desired form of formulating such a combination of combinatorial principles that imply that all automorphisms are trivial and to then prove the consistency, with large values of \mathfrak{c} , of this combination. Our

solution is to formulate a version of Martin's Axiom that we call ω_1 -strategic MA (Definition 1.3). We are pleased with the reward of this effort as it led to a result in the spirit of Velickovic's inspiring result concerning σ -Borel liftings of automorphisms (see [10] and [11]). This result is presented in Theorem 1.1 and the final formulation of our main result with the hoped for more combinatorial flavor is Theorem 1.2.

For a function $F : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$, we say that F induces an automorphism (of $\mathcal{P}(\omega)/\text{fin}$) if $F(x) =^* F(y)$ whenever $x =^* y$ and the function sending the equivalence class of x (mod finite) to that of $F(x)$ (mod finite) is indeed an automorphism of $\mathcal{P}(\omega)/\text{fin}$. When F does not induce a trivial automorphism, the family $\text{triv}(F)$ is the ideal of sets $a \subset \omega$ such that $F \upharpoonright \mathcal{P}(a)$ is trivial in the usual sense, namely that there is a function h_a from a into ω that is 1-to-1 on a cofinite subset of a and satisfies that $F(x) =^* h_a(x)$ for all $x \subset a$. It is shown in [7] that $\text{triv}(F)$ is an ideal on ω . As usual, $\text{triv}(F)^+$ will denote (infinite) subsets of ω that are not elements of $\text{triv}(F)$.

Let BOR denote the set of Borel self-maps on $\mathcal{P}(\omega)$ where the topology on $\mathcal{P}(\omega)$ is generated by the clopen subbase of sets $\{\{a \subset \omega : n \in a\} : n \in \omega\}$. If F is any self-map on $\mathcal{P}(\omega)$ then F can be covered by $\mathcal{E} \subset \text{BOR}$ simply means that, for all $x \in \mathcal{P}(\omega)$, there is an $E \in \mathcal{E}$ satisfying that $F(x) =^* E(x)$. Similarly we will say, for $\mathcal{E} \subset \text{BOR}$, that F is \mathcal{E} -Borel if it can be covered by \mathcal{E} . As is well-known, if F is a lifting of an automorphism that can be covered by a countable $\mathcal{E} \subset \text{BOR}$, then F is trivial. We wish to consistently generalize this to coverings by larger subsets of BOR. We will say that F is λ -Borel or $< \mathfrak{c}$ -Borel if it can be covered by \mathcal{E} for some $\mathcal{E} \subset \text{BOR}$ with $|\mathcal{E}| \leq \lambda$ or $|\mathcal{E}| < \mathfrak{c}$ respectively.

Recall that \mathfrak{p} is the well-known pseudo-intersection number and that the statement $\mathfrak{p} = \mathfrak{c}$ is equivalent, by M. Bell's theorem, to Martin's Axiom for σ -centered posets. The following is really the major new result in the paper. We restate and prove it later as Theorem 4.5.

Theorem 1.1 ($\mathfrak{p} = \mathfrak{c}$). *Every automorphism of $\mathcal{P}(\omega)/\text{fin}$ that has a $< \mathfrak{c}$ -Borel lifting is trivial.*

Theorem 1.1 is a critical piece in adapting the Shelah-Steprans PFA technique from [8] to models with $\mathfrak{c} > \omega_2$.

Theorem 1.2. *The combinatorial principle ω_1 -strategic MA implies that all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial.*

We next define the ω_1 -strategic MA principle. The model component is attempting to capture the key properties corresponding to having a forcing extension, $V[G_\kappa]$, for a generic filter G_κ for a standard finite

support ccc Martin's Axiom iteration in which each P_α has cardinality less than κ for all $\alpha < \kappa$, and $M_\eta = H(\kappa)[G_\kappa \cap M_\eta]$. Secondly $\diamond(S_1^\kappa)$ in the ground model is strategically utilized to choose \dot{Q}_λ for all $\lambda \in S_1^\kappa$. S_1^κ denotes the stationary set of ordinals in κ that have cofinality ω_1 . The idea is that if we have a viable ω_1 -length recursive strategy for constructing a ccc poset, then at least once, we will force with a poset that was constructed using that strategy. We formulate key combinatorial principles in Section 2 that are more routine consequences, as shown in Section 3. Then the most difficult extension of [9] in Section 3 that requires the new result about Borel liftings. Some specialists may prefer to ignore ω_1 -strategic MA in favor the forcing formulation in Theorem 1.5.

Definition 1.3. ω_1 -strategic MA is the following principle: \mathfrak{c} is a regular cardinal and $H(\mathfrak{c})$ can be covered by a strictly increasing chain, $\mathfrak{M} = \langle M_\eta : \eta < \mathfrak{c} \rangle$, of elementary submodels satisfying, for each $\lambda \in S_1^\mathfrak{c}$,

- (1) $M_\lambda \cap \mathcal{P}(\omega) = \bigcup \{M_\mu \cap \mathcal{P}(\omega) : \mu < \lambda\}$ and has cardinality $< \mathfrak{c}$,
- (2) there is a continuous strictly increasing cofinal function C_λ from ω_1 to λ satisfying that for each $\rho \in \lambda^{\omega_1} \cap M_\lambda$, there is a stationary set of $\delta \in \omega_1$ such that $\rho \upharpoonright \delta \in M_{C_\lambda(\delta)}$, and
- (3) for every $\mathcal{R} = \langle (R_t, \eta_t) : t \in \mathfrak{c}^{<\omega_1} \rangle$ satisfying that $R_t \in M_{\eta_t}$ and $\eta_t < \eta_s$ for all $t \subsetneq s \in \mathfrak{c}^{<\omega_1}$ (we say that \mathcal{R} is an \mathfrak{M} -system), there is a cub $C(\mathcal{R}) \subset \mathfrak{c}$ satisfying that for all $\lambda \in C(\mathcal{R}) \cap S_1^\mathfrak{c}$, there is a $\rho \in \lambda^{\omega_1}$ such that $\langle R_{\rho \upharpoonright \alpha} : \alpha \in \omega_1 \rangle$ and $\langle \eta_{\rho \upharpoonright \alpha} : \alpha \in \omega_1 \rangle$ are elements of M_λ with the latter being cofinal in λ .

Say that an \mathfrak{M} -system $\mathcal{R} = \langle (R_t, \eta_t) : t \in \mathfrak{c}^{<\omega_1} \rangle$ is a ccc poset \mathfrak{M} -strategy if for every $t \subset s \in \mathfrak{c}^{<\omega_1}$, R_t is a subposet of R_s , and there is a cub $C(\mathcal{R})$ satisfying that for each $\lambda \in C(\mathcal{R}) \cap S_1^\mathfrak{c}$ and each $\rho \in M_\lambda \cap \lambda^{\omega_1}$ as described in item 3, $R_\rho = \bigcup \{R_{\rho \upharpoonright \alpha} : \alpha \in \omega_1\}$ is a ccc poset in the model M_λ . Say that such an (R_ρ, M_λ) is a poset/model pair constructed by \mathcal{R} .

Finally, the principle states that for every ccc poset \mathfrak{M} -strategy \mathcal{R} there is a poset/model pair (R_ρ, M_λ) constructed by the system for which there is an M_λ -generic filter for the poset R_ρ .

The principle ω_1 -strategic MA implies the usual Martin's Axiom. This can be seen by noticing that for any ccc poset $R \in H(\mathfrak{c})$, each R_t can equal R for all $t \in \mathfrak{c}^{<\omega_1}$. It is a routine exercise to verify that this principle is consistent with arbitrarily large values of \mathfrak{c} . More specifically, start with a $\diamond(S_1^\kappa)$ -sequence and use a standard finite support ccc iteration $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ that is designed so that P_κ forces Martin's Axiom. The additional requirement is that given any P_κ -name

of a sequence $\langle (R_t, \eta_t) : t \in \kappa^{<\omega_1} \rangle$ as in the Definition, it is a simple matter, if the λ -th element of the diamond sequence element is suitable, to choose a function $\rho \in \lambda^{\omega_1} \cap V[G_\lambda]$ satisfying that $\{\eta_{\rho|\gamma} : \gamma < \omega_1\}$ is cofinal in λ , and to set \dot{Q}_λ in the iteration sequence to equal R_ρ . Then it is clear that we have proven the next theorem.

Theorem 1.4. *It is consistent with arbitrarily large values of \mathfrak{c} that ω_1 -strategic MA holds, and therefore that all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial.*

The remainder of the paper is devoted to proving Theorem 1.1 and Theorem 1.2 but anyone more interested in a forcing approach can easily modify our proofs so as to prove the following, largely equivalent, result since ω_1 -strategic MA is a forcing axiom. The discussion in the preceding few paragraphs concerning the formulation of ω_1 -strategic MA should serve in lieu of proofs of the previous and this next Theorem.

Theorem 1.5. *Assume GCH and that $\kappa > \omega_1$ is a cardinal such that $\diamond(S_1^\kappa)$ holds. Let $e : \kappa \rightarrow H(\kappa)$ be an enumeration of $H(\kappa)$ and let $\langle D_\lambda : \lambda \in S_1^\kappa \rangle$ be a $\diamond(S_1^\kappa)$ -sequence.*

The poset P_κ forces that MA (even ω_1 -strategic MA) holds and that all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial providing the finite support iteration $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ satisfies these simple conditions:

- (1) *for each $\beta < \kappa$, \dot{Q}_β is a P_β -name in $H(\kappa)$ of a ccc poset,*
- (2) *for each $\lambda \in S_1^\kappa$, if $e(D_\lambda) = \{e(\xi) : \xi \in D_\lambda\}$ is a P_λ -name of a ccc poset, then this is the value of \dot{Q}_λ ,*
- (3) *if $\lambda \in S_1^\kappa$ and $e(D_\lambda)$ is a P_λ -name that is forced by 1_{P_λ} to be a system $\langle R_t : t \in \lambda^{<\omega_1} \rangle$ satisfying that for $t \subset s \in \lambda^{<\omega_1}$, R_t is a subposet of R_s , and if there is a $\rho \in \lambda^{\omega_1}$ such that $\langle \rho(\xi) : \xi < \omega_1 \rangle$ is strictly increasing and cofinal in λ and $R_\rho = \bigcup \{R_{\rho|\xi} : \xi \in \omega_1\}$ is ccc, then \dot{Q}_λ is forced to be one such R_ρ .*

Here is a brief outline of the paper. In Section 2 we introduce the combinatorial notions leading to the formulation of ω^ω -cohere and P -cohere and prove that these imply that an automorphism is trivial if it has a lifting F such that $\text{triv}(F)$ is ccc over fin . In Section 3 we prove the principles from Section 2 hold in models of ω_1 -strategic MA. In Section 4 we prove the result on $<\mathfrak{c}$ -Borel. In Section 5 we finish the proof of Theorem 1.2 by generalizing the Shelah-Steprans method from [8] of splitting and freezing ω_1 -gaps related to the structure of $\text{triv}(F)$.

2. TOOLS FOR YOUR FORCING ALL AUTOMORPHISMS ARE TRIVIAL CONSTRUCTION

In this section we review some familiar combinatorial principles that greatly assist in proving that all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial. We prove in Section 3 that they are consequences of ω_1 -strategic MA. This first result is true in ZFC.

Proposition 2.1. *If F induces an automorphism and $a \in \text{triv}(F)^+$, then for any 1-to-1 function $h \in \omega^a$, there is an infinite $y \subset a$ such that $h(y) \cap F(y)$ is finite.*

Proof. Since $a \notin \text{triv}(F)$, the function h does not induce $F \upharpoonright \mathcal{P}(a)$. Choose any $x \subset a$ so that $h(x) \Delta F(x)$ is infinite. If $h(x) \setminus F(x)$ is infinite, then let $y \subset x$ be chosen so that $h(y) = h(x) \setminus F(x)$. Since $F(y) \subset^* F(x)$, we have $h(y) \cap F(y)$ is finite as required. In the other case when $F(x) \setminus h(x)$ is infinite, choose $y \subset x$ so that $F(y) =^* F(x) \setminus h(x)$. Here again we have that $h(y) \subset h(x)$ and $h(x) \cap F(y) =^* \emptyset$. \square

Definition 2.2. A family \mathcal{H} of possibly partial functions from a countable set D to ω is coherent if for $h_1, h_2 \in \mathcal{H}$, the set $\{n \in \text{dom}(h_1) \cap \text{dom}(h_2) : h_1(n) \neq h_2(n)\}$ is finite.

Such a coherent family \mathcal{H} is maximal if whenever $\mathcal{H} \cup \{\bar{h}\}$ is coherent, the domain of \bar{h} is in the ideal generated by $[D]^{<\aleph_0} \cup \{\text{dom}(h) : h \in \mathcal{H}\}$. A coherent family will be called trivial if D is in this ideal.

For a function $f \in \omega^\omega$, let f^\downarrow denote the set $\{(n, m) \in \omega \times \omega : m < f(n)\}$. A family \mathcal{H} of functions is an ω^ω -family if $\mathcal{H} = \{h_f : f \in \omega^\omega\}$ and, for each $f \in \omega^\omega$, h_f is a function from f^\downarrow to ω .

We note that every subfamily of a coherent family is also coherent.

Definition 2.3. Say that the principle ω^ω -cohere holds if each ω^ω -family \mathcal{H} that is coherent, there is a function $h : \omega \times \omega \rightarrow \omega$ such that $\mathcal{H} \cup \{h\}$ is also coherent.

The principle ω^ω -cohere (not so named) is a well-known consequence of OCA due to Todorćević (see [2, 2.2.7]). This next result is similar to [11, proof of Lemma 2.5].

Lemma 2.4. *Suppose that F induces an automorphism on $\mathcal{P}(\omega)/\text{fin}$ and suppose that $\text{triv}(F)$ is proper dense ideal. The principle ω^ω -cohere implies that if $\{a_n : n \in \omega\}$ is a mod finite increasing sequence of subsets of ω then either there is an n so that $\omega \setminus a_n \in \text{triv}(F)$, or there is an $b \in \text{triv}(F)^+$ that is almost disjoint from each a_n .*

Proof. Let $\{a_n : n \in \omega\} \subset [\omega]^{\aleph_0}$ be increasing. There is nothing to prove if the family $\{a_n : n \in \omega\}$ is eventually constant mod finite, so we may assume that it is strictly increasing mod finite. As usual, let $\{a_n\}_n^\perp$ denote the ideal of sets b that are almost disjoint from each a_n . Assume that $b \in \text{triv}(F)$ for all $b \in \{a_n\}_n^\perp$, and as usual, let h_b be the function on b inducing $F \upharpoonright \mathcal{P}(b)$. Since the family $\mathcal{H} = \{h_b : b \in \{a_n\}_n^\perp\}$ is a coherent family, it is an obvious consequence of ω^ω -cohere that there is a function $h \in \omega^\omega$ such that $\mathcal{H} \cup \{h\}$ is coherent.

We show that there is an n_0 so that $h \upharpoonright (\omega \setminus a_{n_0})$ is 1-to-1. Otherwise we may choose a sequence of pairs $\{(i_n, j_n) : n \in \omega\}$ so that $\{i_n, j_n\} \subset \omega \setminus a_n$ and $h(i_n) = h(j_n)$. By construction the set $b = \{i_n : n \in \omega\} \cup \{j_n : n \in \omega\}$ is an element of $\{a_n\}_n^\perp$. Since h_b is 1-to-1, this contradicts that $h_b \subset^* h$.

If there is an $n \geq n_0$ so that $h \upharpoonright (\omega \setminus a_n)$ induces F on $\mathcal{P}(\omega \setminus a_n)$ then the Lemma is proven. Otherwise, we may choose, using Lemma 2.1, for each $n_0 \leq n \in \omega$ an infinite set $x_n \subset \omega \setminus a_n$ so that $h(x_n) \cap F(x_n)$ is finite. Since x_n cannot be an element of $\{a_n\}_n^\perp$, we may shrink x_n so that there is an $k_n \geq n$ such that $x_n \subset a_{k_{n+1}} \setminus a_{k_n}$. Since $\text{triv}(F)$ is assumed to be dense, we may also assume that $x_n \in \text{triv}(F)$. Choose an infinite $J \subset \omega$ so that for $n < m \in J$, $k_{n+1} \leq m$. For each $n \in J$, let h_n denote the function h_{x_n} . Note that $h(x_n) \cap h_n(x_n) =^* h(x_n) \cap F(x_n)$ is finite. By a simple recursion we can further arrange that $h(i) \neq h_n(j)$ for all $i \in \bigcup \{x_k : k \in J\}$ and $j \in x_n$. Let $x = \bigcup \{x_n : n \in J\}$. For each $n \in J$, choose a finite $H_n \subset x_n$ so that $h_n(x_n \setminus H_n) \subset F(x)$. By recursion on $n \in J$, choose $i_n \in x_n \setminus H_n$ so that $h_n(i_n) \notin \bigcup \{F(a_k) : k \leq k_n\}$. Choose $y \subset \omega$ so that $F(y) =^* \{h_n(i_n) : n \in J\}$ and note that $y \cap a_k$ is finite for all $k \in \omega$. Since $F(y) \subset F(x)$ we may assume that $y \subset x$. But now, $F(y) =^* h(y)$ and yet $h(y)$ is disjoint from $\{h_n(i_n) : n \in J\}$. \square

Definition 2.5. An ideal \mathcal{I} on a set $A \subset \omega$ is ccc over fin if $\mathcal{A} \cap \mathcal{I}$ is not empty for every uncountable almost disjoint family \mathcal{A} of subsets of A .

The notion of an ideal being ccc over fin was introduced in [2].

Definition 2.6. Say that the principle P-cohere holds providing there is no non-trivial coherent family of functions whose domains form a ccc over fin P -ideal.

Lemma 2.7. *Assume that the principles ω^ω -cohere and P-cohere hold. If F induces an automorphism on $\mathcal{P}(\omega)/\text{fin}$ then $A \in \text{triv}(F)$ for any $A \subset \omega$ such that $\text{triv}(F) \cap \mathcal{P}(A)$ is ccc over fin.*

Proof. Again assume that we have a fixed assignment of $h_a \in \omega^a$ for all $a \in \text{triv}(F)$ such that $h_a(x) =^* F(x)$ for all $x \subset a$. Since F induces an automorphism, it follows that the family $\mathcal{H} = \{h_a : a \in \text{triv}(F)\}$ is a coherent family. Since a ccc over fin ideal is a dense ideal, we assume for the remainder of the proof that $\text{triv}(F)$ is a proper dense ideal. We prove that it can not be ccc over fin.

We first check that if $h \in \omega^\omega$, then $\mathcal{H} \cup \{h\}$ is not coherent. Since we are assuming that $\omega \notin \text{triv}(F)$, we have, by Lemma 2.1, that there is an infinite $x \subset \omega$ such that $h(x) \cap F(x)$ is finite. We are also assuming that $\text{triv}(F)$ is a dense ideal, so, by possibly shrinking x , we can assume that $x \in \text{triv}(F)$. Then $h_x \in \mathcal{H}$ and evidently, $\{h_x, h\}$ is not coherent.

It is immediate from the assumption that P-cohere holds that $\text{triv}(F)$ is not a P -ideal. We complete the proof by considering the case where $\text{triv}(F)$ is not a P -ideal. Fix any increasing sequence $\{a_n : n \in \omega\} \subset \text{triv}(F)$ with the property that $a \notin \text{triv}(F)$ for any $a \subset \omega$ that mod finite contains each a_n . Since $\omega \notin \text{triv}(F)$, it follows that $\omega \setminus a_n \notin \text{triv}(F)$ for all $n \in \omega$. Recall that $\{a_n\}_n^\perp$ denotes the ideal of sets $b \subset \omega$ that are almost disjoint from a_n for each n . For each $b \in \{a_n\}_n^\perp$, the ideal $\{b \cup a_n\}_n^\perp$ is a P -ideal. Using ω^ω -cohere and Lemma 2.4, for every $b_0 \in \{a_n\}_n^\perp$, there is a $b_1 \in \text{triv}(F)^+ \cap \{b_0 \cup a_n\}_n^\perp$. Therefore, we can by recursion, construct a mod finite increasing sequence $\{b_\alpha : \alpha < \omega_1\} \subset \{a_n\}_n^\perp$ so that, for each $\alpha < \omega_1$, $b_{\alpha+1} \setminus b_\alpha \in \text{triv}(F)^+ \cap \{b_\alpha \cup a_n\}_n^\perp$. This clearly shows that $\text{triv}(F)$ is not then ccc over fin. \square

Definition 2.8. Say that a family $\{h_\alpha : \alpha \in \omega_1\}$ of partial functions from a countable set A to ω is Luzin incoherent if for all $\alpha < \beta$, $h_\alpha \cup h_\beta$ is not a function.

Proposition 2.9. *If $\mathcal{H} = \{h_\alpha : \alpha \in \omega_1\}$ is a Luzin incoherent family of partial functions from a countable set A to ω , then for all $h \in \omega^A$, $\mathcal{H} \cup \{h\}$ is not coherent.*

Proof. Identify A with ω . Let $h \in \omega^\omega$ and assume, for a contradiction, that $\mathcal{H} \cup \{h\}$ is coherent. Choose $m \in \omega$ and function $t : m \rightarrow \omega$ so that, there is an uncountable $\Gamma \subset \omega_1$ so that, for all $\alpha \in \Gamma$, $h_\alpha \upharpoonright (\text{dom}(h_\alpha) \setminus m) \subset h$ and $h_\alpha \upharpoonright m \subset t$. Of course we now find that for all $\alpha < \beta$ both in Γ , $h_\alpha \cup h_\beta \upharpoonright (\omega \setminus m)$ is a subfunction of $t \cup h \upharpoonright (\omega \setminus m)$. \square

Definition 2.10. Let $\mathcal{H} = \langle h_\alpha : \alpha \in \delta \rangle$ be a coherent family of partial functions on ω . Define the poset $Q(\mathcal{H})$ to be the set of pairs $q = (\{s_\rho^q : \rho \in 2^{\leq n_q}\}, \Gamma_q)$ satisfying:

- (1) $n_q \in \omega$ and $\{s_\rho^q : \rho \in 2^{\leq n_q}\} \subset [\omega]^{< \aleph_0}$,
- (2) $\Gamma_q \in [\delta]^{< \aleph_0}$,

- (3) for $\alpha \neq \beta \in \Gamma_q$ and $\rho \in 2^{n_q}$, the union $(h_\alpha \upharpoonright s_\rho^q) \cup (h_\beta \upharpoonright s_\rho^q)$ is not a function,
- (4) for each $\rho \in 2^{n_q}$, the sequence $\{s_{\rho \upharpoonright j}^q : j \leq n_q\}$ is increasing,
- (5) for $\rho, \psi \in 2^{n_q}$, $s_\rho^q \cap s_\psi^q = s_{\rho \upharpoonright j}^q$ where j is maximal such that $\rho \upharpoonright j = \psi \upharpoonright j$.

The ordering on $Q(\mathcal{H})$ is that $q \leq r$ providing $n_q \geq n_r$, $\Gamma_q \supset \Gamma_r$, and $s_\rho^q = s_\rho^r$ for all $\rho \in 2^{n_r}$.

Let $Q^+(\langle h_\xi : \xi \in \delta \rangle)$ be the isomorphic poset where $q \in Q(\langle h_\xi : \xi \in \delta \rangle)$ is sent to $(s^q, \{h_\xi : \xi \in \Gamma_q\})$.

Fix any bijection $\varphi : \omega \times \omega \rightarrow \omega$. If $\langle h_\alpha : \alpha \in \omega_1 \rangle$ is a coherent subfamily of an ω^ω -family of functions, then let $Q_\varphi(\langle h_\alpha : \alpha \in \omega_1 \rangle)$ denote the poset $Q(\langle h_\alpha \circ \varphi : \alpha \in \omega_1 \rangle)$.

The idea for the poset $Q(\mathcal{H})$ can be found in [2, proof of 3.8.1]. The alternate version $Q^+(\langle h_\xi : \xi \in \delta \rangle)$ has the property that $\{h_\xi : \xi \in \delta\}$ is in its transitive closure. This will be useful in the context of ω_1 -strategic MA.

Lemma 2.11. *Assume that $\mathcal{H} = \langle h_\alpha : \alpha \in \omega_1 \rangle$ is a coherent family of partial functions on ω such that $\{\text{dom}(h_\alpha) : \alpha \in \omega_1\}$ is mod finite increasing, and there is no $h \in \omega^\omega$ such that $\mathcal{H} \cup \{h\}$ is coherent. Then the poset $Q(\mathcal{H})$ is ccc. Furthermore, there is a condition $q \in Q(\mathcal{H})$ that forces there is an uncountable $\Gamma \subset \omega_1$ and an uncountable almost disjoint family $\mathcal{A} \subset [\omega]^{\aleph_0}$, such that, for each $a \in \mathcal{A}$, the sequence $\langle h_\alpha \upharpoonright a : \alpha \in \Gamma \rangle$ is a Luzin incoherent family.*

Proof. Let $\{q_\xi : \xi \in \omega_1\} \subset Q(\mathcal{H})$. By passing to an uncountable subcollection, we can assume that there is a single sequence $\{s_\rho : \rho \in 2^{\leq n}\}$ such that, for all $\xi \in \omega_1$ and $\rho \in 2^{\leq n}$, $n_{q_\xi} = n$ and $s_\rho^{q_\xi} = s_\rho$. For each $\xi \in \omega_1$, let $\Gamma_\xi = \Gamma_{q_\xi}$. By again passing to an uncountable subcollection we may assume that the family $\{\Gamma_\xi : \xi \in \omega_1\}$ is a Δ -system with root Γ' . For each $\xi \in \omega_1$, let α_ξ be the minimum element of $\Gamma_\xi \setminus \Gamma'$. With yet another such reduction, we may assume that there is an integer m so that, for all ξ , $h_{\alpha_\xi} \upharpoonright (\text{dom}(h_{\alpha_\xi}) \setminus m)$ is a subset of h_β for all $\beta \in \Gamma_\xi \setminus \Gamma'$.

Fix any countable elementary submodel M of $H(\mathfrak{c}^+)$ with the family $\{h_{\alpha_\xi} : \xi \in \omega_1\}$ as an element. Let $\delta = M \cap \omega_1$. Fix any $\xi \in \omega_1 \setminus \delta$ and finite set $I \subset \text{dom}(h_{\alpha_\xi})$. By simple elementary we have that the set $\{\eta \in \omega_1 : h_{\alpha_\eta} \upharpoonright I = h_{\alpha_\xi} \upharpoonright I\}$ is an element of M and is uncountable. Since the family \mathcal{H} is mod finite increasing and is not coherently extendable, it follows that $\bigcup \{h_{\alpha_\xi} \upharpoonright \text{dom}(h_{\alpha_\xi}) \setminus m : \delta \leq \xi\}$ is not a function. Choose any $i_0 > m$ so that there are $\delta \leq \xi_0, \eta_0$ with

$h_{\alpha_{\xi_0}}(i_0) = j_0 \neq k_0 = h_{\alpha_{\eta_0}}(i_0)$. Let $J_0 = \{\xi : h_{\alpha_\xi}(i_0) = j_0\}$ and $K_0 = \{\eta : h_{\alpha_\eta}(i_0) = k_0\}$. Again, for every $i \geq i_0$, $\bigcup\{h_{\alpha_\xi} \upharpoonright \text{dom}(h_{\alpha_\xi}) \setminus i : \xi \in J_0 \setminus \delta\}$ is not a function but the domain of this relation will mod finite contain the domain of $\bigcup\{h_{\alpha_\eta} \upharpoonright \text{dom}(h_{\alpha_\eta}) \setminus i : \eta \in K_0 \setminus \delta\}$. Therefore we may choose an $i_1 > i_0$ so that there are $\xi_1 \in J_0$ and $\eta_1 \in K_0$ so that $h_{\alpha_{\xi_1}}(i_1) = j_1 \neq k_1 = h_{\alpha_{\eta_1}}(i_1)$. It should be clear that continuing with such a recursive construction we can find a sequence $\{i_\ell : \ell < 2^{n+1}\}$ so that there is a pair ξ, η satisfying that $h_{\alpha_\xi}(i_\ell) \neq h_{\alpha_\eta}(i_\ell)$ for all $\ell < 2^{n+1}$.

We are ready to find a condition r that is below each of q_ξ and q_η . We let $n_r = n + 1$ and we re-index $\{i_\ell : \ell < 2^{n+1}\}$ as $\{i_\rho : \rho \in 2^{n+1}\}$. For each $\rho \in 2^{n+1}$, the definition of s_ρ^r is $s_{\rho \upharpoonright n} \cup \{i_\rho\}$. For $\rho \in 2^{\leq n}$, $s_\rho^r = s_\rho$. Clearly, for each $\rho \in 2^{n+1}$, we have that $(h_{\alpha_\xi} \upharpoonright s_\rho^r) \cup (h_{\alpha_\eta} \upharpoonright s_\rho^r)$ is not a function because $h_{\alpha_\xi}(i_\rho) \neq h_{\alpha_\eta}(i_\rho)$. By the choice of m it also therefore follows that for all $\beta_0 \in \Gamma_\xi \setminus \Gamma'$ and $\beta_1 \in \Gamma_\eta \setminus \Gamma'$, $(h_{\beta_0} \upharpoonright s_\rho^r) \cup (h_{\beta_1} \upharpoonright s_\rho^r)$ is not a function. If $\beta_0 \neq \beta_1$ are both elements of Γ_ξ or elements of Γ_η , then we already have that $(h_{\beta_0} \upharpoonright s_{\rho \upharpoonright n}^r) \cup (h_{\beta_1} \upharpoonright s_{\rho \upharpoonright n}^r)$ is not a function.

Since $Q(\mathcal{H})$ is ccc, there is a condition q that forces that the generic filter G is uncountable. Consider any such generic filter G . It is easily checked that, for each $n \in \omega$, $D_n = \{r \in Q(\mathcal{H}) : n_r \geq n\}$ is dense. Let $\Gamma = \bigcup\{\Gamma_r : r \in G\}$ and, for each $\rho \in 2^\omega$, let $a_\rho = \bigcup\{a_{\rho \upharpoonright n_r}^r : r \in G\}$. The family $\mathcal{A} = \{a_r : r \in 2^\omega\}$ is almost disjoint. We check that for $r \in 2^\omega$, the family $\langle h_\beta \upharpoonright a_r : \beta \in \Gamma \rangle$ is Luzin incoherent. Let $\alpha \neq \beta \in \Gamma$ and choose $r \in G$ so that $\{\alpha, \beta\} \subset \Gamma_r$. Since $(h_\alpha \upharpoonright s_{\rho \upharpoonright n}^r) \cup (h_\beta \upharpoonright s_{\rho \upharpoonright n}^r)$ is not a function, it of course follows that $(h_\alpha \upharpoonright a_r) \cup (h_\beta \upharpoonright a_r)$ is not a function. \square

3. ENSURING $\text{TRIV}(F)$ IS NOT CCC OVER FIN

To complete the task of this section, it suffices, by Theorem 2.7, to prove that ω^ω -cohere and P-cohere are consequences of ω_1 -strategic MA. We prove each of these by constructing suitable ccc poset \mathfrak{M} -strategies based on the poset $Q(\mathcal{H})$ as defined in Definition 2.10. For P-cohere we also use the poset ADF_{ω_1} , defined next, and for the purposes illustrated in the Lemma immediately following.

Definition 3.1. Let ADF_{ω_1} denote the poset of finite partial functions p from $\omega_1 \times \omega$ to 2, such that, for some $n_p \in \omega$, $\text{dom}(p) = F_p \times n_p$.

Since the intention is to add a canonical almost disjoint family, the ordering on ADF_{ω_1} is that $p < q$ providing $n_p \geq n_q$, $F_p \supset F_q$, and for all $\alpha \neq \beta \in F_q$ and $n_q \leq j < n_p$, $p(\alpha, j) \cdot p(\beta, j) = 0$ (i.e. at least one has value 0). For each $\iota \in \omega_1$, we let $\dot{a}(\iota)$ be the ADF_{ω_1} name for the set $\{j \in \omega : (\exists p \in G) p(\iota, j) = 1\}$.

Lemma 3.2 (MA). *Suppose that \mathcal{I} is a ccc over fin ideal and that $\{h_a \in \omega^a : a \in \mathcal{I}\}$ is a maximal coherent family of functions. Let $\mathcal{F} \subset \omega^\omega$ have cardinality less than \mathfrak{c} . Then there is an $a \in \mathcal{I}$ such that $\{f, h_a\}$ is not coherent for all $f \in \mathcal{F}$.*

Proof. Let $\mathcal{F} \subset \omega^\omega$ have cardinality less than \mathfrak{c} . Since $\{h_a : a \in \mathcal{I}\}$ is a maximal coherent family, there is a $\mathcal{A} \subset \mathcal{I}$ of cardinality at most $|\mathcal{F}|$ satisfying that for all $f \in \mathcal{F}$ there is an $a \in \mathcal{A}$ such that $b(f) = \{n \in a : f(n) \neq h_a(n)\}$ is infinite. For each $f \in \mathcal{F}$, $\xi \in \omega_1$ and $m \in \omega$, the set $D(f, \xi, m) = \{p \in \text{ADF}_{\omega_1} : (\exists n \in b(f) \setminus m) p(\xi, n) = 1 \text{ and } \{\xi\} \times n \subset \text{dom}(p)\}$ is evidently a dense subset of ADF_{ω_1} . Let $G \subset \text{ADF}_{\omega_1}$ be a filter satisfying that $G \cap D(f, \xi, m) \neq \emptyset$ for all $f \in \mathcal{F}$, $\xi \in \omega_1$, $m \in \omega$. Since \mathcal{I} is assumed to be ccc over fin, there is an $\iota \in \omega_1$ such that $a(\iota) = \{n \in \omega : (\exists p \in G) p(\iota, n) = 1\}$ is in \mathcal{I} . Let $f \in \mathcal{F}$ and we prove that $\{f, h_{a(\iota)}\}$ is not coherent. Choose the $a \in \mathcal{A}$ satisfying that $b(f) = \{n \in a : f(n) \neq h_a(n)\}$. Since $h_{a(\iota)}$ and h_a are in the coherent system it follows that $\{n \in a(\iota) \cap a : h_{a(\iota)}(n) \neq h_a(n)\}$ is finite. Since $a(\iota) \cap b(f)$ is infinite it follows that $\{n \in a(\iota) \cap a : f(n) \neq h_{a(\iota)}(n) = h_a(n)\}$ is infinite. \square

Lemma 3.3. *ω_1 -strategic MA implies P-cohere.*

Proof. Let \mathfrak{M} be the sequence of models as in ω_1 -strategic MA. Let \mathcal{H} be a maximal coherent family of partial functions on ω and assume that the ideal, \mathcal{I} , generated by $\{\text{dom}(h) : h \in \mathcal{H}\}$ is a P-ideal. Since \mathcal{H} is maximal, it follows that \mathcal{I} is dense. Let us note that by Lemma 3.2 we may assume that for any $\mathcal{F} \subset \omega^\omega$ of cardinality less than \mathfrak{c} , there is an $a(\mathcal{F}) \in \mathcal{I}$ satisfying that $\{f, h_{a(\mathcal{F})}\}$ is not coherent for any $f \in \mathcal{F}$. With this additional assumption we now define our ccc poset \mathfrak{M} -strategy.

Choose any enumeration $\{h_\xi : \xi \in \mathfrak{c}\}$ of \mathcal{H} . By recursion on $\text{dom}(t)$ for $t \in \mathfrak{c}^{<\omega_1}$ choose a pair (ξ_t, η_t) satisfying the following conditions:

- (1) for all $s \subsetneq t$, $\eta_s < \eta_t$,
- (2) $\langle h_{\xi_s}, \xi_s : s \subseteq t \rangle \in M_{\eta_t}$,
- (3) $\langle \text{dom}(h_{\xi_s}) : s \subseteq t \rangle$ is mod finite increasing,
- (4) for all $s \subsetneq t$, $\{f, h_{\xi_t}\}$ is not coherent for any $f \in M_{\eta_s}$.

Finally we define, for $t \in \mathfrak{c}^{<\omega_1}$, $R_t = Q^+(\langle h_{\xi_s}, \xi_s : s \subseteq t \rangle)$. Since $\langle h_{\xi_s}, \xi_s : s \subseteq t \rangle \in M_{\eta_t}$, we have that $R_t \in M_{\eta_t}$. Let $\mathcal{R} = \langle (R_t, \eta_t) : t \in \mathfrak{c}^{<\omega_1} \rangle$ and, since \mathcal{R} is an \mathfrak{M} -system there is a cub $C(\mathcal{R}) \subset \mathfrak{c}$ as in the statement of ω_1 -strategic MA (Definition 1.3). Let us also note that for $t \subset s \in \mathfrak{c}^{<\omega_1}$, R_t is a subset of R_s .

Consider any $\lambda \in S_1^c \cap C(\mathcal{R})$ and let $\rho \in \mathfrak{c}^{\omega_1} \cap M_\lambda$ satisfy that $\langle \eta_{\rho \upharpoonright \alpha} : \alpha \in \omega_1 \rangle \in M_\lambda$ is cofinal in λ and $\langle R_{\rho \upharpoonright \alpha} : \alpha \in \omega_1 \rangle \in M_\lambda$. For each $\alpha < \omega_1$, let $t_\alpha = \rho \upharpoonright \alpha$.

It follows also that $\{h_{\xi_{t_\alpha}} : \alpha \in \omega_1\}$ is in M_λ . Since the sequence $\langle \text{dom}(h_{\xi_{t_\alpha}}) : \alpha \in \omega_1 \rangle$ is well-ordered by mod finite inclusion, this sequence is also in M_λ . For each $f \in \omega^\omega \cap M_\lambda$, there is an $\alpha < \omega_1$ such that $f \in M_{\eta_{t_\alpha}}$. By construction $\{f, h_{\xi_{t_{\alpha+1}}}\}$ is not coherent. It follows then from Lemma 2.11, that R_ρ is ccc in M_λ .

This proves that \mathcal{R} is a ccc poset \mathfrak{M} -system and therefore there is a (R_ρ, M_λ) constructed by the system as in the previous paragraph for which there is an uncountable filter G on R_ρ that is M_λ -generic. It follows again from Lemma 2.11 that \mathcal{I} is not ccc over fin. \square

The proof of this next result is quite similar.

Lemma 3.4. *ω_1 -strategic MA implies ω^ω -cohere.*

Proof. In this proof we use the poset $Q_\varphi(\mathcal{H})$ or rather the isomorphic copy $Q_\varphi^+(\mathcal{H})$ analogous to $Q^+(\mathcal{H})$. Let $\{h_f : f \in \omega^\omega\}$ be a coherent ω^ω family. Fix a sequence $\langle f_\xi : \xi < \mathfrak{c} \rangle$ that is $<^*$ -increasing and $<$ -cofinal in ω^ω . For each $\xi < \mathfrak{c}$, let $h_\xi = h_{f_\xi}$.

Let \mathfrak{M} be the sequence of models as in the statement of ω_1 -strategic MA. Consider any $M_\eta \in \mathfrak{M}$ and since $|M_\eta \cap \mathcal{P}(\omega)| = |M_\eta \cap \omega^\omega|$, we may choose $\zeta \in \mathfrak{c}$ so that $f \leq^* f_\zeta$ for all $f \in M_\eta \cap \omega^\omega$. Of course this means that $h_f \leq^* h_\zeta$ for all $f \in M_\eta \cap \omega^\omega$.

Now consider any $h \in M_\eta$ that is a function from $\omega \times \omega$ to ω . By elementary, if $\{h\} \cup \{h_f : f \in M_\mu \cap \omega^\omega\}$ is coherent, then $\{h\} \cup \{h_f : f \in \omega^\omega\}$ is coherent. Therefore we will continue with the assumption that, for any $\eta < \mathfrak{c}$, there is a $\zeta < \mathfrak{c}$ such that $\{h_\zeta, h_f\}$ is not coherent for any $f \in M_\eta \cap \omega^\omega$.

By recursion on $\text{dom}(t)$ for $t \in \mathfrak{c}^{<\omega_1}$ choose a pair (ξ_t, η_t) satisfying the following conditions:

- (1) for all $s \subsetneq t$, $\eta_s < \eta_t$ and $\xi_s < \xi_t$,
- (2) $\langle h_{\xi_s}, \xi_s : s \subseteq t \rangle \in M_{\eta_t}$,
- (3) for all $s \subsetneq t$, $\{h_f, h_{\xi_t}\}$ is not coherent for any $f \in M_{\eta_s}$.

We define, for $t \in \mathfrak{c}^{<\omega_1}$, $R_t = Q_\varphi^+(\langle h_{\xi_s} : s \subseteq t \rangle)$. The proof that $\mathcal{R} = \langle (R_t, \eta_t) : t \in \mathfrak{c}^{<\omega_1} \rangle$ is a ccc poset \mathfrak{M} -system proceeds as in Lemma 3.3. Similarly, it also follows as in Lemma 3.3, that there is a $\Gamma \subset \mathfrak{c}$ of cardinality \aleph_1 such that $\{h_\xi : \xi \in \Gamma\}$ is Luzin incoherent. Choose any $\zeta < \mathfrak{c}$ such that $\Gamma \subset \zeta$. It follows that $\text{dom}(h_\zeta) = f_\zeta^\perp \text{ mod finite}$ contains $\text{dom}(h_\xi)$ for all $\xi \in \Gamma$. Let $A = \text{dom}(h_\zeta)$ and notice the fact that $\{h_\zeta\} \cup \{h_\xi \upharpoonright A : \xi \in \Gamma\}$ is coherent is contradicted by Lemma 2.9. \square

4. ON $\langle \mathfrak{p} \rangle$ -BOREL AUTOMORPHISMS

In this section we prove Theorem 1.1 as Theorem 4.5, namely that $\mathfrak{p} = \mathfrak{c}$ implies that any $\langle \mathfrak{c} \rangle$ -Borel automorphism is trivial. The very rough idea, since it might help, is that since we have \mathfrak{c} -many mutual Cohen subsets of ω there will be non-meager many pairs of such Cohen subsets that will lift to the same Borel function E together with all the unions of those pairs also lifting to a common Borel function. We then uncover an added trick (extra condition on those Cohen subsets relative to special submodels) utilizing Proposition 2.1 that allows us to prove, following Velickovic's methods, that the function E actually corresponds to a trivial automorphism on all those pairs. This same trick assuming that the automorphism with the Borel lifting is non-trivial, at a couple of key new steps, and taking great advantage of the conflicted trivial behavior with respect to Proposition 2.1 at the Cohen generics, leads to the desired contradiction.

We make use of the following easy Lemma that we will apply to Borel liftings. Note that the elements $[s; n]$ of the poset \mathcal{C}_ω can also be identified with the standard clopen basis for $\mathcal{P}(\omega)$ given by $[s; n] = \{x \subset \omega : x \cap n = s\}$. Given an infinite subset $A \subset \omega$, let \mathcal{C}_A correspond to the subposet consisting of those $[s; n]_A = [s; n] \cap \mathcal{P}(A)$ with $s \subset A$. Recall that each element of BOR is continuous on a dense G_δ subset of $\mathcal{P}(\omega)$ (see [11]) and that there is a 1-to-1 correspondence between elements of BOR and \mathcal{C}_ω -names of subsets of ω as explained in [4, §3] and [3].

Lemma 4.1. *Let $\mathcal{E} \subset \text{BOR}$ have cardinality less than \mathfrak{p} . Then there is a sequence $\langle t_\ell : \ell \in \omega \rangle$ of pairwise disjoint non-empty finite subsets of ω satisfying:*

- (1) for each $\ell \in \omega$, let $k_\ell = \min(t_\ell)$ and $m_\ell = \max(t_\ell) < k_{\ell+1}$,
- (2) let $T = \bigcup \{t_\ell : \ell \in \omega\}$ and $A = \omega \setminus \bigcup \{[k_\ell, m_\ell] : \ell \in \omega\}$,
- (3) for each $E \in \mathcal{E}$, let E_T denote the function with domain $\mathcal{P}(A)$ defined by $E_T(x) = E(x \cup T)$ for each $x \subset A$,
- (4) for each $E \in \mathcal{E}$, E_T is continuous and, for cofinitely many ℓ , and all $x \subset A$,

$$E_T([x \cap k_\ell; k_\ell]_A) \subset [E_T(x) \cap k_\ell; k_\ell]$$

in other words the first k_ℓ many values of $E_T(x)$ are determined by the first k_ℓ many values of $x \subset A$.

For any infinite $L \subset \omega$, replacing $\langle t_\ell : \ell \in \omega \rangle$ by $\langle t_\ell : \ell \in L \rangle$ also satisfies the requirements.

Proof. Choose any elementary submodel M of $H(\mathfrak{c})$ of cardinality less than \mathfrak{p} such that $\mathcal{E} \subset M$. For each $E \in \mathcal{E}$, let $D_E \subset \mathcal{P}(\omega)$ denote a dense G_δ in M on which E is continuous. Also fix a descending sequence $\{U(E, m) : m \in \omega\} \in M$ of dense open subsets of $\mathcal{P}(\omega)$ satisfying that $\bigcap\{U(E, m) : m \in \omega\} = D_E$.

Let \dot{c} denote the generic subset of ω added by \mathcal{C}_ω and let $g = \text{val}_{G_M}(\dot{c})$ where G_M is an \mathcal{C}_ω filter that is M -generic. For each $s \subset k \in \omega$, the set $s \cup (g \setminus k)$ is also \mathcal{C}_ω -generic over M and so is an element of D_E for all $E \in \mathcal{E}$. For each $s \subset k \in \omega$ and $E \in \mathcal{E}$, there is a $m = m(E, s, k) \in \omega$ such that $E([s \cup (g \cap m \setminus k)] \cap D_E) \subset [E(s \cup (g \setminus k)); k]$ (i.e. E is continuous at $s \cup (g \setminus k)$ in D_E). Therefore there is an $f_E \in \omega^\omega$ satisfying that, for all $s \subset k \in \omega$

$$E([s \cup (g \cap f_E(k) \setminus k); f_E(k)] \cap D_E) \subset [E(s \cup (g \setminus k)) \cap k; k]$$

i.e. the first $f_E(k)$ many values of $s \cup (g \setminus k)$ determines the first k many values of $E(s \cup (g \setminus k))$. We also ensure that $f_E(k)$ is large enough so that $[s \cup (g \cap f_E(k) \setminus k); f_E(k)] \subset U(E, k)$ for all $s \subset k$.

Now choose any strictly increasing function $\tilde{f} \in \omega^\omega$ satisfying that $f_E <^* \tilde{f}$ for all $E \in \mathcal{E}$. Let $k_0 = 0$ and $m_0 = \tilde{f}(k_0)$, and by recursion, $k_{\ell+1} = \tilde{f}(m_\ell)$ and $m_{\ell+1} = \tilde{f}(k_{\ell+1})$. Set $t_\ell = g \cap m_\ell \setminus k_\ell$ for all $\ell \in \omega$ and let $T = \bigcup\{t_\ell : \ell \in \omega\}$ and $A = \omega \setminus \bigcup\{m_\ell \setminus k_\ell : \ell \in \omega\}$. Fix any $E \in \mathcal{E}$ and $x \subset A$ and choose $n \in \omega$ satisfying that $f_E(k) < \tilde{f}(k)$ for all $n \leq k \in \omega$. Then, for each $k < \ell$, $[(x \cap k_\ell) \cup (g \cap m_\ell \setminus k_\ell); m_\ell]$ is equal to $[(x \cap k_\ell) \cup t_\ell; m_\ell]$ and $[(x \cap k_\ell) \cup (g \cap m_\ell \setminus k_\ell); m_\ell]$ is a subset of $[(x \cap k_\ell) \cup (g \cap f_E(k_\ell) \setminus k_\ell); F_E(k_\ell)] \subset U(E, k_\ell)$. Therefore, for all $x \subset A$, $x \cup T \in D_E$ and $\mathcal{P}(A) \times \{T\} \subset D_E$. Therefore we have that

$$E_T([x \cap k_\ell; k_\ell]_A) = E_T([x \cap k_\ell, k_\ell]_A) \subset [E_T(x) \cap k_\ell; k_\ell]$$

as required. \square

Next we define a family of standard σ -centered posets.

Definition 4.2. Let $\mathcal{Y} \subset \mathcal{P}(\omega)$ and let $\Pi\{\mathcal{C}_y : y \in \mathcal{Y}\}$ denote the finite support product. For each $y \in \mathcal{Y}$, let \dot{c}_y denote the standard name for the Cohen generic subset of y added by the coordinate poset \mathcal{C}_y . Let $\dot{Q}(\mathcal{Y})$ denote the $\Pi\{\mathcal{C}_y : y \in \mathcal{Y}\}$ -name for a standard σ -centered poset that adds a generic subset of $A = \bigcup \mathcal{Y}$ that is almost disjoint from every member of the family $\{\dot{c}_y : y \in \mathcal{Y}\}$. That is, $q \in \dot{Q}(\mathcal{Y})$ has the form $q = (s_q, n_q, L_q) \in [A]^{<\aleph_0} \times \omega \times [\mathcal{Y}]^{<\aleph_0}$ ordered by $r < q$ providing $n_q \leq n_r$, $s_r \cap n_q = s_q$, $L_r \supset L_q$, and $s_r \cap c_y = s_q \cap c_y$ for all $y \in L_q$.

Let $\mathcal{K}(\mathcal{Y}) = \Pi\{\mathcal{C}_y : y \in \mathcal{Y}\} * \dot{Q}(\mathcal{Y})$ and let $\dot{x}(\mathcal{Y})$ denote the canonical $\mathcal{K}(\mathcal{Y})$ -name for the subset of $\bigcup \mathcal{Y}$ added by $\dot{Q}(\mathcal{Y})$.

The next Lemma is an easy consequence of the familiar facts that the members of $\{c_y : y \in \mathcal{Y}\}$ are mutually Cohen generic and that, in the $\Pi\{\mathcal{C}_y : y \in \mathcal{Y}\}$ -generic extension, the generic added by $\dot{Q}(\mathcal{Y})$ meets every subset of $\bigcup \mathcal{Y}$ that is not in the ideal generated by $\{c_y : y \in \mathcal{Y}\}$.

Lemma 4.3. *Let $\mathcal{Y} \subset \mathcal{P}(\omega)$ be a subset of an elementary submodel M of $H(\mathfrak{c}^+)$. Let G be a filter on $\mathcal{K}(\mathcal{Y})$ that is M -generic. For each $y \in \mathcal{Y}$, let $c_y = \text{val}_G(\dot{c}_y)$. Let $M[c_y]$ denote, as usual, the set of all valuations of \mathcal{C}_y -names in M by the filter generated by c_y . If $x \in M[c_y]$ and $x \cap y$ is not mod finite contained in c_y , then $x \cap \text{val}_G(\dot{x}(\mathcal{Y}))$ is infinite.*

Lemma 4.4. *Let $\mathcal{Y} \subset \mathcal{P}(\omega)$ be a subset of an elementary submodel M of $H(\mathfrak{c}^+)$. Let G be a filter on $\mathcal{K}(\mathcal{Y})$ that is M -generic. Then $\text{val}_G(\dot{x}(\mathcal{Y}))$ is an element of U for every $U \in M$ that is a dense open subset of $\mathcal{P}(\bigcup \mathcal{Y})$.*

Proof. Since $\mathcal{K}(\mathcal{Y}) = \Pi\{\mathcal{C}_y : y \in \mathcal{Y}\} * \dot{Q}(\mathcal{Y})$, we can treat members of $\mathcal{K}(\mathcal{Y})$ as pairs (p, q) from a dense subset with $p \in \Pi\{\mathcal{C}_y : y \in \mathcal{Y}\}$ and $q = (\sigma_q, n_q, L_q) \in [\bigcup \mathcal{Y}]^{<\aleph_0} \times \omega \times [\mathcal{Y}]^{<\aleph_0}$ and with L_q a subset of the support of p and for each y in the support of p , $p(y)$ has the form $[s_y; n_q]_y$ for some $s_y \subset n_q \cap y$. Of course it suffices to prove that

$$D(U) = \{(p, q) \in \mathcal{K}(\mathcal{Y}) : [s_q, n_q]_{\bigcup \mathcal{Y}} \subset U\}$$

is a dense subset of $\mathcal{K}(\mathcal{Y})$. This argument is routine so we omit it. \square

Now we restate and prove the main result of this section. The combinatorial details at the heart of the proof are numerous, but are similar to those presented by Velickovic [10] in his proof that all Borel automorphisms are trivial and those in [7] that inspired it.

Theorem 4.5. *$\mathfrak{p} = \mathfrak{c}$ implies that if $\lambda < \mathfrak{c}$ then any automorphism of $\mathcal{P}(\omega)/\text{fin}$ that is λ -Borel is trivial.*

Proof. Let F be a lifting of an automorphism of $\mathcal{P}(\omega)/\text{fin}$. Assume that $\mathcal{E} \subset \text{BOR}$ has cardinality $< \mathfrak{c}$ and that F is covered by \mathcal{E} in the sense that, for all $x \subset \mathcal{P}(\omega)$, there is an $E \in \mathcal{E}$ such that $F(x) =^* E(x)$. We will assume that $\omega \notin \text{triv}(F)$ and prove the theorem by contradiction.

Choose an increasing chain $\{M_\mu : \mu \in \mathfrak{c}\}$ of elementary submodels of $H(\mathfrak{c}^+)$, each of cardinality less than \mathfrak{c} and with $F, \mathcal{E} \in M_0$ and $\mu \subset M_\mu \in M_{\mu+1}$ for all $\mu < \mathfrak{c}$. Choose a sequence $\langle t_\ell : \ell \in \omega \rangle \in M_1$ as in Lemma 4.1. Additionally, let $k_\ell = \min(t_\ell)$ and $m_\ell = \max(t_\ell)$ for all $\ell \in \omega$. Choose any $L \subset \omega$ such that $A = \omega \setminus \bigcup \{t_\ell : \ell \in L\}$ is not in $\text{triv}(F)$. By elementarity we may choose such an $L \in M_1$. By reindexing, we may assume for simplicity that $L = \omega$, and that A, T

are in M_1 and satisfy that, for all $E \in \mathcal{E}$,

$$E_T([x \cap k_\ell; k_\ell]_A) \subset [E_T(x) \cap k_\ell; k_\ell]$$

for almost all ℓ ; in other words the first k_ℓ many values of $E_T(x)$ are determined by the first k_ℓ many values of $x \subset A$. There is no loss of generality if we henceforth assume that for all $x \subset A$, $F(x) \subset F(A)$ rather than just $F(x) \subset^* F(A)$. Also, for notational convenience, let, for $E \in \mathcal{E}$, E_T^A denote the Borel function $E_T(x) \cap F(A)$ for all $x \subset A$.

Let $\mathcal{Y} = M_1 \cap \mathcal{P}(A)$, hence $\bigcup \mathcal{Y} = A$. For every $1 < \mu \in \mathfrak{c}$, let $G_\mu \in M_{\mu+1}$ be a filter on $\mathcal{K}(\mathcal{Y})$ that is M_μ -generic and let \bar{x}_μ denote the set $\text{val}_{G_\mu}(\dot{x}(\mathcal{Y})) \in M_{\mu+1}$. Also, by assumption, we may choose

$$E_\mu \in \mathcal{E} \text{ such that } F(\bar{x}_\mu \cup T) =^* E_\mu(\bar{x}_\mu \cup T) .$$

Therefore, we can also choose $m_\mu \in \omega$ such that

$$E_{\mu,T}(\bar{x}_\mu) \cap F(A) \setminus m_\mu = F(\bar{x}_\mu) \setminus m_\mu = F(\bar{x}_\mu \cup T) \cap F(A) \setminus m_\mu$$

where $E_{\mu,T}$ is a notation for $(E_\mu)_T$. Choose any $E \in \mathcal{E}$ and $\bar{m}_1 \in \omega$ satisfying that Γ is cofinal in \mathfrak{c} where $\Gamma = \{\mu \in \mathfrak{c} : 1 < \mu, m_\mu = \bar{m}_1 \text{ and } E = E_\mu\}$. It follows from Lemma 4.4 that $X_E = \{\bar{x}_\mu : \mu \in \Gamma\}$ is a non-meager subset of $\mathcal{P}(A)$.

For the next step, for each $\mu_1 < \mu_2$ both in Γ , there is some $\tilde{E} \in \mathcal{E}$, which we briefly denote as $E(\mu_1, \mu_2)$, and some $k = k(\mu_1, \mu_2)$ that witnesses the lifting assumption for $\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2}$ in the sense that

$$(4.1) \quad (F(\bar{x}_{\mu_1}) \cup F(\bar{x}_{\mu_2})) \setminus k = F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2} \cup T) \cap F(A) \setminus k \\ = \tilde{E}_T(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2}) \cap F(A) \setminus k .$$

Again for each $\mu_1 \in \Gamma$, we may fix a single pair $\tilde{E}^{\mu_1}, k(\mu_1)$ and a single basic clopen set $[\bar{s}_{2,\mu_1}, \bar{n}_{2,\mu_1}]_A$ satisfying that $X(\bar{x}_{\mu_1})$ meets $[\bar{s}_{2,\mu_1}, \bar{n}_{2,\mu_1}]_A$ in a nowhere meager set, where $X(\bar{x}_{\mu_1})$ is the set

$$\{\bar{x}_{\mu_2} : \mu_1 < \mu_2 \in \Gamma, k(\mu_1, \mu_2) = k(\mu_1), \text{ and } \tilde{E}^{\mu_1} = E(\mu_1, \mu_2)\} .$$

Working in reverse then, there is a single choice $\tilde{E}, \bar{k}_2, [\bar{s}_2, \bar{n}_2]_A$ such that for a cofinal subset $\Gamma_1 \subset \Gamma$ and all $\mu_1 \in \Gamma_1$, $\tilde{E} = \tilde{E}^{\mu_1}$, $k(\mu_1) = \bar{k}_2$ and $[\bar{s}_{2,\mu_1}, \bar{n}_{2,\mu_1}]_A = [\bar{s}_2, \bar{n}_2]_A$. Again there is a clopen set $[\bar{s}_1, \bar{n}_1]_A$ satisfying that $[\bar{s}_1, \bar{n}_1]_A \cap \{\bar{x}_\mu : \mu \in \Gamma_1\}$ is nowhere meager .

Choose $\bar{m} = k_{\bar{\ell}}$ large enough so that $\max(\bar{n}_1, \bar{n}_2, \bar{m}_1, \bar{k}_2) \leq \bar{m}$ and for all $\bar{\ell} \leq \ell \in \omega$ each of the following hold $E_T([x \cap k_\ell; k_\ell]_A) \subset [E_T(x) \cap k_\ell; k_\ell]$ and $\tilde{E}_T([x \cap k_\ell; k_\ell]_A) \subset [\tilde{E}_T(x) \cap k_\ell; k_\ell]$. For each $\bar{\ell} \leq \ell \in \omega$, let $A_\ell = A \cap k_\ell \setminus \bar{m}$ and let $\bar{s}_3 = \bar{s}_1 \cup \bar{s}_2$.

We can observe that we have the following forcing like statement: for all $\bar{\ell} < \ell$ and $s_1, s_2 \subset A_\ell$ there are $\mu_1 < \mu_2$ satisfying

- (1) $\bar{x}_{\mu_1} \in [\bar{s}_1 \cup s_1; k_\ell]_A$ and $\bar{x}_{\mu_2} \in [\bar{s}_2 \cup s_2; k_\ell]_A$,
- (2) $F(\bar{x}_{\mu_1}) \setminus \bar{m} = E_T(\bar{x}_{\mu_1}) \cap F(A) \setminus \bar{m}$
- (3) $F(\bar{x}_{\mu_2}) \setminus \bar{m} = E_T(\bar{x}_{\mu_2}) \cap F(A) \setminus \bar{m}$,
- (4) $(F(\bar{x}_{\mu_1}) \cup F(\bar{x}_{\mu_2})) \setminus \bar{m} = F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2}) \setminus \bar{m}$
- (5) $F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2}) \setminus \bar{m} = \tilde{E}_T(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2}) \cap F(A) \setminus \bar{m}$.

Let us refer to items (1)-(5) as the (F, E) -relations. We have shown that every pair $\mu_1 \in \Gamma_1$ and μ_2 with $\bar{x}_{\mu_2} \in X(\bar{x}_{\mu_1})$ and $s_i = \bar{x}_{\mu_i} \cap k_\ell \setminus \bar{m}$ ($i = 1, 2$), the (F, E) -relations hold. For any finite $t \subset k_\ell$, recall that $[t; k_\ell] \cap \mathcal{P}(F(A))$ can be denoted as $[t; k_\ell]_{F(A)}$.

For all $\bar{\ell} < \ell \in \omega$ and $s \subset A_\ell$ define $H_s^1(k_\ell)$, $H_s^2(k_\ell)$ and $H_s^3(k_\ell)$ to be those subsets of $F(A) \cap k_\ell$ satisfying, by the continuity properties of E_T and \tilde{E}_T , that for all $x_i \in [\bar{s}_i \cup s; k_\ell]$ ($i = 1, 2, 3$) and for $i = 1, 2$:

$$(4.2) \quad E_T^A(x_i) \in [H_s^i(k_\ell); k_\ell]_{F(A)} \quad \text{and} \quad \tilde{E}_T^A(x_3) \in [H_s^3(k_\ell); k_\ell]_{F(A)}.$$

Claim 4.5.1. $H_{s_1 \cup s_2}^3(k_\ell) = H_{s_1}^1(k_\ell) \cup H_{s_2}^2(k_\ell)$ for all $\bar{\ell} < \ell \in \omega$ and $s_1, s_2 \subset A_\ell$.

Proof of Claim. Choose $\mu_1 < \mu_2$ so that the (F, E) -relations hold for $s_1, s_2 \subset A_\ell$. Letting $x_1 = x_{\mu_1}$, $x_2 = x_{\mu_2}$, and $x_3 = x_1 \cup x_2$, we have by equation 4.2, $F(x_1) \cap k_\ell \setminus \bar{m} = H_{s_1}^1(k_\ell)$, $F(x_2) \cap k_\ell \setminus \bar{m} = H_{s_2}^2(k_\ell)$, and $F(x_1 \cup x_2) \cap k_\ell \setminus \bar{m} = H_{s_1}^1(k_\ell) \cup H_{s_2}^2(k_\ell)$. Since, in addition $F(x_1 \cup x_2) \cap k_\ell \setminus \bar{m}$ is equal to $H_{s_1 \cup s_2}^3(k_\ell)$, this completes the proof of the claim. \square

Claim 4.5.2. For all $\bar{\ell} < \ell \in \omega$, each of $H_\emptyset^1(k_\ell)$ and $H_\emptyset^2(k_\ell)$ are subsets of $F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2})$ for all $\mu_1, \mu_2 \in \Gamma_1$ such that $\bar{x}_{\mu_2} \in X(x_{\mu_1})$.

Proof of Claim. Fix the pair $\mu_1 < \mu_2$ as in the statement of the Claim. Let $s_1 = \bar{x}_{\mu_1} \cap k_\ell \setminus \bar{m}$ and $s_2 = \bar{x}_{\mu_2} \cap k_\ell \setminus \bar{m}$. Also let $s = s_1 \cup s_2$. By Claim 4.5.1, $H_s^3(k_\ell) = H_\emptyset^1(k_\ell) \cup H_s^2(k_\ell) = H_s^1(k_\ell) \cup H_\emptyset^2(k_\ell)$. Finally, by the (F, E) -relations, $F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2}) \cap k_\ell \setminus \bar{m} = \tilde{E}(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2}) \cap k_\ell \setminus \bar{m} = H_s^3(k_\ell)$. This proves the claim. \square

With a very similar proof, but now using the density assumptions on Γ_1 we have the next claim.

Claim 4.5.3. For all $\bar{m} < \ell \in \omega$ and $s \subset A_\ell$ and $\mu_1 \in \Gamma_1$ such that $\bar{x}_{\mu_1} \in [\bar{s}_1 \cup s; k_\ell]_A$ (which exists) and any $\mu_2 \in \Gamma$ such that \bar{x}_{μ_2} is an element of $[\bar{s}_2 \cup s; k_\ell]_A \in X(\bar{x}_{\mu_1})$ (which also exists), each of $H_s^1(k_\ell)$ and $H_s^2(k_\ell)$ are subsets of $F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2})$.

The next claim follows immediately from Lemma 4.4 and the fact that $F \in M_0$.

Claim 4.5.4. For all $\mu_1, \mu_2 \in \Gamma$, none of $F(\bar{x}_{\mu_1})$, $F(\bar{x}_{\mu_2})$, and $F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2})$ contain an infinite subset of $F(A)$ that is an element of M_1 .

Therefore, we have that, by Claim 4.5.3, for all finite $s \subset A \setminus \bar{m}$ the union of the collection $\{H_s^1(k_\ell) \cup H_s^2(k_\ell) : s \subset \ell\}$ is finite. It will be convenient to now assume that $\bar{m} = k_{\bar{\ell}}$ was chosen large enough so that $H_{\bar{\theta}}^1(\ell) \cup H_{\bar{\theta}}^2(\ell)$ is a subset of \bar{m} for all $\bar{\ell} < \ell$. In fact, by then applying the definition with the updated value of \bar{m} , this means that $H_{\bar{\theta}}^1(k_\ell)$ and $H_{\bar{\theta}}^2(k_\ell)$ are empty for all $\bar{\ell} < \ell$.

Claim 4.5.5. *For all finite $s \subset A \setminus \bar{m}$, the sequence $\{H_s^1(k_\ell) : s \subset \ell\}$ is eventually constant.*

Proof of Claim. Choose any ℓ_s such that $s \subset \ell_s$. Notice that the sequence $\{H_s^1(k_\ell) : \ell_s < \ell\}$ is increasing simply because the sequence of sets $\{[\bar{s}_1 \cup s; k_\ell]_A : \ell_s < \ell\}$ are getting smaller and $H_s^1(k_\ell)$ is the subset of $F(A) \cap k_\ell \setminus \bar{m}$ determined by $E_T^A([\bar{s}_1 \cup s; k_\ell]_A)$. Since the union of the chain $\{H_s^1(k_\ell) : \ell_s < \ell \in \omega\} \in M_1$ a subset of $F(\bar{x}_{\mu_1} \cup \bar{x}_{\mu_2})$ for suitable $\mu_1, \mu_2 \in \Gamma$ as in Claim 4.5.3, it must be finite by Claim 4.5.4. \square

We now choose an infinite subset L of $\omega \setminus \bar{\ell}$ such that $\bar{\ell} \in L \in M_1$ and, for all $\ell \in \omega$ and $s \subset A_\ell$, the value of $H_s^1(k_{\ell'})$ is the same for all $\ell < \ell' \in L$. By the same argument we may assume the same to be true for the value of $H_s^2(k_{\ell'})$ and $s \subset A_\ell$ with $\ell < \ell' \in L$. Henceforth, we let H_s^1 and H_s^2 denote this *eventual* value of $H_s^1(k_{\ell'})$ and $H_s^2(k_{\ell'})$ respectively. Similarly, the value of H_s^3 will simply be defined as $H_s^1 \cup H_s^2$.

This next claim is an immediate consequence of Claim 4.5.1.

Claim 4.5.6. *For all finite $s \subset A \setminus \bar{m}$, $H_s^1 = H_s^2 = H_s^3$.*

Claim 4.5.7. *For all finite $s, t \subset A \setminus \bar{m}$, $H_s^1 \cup H_t^1 = H_{s \cup t}^1$.*

Proof of Claim. We have $H_t^1 = H_t^2$ and $H_s^1 \cup H_t^2 = H_{s \cup t}^3 = H_{s \cup t}^1$. \square

For each $j \in A$, let $H_j^1 = H_{\{j\}}^1$, and it is immediate by induction on $|s|$ that:

Claim 4.5.8. *For all finite $s \subset A \setminus \bar{m}$, $H_s^1 = \bigcup_{j \in s} H_j^1$.*

Claim 4.5.9. *For all $\mu \in \Gamma_1$, $F(\bar{x}_\mu) \setminus \bar{m}$ equals $\bigcup \{H_j^1 : j \in \bar{x}_\mu \setminus \bar{m}\}$.*

Proof of Claim. Fix any $\mu \in \Gamma_1$. Clearly, by the genericity condition in Lemma 4.4, there are infinitely many $\ell \in L$ such that \bar{x}_{μ_1} is disjoint from $k_{\ell^+} \setminus k_\ell$ where ℓ^+ is the minimum element of L that is greater than ℓ . This implies that, with $s_\ell = \bar{x}_{\mu_1} \cap k_\ell$, $E_T^A(\bar{x}_1) \cap k_\ell \setminus \bar{m}$ is equal to $H_{s_\ell}^1 = \bigcup \{H_j^1 : j \in s_\ell\}$. \square

Claim 4.5.10. $F(A) =^* \bigcup \{H_j^1 : j \in A\}$

Proof of Claim. By definition $H_j^1 \subset F(A)$ for all $j \in A$. Let $b = F(A) \setminus \bigcup\{H_j^1 : j \in A\}$ and assume that $b \in M_1$ is infinite. Choose $a \subset A$ so that $F(a) = b$. Clearly, by the genericity condition in Lemma 4.4, for all $\mu \in \Gamma_1$, $\bar{\mu}_1 \cap a$ is infinite, and so is $F(\bar{x}_{\mu_1}) \cap F(a)$. Since $F(\bar{x}_{\mu_1}) \subset \bigcup\{H_j^1 : j \in A\}$, we have the stated conclusion by Claim 4.5.9. \square

Claim 4.5.11. *For each $m \in \omega$, the set $z_m = \{j \in A : H_j^1 \subset m\}$ is finite.*

Proof of Claim. Assume otherwise that z_m is infinite and recursively choose an infinite sequence of pairs $\{j_i, k_i : i \in \omega\}$ with $j_i \in A$ and so that $m < k_i \in F_A(z_m) \cap H_{j_i}^1 \setminus \bigcup\{H_{j_n}^1 : n < i\}$. This we may do since $F_A(z_m)$ is infinite and contained in $F(A)$. Also notice that $j_i \notin z_m$ since $k_i \in H_{j_i}^1 \setminus m$. We have two sets in M_1 : $x = \{j_i : i \in \omega\}$ and $y \subset A$ such that $F(y) =^* \{k_i : i \in \omega\}$. By construction $\{k_i : i \in \omega\} \subset F(z_m)$, hence $y \subset^* z_m$. Since $x \cap z_m$ is empty, $x \cap y$ is finite. Fix any $\mu \in \Gamma_1$ and recall the M_μ -generic filter G_μ that gave us $\bar{x}_\mu = \text{val}_{G_\mu}(\dot{x}(\mathcal{Y}))$. Now let c_y denote the generic subset of y given by $G_\mu \cap \mathcal{C}_y$ as in Lemma 4.3. Pass to the submodel $M_\mu[c_y]$ also as in Lemma 4.3. Since $F(c_y) =^* E_y(c_y)$ for some $E_y \in \mathcal{E} \subset M_\mu$, we have that $\tilde{x} = \{j_i : k_i \in F(c_y)\} \subset x$ is an infinite set and is in $M_\mu[c_y]$. Since $\tilde{x} \subset x$ and x is disjoint from y , it follows from Lemma 4.3 that $\tilde{x} \cap \bar{x}_\mu$ is infinite. Thus we finally have that $F(\tilde{x} \cap \bar{x}_\mu)$ is an infinite subset of $F(c_y)$ while, with $c_y \cap \bar{x}_\mu$ being finite, we have that $F(\tilde{x} \cap \bar{x}_\mu)$ is almost disjoint from $F(c_y)$. \square

Claim 4.5.12. *There is $\tilde{m} \in \omega$ such that the elements of the family $\{H_j^1 \setminus \tilde{m} : j \in A\}$ are pairwise disjoint.*

Proof of Claim. If this Claim fails to hold then, working in M_1 , we may recursively choose a sequence $\{i, j_i^1, j_i^2 : i \in I\}$ for some infinite set $I \subset A$ so that for each $i < i' \in I$

- (1) $j_i^1 \neq j_{i'}^2$ and $\max(\{i, j_i^1, j_i^2\}) < \min(\{i', j_{i'}^1, j_{i'}^2\})$
- (2) $i \in H_{j_i^1}^1 \cap H_{j_i^2}^1$.

Since $F(\{j_i^1 : i \in I\})$ and $F(\{j_i^2 : i \in I\})$ are almost disjoint, we may suppose that $I_2 = I \setminus F(\{j_i^1 : i \in I\})$ is infinite. Choose $y \in \mathcal{Y}$ so that $F(y) =^* I_2$. It follows that $y \cap \{j_i^1 : i \in I\}$ is finite. Choose any $\mu \in \Gamma$, and let c_y be $\text{val}_{G_\mu}(\dot{c}_y)$. Set $\tilde{x} = \{j_i^1 : i \in F(c_y)\}$. We again have that $\tilde{x} \in M_\mu[c_y]$ and is disjoint from c_y . Therefore, by Lemma 4.3, $\tilde{x} \cap \bar{x}_\mu$ is infinite. Let \tilde{I} denote the infinite set of $i \in I$ such that $i \in F(c_y)$ and $j_i^1 \in \bar{x}_\mu$. Evidently, \tilde{I} is an infinite subset of $F(c_y)$. However, by Claim 4.5.9, $F(\bar{x}_\mu) \setminus \tilde{m} = \bigcup\{H_j^1 : j \in \bar{x}_\mu\} \supset \bigcup\{H_{j_i^1}^1 : j_i^1 \in \bar{x}_\mu \cap \tilde{x}\} =$

$\bigcup\{H_{j_i^1}^1 : j_i^1 \in \bar{x} \text{ and } i \in F(c_y)\}$, and so \tilde{I} is also an infinite subset of $F(\bar{x}_\mu)$. This contradicts that $F(\bar{x}_\mu)$ and $F(c_y)$ are almost disjoint. \square

Claim 4.5.13. *There is a $\tilde{m} \in \omega$ such that the family $\{H_j^1 \setminus \tilde{m} : \tilde{m} < j \in \bar{a}\}$ are disjoint singleton sets.*

Proof of Claim. The failure of this Claim, together with the conclusion of Claim 4.5.11, would have that there is a strictly increasing sequence $\{j_k : k \in \omega\}$ so that, for each $k \in \omega$, $H_{j_k}^1$ contains a pair $\{i_k^1, i_k^2\}$ with $\max(\bigcup\{H_{j_\ell}^1 : \ell < k\}) < \min(\{i_k^1, i_k^2\})$. By elementarity, there is an almost disjoint pair $y_1, y_2 \in \mathcal{Y}$ satisfying that $F(y_1) =^* \{i_k^1 : k \in \omega\}$ and $F(y_2) =^* \{i_k^2 : k \in \omega\}$.

By symmetry we may assume that $\{j_k : k \in \omega\} \setminus y_1$ is infinite. Let $I_2 = \{k : j_k \notin y_1\}$ and choose $y \in \mathcal{Y}$ so that $F(y) =^* \{i_k^1 : k \in I_2\} = F(y_1)$. Note that $y \subset^* y_1$ and for $k \in I_2$, $j_k \notin y_1$.

As usual, choose $\mu \in \Gamma$ and let $c_y = \text{val}_{G_\mu}(\dot{c}_y) \subset y$. Again note that for $i_k^1 \in F(c_y)$, $j_k \notin y_1$. Set $\tilde{x} = \{j_k : i_k^1 \in F(c_y)\} \in M_\mu[c_y]$ and we have that \tilde{x} is almost disjoint from y_1 , and therefore also from c_y . By Lemma 4.3, $\tilde{x} \cap \bar{x}_\mu$ is infinite. This leads to a contradiction because each $j_k \in \tilde{x} \cap \bar{x}_\mu$ gives rise to another value $i_k^1 \in H_{j_k}^1$ in the finite set $F(c_y) \cap F(\bar{x})$. \square

Now that we have established the above claims, we choose \tilde{m} as in Claim 4.5.13 and let $h \in M_1$ be a bijection between a cofinite subset of A and $F(A) \setminus \tilde{m}$ such that $\{h(j)\} = H_j^1 \setminus \tilde{m}$ for all $j \in \text{dom}(h)$.

We are not claiming that h induces F on A but rather that h induces E on all $x_\mu \in [\bar{s}; k_{\bar{\ell}}]_A$ for $\mu \in \Gamma$. But this leads to the following contradiction. Since $A \notin \text{triv}(F)$, by Lemma 2.1, there is an infinite $z \subset \text{dom}(h)$ satisfying that $h(z) \cap F(z)$ is finite. Notice that $h(x) = \bigcup\{H_j^1 \setminus \tilde{m} : j \in x\}$ for all $x \subset \text{dom}(h)$. Choose $y \in \mathcal{Y}$ so that $F(y) =^* h(z)$ and, since we may remove any finite set from z , we may assume that $y \cap z$ is empty. Choose $\mu \in \Gamma$ and let $c_y = \text{val}_{G_\mu}(\dot{c}_y)$ and let $\tilde{x} = \{j \in z : H_j^1 \cap F(c_y) \neq \emptyset\} \in M_\mu[c_y]$.

Since $\tilde{x} \subset z$ we have that \tilde{x} and c_y are disjoint. As in the previous cases, $\tilde{x} \cap \bar{x}_\mu$ is infinite. It follows that $F(\bar{x}_\mu) \supset H_j^1$ for infinitely many $j \in \tilde{x}$, which in turn implies that $F(\bar{x}_\mu) \cap F(c_y)$ is infinite. This contradicts that $\bar{x}_\mu \cap c_y$ is finite.

This completes the proof. \square

5. PROOF OF THEOREM 1.2

In this section we discuss Shelah's method of constructing an (ω_1, ω_1) -gap together with a ccc poset that will generically split the gap while

ensuring that the image of the gap by a non-trivial automorphism is not split by that poset, and can therefore be *frozen*.

To avoid possible confusion when stating the results we will formulate a statement of the hypotheses that are being assumed in their statements.

Definition 5.1. Let $\text{HYP}(F, \mathfrak{M})$ denote the following assumptions:

- (1) ω_1 -strategic MA holds and \mathfrak{M} is the system of models witnessing that it does,
- (2) $\langle M_\eta : \eta < \mathfrak{c} \rangle$ is used to denote the models in \mathfrak{M} ,
- (3) F is a lifting of an automorphism of $\mathcal{P}(\omega)/\text{fin}$
- (4) for every $A \in \text{triv}(F)^+$, $F \upharpoonright \mathcal{P}(A)$ not $< \mathfrak{c}$ -Borel,
- (5) for every $A \in \text{triv}(F)^+$, $\text{triv}(F) \cap \mathcal{P}(A)$ is not ccc over fin (a proven consequence of ω_1 -strategic MA)

Oddly the purpose of this subsection is to prove that $\text{HYP}(F, \mathfrak{M})$ is false. We will need the notion of relative complete embedding of posets. Let P and Q be posets and let M be a set (usually a model). We say that P is an M -complete subposet of Q , written as $P <_M Q$ if P is a subposet of Q (inheriting the extension and incompatibility relations) and every $D \in M$ that is a pre-dense subset of P is also pre-dense in Q . A set is pre-dense in a poset if its downward closure is dense. We will need this method of Shelah's to build ccc posets.

Proposition 5.2 ($\text{HYP}(F, \mathfrak{M})$). *Suppose that $\lambda \in S_1^{\mathfrak{c}}$ and with C_λ as in Definition 1.3, we have that $\eta_\alpha = C_\lambda(\alpha)$ for all $\alpha \in \omega_1$. Then $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ is a continuous increasing sequence of ordinals from \mathfrak{c} and λ is the supremum. Thus $\langle M_{\eta_\alpha} : \alpha < \omega_1 \rangle$ is an increasing chain of models and we have that*

$$M_\lambda \cap \mathcal{P}(\omega) = \bigcup \{ M_{\eta_\alpha} \cap \mathcal{P}(\omega) : \alpha < \omega_1 \} \quad \text{and that}$$

for any $\rho \in M_\lambda \cap 2^{\omega_1}$, the set of $\{ \delta \in \omega_1 : \rho \upharpoonright \delta \in M_{\rho_\delta} \}$ is stationary.

Suppose that $\{ P_\alpha : \alpha \in \omega_1 \}$ is a continuous increasing chain of countable posets $P_\alpha \in M_{\eta_\alpha}$ and that $\langle P_\alpha : \alpha < \omega_1 \rangle \in M_\lambda$. If for all $\alpha < \omega_1$, $P_\alpha <_{M_{\eta_\alpha}} P_{\alpha+1}$, then $P_{\omega_1} = \bigcup_{\alpha \in \omega_1} P_\alpha$ is ccc in M_λ .

Proof. In M_λ choose an enumeration $\{ p_\xi : \xi \in \omega_1 \}$ of P_{ω_1} . Choose a cub $C \subset \omega_1$ satisfying that for $\delta \in C$, $\{ p_\xi : \xi \in \delta \} = P_\delta$. Assume that $D \subset \omega_1$ is in M_{ω_1} and that $D(P) = \{ p_\xi : \xi \in D \}$ is a maximal antichain of P_{ω_1} . Choose an increasing function $g : \omega_1 \rightarrow \omega_1$ satisfying that for all $p \in P_\alpha$, there is a $d \in D(P) \cap P_{g(\alpha)}$ that is compatible with p . Let $C_1 \subset C$ be a cub such that $g(\alpha) < \delta$ for all $\delta \in C_1$. Observe that $\{ p_\xi : \xi \in D \cap \delta \}$ is pre-dense in P_δ for all $\delta \in C_1$.

Now let S be the stationary set of δ such that $D \cap \delta \in M_\delta$. If we choose $\delta \in S \cap C_1$, we have that $D_\delta(P) = \{p_\xi : \xi \in D \cap \delta\}$ is pre-dense subset of P_δ that is in M_{η_δ} . By induction on $\delta \leq \alpha < \omega_1$, $D_\delta(P)$ is pre-dense in P_α for all $\alpha \geq \delta$. This proves that $D = D \cap \delta$ is countable. \square

It will be convenient to adopt the following non-standard presentation of a gap structure.

Definition 5.3. For an ordinal $\delta \leq \omega_1$, a δ -gap is a sequence $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta \rangle$ such that, for all $\alpha \leq \beta < \delta$:

- (1) $a_\alpha \subset^* a_\beta \subset \omega$,
- (2) $x_\alpha \subset a_\alpha$,
- (3) $x_\beta \cap a_\alpha =^* x_\beta$.

A set Y splits the δ -gap $\langle a_\alpha, x_\alpha : \alpha < \delta \rangle$, if $Y \cap a_\alpha =^* x_\alpha$ for all $\alpha < \delta$. We say that a gap can not be split if there is no such Y . A helpful notion is to define that an n -gap $\langle a_i, x_i : i < n \rangle$ is a pure gap if $a_i \subset a_j$ and $x_i = x_j \cap a_i$ for $i < j < n$.

A Hausdorff-Luzin gap is Todorćević's variant of a Hausdorff gap, and, just as with a Hausdorff gap, a Hausdorff-Luzin gap can not be split.

Definition 5.4. A Hausdorff-Luzin gap is an ω_1 -gap $\langle a_\alpha, x_\alpha : \alpha < \omega_1 \rangle$ that has the property that for each $\alpha \neq \beta \in \omega_1$

$$((a_\alpha \setminus x_\alpha) \cap x_\beta) \cup (x_\alpha \cap (a_\beta \setminus x_\beta)) \text{ is not empty.}$$

It is a well-known unpublished result of Kunen that if an ω_1 -gap is not split, then there is a ccc poset that will freeze the gap in the sense that it has a Hausdorff subgap. The same is true, due to Todorćević, for Hausdorff-Luzin subgaps. We will use these since the poset is a better fit for ω_1 -strategic MA. In anticipation of those arguments we change the letter choices.

Definition 5.5. Let $\vec{c} = \langle c_\alpha, d_\alpha : \alpha < \delta \rangle$ be a δ -gap. The poset $HL(\vec{c})$ will consist of those finite subsets s of δ that satisfy that, for all $\alpha \neq \beta$ in s

$$((c_\alpha \setminus d_\alpha) \cap d_\beta) \cup (d_\alpha \cap (c_\beta \setminus d_\beta)) \text{ is not empty}$$

and is ordered by simple set inclusion.

For a δ -gap \vec{a} , let $F(\vec{a})$ denote the gap $\vec{c} = \langle F(a_\alpha), F(x_\alpha) \cap F(a_\alpha) : \alpha < \delta \rangle$. Define $HL_F(\vec{a})$ to be isomorphic to $HL(\vec{c})$ by the map sending $s \in HL(\vec{c})$ to $\{(c_\alpha, d_\alpha) : \alpha \in s\}$.

Proposition 5.6. *If \vec{c} is an ω_1 -gap that can not be split, then $HL(\vec{c})$ is ccc.*

Proof. Suppose that $\mathcal{S} = \{s_\xi : \xi \in \omega_1\} \subset HL(\vec{c}) \subset [\omega_1]^{<\aleph_0}$. We prove that it is not an antichain in $HL(\vec{c})$. By passing to an uncountable subset we can assume that it is a Δ -system. We will skip the argument that we can assume that the root is empty. For each ξ , let $\alpha_\xi = \min(s_\xi)$ and choose $m_\xi \in \omega$ so that $(c_{\alpha_\xi} \setminus d_{\alpha_\xi})$ and d_{α_ξ} are contained in $(c_\beta \setminus d_\beta) \cup m_\xi$ and $d_\beta \cup m_\xi$ respective for all $\beta \in s_\xi$. Again by passing to an uncountable (and re-indexed) subset we can assume that $m_\xi = m$ for all $\xi \in \omega_1$. Let $D = \bigcup \{d_{\alpha_\xi} \setminus m : \xi \in \omega_1\}$. Since D does not split the gap, there is an $\eta \in \omega_1$ such that $D \cap (c_{\alpha_\eta} \setminus d_{\alpha_\eta})$ is not empty. Fix such an η and choose $k \in D \cap (c_{\alpha_\eta} \setminus d_{\alpha_\eta})$. Since $k \in D$, there is a ξ such that $k \in d_{\alpha_\xi}$. Since $k > m$, it follows that $k \in (c_\beta \setminus d_\beta) \cap d_\gamma$ for all $\beta \in s_\eta$ and $\gamma \in s_\xi$. Since each of s_ξ and s_η are elements of $HL(\vec{c})$, it is then easily checked that $s_\xi \cup s_\eta$ is also in $HL(\vec{c})$. Since $HL(\vec{c})$ is ordered by extension, the family \mathcal{S} was not an antichain. \square

Splitting an ω_1 -gap with a ccc poset can not be done in general (e.g. a Haudorff-Luzin gap simply can not be split).

Definition 5.7. For a δ -gap $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta \rangle$, we define the poset $P_{\vec{a}} = P_{\langle a_\alpha, x_\alpha \rangle_{\alpha < \delta}}$ (or $P_{\langle a_\alpha, x_\alpha : \alpha < \delta \rangle}$) to be the set of conditions $p = \langle x_p, n_p, L_p \rangle \in \mathcal{P}(\omega) \times \omega \times [\delta]^{<\aleph_0}$, where:

- (1) $x_p \setminus n_p = \bigcup \{x_\alpha \setminus n_p : \alpha \in L_p\}$,
- (2) $\langle (a_\alpha \setminus n_p), (x_\alpha \setminus n_p) : \alpha \in L_p \rangle$, when re-indexed, is a pure gap.

The ordering on $P_{\vec{a}}$ is given by $p < q$ providing

- (4) $x_q \subset x_p$, $n_q \leq n_p$, $L_q \subset L_p$,
- (5) $x_p \cap n_q = x_q$,
- (6) $x_p \cap (a_\alpha \setminus n_q) = x_\alpha \setminus n_q$ for all $\alpha \in L_q$.

For each $\omega \leq \delta \in \omega_1$, fix an enumeration $\{\alpha_i^\delta : i \in \omega\}$ of δ .

Proposition 5.8. For any δ -gap \vec{a} , the poset $P_{\vec{a}}$ satisfies

- (1) for each $\beta < \delta$, the poset $P_{\vec{a} \upharpoonright \beta}$ is a subposet of $P_{\vec{a}}$,
- (2) for each $\beta < \delta$, the set of $p \in P_{\vec{a}}$ such that $\beta \in L_p$ is dense,
- (3) for each $p \in P$ and $n_p < m$, the condition $\langle x_p, m, L_p \rangle$ is an extension of p .

The canonical $P_{\vec{a}}$ -name $\dot{Y}_{\vec{a}}$ given by $\{(k, p) : p \in P_{\vec{a}}, k \in x_p\}$ satisfies that $\mathbb{1}$ forces that $\dot{Y}_{\vec{a}}$ splits the gap \vec{a} .

Proposition 5.9. Let $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta \rangle$ ($\delta \leq \omega_1$) be any δ -gap and let $p \in P_{\vec{a}}$. Then for every $s \subset n_p$ and every $m \geq n_p$, each of $p \downarrow s$ and $p \uparrow m$ are in $P_{\vec{a}}$ where

- (1) $p \downarrow s = \langle s \cup (x_p \setminus n_p), n_p, L_p \rangle$, and
- (2) $p \uparrow m = \langle x_p, m, L_p \rangle$.

In addition $p^\uparrow m \leq p$.

Proof. The verification that $p^\uparrow m \in P_{\vec{a}}$ and $p^\uparrow m \leq p$ is completely routine so we skip the details. Now let $r = p^\downarrow s$ so that we can refer easily to x_r . Clearly $x_r \setminus n_r = x_r \setminus n_p = x_p \setminus n_p$. Each of the conditions in the definition of $P_{\vec{a}}$ really only depend on the value of $x_p \setminus n_p$ and so the condition r also satisfies those conditions. \square

Definition 5.10. Let M be an elementary submodel of $H(\mathfrak{c})$ and let a δ -gap $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta \rangle$ be in M . A $\delta+1$ -gap, $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$, is an M -sealing extension of \vec{a} providing there is an infinite set $J \subset \omega$ such that for every $D \in M$ that is a dense subset of $P_{\langle a_\alpha, x_\alpha : \alpha < \delta \rangle}$, there is an m_D such that for all $p \in P_{\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle}$, with $m_D \leq n_p \in J$ and $\delta \in L_p \subset \{\alpha_i^\delta : i < n_p\} \cup \{\delta\}$, there is a $d \in D$ such that, with $\beta = \max(L_p \cap \delta)$, $d \leq ((x_p \cap n_p) \cup (x_\beta \setminus n_p), n_p, L_p \cap \delta)$ and $x_d \setminus n_p \subset x_\delta$, and for all $\alpha \in L_d$, $x_\delta \cap (a_\alpha \setminus n_d) = x_\alpha \setminus n_d$.

Proposition 5.11. *If $\vec{a} \cup \langle a_\delta, x_\delta \rangle$ is an M -sealing for an elementary submodel M of $H(\mathfrak{c})$, then for any $\delta < \gamma$ and γ -gap extension $\vec{a}(\gamma)$, $P_{\vec{a}} <_M P_{\vec{a}(\gamma)}$.*

Proof. Let $\langle a_\alpha, x_\alpha : \alpha < \gamma \rangle$ be any γ -gap with $\delta < \gamma$. Let $J \subset \omega$ be the infinite set as postulated in Definition 5.10. Let $D \in M$ be any dense subset of $P_{\vec{a}}$ and let m_D be as given in Definition 5.10. Consider any $p \in P_{\vec{a}(\gamma)}$. By extending p we can, by Proposition 5.8, assume that $\delta \in L_p$. By further extending, only the value of n_p as in Proposition 5.8, we can assume that $m_D \leq n_p \in J$ and that $L_p \cap \delta \subset \{\alpha_i^\delta : i < n_p\}$.

Let $\tilde{p} = ((x_p \cap n_p) \cup (x_\delta \setminus n_p), n_p, L_p \cap \delta)$ and notice that $p \leq \tilde{p}$ and that $\tilde{p} \in P_{\langle a_\alpha, x_\alpha : \alpha \leq \delta \rangle}$. Now let $\beta = \max(L_p \cap \delta)$. By the assumption on M -sealing, there is a $d \in D$ such that $d < ((x_p \cap n_p) \cup (x_\beta \setminus n_p), n_p, L_p \cap \delta)$ and $x_d \subset x_{\tilde{p}}$. It is nearly immediate that $(x_d \cup (x_\delta \setminus n_d), n_d, L_{\tilde{p}})$ is an extension of \tilde{p} . This is because $x_d \subset x_{\tilde{p}}$ and for all $\alpha \in L_d$, $x_\delta \cap (a_\alpha \setminus n_d) = x_\alpha \setminus n_d$.

We check that $q = (x_d \cup x_p, n_d, L_p \cup L_d)$ is in $P_{\vec{a}(\gamma)}$ and is an extension of both p and d . The proof that $q \in P_{\vec{a}(\gamma)}$ follows easily by observing that $\langle (a_\alpha \setminus n_d), (x_\alpha \setminus n_d) : \alpha \in L_d \cup (L_p \cap \delta + 1) \rangle$ and $\langle (a_\beta \setminus n_d), (x_\beta \setminus n_d) : \beta \in L_p \rangle$ are pure gaps.

Similarly, the fact that $q < p$ follows easily from the fact that $x_d \subset x_{\tilde{p}}$. Finally, the fact that $q < p$ is because $x_q \setminus a_\delta = x_p \setminus a_\delta$ and that $a_\alpha \subset a_\delta \cup n_d$ for all $\alpha \in L_d$. \square

Now we will say that a δ -gap $\langle a_\alpha, x_\alpha : \alpha < \delta \rangle$ is an (F, δ) -gap if $\omega \setminus a_\alpha \notin \text{triv}(F)$ for all $\alpha < \delta$. Note that by Lemma 2.4, any countable (F, δ) -gap can be extended.

This next result is really just the Cohen forcing version of Lemma 4.1.

Lemma 5.12 (HYP(F, \mathfrak{M})). *Let $M \in \mathfrak{M}$. Assume that $\delta < \omega_1$ and that $\bar{a} = \langle a_\alpha, x_\alpha : \alpha < \delta \rangle$ is a (F, δ) -gap in M .*

There is a pair $a_\delta \supset x_\delta$ such that

- (1) $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$ is an $(F, \delta+1)$ -gap,
- (2) for every $\delta < \gamma$ and γ -gap $\bar{a}(\gamma)$ extending $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$,

$$P_{\bar{a}} = P_{\langle a_\alpha, x_\alpha : \alpha < \delta \rangle} <_M P_{\bar{a}(\gamma)}$$

i.e. that $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$ is an M -sealing of \bar{a} .

Proof. Recall that every (F, δ) -gap can be extended. That is, we may choose some \bar{a}_δ so that $a_\alpha \subset^* \bar{a}_\delta$ for all $\alpha < \delta$ and $\omega \setminus \bar{a}_\delta \notin \text{triv}(F)$.

Next, choose a filter $G \subset P_{\bar{a}}$ that is M -generic for $P_{\bar{a}}$ and let \bar{x}_δ be the generic splitting of the gap $\langle a_\alpha, x_\alpha : \alpha < \delta \rangle$ in the sense that

$$\bar{x}_\delta = \bigcup \{x_p : p \in G\}.$$

Fix any dense set $D \subset P_{\bar{a}}$.

Fact 5.12.1. *For each $m \in \omega$ the set $D(m)$ is dense in $P_{\bar{a}}$ where $d \in D(m)$ if $L_d \supset \{\alpha_i^\delta : i < m\}$, $m \leq n_d$ and for all $s \subset m$, $(s \cup (x_d \setminus m), n_d, L_d) \in D$.*

Proof of Fact: We begin a recursion by choosing any d_0 in D such that $x_{d_0} \cap m = \emptyset$ and $L_{d_0} \supset \{\alpha_i^\delta : i < m\}$. Fix an enumeration $\{s_i : i < 2^m\}$ of $\mathcal{P}(m)$ so that $s_0 = \emptyset = x_{d_0} \cap m$. Recursively choose a descending sequence $\{d_i : i < 2^m\} \subset D$. Having chosen d_i , let $s = s_i \cup (x_{d_i} \cap n_{d_i} \setminus m)$. Note that $d_i^\downarrow s$ is equal to $(s_i \cup (x_{d_i} \setminus m), n_{d_i}, L_{d_i})$. Naturally we choose $d_{i+1} < d_i$ in D . It should be clear that d_{2^m} is an element of $D(m)$. \square

It follows from Fact 5.12.1 that there is a function $f_D \in \omega^\omega$ satisfying that for each $m \in \omega$ and each $\beta \in \{\alpha_i^\delta : i < m\}$, there is a $d \in D(m, \beta) \cap G$ such that $n_d < f_D(m)$ and $a_\alpha \setminus \bar{a}_\delta \subset f_D(m)$ for all $\alpha \in L_d$.

Since G is a filter, this means that $x_d \subset \bar{x}_\delta$ and $\bar{x}_\delta \cap a_\alpha \setminus n_d = x_\alpha \setminus n_d$ for all $\alpha \in L_d$.

Since MA is assumed, there is a sequence of integers $\langle k_\ell : \ell \in \omega \rangle$ satisfying that, for all dense $D \subset P_{\bar{a}}$ in M , there is an m_D so that for all $k_\ell \geq m_D$, $f_D(k_\ell) < k_{\ell+1}$. By Lemma 2.7, we may choose an infinite $L \subset \omega$ so $(\omega \setminus \bar{a}_\delta) \cup \bigcup \{k_{\ell+1} \setminus k_\ell : \ell \notin L\}$ is not in $\text{triv}(F)$.

Set $a_\delta = \bar{a}_\delta \cup \bigcup \{k_{\ell+1} \setminus k_\ell : \ell \in L\}$, hence $\omega \setminus a_\delta$ is not in $\text{triv}(F)$. Set $x_\delta = \bar{x}_\delta \cap a_\delta$. It should be clear that $\langle a_\alpha, x_\alpha : \alpha \leq \delta \rangle$ is an $(F, \delta+1)$ -gap.

Now we verify that

Fact 5.12.2. $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$ is an M -sealing extension of \vec{a} .

Proof of Fact: Naturally we prove that $J = \{k_\ell : \ell \in L\}$ is the set required in Definition 5.10. Fix any dense set $D \subset P_{\vec{a}}$ with $D \in V[G_\mu]$ and choose any m_D so that $f_D(k_\ell) < k_{\ell+1}$ for all $m_D \leq k_\ell$. Fix any $p \in P_{\langle a_\alpha, x_\alpha : \alpha \leq \delta \rangle}$ and $k_\ell \in J$, such that $n_p \leq k_\ell$ and $L_p \subset \{\alpha_i^\delta : i < k_\ell\} \cup \delta$. Let $\beta = \max(L_p \cap \delta) \in \{\alpha_i^\delta : i < k_\ell\}$. Note that $x_\delta \cap (a_\alpha \setminus n_p) = x_\beta \cap (a_\alpha \setminus n_p) = x_\alpha \setminus n_p$ for all $\alpha \in L_p \cap \delta$.

Let \tilde{p} be the condition $((x_p \cap n_p) \cup (x_p \cap a_\beta), n_p, L_p \cap \delta)$ where $\beta = \max(L_p \cap \delta)$. Thus $x_{\tilde{p}} = (x_p \cap n_p) \cup (x_p \cap a_\beta)$ and $x_p \setminus (n_p \cup a_\beta) \subset x_\delta$.

Fix any $d \in D(k_\ell) \cap G$ such that $n_d < k_{\ell+1}$. Therefore, $x_d \subset \bar{x}_\delta$ and, since $d > p \in G$ implies that $x_p \cap n_d = x_d \cap n_d$, we have that $\bar{x}_\delta \cap n_d = x_d$. Also $\bar{x}_\delta \cap a_\alpha \setminus n_d = x_\alpha \setminus n_d$ for all $\alpha \in L_d$. Furthermore, since $a_\alpha \setminus \bar{a}_\delta \subset k_{\ell+1}$ for all $\alpha \in L_d$, we have that $x_\delta \cap a_\alpha \setminus n_d = x_\alpha \setminus n_d$ for all $\alpha \in L_d$.

Now let $s = x_p \cap k_\ell$ and the condition $\tilde{d} = d \upharpoonright s \in D$. We must prove that $\tilde{d} \leq \tilde{p}$ to finish the proof. The only condition that is not obvious is that for each $\alpha \in L_{\tilde{p}}$, $x_{\tilde{d}} \cap (a_\alpha \setminus k_\ell) = x_\alpha \setminus k_\ell$. Notice simple $\alpha \in L_{\tilde{d}}$ and $n_{\tilde{d}} = n_d$ ensures that $x_{\tilde{d}} \cap (a_\alpha \setminus n_d)$ equals $x_\alpha \setminus n_d$. Next we have that $x_d \cap n_d \setminus n_p = x_\delta \cap n_d \setminus n_p$ and that $\alpha \in L_p$ implies that $x_\delta \cap (a_\alpha \cap k_\ell) = x_\alpha \cap k_\ell$. \square

This completes the proof of the Lemma. \square

Next we consider how to extend an $(F, \delta+1)$ -gap \vec{a} in such a way that we make progress on ensuring that the corresponding gap that we loosely describe as $F(\vec{a})$ will not be split if we force with (F, ω_1) -gap poset that the ccc poset \mathfrak{M} -strategy constructs.

Lemma 5.13 (HYP(F, \mathfrak{M})). *Given an $(F, \delta+1)$ -gap $\vec{a} = \langle a_\alpha, x_\alpha : \alpha \leq \delta \rangle$ and a model $M \subset H(\mathfrak{c})$ of cardinality less than \mathfrak{c} , there is a pair $a_{\delta+1}, x_{\delta+1}$ such that $\langle a_\alpha, x_\alpha : \alpha < \delta+2 \rangle$ is an $(F, \delta+2)$ -gap and for every $P_{\vec{a}}$ -name $\dot{Y} \in M$ of a subset of ω ,*

$$1 \Vdash_{P_{\langle a_\alpha, x_\alpha : \alpha < \delta+2 \rangle}} \dot{Y} \cap F(a_{\delta+1}) \neq^* F(x_{\delta+1}) .$$

Furthermore $a_{\delta+1} \supset a_\delta$ and $x_{\delta+1} \cap a_\delta = x_\delta$.

Proof. Here we must use that $\text{triv}(F)$ is not ccc over fin on any member of $\text{triv}(F)^+$. Since $\omega \setminus a_\delta \in \text{triv}(F)^+$, there is a set $a_{\delta+1} \supset a_\delta$ such that $a_{\delta+1} \setminus a_\delta$ and $\omega \setminus a_{\delta+1}$ are not members of $\text{triv}(F)$. This means that whatever we may choose for $x_{\delta+1} \subset a_{\delta+1}$ satisfying that $x_{\delta+1} \cap a_\delta = x_\delta$, the sequence $\langle a_\alpha, x_\alpha : \alpha < \delta+2 \rangle$ is an $(F, \delta+2)$ -gap.

By enlarging M we may assume that \vec{a} and $a_{\delta+1}$ are elements of M . Let G be any filter on $P_{\vec{a}}$ that is generic over M . For $x \subset a_{\delta+1} \setminus a_\delta$ and

$p \in P_{\vec{x}}$, let $p(x)$ be the condition $(x \cup (x_p \setminus (a_{\delta+1} \setminus a_\delta)), n_p, L_p)$. In essence $p(x)$ is a condition in $P_{\vec{a}(x)}$ where $\vec{a}(x)$ is the extension of \vec{a} by appending $(a_{\delta+1}, x_\delta \cup x)$. It is immediate that $G(x) = \{p(x) : p \in G\}$ is a filter on $P_{\vec{a}(x)}$ and we may define $E_{\dot{Y}}(x)$ to equal $\text{val}_{G(x)}(\dot{Y}) \cap F(a_{\delta+1}) \setminus F(a_\delta)$. It follows by well-known ([11], [4, §3], [3]) arguments that $E_{\dot{Y}}$ is a Borel function. Again our assumptions are that $F \upharpoonright \mathcal{P}(a_{\delta+1} \setminus a_\delta)$ is not covered by the family $\{E_{\dot{Y}} : \dot{Y} \in M\}$.

Therefore there is an $x \subset a_{\delta+1} \setminus a_\delta$ such that, for all $\dot{Y} \in M$, $F(x) \neq^* E_{\dot{Y}}(x)$. For each $\dot{Y} \in M$, there is a countable set $\mathcal{D}_{\dot{Y}}$ of dense subsets of $P_{\vec{a}}$ satisfying that if G is any filter of $P_{\vec{a}}$ meeting every member of $\mathcal{D}_{\dot{Y}}$, then $F(x) \neq \text{val}_{G(x)}(\dot{Y}) \cap F(a_{\delta+1}) \setminus F(a_\delta)$. It follows then, that for this choice of $x \subset a_{\delta+1} \setminus a_\delta$,

$$1 \Vdash_{P_{\vec{a}(x)}} F(x_\delta \cup x) \neq^* \dot{Y} \cap F(a_{\delta+1})$$

for all $\dot{Y} \in M$. □

It will be more convenient to combine Lemma 5.12 and Lemma 5.13 into a single step.

Corollary 5.14 (HYP(F, \mathfrak{M})). *Let $\eta < \mathfrak{c}$ and suppose that $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta \rangle$ is an (F, δ) -gap in M_η with $\delta \in \omega_1$. There is a pair $a_\delta \supset x_\delta$ such that*

- (1) $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$ is an $(F, \delta+1)$ -gap,
- (2) for every $\delta < \gamma$ and γ -gap $\vec{a}(\gamma)$ extending $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$,

$$P_{\vec{a}} = P_{\langle a_\alpha, x_\alpha : \alpha < \delta \rangle} <_M P_{\vec{a}(\gamma)}$$

i.e. that $\langle a_\alpha, x_\alpha : \alpha < \delta+1 \rangle$ is an M_η -sealing of \vec{a} .

- (3) for every $P_{\vec{a}}$ -name $\dot{Y} \in M_\eta$, $P_{\langle a_\alpha, x_\alpha : \alpha \leq \delta \rangle}$ forces that $F(x_\delta) \neq^* \dot{Y} \cap F(a_\delta)$.

Proof. Choose a_δ, x_δ as in Lemma 5.12, then choose $a_{\delta+1}, x_{\delta+1}$ as in Lemma 5.13. The extra assumption in Lemma 5.13 that $a_{\delta+1} \supset a_\delta$ and $x_{\delta+1} \cap a_\delta = x_\delta$ makes it very routine to prove that $\vec{\mathfrak{a}} = \langle \vec{a} \cup \langle \bar{a}_\delta, \bar{x}_\delta \rangle \rangle$ is M_η -sealing where $\bar{a}_\delta = a_{\delta+1}$ and $\bar{x}_\delta = x_{\delta+1}$. Similarly, the set of conditions $p \in P_{\langle a_\alpha, x_\alpha : \alpha < \delta+2 \rangle}$ such that L_p does not split $\{\delta, \delta+1\}$ is dense and is isomorphic to $P_{\vec{\mathfrak{a}}}$. Therefore $P_{\vec{\mathfrak{a}}}$ also forces that $F(\bar{x}_\delta) \neq^* \dot{Y} \cap F(\bar{a}_\delta)$ for all $\dot{Y} \in M_\eta$. Thus the Corollary is proven by choosing \bar{a}_δ and \bar{x}_δ . □

Lemma 5.15 (HYP(F, \mathfrak{M})). *There is a ccc poset \mathfrak{M} -strategy such that every ccc poset/model pair (R_ρ, M_λ) produced by the strategy satisfies that*

- (1) there is an ω_1 -gap $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \omega_1 \rangle$ in the model M_λ ,
- (2) $\vec{c} = F(\vec{a})$ is an ω_1 -gap that is also in M_λ ,

(3) R_ρ is the (ccc in M_λ) poset $P_{\vec{a}} * HL_F(\vec{a})$.

Proof. We recursively define, for $t \in \mathfrak{c}^{<\omega_1}$, a triple (a_t, x_t, η_t) with $x_t \subset a_t \subset \omega$ and $\eta_t < \mathfrak{c}$ with $\{a_t, x_t, F(a_t), F(x_t)\} \subset M_{\eta_t}$.

In this recursion, for each $t \in \mathfrak{c}^{<\omega_1}$ and $\alpha \in \text{dom}(t)$, let $(a_\alpha^t, x_\alpha^t, \eta_\alpha^t) = (a_{t \upharpoonright \alpha}, x_{t \upharpoonright \alpha}, \eta_{t \upharpoonright \alpha})$ and let $\vec{a}_t = \langle a_\alpha^t, x_\alpha^t : \alpha \in \text{dom}(t) \rangle$. Similarly define \vec{a}_ρ for $\rho \in \mathfrak{c}^{\omega_1}$ and the sequence $\langle \eta_\alpha^\rho : \alpha < \omega_1 \rangle$. We shall assume, by induction on $\text{dom}(t)$ that \vec{a}_t is an $(F, \text{dom}(t))$ -gap. Note that this is a simple requirement of mod finite increasing and having that $x_\beta^t =^* a_\beta^t \cap x_\alpha^t$ for all $\beta < \alpha \in \text{dom}(t)$.

Let $\nu_t \in \mathfrak{c}$ denote the minimal ordinal such that $\eta_\alpha^t \leq \nu_t$ for all $\alpha \in \text{dom}(t)$, and let $M_t = M_{\nu_t}$.

By recursion on $\text{dom}(t)$ for $t \in \mathfrak{c}^{<\omega_1}$, choose a_t, x_t, η_t so as to satisfy:

- (1) $\eta_t \in \mathfrak{c}$, $x_t \subset a_t$ are subsets of ω ,
- (2) $\{a_t, x_t, F(a_t), F(x_t)\}$ are in the model M_{η_t} ,
- (3) $\nu_t < \eta_t$,
- (4) $\omega \setminus a_t$ is not in $\text{triv}(F)$,
- (5) for all $\beta \in \text{dom}(t)$, $a_\beta^t \subset^* a^t$ and $x_\beta^t =^* a_\beta^t \cap x_t$,
- (6) $\vec{a}_t \cup \langle a_t, x_t \rangle$ is M_t -sealing,
- (7) $P_{\vec{a}_t \cup \langle a_t, x_t \rangle}$ forces that $F(x_t)$ is not mod finite equal to $\dot{Y} \cap F(a_t)$ for any $P_{\vec{a}_t}$ -name \dot{Y} in M_t .

By Corollary 5.14, the recursion succeeds.

For each $t \in \mathfrak{c}^{<\omega_1}$ and $\alpha \in \text{dom}(t)$, let $c_\alpha^t = F(a_\alpha^t)$ and $d_\alpha^t = F(x_\alpha^t) \cap F(a_\alpha^t)$. Also let $c_t = F(a_t)$ and $d_t = F(x_t) \cap F(a_t)$, and \vec{c}_t

The poset R_t is $P_{\vec{a}_t \cup \langle a_t, x_t \rangle} * HL_F(\vec{a}_t \cup \langle a_t, x_t \rangle)$. Note that $R_t \in M_{\eta_t}$. Let \mathcal{R} be the \mathfrak{M} -system $\langle (R_t, \eta_t) : t \in \mathfrak{c}^{<\omega_1} \rangle$ and λ be any element of $C(\mathcal{R}) \cap S_1^{\mathfrak{c}}$. To prove that \mathcal{R} is a ccc poset \mathfrak{M} -system, we must prove that if $\rho \in \mathfrak{c}^{<\omega_1}$, then $R_\rho = \bigcup \{R_{\rho \upharpoonright \alpha} : \alpha \in \omega_1\}$ is a ccc poset in M_λ .

Since $\langle R_{\rho \upharpoonright \alpha} : \alpha \in \omega_1 \rangle$ is in M_λ , it follows that $\vec{a}_\rho = \langle a_\alpha^\rho, x_\alpha^\rho : \alpha \in \omega_1 \rangle$ is also in M_λ . Letting, for $\alpha \in \omega_1$, $P_\alpha = P_{\vec{a}_\rho \upharpoonright C_\lambda(\alpha)}$, it follows from Lemma 5.2 that $P_{\vec{a}_\rho}$ is ccc. For each countable $P_{\vec{a}_\rho}$ -name \dot{Y} in M_λ , there is a $\delta \in \omega_1$ such that $\dot{Y} \in M_t$ for $t = \rho \upharpoonright \delta$. By construction, $P_{\vec{a}_\rho \upharpoonright \delta+1}$ forces that $F(x_{\delta+1}^\rho) \neq^* \dot{Y} \cap F(a_{\delta+1}^\rho)$. Similarly the construction ensures that $P_{\vec{a}_{\delta+1}^\rho} <_{M_{\eta_{\delta+1}^\rho}} P_{\vec{a}}$.

Finally, it is also clear from the definition of $HL_F(\vec{a})$, as opposed to $HL(F(\vec{a}))$, that $F(\vec{a}_\rho)$ is also in M_λ and, by Proposition 5.6, even after forcing with $P_{\vec{a}_\rho} HL_F(\vec{a}_\rho)$, equivalently $HL(F(\vec{a}_\rho))$, is ccc. Therefore $R_\rho = P_{\vec{a}_\rho} * HL_F(\vec{a}_\rho)$ is ccc. \square

This completes the proof that $\text{HYP}(F, \mathfrak{M})$ can not hold, and therefore the proof of Theorem 1.2.

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