AUTOMORPHISMS OF $\mathcal{P}(\omega)/\mathbf{fin}$ AND LARGE CONTINUUM

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ABSTRACT. We prove that it is consistent with $\mathfrak{c} > \aleph_2$ that all automorphisms of $\mathcal{P}(\omega)/\text{fin}$ are trivial.

1. INTRODUCTION

The study of automorphisms of $\mathcal{P}(\omega)/\text{fin}$ has, by now, an extensive and fascinating history. Naturally $\mathcal{P}(\omega)/\text{fin}$ is the quotient Boolean algebra of the complete Boolean power set algebra $\mathcal{P}(\omega)$ by the ideal fin of finite sets. Every bijection between cofinite subsets of ω induces an automorphism of $\mathcal{P}(\omega)/\text{fin}$ and such automorphisms are said to be trivial. W. Rudin [6] established that the continuum hypothesis implied that there were non-trivial automorphisms. S. Shelah [7] established that it was consistent that all automorphisms were trivial and Shelah and Steprans [8] proved that this was a consequence of PFA. Our results follow the basic approach of both [7, 8] but also benefit from the considerable contributions in [3, 11, 12] and others. Shelah and Steprans [9] have shown that it is consistent with $\mathfrak{c} > \aleph_2$ there are non-trivial, even nowhere trivial, automorphisms. In this paper we establish that it is consistent with $\mathfrak{c} > \aleph_2$ that all automorphisms are trivial.

For a function $F : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, we say that F induces an automorphism (of $\mathcal{P}(\omega)/\text{fin}$) if $F(x) =^* F(y)$ whenever $x =^* y$ and the function sending the equivalence class of x (mod finite) to that of $F(x)$ (mod finite) is indeed an automorphism of $\mathcal{P}(\omega)/\text{fin}$. When F does not induce a trivial automorphism, the family $\text{triv}(F)$ is the ideal of sets $a \subset \omega$ such that $F \restriction \mathcal{P}(a)$ is trivial in the usual sense, namely that there is a 1-to-1 function h_a from a into ω satisfying that $F(x) =^* h_a(x)$ for all $x \subset a$. We will, by default, let h_a denote a 1-to-1 function that induces F on $\mathcal{P}(a)$ when a is an infinite element of $\text{triv}(F)$. It is easily seen that $\text{triv}(F)$ is an ideal on ω [7]. As usual, $\text{triv}(F)^+$ will denote (infinite) subsets of ω that are not elements of $\text{triv}(F)$.

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We adopt the P-name convention, for each poset P, that a P-name is a set of ordered pairs where the first coordinate is a P-name and the second coordinate is an element of P. We will also abuse notation in a standard way and treat ordinals and finite tuples of ordinals as P-names for themselves. Hence, for example, a subset of $\omega \times P$ will be regarded as a P-name for a subset of ω. We typically use the dot notation $\dot{y}$ to denote a P-name of a set. In the context of an argument with a generic filter in the discussion, removing the dot will denote the set that results from evaluating the name using that generic filter.

2. Tools for your forcing all automorphisms are trivial construction

In this section, analogous to the results in [4], we collect some of the consistency results and combinatorial tools that have been developed that will help control the behavior on automorphism of $\mathcal{P}(\omega)/\text{fin}$. We will also arrange that these principles will hold in our final model.

Proposition 2.1. *If F induces an automorphism and $a \in triv(F)^+$, then for any 1-to-1 function $h \in {}^a\omega$, there is an infinite $x \subset a$ such that $h(x) \cap F(x)$ is finite.*

Proof. Since $a \notin \text{triv}(F)$, the function h does not induce $F \upharpoonright \mathcal{P}(a)$. Choose any $x_1 \subset a$ so that $h(x_1)\Delta F(x_1)$ is infinite. If $h(x_1) \setminus F(x_1)$ is infinite, then let $x \subset x_1$ be chosen so that $h(x) = h(x_1) \setminus F(x_1)$. Since $F(x) \subset^* F(x_1)$, we have $h(x) \cap F(x)$ is finite as required. In the other case when $F(x_1) \setminus h(x_1)$ is infinite, choose $x \subset x_1$ so that $F(x) =^* F(x_1) \setminus h(x_1)$. Here again we have that $h(x) \subset h(x_1)$ and $h(x_1) \cap F(x) =^* \emptyset$. □

Definition 2.2. A family $\mathcal{H}$ of possibly partial functions from a countable set D to ω is coherent if for $h_1, h_2 \in \mathcal{H}$, the set $\{n \in dom(h_1) \cap dom(h_2) : h_1(n) \neq h_2(n)\}$ is finite.

Such a coherent family $\mathcal{H}$ is maximal if whenever $\mathcal{H} \cup \{\bar{h}\}$ is coherent, the domain of $\bar{h}$ is in the ideal generated by $[D]^{<\aleph_0} \cup \{dom(h) : h \in \mathcal{H}\}$. A coherent family will be called trivial if D is in this ideal.

For a function $f \in \omega^\omega$, let $f^\downarrow$ denote the set $\{(n,m) \in \omega \times \omega : m < f(n)\}$. A family $\mathcal{H}$ of functions is an ω^ω-family if $\mathcal{H} = \{h_f : f \in \omega^\omega\}$ and, for each $f \in \omega^\omega$, h_f is a function from $f^\downarrow$ to ω.

We note that every subfamily of a coherent family is also coherent.

Definition 2.3. Say that the principle ω^ω-cohere holds if each ω^ω-family $\mathcal{H}$ that is coherent, there is a function $h : \omega \times \omega \to \omega$ such that $\mathcal{H} \cup \{h\}$ is also coherent.

The principle ω^ω-cohere (not yet named) is a well-known consequence of OCA due to Todorcevic (see [3, 2.2.7]). This next result is similar to [11, proof of Lemma 2.5].

Lemma 2.4. *Suppose that F induces an automorphism on $\mathcal{P}(\omega)/fin$ and suppose that triv(F) is proper dense ideal. The principle ω^ω-cohere implies that if $\{a_n : n \in \omega\}$ is a mod finite increasing sequence of subsets of ω then either there is an n so that $\omega \setminus a_n \in$ triv(F), or there is an $b \in$ triv$(F)^+$ that is almost disjoint from each a_n.*

Proof. Let $\{a_n : n \in \omega\} \subset [\omega]^{\aleph_0}$ be increasing. There is nothing to prove if the family $\{a_n : n \in \omega\}$ is eventually constant mod finite, so we may assume that it is strictly increasing mod finite. As usual, let $\{a_n\}_n^\perp$ denote the ideal of sets b that are almost disjoint from each a_n. Assume that $b \in \text{triv}(F)$ for all $b \in \{a_n\}_n^\perp$, and as usual, let h_b be the function on b inducing $F \restriction \mathcal{P}(b)$. Since the family $\mathcal{H} = \{h_b : b \in \{a_n\}_n^\perp\}$ is a coherent family, it is an obvious consequence of ω^ω-cohere that there is a function $h \in \omega^\omega$ such that $\mathcal{H} \cup \{h\}$ is coherent.

We show that there is an n_0 so that $h \restriction (\omega \setminus a_{n_0})$ is 1-to-1. Otherwise we may choose a sequence of pairs $\{(i_n, j_n) : n \in \omega\}$ so that $\{i_n, j_n\} \subset \omega \setminus a_n$ and $h(i_n) = h(j_n)$. By construction the set $b = \{i_n : n \in \omega\} \cup \{j_n : n \in \omega\}$ is an element of $\{a_n\}_n^\perp$. Since h_b is 1-to-1, this contradicts that $h_b \subset^* h$.

If there is an $n \geq n_0$ so that $h \restriction (\omega \setminus a_n)$ induces F on $\mathcal{P}(\omega \setminus a_n)$ then the Lemma is proven. Otherwise, we may choose, using Lemma 2.1, for each $n_0 \leq n \in \omega$ an infinite set $x_n \subset \omega \setminus a_n$ so that $h(x_n) \cap F(x_n)$ is finite. Since x_n cannot be an element of $\{a_n\}_n^\perp$, we may shrink x_n so that there is an $k_n \geq n$ such that $x_n \subset a_{k_n+1} \setminus a_{k_n}$. Since $\text{triv}(F)$ is assumed to be dense, we may also assume that $x_n \in \text{triv}(F)$. Choose an infinite $J \subset \omega$ so that for $n < m \in J$, $k_{n+1} \leq m$. For each $n \in J$, let h_n denote the function h_{x_n}. Note that $h(x_n) \cap h_n(x_n) =^* h(x_n) \cap F(x_n)$ is finite. By a simple recursion we can further arrange that $h(i) \neq h_n(j)$ for all $i \in \bigcup\{x_k : k \in J\}$ and $j \in x_n$. Let $x = \bigcup\{x_n : n \in J\}$. For each $n \in J$, choose a finite $H_n \subset x_n$ so that $h_n(x_n \setminus H_n) \subset F(x)$. By recursion on $n \in J$, choose $i_n \in x_n \setminus H_n$ so that $h_n(i_n) \notin \bigcup\{F(a_k) : k \leq k_n\}$. Choose $y \subset \omega$ so that $F(y) =^* \{h_n(i_n) : n \in J\}$ and note that $y \cap a_k$ is finite for all $k \in \omega$. Since $F(y) \subset F(x)$ we may assume that $y \subset x$. But now, $F(y) =^* h(y)$ and yet $h(y)$ is disjoint from $\{h_n(i_n) : n \in J\}$. □

Definition 2.5. Say that a family $\{h_\alpha : \alpha \in \omega_1\}$ of partial functions from ω to ω is Luzin incoherent if for all $\alpha < \beta$, $h_\alpha \cup h_\beta$ is not a function.

Proposition 2.6. *If $\mathcal{H} = \{h_\alpha : \alpha \in \omega_1\}$ is a Luzin incoherent family, then for all $h \in \omega^\omega$, $\mathcal{H} \cup \{h\}$ is not coherent.*

Proof. Let $h \in \omega^\omega$ and assume, for a contradiction, that $\mathcal{H} \cup \{h\}$ is coherent. Choose $m \in \omega$ and function $t : m \to \omega$ so that, there is an uncountable $\Gamma \subset \omega_1$ so that, for all $\alpha \in \Gamma$, $h_\alpha \restriction (dom(h_\alpha) \setminus m) \subset h$ and $h_\alpha \restriction m \subset t$. Of course we now find that for all $\alpha < \beta$ both in Γ, $h_\alpha \cup h_\beta(n)$ is a subfunction of $t \cup h \restriction (\omega \setminus m)$. ☐

Definition 2.7. An ideal $\mathcal{I}$ on a set $A \subset \omega$ is ccc over fin if $\mathcal{A} \cap \mathcal{I}$ is not empty for every uncountable almost disjoint family $\mathcal{A}$ of subsets of A.

The notion of an ideal being ccc over fin was introduced in [3].

Definition 2.8. Say that the principle P-cohere holds providing there is no non-trivial coherent family of functions whose domains form a ccc over fin P-ideal.

Lemma 2.9. *Assume that the principles ω^ω-cohere and P-cohere hold. If F induces an automorphism on $\mathcal{P}(\omega)/fin$ then $A \in triv(F)$ for any $A \subset \omega$ such that $triv(F) \cap \mathcal{P}(A)$ is ccc over fin.*

Proof. Recall that we have assumed a fixed assignment of $h_a \in {}^a\omega$ for all $a \in \mathrm{triv}(F)$ such that $h_a(x) =^* F(x)$ for all $x \subset a$. Since F induces an automorphism, it follows that the family $\mathcal{H} = \{h_a : a \in \mathrm{triv}(F)\}$ is a coherent family. Since a ccc over fin ideal is a dense ideal, we assume for the remainder of the proof that $\mathrm{triv}(F)$ is a proper dense ideal. We prove that it can not be ccc over fin.

We first check that if if $h \in {}^\omega\omega$, then $\mathcal{H} \cup \{h\}$ is not coherent. Since we are assuming that $\omega \notin \mathrm{triv}(F)$, we have, by Lemma 2.1, that there is an infinite $x \subset \omega$ such that $h(x) \cap F(x)$ is finite. We are also assuming that $\mathrm{triv}(F)$ is a dense ideal, so, by possibly shrinking x, we can assume that $x \in \mathrm{triv}(F)$. Then $h_x \in \mathcal{H}$ and evidently, $\{h_x, h\}$ is not coherent.

It is immediate from the assumption that P-cohere holds that $\mathrm{triv}(F)$ is not a P-ideal. We complete the proof by considering the case where $\mathrm{triv}(F)$ is not a P-ideal. Fix any increasing sequence $\{a_n : n \in \omega\} \subset \mathrm{triv}(F)$ with the property that $a \notin \mathrm{triv}(F)$ for any $a \subset \omega$ that mod finite contains each a_n. Since $\omega \notin \mathrm{triv}(F)$, it follows that $\omega \setminus a_n \notin \mathrm{triv}(F)$ for all $n \in \omega$. Recall that $\{a_n\}_n^\perp$ denotes the ideal of sets $b \subset \omega$ that are almost disjoint from a_n for each n. For each $b \in \{a_n\}_n^\perp$, the ideal $\{b \cup a_n\}_n^\perp$ is a P-ideal. Using ω^ω-cohere and Lemma 2.4, for every $b_0 \in \{a_n\}^\perp$, there is a $b_1 \in \mathrm{triv}(F)^+ \cap \{b_0 \cup a_n\}_n^\perp$. Therefore, we can by recursion, construct a mod finite increasing sequence $\{b_\alpha : \alpha < \omega_1\} \subset \{a_n\}_n^\perp$ so that, for each $\alpha < \omega_1$, $b_{\alpha+1} \setminus b_\alpha \in \mathrm{triv}(F)^+ \cap \{b_\alpha \cup a_n\}_n^\perp$. This clearly shows that $\mathrm{triv}(F)$ is not then ccc over fin. ☐

Definition 2.10. Let $\mathcal{H} = \{h_\alpha : \alpha \in \omega_1\}$ be a coherent family of partial functions on ω. Define the poset $Q(\mathcal{H})$ to be the set of pairs $q = (\{s^q_\rho : \rho \in 2^{\leq n_q}\}, \Gamma_q)$ where

1. $n_q \in \omega$ and $\{s^q_\rho : \rho \in 2^{n_q}\} \subset [\omega]^{<\aleph_0}$,
2. $\Gamma_q \in [\omega_1]^{<\aleph_0}$,
3. for $\alpha \neq \beta \in \Gamma_q$ and $\rho \in 2^{n_q}$, the union $(f_\alpha \restriction s^q_\rho) \cup (f_\beta \restriction s^q_\rho)$ is not a function,
4. for each $\rho \in 2^{n_q}$, the sequence $\{s^q_{\rho\restriction j} : j \leq n_q\}$ is increasing,
5. for $\rho, \psi \in 2^{n_q}$, $s^q_\rho \cap s^q_\psi = s^q_{\rho\restriction j}$ where j is maximal such that $\rho \restriction j = \psi \restriction j$.

The ordering on $Q(\mathcal{H})$ is that $q \leq r$ providing $n_q \geq n_r$, $\Gamma_q \supset \Gamma_r$, and $s^q_\rho = s^r_\rho$ for all $\rho \in 2^{n_r}$.

Fix any bijection $\varphi : \omega \times \omega \to \omega$. If $\{h_\alpha : \alpha \in \omega_1\}$ is coherent subfamily of an ω^ω-family of functions, then let $Q_\varphi(\{h_\alpha : \alpha \in \omega_1\})$ denote the poset $Q(\{h_\alpha \circ \varphi : \alpha < \omega_1\})$.

The idea for the poset $Q(\mathcal{H})$ can be found in [3, proof of 3.8.1]. The intention of this next definition of a poset is to add a canonical almost disjoint family and this is reflected in the definition of the ordering.

Definition 2.11. Let ADF_{ω_1} denote the poset of finite partial functions p from $\omega_1 \times \omega$ to 2, such that, for some $n_p \in \omega$ $dom(p) = F_p \times n_p$. The ordering on ADF_{ω_1} is that $p < q$ providing $n_p \geq n_q$, $F_P \supset F_q$, and for all $\alpha \neq \beta \in F_q$ and $n_q \leq j < n_p$, $p(\alpha, j) \cdot p(\beta, j) = 0$ (i.e. at least one has value 0).

For each $\iota \in \omega_1$, we let $\dot{a}(\iota)$ be the ADF_{ω_1} name for the set $\{j \in \omega : (\exists p \in G)\ p(\iota, j) = 1\}$.

Lemma 2.12. *Assume that $\mathcal{H} = \{h_\alpha : \alpha \in \omega_1\}$ is a coherent family of partial functions on ω such that $\{dom(h_\alpha) : \alpha \in \omega_1\}$ is mod finite increasing, and there is no $h \in \omega^\omega$ such that $\mathcal{H} \cup \{h\}$ is coherent. Then the poset $Q(\mathcal{H})$ is ccc. Furthermore, there is a condition $q \in Q(\mathcal{H})$ that forces there is an uncountable $\Gamma \subset \omega_1$ and an uncountable almost disjoint family $\mathcal{A} \subset [\omega]^{\aleph_0}$, such that, for each $a \in \mathcal{A}$, the sequence $\{h_\alpha \restriction a : \alpha \in \Gamma\}$ is a Luzin incoherent family.*

Proof. Let $\{q_\xi : \xi \in \omega_1\} \subset Q(\mathcal{H})$. By passing to an uncountable subcollection, we can assume that there is a single sequence $\{s_\rho : \rho \in 2^{\leq n}\}$ such that, for all $\xi \in \omega_1$ and $\rho \in 2^{\leq n}$, $n_{q_\xi} = n$ and $s^{q_\xi}_\rho = s_\rho$. For each $\xi \in \omega_1$, let $\Gamma_\xi = \Gamma_{q_\xi}$. By again passing to an uncountable subcollection we may assume that the family $\{\Gamma_\xi : \xi \in \omega_1\}$ is a Δ-system with root Γ'. For each $\xi \in \omega_1$, let α_ξ be the minimum element of $\Gamma_\xi \setminus \Gamma'$. With yet another such reduction, we may assume that there is an integer m so that, for all ξ, $h_{\alpha_\xi} \restriction (dom(h_{\alpha_\xi}) \setminus m$ is a subset of h_β for all $\beta \in \Gamma_\xi \setminus \Gamma'$.

Fix any countable elementary submodel M of $H(\mathfrak{c}^+)$ with the family $\{h_{\alpha_\xi} : \xi \in \omega_1\}$ as an element. Let $\delta = M \cap \omega_1$. Fix any $\xi \in \omega_1 \setminus \delta$ and finite set $I \subset dom(h_{\alpha_\xi})$. By simple elementary we have that the set $\{\eta \in \omega_1 : h_{\alpha_\eta} \restriction I = h_{\alpha_\xi} \restriction I\}$ is an element of M and is uncountable. Since the family $\mathcal{H}$ is mod finite increasing and is not coherently extendable, it follows that $\bigcup\{h_{\alpha_\xi} \restriction dom(h_{\alpha_\xi}) \setminus m : \delta \leq \xi\}$ is not a function. Choose any $i_0 > m$ so that there are $\delta \leq \xi_0, \eta_0$ with $h_{\alpha_{\xi_0}}(i_0) = j_0 \neq k_0 = h_{\alpha_{\eta_0}}(i_0)$. Let $J_0 = \{\xi : h_{\alpha_\xi}(i_0) = j_0\}$ and $K_0 = \{\eta : h_{\alpha_\eta}(i_0) = k_0\}$. Again, for every $i \geq i_0$, $\bigcup\{h_{\alpha_\xi} \restriction dom(h_{\alpha_\xi}) \setminus i : \xi \in J_0 \setminus \delta\}$ is not a function but the domain of this relation will mod finite contain the domain of $\bigcup\{h_{\alpha_\eta} \restriction dom(h_{\alpha_\xi}) \setminus i : \eta \in K_0 \setminus \delta\}$. Therefore we may choose an $i_1 > i_0$ so that there are $\xi_1 \in J_0$ and $\eta_1 \in K_0$ so that $h_{\alpha_{\xi_1}}(i_1) = j_1 \neq k_1 = h_{\alpha_{\eta_1}}(i_1)$. It should be clear that continuing with such a recursive construction we can find a sequence $\{i_\ell : \ell < 2^{n+1}\}$ so that there is a pair ξ, η satisfying that $h_{\alpha_\xi}(i_\ell) \neq h_{\alpha_\eta}(i_\ell)$ for all $\ell < 2^{n+1}$.

We are ready to find a condition r that is below each of q_ξ and q_η. We let $n_r = n+1$ and we re-index $\{i_\ell : \ell < 2^{n+1}\}$ as $\{i_\rho : \rho \in 2^{n+1}\}$. For each $\rho \in 2^{n+1}$, the definition of s^r_ρ is $s_{\rho\restriction n} \cup \{i_\rho\}$. For $\rho \in 2^{\leq n}$, $s^r_\rho = s_\rho$. Clearly, for each $\rho \in 2^{n+1}$, we have that $(h_{\alpha_\xi} \restriction s^r_\rho) \cup (h_{\alpha_\eta} \restriction s^r_\rho)$ is not a function because $h_{\alpha_\xi}(i_\rho) \neq h_{\alpha_\eta}(i_\rho)$. By the choice of m it also therefore follows that for all $\beta_0 \in \Gamma_\xi \setminus \Gamma'$ and $\beta_1 \in \Gamma_\eta \setminus \Gamma'$, $(h_{\beta_0} \restriction s^r_\rho) \cup (h_{\beta_1} \restriction s^r_\rho)$ is not a function. If $\beta_0 \neq \beta_1$ are both elements of Γ_ξ or elements of Γ_η, then we already have that $(h_{\beta_0} \restriction s^r_{\rho\restriction n}) \cup (h_{\beta_1} \restriction s^r_{\rho\restriction n})$ is not a function.

Since $Q(\mathcal{H})$ is ccc, there is a condition q that forces that the generic filter G is uncountable. Consider any such generic filter G. It is easily checked that, for each $n \in \omega$, $D_n = \{r \in Q(\mathcal{H}) : n_r \geq n\}$ is dense. Let $\Gamma = \bigcup\{\Gamma_r : r \in G\}$ and, for each $\rho \in 2^\omega$, let $a_\rho = \bigcup\{a^r_{\rho\restriction n_r} : r \in G\}$. The family $\mathcal{A} = \{a_r : r \in 2^\omega\}$ is almost disjoint. We check that for $r \in 2^\omega$, the family $\{h_\beta \restriction a_r : \beta \in \Gamma\}$ is Luzin incoherent. Let $\alpha \neq \beta \in \Gamma$ and choose $r \in G$ so that $\{\alpha, \beta\} \subset \Gamma_r$. Since $(h_\alpha \restriction s^r_{\rho\restriction n}) \cup (h_\beta \restriction s^r_{\rho\restriction n})$ is not a function, it of course follows that $(h_\alpha \restriction a_r) \cup (h_\beta \restriction a_r)$ is not a function. $\square$

For a regular cardinal $\kappa > \omega_1$, S^κ_1 denotes the set of $\lambda < \kappa$ that have cofinality ω_1. For the remainder of the paper we assume that there is a $\diamondsuit(S^\kappa_1)$-sequence $\{A_\lambda : \lambda \in S^\kappa_1\}$. Of course what this means is that each $A_\lambda \subset \lambda$ and for all $A \subset \kappa$, the set $\{\lambda \in S^\kappa_1 : A \cap \lambda = A_\lambda\}$ is stationary. This however is not the sequence that we will call our $\diamondsuit(S^\kappa_1)$-sequence.

We also assume that GCH holds below κ and so $H(\kappa)$ has cardinality κ.

Definition 2.13. Fix any enumeration $\{d_\xi : \xi \in \kappa\}$ of $H(\kappa)$. For each $\lambda \in S_1^\kappa$, let $D_\lambda = \{d_\xi : \xi \in A_\lambda\}$. We refer to $\{D_\lambda : \lambda \in S_1^\kappa\}$ as our $\diamondsuit(S_1^\kappa)$-sequence.

Lemma 2.14. *Let $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ denote a finite support ccc iteration of posets of cardinality less than κ. In addition, make the following assumptions about this sequence.*

1. *For each $\omega_1 \leq \lambda < \kappa$ and each P_λ-name of a σ-centered poset $\dot{Q}$, there is a $\beta < \lambda + \lambda$ such that $\dot{Q}_\beta = \dot{Q}$,*
2. *for each $\lambda \in S_1^\kappa$, if D_λ is a P_λ-name that is forced, by 1, to be a maximal coherent set $\mathcal{H}$ of functions whose domains form a ccc over fin P-ideal, then, if there is a sequence $\mathcal{H}_\lambda = \{\dot{h}_\alpha : \alpha \in \omega_1\}$ such that $1 \Vdash \mathcal{H}_\lambda \subset \mathcal{H}$, 1 forces that $\mathcal{H}_\lambda$ is non-extendable and the domains are mod finite increasing, then $\dot{Q}_{\lambda+1} = Q(\mathcal{H}_\lambda)$,*
3. *for each $\lambda \in S_1^\kappa$, if D_λ is a P_λ-name of a non-extendable coherent ω^ω-family, $\mathcal{H}$, then (since $P_\lambda \Vdash \mathfrak{b} = \aleph_1$), we let $\dot{Q}_\lambda = Q_\varphi(\mathcal{H}_\lambda)$ where $\mathcal{H}_\lambda \subset \mathcal{H}$ is a cofinal subsequence of order type ω_1.*

Then P_κ forces that the principles ω^ω-cohere and P-cohere hold.

Proof. We begin with ω^ω-cohere. It is clear that P_κ forces that $\mathfrak{d} = \mathfrak{b} = \kappa$. Fix a sequence $\{\dot{f}_\xi : \xi < \kappa\}$ of (countable) P_κ-names of elements of ω^ω such that the sequence is forced to be an $<^*$-increasing and $<$-cofinal subset of ω^ω. To prove that ω^ω-cohere holds it suffices to consider a sequence $\{\dot{h}_\xi : \xi < \kappa\}$ of countable P_κ-names such that 1 forces that $\dot{h}_\xi$ is a function into ω with domain equal to $\dot{f}_\xi^\downarrow$. Fix any sequence $\{M_\mu : \mu < \kappa\}$ of elementary submodels of $H(\kappa^+)$ such that $\{\dot{h}_\xi : \xi < \kappa\} \in M_0$, each M_μ has cardinality less than κ, $M_\mu^\omega \subset M_{\mu+1}$ for all μ, and $M_\lambda = \bigcup\{M_\mu : \mu < \lambda\}$ for all limit λ. Let $C = \{\mu < \kappa : M_\mu \cap \kappa = \mu\}$. It is well-known that C is a cub subset of κ. Choose any $\lambda \in C$ with uncountable cofinality. Then it follows that $\{\dot{f}_\xi : \xi < \lambda\} \cup \{\dot{h}_\xi : \xi < \lambda\}$ are P_λ-names, and that $1 \Vdash_{P_\lambda}$ "$\{\dot{f}_\xi : \xi < \lambda\}$ is cofinal in ω^ω". In addition, by elementarity, if there is P_λ-name $\dot{h} \in \omega^\omega$ that is forced to coherently extend the sequence $\{\dot{h}_\xi : \xi < \lambda\}$, then there is such a name in M_λ which will then be forced by P_κ to coherently extend the entire sequence $\{\dot{h}_\xi : \xi < \kappa\}$. Now there must be such a $\dot{h}$, since otherwise the poset $Q_{\lambda+1}$ renders the initial segment $\{\dot{h}_\xi : \xi < \lambda\}$ non-extendable – contradicting the existence of $\dot{h}_{\lambda+1}$.

Now assume that $\{\dot{h}_\xi : \xi < \kappa\}$ is a sequence of countable P_κ-names that are forced to be a coherent family of partial functions on ω and that the family $\{dom(\dot{h}_\xi) : \xi < \kappa\}$ is forced to generate an ideal $\dot{\mathcal{I}}$ that is ccc over fin P-ideal. By enriching the list of names, there is no loss to assume that for each countable $S \subset \kappa$, there is an $\eta < \kappa$ satisfying that $1 \Vdash dom(\dot{h}_\xi) \subset^* dom(\dot{h}_\eta)$ for all $\xi \in S$. Again choose a sequence $\{M_\mu : \mu < \kappa\}$ of elementary submodels of $H(\kappa^+)$ such that each M_μ has cardinality less than κ, $\{\dot{h}_\xi : \xi \in \kappa\} \in M_0$, $M_{\mu+1}^\omega \subset M_{\mu+1}$ for all μ, and $M_\lambda = \bigcup\{M_\mu : \mu < \lambda\}$ for all limit λ. Let $C = \{\mu < \kappa : M_\mu \cap \kappa = \mu\}$. Again choose any $\lambda \in C \cap S^\kappa_1$ so that D_λ is the sequence $\{\dot{h}_\xi : \xi < \lambda\}$. Choose any sequence $\{\mu_\beta : \beta < \omega_1\} \subset \lambda$ of successor ordinals that is cofinal. Recall that $M_{\mu_\beta}^\omega \subset M_{\mu_\beta}$ for all $\beta < \omega_1$. For each $\beta < \omega_1$, choose $\xi_\beta \in M_{\mu_\beta} \setminus \bigcup\{M_{\mu_\zeta} : \zeta < \beta\}$ so that $\dot{Q}_{\xi_\beta}$ is the P_{ξ_β}-name of the poset ADF_{ω_1}. For each β, let $\{\dot{a}(\xi_\beta, \iota) : \iota < \omega_1\}$ denote the generic almost disjoint family added by ADF_{ω_1}. For each $\beta < \omega_1$, fix a choice $\iota_\beta < \omega_1$ such that $1 \Vdash_{P_\kappa} \dot{a}(\xi_\beta, \iota_\beta) \in \dot{\mathcal{I}}$. We note that, by elementarity, $h_{\dot{a}(\xi_\beta,\iota_\beta)}$ has a P_{μ_β}-name.

Now let G_λ be a P_λ-generic filter and we finish the argument in $V[G_\lambda]$. By recursion on $\gamma < \omega_1$, we can choose $a_\gamma \in \mathrm{val}_{G_\lambda}(\dot{\mathcal{I}})$ so that, for all $\beta < \gamma$, each of a_β and $\mathrm{val}_{G_\lambda}(\dot{a}(\xi_\beta, \iota_\beta))$ are mod finite contained in a_γ. For each $\gamma < \omega_1$, let $\bar{h}_\gamma$ denote the usual h_{a_γ}. We obtain a contradiction by assuming that, in $V[G_\lambda]$, no $h \in \omega^\omega$ satisfies that $\{\mathrm{val}_{G_\lambda}(\dot{h}_\xi) : \xi < \lambda\} \cup \{h\}$ is coherent. Consider any $h \in \omega^\omega \cap V[G_\lambda]$. Choose $\beta < \omega_1$ so that $h \in V[G_{\mu_\zeta}]$ for some $\zeta < \beta$, and fix such a ζ. Let $p \in G_{\mu_\beta}$ be arbitrary and let $m < \omega$ be arbitrary. Choose $\xi < \mu_\zeta$ such that, in $V[G_{\mu_\zeta}]$, $y = \{n \in dom(\mathrm{val}_{G_{\mu_\zeta}}(\dot{h}_\xi)) : h(n) \neq \mathrm{val}_{G_{\mu_\zeta}}(\dot{h}_\xi)(n)\}$ is infinite. If we jump to $V[G_{\xi_\beta}]$ it is clear that the set of $r \in \mathrm{ADF}_{\omega_1}$ such that $r < p(\xi_\beta)$ and $r((\iota_\beta, j)) = 1$ for some $j \in y \setminus m$, is pre-dense below $p(\xi_\beta)$. This proves that $\{h, \bar{h}_{\beta+1}\}$ is not coherent. Therefore we have proven that the desired sequence $\mathcal{H}_\lambda = \{\bar{h}_\alpha : \alpha < \omega_1\}$ for defining $\dot{Q}_\lambda = Q(\mathcal{H}_\lambda)$ exists in $V[G_\lambda]$. This completes the proof since, by Lemma 2.12, there is clearly an uncountable almost disjoint family of sets that are forced by $P_{\lambda+1}$ to not be elements of $\dot{\mathcal{I}}$. □

3. Cohen reals and Hausdorff gaps

In this section we discuss Shelah's method of constructing an (ω_1, ω_1)-gap with the property that one can generically split the gap while ensuring that the image of the gap by a non-trivial automorphism is not split in the forcing extension. To be more precise, Shelah [7], and again in [8], rather constructs a Luzin type gap and the idea to utilize a Hausdorff style gap is motivated by the results in [2]. The benefit to doing so will be that the ccc poset in Lemma 3.4 can be applied to the gap produced as in Lemma 3.12. In the Shelah-Steprans paper [8], a proper poset that was not ccc was needed for this purpose.

It will be convenient to adopt the following non-standard presentation of a gap structure.

Definition 3.1. For an ordinal $\delta \leq \omega_1$, a δ-gap is a sequence $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta\rangle$ such that, for all $\alpha \leq \beta < \delta$:

(1) $a_\alpha \subset^* a_\beta \subset \omega$,
(2) $x_\alpha \subset a_\alpha$,
(3) $x_\beta \cap a_\alpha =^* x_\beta$.

A set Y splits the δ-gap $\langle a_\alpha, x_\alpha : \alpha < \delta\rangle$, if $Y \cap a_\alpha =^* x_\alpha$ for all $\alpha < \delta$. We say that a gap can not be split if there is no such Y.

Definition 3.2. A Hausdorff gap is an ω_1-gap $\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle$ that has the property that for each $\alpha \in \omega_1$ and $n \in \omega$, the set $\{\beta < \alpha : x_\alpha \cap (a_\beta \setminus x_\beta) \subset n\}$ is finite.

Proposition 3.3 ([5, II Exercise (24)])**.** *A Hausdorff ω_1-gap is not split.*

There are two natural things one may be interested in doing when presented with an ω_1-gap: *splitting* the gap or *freezing* an unsplit gap. It is a well-known unpublished result of Kunen that if an ω_1-gap is not split, then there is a ccc poset which will freeze the gap in the sense that it has a Hausdorff subgap.

Proposition 3.4 ([1, 4.2])**.** *If $\vec{a} = \langle a_\alpha, x_\alpha : \alpha \in \omega_1\rangle$ is an ω_1-gap that is not split, then there is a ccc poset $Q(\vec{a})$ that introduces an increasing function $f \in {}^{\omega_1}\omega_1$ such that $\langle a_{f(\alpha)}, x_{f(\alpha)} : \alpha \in \omega_1\}$ is a Hausdorff gap.*

Splitting an ω_1-gap with a ccc poset can not be done in general (e.g. a Haudorff gap simply can not be split).

Definition 3.5. For a δ-gap $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta\rangle$, we define the poset $P_{\vec{a}} = P_{\langle a_\alpha, x_\alpha\rangle_{\alpha<\delta}}$ (or $P_{\langle a_\alpha, x_\alpha : \alpha<\delta\rangle}$) to be the set of conditions $p = (x_p, n_p, L_p) \in \mathcal{P}(\omega) \times \omega \times [\delta]^{<\aleph_0}$, where:

(1) $x_p \subset n_p \cup \bigcup\{a_\alpha : \alpha \in L_p\}$,
(2) $a_\alpha \setminus a_\beta \subset n_p$ for all $\alpha \leq \beta$ both in L_p,
(3) $x_\alpha \Delta (a_\alpha \cap x_p) \subset n_p$ for each $\alpha \in L_p$.

The ordering on $P_{\vec{a}}$ is given by $p < q$ providing

(4) $x_q \subset x_p$, $n_q \leq n_p$, $L_q \subset L_p$,

(5) $x_p \cap n_q = x_q$,
(6) $x_p \cap a_\alpha = x_q \cap a_\alpha$ for all $\alpha \in L_q$.

Proposition 3.6. *For any δ-gap $\vec{a}$ and any $\beta < \delta$, the poset $P_{\vec{a}\restriction\beta}$ is a subposet of $P_{\vec{a}}$. The canonical $P_{\vec{a}}$-name $\dot{Y}_{\vec{a}}$ given by $\{(k,p) : p \in P_{\vec{a}}\ ,\ \ k \in x_p\}$ satisfies that 1 forces that $\dot{Y}_{\vec{a}}$ splits the gap $\vec{a}$.*

The poset $P_\emptyset$ can be identified with a standard poset $\mathcal{C}_\omega \subset [\omega]^{<\aleph_0} \times \omega$ for adding a Cohen subset of ω. A condition $(s,n) \in \mathcal{C}_\omega$ providing $s \subset n \in \omega$ and $(s,n) \leq (t,m)$ providing $m \leq n$ and $s \cap m = t$. The canonical name, $\dot{c}$, for the Cohen real added is the set $\{(j,(s,n)) : j \in s\}$. Another standard poset we mention here is the dominating real poset $\mathbb{D}$. Just to be specific we define it in such a way that the generic dominating function added is strictly increasing.

Definition 3.7. The poset $\mathbb{D}$ is the poset consisting of pairs (σ, f) where $\sigma \in \bigcup_n {}^n\omega$ is a strictly increasing function and $f \in \omega^\omega$. A condition (σ, f) extends (τ, h) providing $\tau \subset \sigma$, $h \leq f$, and $\sigma(k) > f(k)$ for all $k \in dom(\sigma) \setminus dom(\tau)$.

For each $\omega \leq \delta \in \omega_1$, fix, in the ground model, an enumeration $\{\alpha_i^\delta : i \in \omega\}$ of δ.

Proposition 3.8. *Let $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta\rangle$ ($\delta \leq \omega_1$) be any δ-gap and let $p \in P_{\vec{a}}$. Then for every $s \subset n_p$ and every $m \geq n_p$, each of $p^\downarrow s$ and $p^\uparrow m$ are in $P_{\vec{a}}$ where*
(1) $p^\downarrow s = (s \cup (x_p \setminus n_p), n_p, L_p)$, and
(2) $p^\uparrow m = (x_p, m, L_p)$.
In addition $p^\uparrow m \leq p$.

Proof. The verification that $p^\uparrow m \in P_{\vec{a}}$ and $p^\uparrow m \leq p$ is completely routine so we skip the details. Now let $r = p^\downarrow s$ so that we can refer easily to x_r. Clearly $x_r \setminus n_r = x_r \setminus n_p = x_p \setminus n_p$. Each of the conditions (1), (2), and (3) in the definition of $P_{\vec{a}}$ really only depend on the value of $x_p \setminus n_p$ and so the condition r also satisfies those conditions. □

Definition 3.9. A δ-gap, $\langle a_\alpha, x_\alpha : \alpha < \delta\rangle$, is a W-sealing extension of $\langle a_\alpha, x_\alpha : \alpha < \beta\rangle$ providing
(1) W is a transitive submodel of a sufficient fragment of ZF,
(2) $\beta < \delta \leq \omega_1$,
(3) $\{a_\alpha, x_\alpha : \alpha < \beta\} \in W$,
(4) there is an infinite set $J \subset \omega$ such that for every ω-sequence $\mathcal{D} = \langle D_n : n \in \omega\rangle$ in W consisting of dense subsets of $P_{\{a_\alpha, x_\alpha : \alpha < \beta\}}$, there is an $m_{\mathcal{D}}$ such that for all $p \in P_{\langle a_\alpha, x_\alpha : \alpha < \beta\rangle}$, with $m_{\mathcal{D}} \leq n_p \in J$ and $L_p \subset \{\alpha_i^\delta : i < n_p\}$, there is a $d < p$ such that

$d \in \bigcap\{D_k : k < n_p\}$, $x_d \cap n_d \setminus n_p = x_\delta \cap n_d \setminus n_p$, $n_d \setminus n_p \subset a_\delta$, and, for all $\alpha \in L_d$, $a_\alpha \setminus a_\delta \subset n_d$ and $x_\delta \cap a_\alpha \setminus n_d = x_\alpha \setminus n_d$.

Proposition 3.10. *For $\beta < \delta < \omega_1$, a δ-gap $\vec{a}$ is a W-sealing of $\vec{a} \restriction \beta$ if and only if $\vec{a} \restriction (\beta{+}1)$ is a W-sealing of $\vec{a} \restriction \beta$.*

Lemma 3.11. *Let $\langle P_\xi, \dot{Q}_\zeta : \xi \leq \omega_1, \zeta < \omega_1\rangle$ be a finite support iteration of ccc posets and let G be a P_{ω_1}-generic filter. Assume that $\vec{a} = \langle a_\alpha, x_\alpha : \alpha \in \omega_1\rangle$ is an ω_1-gap in $V[G]$. For $\delta < \omega_1$, let $G_\delta = G \cap P_\delta$. Then $P_{\vec{a}}$ is ccc providing there is a stationary set $S \subset \omega_1$ satisfying that, for all $\delta \in S$, $\vec{a} \restriction \delta = \langle a_\alpha, x_\alpha : \alpha < \delta\rangle$ is an element of $V[G_\delta]$ and $\vec{a} \restriction \delta{+}1$ is a $V[G_\delta]$-sealing extension of $\vec{a} \restriction \delta$.*

Proof. Let D be any dense subset of $P_{\vec{a}}$, and fix a canonical P_{ω_1}-name $\dot{D}$ for D. For each $\beta < \omega_1$, $P_{\vec{a}\restriction\beta}$ is a countable poset and $P_{\vec{a}} = \bigcup\{P_{\vec{a}\restriction\beta} : \beta < \omega_1\}$. Therefore there is a cub $C \subset \omega_1$ (in $V[G]$) satisfying that for all $\gamma_0 < \gamma_1$ from C, the name $\dot{D} \cap P_{\vec{a}\restriction\gamma_0}$ is a P_{γ_1}-name, and for each $p \in P_{\vec{a}\restriction\gamma_0}$, there is a $d \in D \cap P_{\vec{a}\restriction\gamma_1}$ with $d \leq p$. Choose any cub C_1 from V such that $C_1 \subset C'$. Fix any $\delta \in C_1 \cap S$. We check that $D \cap P_{\vec{a}\restriction\delta}$ is pre-dense in $P_{\vec{a}}$. Firstly, since $\delta \in C'$, it follows that $P_{\vec{a}\restriction\delta} = \bigcup\{P_{\vec{a}\restriction\gamma} : \gamma \in C \cap \delta\}$. This implies that $D \cap P_{\vec{a}\restriction\delta}$ is a dense subset of $P_{\vec{a}\restriction\delta}$. Similarly, it is evident that the name for $\dot{D} \cap P_{\vec{a}\restriction\delta}$ is a P_δ-name. By the assumption on S, $\vec{a} \restriction (\delta{+}1)$ is a $V[G_\delta]$-sealing extension of $\vec{a} \restriction \delta$. Let J be the infinite set as in the definition of $\vec{a} \restriction (\delta{+}1)$ being sealing and let $m_{D,\delta}$ be the corresponding lower bound for the dense set $D \cap P_{\vec{a}\restriction\delta}$. Now consider any $p \in P_{\vec{a}}$ and we wish to find $d \in D \cap P_{\vec{a}\restriction\delta}$ that is compatible with p. By extending p we may assume that $\delta \in L_p$, $m_{D,\delta} \leq n_p \in J$, and that $\emptyset \neq L_p \cap \delta \subset \{\alpha_i^\delta : i < n_p\}$. Let $\beta = \max(L_p \cap \delta)$. Set $q = (x_p \cap (n_p \cup a_\beta), n_p, L_p \cap \delta)$. It is routine to check that $q \in P_{\vec{a}\restriction\delta}$ and that $x_q \setminus n_q = x_\beta \setminus n_q$. Using the assumptions on $m_{D,\delta}$ and J, we may choose $d \in D \cap P_{\vec{a}\restriction\delta}$ so that $d \leq q$, $x_d \cap n_p = x_p \cap n_p$, $x_d \cap (n_d \setminus n_p) = x_\delta \cap (n_d \setminus n_p)$, and for all $\alpha \in L_d$, $a_\alpha \setminus n_d \subset a_\delta$ and $x_\delta \cap a_\alpha \setminus n_d = x_\alpha \setminus n_d$. Note also that $n_d \setminus n_p \subset a_\delta$ and this implies that $x_\gamma \cap (n_d \setminus n_p) = x_\delta \cap (n_d \setminus n_p)$ for all $\gamma \in L_p \setminus \delta$. This then implies that $x_p \cap (n_d \setminus n_p)$ is equal to $x_\delta \cap (n_d \setminus n_p)$, which, in turn, is equal to $x_d \cap (n_d \setminus n_p)$.

Define $r = (x_p, n_d, L_d \cup L_p)$. We complete the proof by checking that $r \in P_{\vec{a}}$, $r \leq p$ and $r \leq d$. Since $p \in P_{\vec{a}}$, to show that $r \in P_{\vec{a}}$, it suffices to only consider $\alpha \in L_d \setminus L_p$, $\gamma \in L_p \setminus \beta$ and show that $x_p \cap (a_\alpha \setminus n_d) = x_\alpha \setminus n_d$ and that $a_\alpha \setminus a_\gamma \subset n_d$. By assumption $a_\alpha \setminus a_\delta \subset n_d$ and $a_\delta \setminus a_\gamma \subset n_p \subset n_d$. Therefore we do have that $a_\alpha \setminus a_\delta \subset n_d$. Similarly, $x_p \cap (a_\delta \setminus n_p) = x_\delta \setminus n_p$ and $x_\delta \cap (a_\alpha \setminus n_d) = x_\alpha \setminus n_d$. Therefore $x_p \cap (a_\alpha \setminus n_d) = x_\alpha \setminus n_d$ as required. The fact that $r \leq p$ is trivial since $x_r = x_p$. To check that $r \leq d$ it remains to show that $x_r \cap a_\alpha = x_d \cap a_\alpha$ for all $\alpha \in L_d$. In fact, since $n_r = n_d$, it suffices to show that $x_r \cap a_\alpha \cap n_d = x_d \cap a_\alpha \cap n_d$, and this holds since $x_p \cap n_d = (x_p \cap n_p) \cup (x_p \cap (n_d \setminus n_p)) = (x_d \cap n_p) \cup (x_d \cap (n_d \setminus n_p))$. □

Now we strengthen the previous lemma and incorporate a name of a non-trivial automorphism.

Lemma 3.12. *Let $\langle P_\xi, \dot{Q}_\zeta : \xi \leq \omega_1, \zeta < \omega_1\rangle$ be a finite support iteration of ccc posets and let $\dot{F}$ be a P_{ω_1}-name that is forced to induce an automorphism on $\mathcal{P}(\omega)/fin$. Assume also that, for all successor $\xi \in \omega_1$ and P_ξ-name $\dot{x}$ of a subset of ω, $\dot{F}(\dot{x})$ is a P_ξ-name.*

Let G be a P_{ω_1}-generic filter and $\vec{a} = \langle a_\alpha, x_\alpha : \alpha \in \omega_1\rangle$ an ω_1-gap in $V[G]$ such that, for all limit $\delta < \omega_1$ and $n \in \omega$, $\vec{a} \restriction (\delta + n)$ is an element of $V[G_{\delta+n}]$ and $\vec{a} \restriction (\delta{+}2n{+}1)$ is a $V[G_{\delta+2n}]$-sealing extension of $\vec{a} \restriction (\delta{+}2n)$.

Finally, assume that for all limit $\delta < \omega_1$ and $n \in \omega$, the following forcing statement holds, in $V[G_{\delta+2n+2}]$, for all $P_{\vec{a}\restriction(\delta+2n)}$-names $\dot{Y}$ in $V[G_{\delta+2n}]$,

$$1 \Vdash_{P_{\vec{a}\restriction(\delta+2n+1)}} \dot{Y} \cap F(a_{\delta+2n+1}) \neq^* F(x_{\delta+2n+1}) \ .$$

Then, in the ccc forcing extension by $P_{\omega_1} * P_{\vec{a}}$, the ω_1-gap given by $\langle F(a_\alpha), F(x_\alpha) : \alpha < \omega_1\rangle$ is not split.

Proof. Let $\dot{Y} \in V[G_{\omega_1}]$ be any element of $\mathcal{P}(\omega, P_{\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle})$. Since $P_{\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle}$ is ccc, $\dot{Y}$ is countable. Therefore there is a limit $\delta < \omega_1$ such that $\dot{Y} \in V[G_\delta]$ and such that $\dot{Y} \in \mathcal{P}(\omega, P_{\langle a_\alpha, x_\alpha : \alpha < \delta\rangle})$. By assumption, we then have, in $V[G_{\delta+2}]$,

$$1 \Vdash_{P_{\vec{a}\restriction(\delta+1)}} \dot{Y} \cap F(a_{\delta+1}) \neq^* F(x_{\delta+1}) \ .$$

For each $n \in \omega$, let $D_n \in V[G_{\delta+2n}]$ be the dense set of conditions $p \in P_{\vec{a}\restriction(\delta+1)}$ that satisfy that, for some $n < j \in F(a_{\delta+1})$,

1. $j \notin F(x_{\delta+1})$ implies that $p \Vdash_{P_{\vec{a}\restriction(\delta+1)}} j \in \dot{Y}$,
2. $j \in F(x_{\delta+1})$ implies that $p \Vdash_{P_{\vec{a}\restriction(\delta+1)}} j \notin \dot{Y}$.

Then, since $\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle$ is a $V[G_{\delta+2}]$-sealing extension of $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n\rangle$, each D_n remains a pre-dense subset of $P_{\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle}$. This proves that $\dot{Y}$ is forced to not split the gap $\langle F(a_\alpha), F(x_\alpha) : \alpha < \omega_1\rangle$. □

For easy reference, let us make a definition of *F-avoiding*.

Definition 3.13. For a function $F \in \mathcal{P}(\omega)^{\mathcal{P}(\omega)}$, a $(\delta{+}2)$-gap, $\vec{a} = \langle a_\alpha, x_\alpha : \alpha < \delta + 2\rangle$, *F-avoids* all W-names, if the forcing statement

$$1 \Vdash_{P_{\vec{a}}} \dot{Y} \cap F(a_{\delta+1}) \neq^* F(x_{\delta+1}) \ .$$

holds for all $P_{\vec{a}}$-names $\dot{Y}$ in W.

Say that a δ-gap $\langle a_\alpha, x_\alpha : \alpha < \delta\rangle$ is an (F,δ)-gap so long as $\omega \setminus a_\alpha \notin \mathrm{triv}(F)$ for all $\alpha < \delta$. In the next section we direct our efforts at producing such $(F, \delta+2)$-gap extensions of an (F,δ)-gap that are sufficiently F-avoiding.

4. Adding Cohen reals to F-avoid names

In this section we examine the extra properties to place on the iteration sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ from Lemma 2.14 so as to be able to produce a sequence as in Lemma 3.12 if $\dot{F}$ is a P_κ-name that is forced to induce a non-trivial automorphism. At a basic level, if for $\lambda \in S^\kappa_1$, the $\diamondsuit(S^\kappa_1)$-element D_λ is a P_λ-name $\dot{F}$ that is forced to induce a non-trivial automorphism on $\mathcal{P}(\omega)$/fin, then we will, if possible, choose $\dot{Q}_\lambda$ to be $P_{\vec{a}}$ for an ω_1-gap as in Lemma 3.12. We will review below that, just as in Shelah-Steprans [8], there will then be a suitable $\dot{Q}_{\lambda+1}$ (see Proposition 3.4) so that $P_{\lambda+2}$ will then force that $\dot{F}$ does not extend to all of $\mathcal{P}(\omega)$/fin in the final model. That is, we are following the method of Shelah-Steprans [8] except that in the case when $\kappa = \aleph_2$ it is shown to be sufficient to, loosely speaking, at each step $\delta+2n+1$, seal only countably many pre-dense subsets of $P_{\vec{a}\restriction(\delta+2n+1)}$ and to *F-avoid* only countably many $P_{\vec{a}\restriction(\delta+2n+1)}$-names. There are three basic minimal requirements that must be met at each step of choosing $P_{\mu_\alpha}, P_{\mu_{\alpha+1}}$ and the $P_{\mu_{\alpha+1}}$-names $\dot{a}_\alpha, \dot{x}_\alpha$. The first is related to the fact that it is impossible to *F-avoid* a name $\dot{Y}$ if F is an automorphism induced by $h \in \omega^\omega$ and $\dot{Y}$ is simply the name with the property that every $p \in P_{\vec{a}}$ forces that $\dot{Y} \cap F(a_\alpha) = h(x_\alpha)$ for all $\alpha \in dom(\vec{a})$. Therefore, we are always having to ensure that $\omega \setminus a_\alpha \notin \mathrm{triv}(F)$, and, additionally we are similarly going to need that $a_{\alpha+1} \setminus a_\alpha$ is not in $\mathrm{triv}(F)$. This is handled by the properties already guaranteed in Lemma 2.14. We need a method to properly seal all dense sets from $V[G_{\mu_\alpha}]$. This is handled by adding a generic $\bar{x}_\alpha$ for $P_{\vec{a}\restriction\alpha}$ followed by a dominating real to allow us to define a_α and $x_\alpha = \bar{x}_\alpha \cap a_\alpha$ (still using the properties of Lemma 2.14 so as to ensure $a_\alpha \notin \mathrm{triv}(F)$). Finally, the newest idea is that we uncover some combinatorics to exploit given that failing to find a suitable $x_{\delta+2n+1}$ (or rather $x_{\delta+2n+1} \setminus x_{\delta+2n}$), it will mean that too many names are failing to *F-avoid* the same $\dot{Y}$.

Fix an iteration sequence $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ as in Lemma 2.14. For each $\lambda \leq \kappa$, let $\mathcal{P}(\omega, P_\lambda)$ denote the set of canonical P_λ-names of subsets of ω. A name $\dot{x}$ is in $\mathcal{P}(\omega, P_\lambda)$ if $\dot{x}$ has the form $\bigcup_{n\in\omega}\{n\} \times A_n$ where each A_n is a countable (possibly empty) antichain of P_λ.

Suppose that $\dot F$ is a P_κ-name that induces a non-trivial automorphism on $\mathcal{P}(\omega)/\text{fin}$. Rather than dealing with a name $\dot F$, we assume instead that F is a function from $\mathcal{P}(\omega, P_\kappa)$ to $\mathcal{P}(\omega, P_\kappa)$ and that 1 forces that F induces a non-trivial automorphism on $\mathcal{P}(\omega)/\text{fin}$. This ensures that if $p \Vdash \dot x = \dot y$, then $p \Vdash F(\dot x) =^* F(\dot y)$ (of course we could easily restrict F so as to require that $p \Vdash F(\dot x) = F(\dot y)$).

Choose a sequence $\{M_\mu : \mu < \kappa\}$ of elementary submodels of $H(\kappa^+)$ such that each M_μ has cardinality less than κ, $F \in M_0$, $M_{\mu+1}^\omega \subset M_{\mu+1}$ for all μ, and $M_\lambda = \bigcup\{M_\mu : \mu < \lambda\}$ for all limit λ. Let $C(F) = \{\mu < \kappa : M_\mu \cap \kappa = \mu\}$. Note that, for $\mu < \kappa$ with $\text{cf}(\mu) \neq \omega$, $F_\mu = F \cap M_\mu$ is a function from $\mathcal{P}(\omega, P_\mu)$ to $\mathcal{P}(\omega, P_\mu)$. We also note that, by elementarity, P_μ, for $\text{cf}(\mu) > \omega$, forces that F_μ satisfies that $\text{triv}(F_\mu) \cap \mathcal{P}(A)$ is not ccc over fin for any $A \in \text{triv}(F_\mu)^+$.

Lemma 4.1. *Assume that $\mu \in C(F)$, $\delta < \omega_1$, and that $\vec a \restriction \delta = \langle \dot a_\alpha, \dot x_\alpha : \alpha < \delta\rangle$ is a sequence of elements of $\mathcal{P}(\omega, P_\mu)$ such that* 1 *forces that $\vec a \restriction \delta$ is an (F_μ, δ)-gap. If $\mu < \mu_1 \in C(F)$, then there is a pair $\{\dot a_\delta, \dot x_\delta\} \subset \mathcal{P}(\omega, P_{\mu_1})$ such that*

(1) $\langle \dot a_\alpha, \dot x_\alpha : \alpha < \delta{+}1\rangle$ is forced by 1 to be an $(F_{\mu_1}, \delta{+}1)$-gap,
(2) $\langle \dot a_\alpha, \dot x_\alpha : \alpha < \delta{+}1\rangle$ is forced by 1 to be a W-sealing extension of $\langle \dot a_\alpha, \dot x_\alpha : \alpha < \delta\rangle$ where W is the forcing extension by P_μ.

Proof. Choose coordinates $\mu \leq \xi_0 < \mu_1$ so that $\dot Q_{\xi_0}$ is equal to the P_μ-name for $P_{\langle \dot a_\alpha, \dot x_\alpha : \alpha < \delta\rangle}$. Then choose $\xi_0 < \xi_1 < \mu_1$ so that $\dot Q_{\xi_1}$ is the standard P_{ξ_1}-name for the dominating real poset $\mathbb{D}$. Pass to the forcing extension $V[G_{\xi_1}]$ and let $\bar x_{\xi_0} = \bigcup\{x_{p(\xi_0)} : \ (p \in G_{\xi_1})\}$, i.e. the generic splitting of $\vec a$ added by $P_{\vec q}$. Let $\dot g_{\xi_1}$ be the standard $\mathbb{D}$-name for the dominating real added.

For each $m \in \omega$, let $P_{\vec a}(m)$ be the set of $p \in P_{\vec a}$ such that $n_p \leq m$ and $L_p \subset \{\alpha_i^\delta : i < m\}$. Fix any dense set $D \subset P_{\vec a}$ with $D \in V[G_\mu]$.

Fact 4.1.1. *For each $m \in \omega$, the set $D(m)$ is dense in $P_{\vec a}$ where $d \in D(m)$ if $L_d \supset \{\alpha_i^\delta : i < m\}$, $m \leq n_d$, and for all $s \subset m$, $(s \cup (x_d \setminus m), n_d, L_d) \in D$.*

Proof of Fact: We remind the reader of the notation $p^{\downarrow}s$ from Proposition 3.8. Choose any d_0 in D such that $n_{d_0} \geq m$ and $L_{d_0} \supset \{\alpha_i^\delta : i < m\}$. Fix an enumeration $\{s_i : i < 2^m\}$ of $\mathcal{P}(m)$ so that $s_0 = x_{d_0} \cap m$. We recursively choose a descending sequence $\{d_i : i < 2^m\} \subset D$. Having chosen d_i, let $s = s_i \cup (x_{d_i} \cap n_{d_i} \setminus m)$. Note that $d_i^{\downarrow}s$ is equal to $(s_i \cup (x_{d_i} \setminus m), n_{d_i}, L_{d_i})$. By Lemma 3.8, $d_i^{\downarrow}s$ is an element of $P_{\vec a}$ and so we may choose $d_{i+1} < d_i$ such that $d_{i+1} \in D$. It should be clear that d_{2^m} is an element of $D(m)$. □

It follows from Fact 4.1.1 that there is a function $f_D \in \omega^\omega \cap V[G_{\xi_1}]$ satisfying that for each $m \in \omega$, there is a $d \in D(m)$ such that $n_d < f_D(m)$, $L_d \subset \{\alpha_i^\delta : i < f_D(m)\}$, and there is a condition $p_d \in G_{\xi_1}$, such that $p_d(\xi_0) = d$. Since G_{ξ_1} is a filter, this means that $x_d \subset \bar{x}_{\xi_0}$ and $\bar{x}_{\xi_0} \cap a_\alpha \setminus n_d = x_d \cap a_\alpha \setminus n_d$ for all $\alpha \in L_d$.

Let $\langle \dot{k}_\ell : \ell \in \omega\rangle$ be the $\mathbb{D}$-name of the recursively defined sequence $k_{\ell+1} = \dot{g}_{\xi_1}(k_\ell)$ (recall that $\mathbb{D}$ forces that $\dot{g}_{\xi_1}$ is strictly increasing).

Now pass to the forcing extension $V[G_{\mu_1}]$ and let $\langle k_\ell : \ell \in \omega\rangle$ be the sequence just defined. We break into two cases. The first case is when $\delta = \beta + 1$ is a successor ordinal. Since $\omega \setminus a_\beta \notin \text{triv}(F_{\mu_1})$, we can, by Lemma 2.9 and elementarity, choose an infinite $L \subset \omega$ in $V[G_{\mu_1}]$ so that that $a_\delta = a_\beta \cup \bigcup\{k_{\ell+1} \setminus k_\ell : \ell \in L\}$ satisfies that $\omega \setminus a_\delta \notin \text{triv}(F_{\mu_1})$. Also let $x_\delta = \bar{x}_{\xi_0} \cap a_\delta$. By the genericity of $\bar{x}_{\xi_0}$ it follows that $x_\delta \cap a_\beta =^* x_\beta$.

Fact 4.1.2. *$\langle a_\alpha, x_\alpha : \alpha < \delta + 1\rangle$ is a $V[G_\mu]$-sealing extension of $\vec{a}$.*

Proof of Fact: Naturally we prove that $J = \{k_\ell : \ell \in L\}$ is the set required in item (4) in Definition 3.9 of sealing. Fix any dense set $D \subset P_{\vec{a}}$ with $D \in V[G_\mu]$ and choose any m_D so that $f_D(m) < g_{\xi_1}(m)$ for all $m_D \leq m$. Fix any $p \in P_{\vec{a}}$ and $k_\ell \in J$, such that $n_p \leq k_\ell$ and $L_p \subset \{\alpha_i^\delta : i < k_\ell\}$. We may assume that $\beta \in \{\alpha_i^\delta : i < k_\ell\}$. Again, by the choice of f_D, there is a $d \in D_(k_\ell)$ such that $n_d < k_{\ell+1}$, $x_d \subset \bar{x}_{\xi_0}$, $\bar{x}_{\xi_0} \cap a_\alpha \setminus n_d = x_d \cap a_\alpha \setminus n_d$ for all $\alpha \in L_d$. By the condition on $d \in P_{\vec{a}}$, we also have that $x_\alpha \setminus n_d = x_d \cap (a_\alpha \setminus n_d)$ and $a_\alpha \setminus n_d \subset a_\beta$ for all $\alpha \in L_d$. Since $k_{\ell+1} \setminus k_\ell \subset a_\delta \cap \bar{x}_{\xi_0}$, it follows that $x_d \setminus x_\delta$ as required. The remaining requirements in item (4) of Definition 3.9 are routine. □

Now we consider the case that δ is a limit ordinal. First, by Lemma 2.4 and elementarity, we may choose a set $\bar{a}_\delta$ in $V[G_{\mu_1}]$ so that $\omega \setminus \bar{a}_\delta \notin \text{triv}(F_{\mu_1})$ and $a_\alpha \subset^* \bar{a}_\delta$ for all $\alpha < \delta$. Then, just as in the successor case, we can choose an infinite $L \subset \omega$ in $V[G_{\mu_1}]$ so that $\omega \setminus a_\delta \notin \text{triv}(F_{\mu_1})$, where $a_\delta = \bar{a}_\delta \cup \bigcup\{k_{\ell+1} \setminus k_\ell : \ell \in L\}$. We set $x_\delta = \bar{x}_{\xi_0} \cap a_\delta$ and the proof that $\langle a_\alpha, x_\alpha : \alpha < \delta + 1\rangle$ is $V[G_\mu]$-sealing of $\langle a_\alpha, x_\alpha : \alpha < \delta\rangle$ proceeds just as it did in the successor case. □

Now we turn our attention to proving there are suitable *F-avoiding* extensions of an (F, δ)-gap.

We will make use of a standard poset.

Definition 4.2. For any ordinal $\mu < \kappa$ and indexed family $\mathcal{I} = \{I_\xi : \xi < \mu\} \subset [\omega]^{\aleph_0}$ such that $\mathcal{I}$ generates a proper ideal on ω, let $Q(\mathcal{I})$ denote σ-centered poset of conditions $q = (s_q, L_q) \in [\omega]^{<\aleph_0} \times [\mu]^{<\aleph_0}$ ordered by $r < q$ providing $s_r \supset s_q$, $L_r \supset L_q$, and $s_r \cap I_\xi = s_q \cap I_\xi$ for all $\xi \in L_q$.

Evidently the poset $Q(\mathcal{I})$ adds an infinite set that is almost disjoint from every element of $\mathcal{I}$ which also meets every subset of ω that is not in the ideal generated by $\mathcal{I}$.

We will assume that $\mu_0 < \mu_1$ are members of $C(F)$ with μ_1 a successor. We will consider, in $V[G_{\mu_1}]$, an $(F_{\mu_1}, \delta+2)$-gap $\langle a_\alpha, x_\alpha : \alpha < \delta+2\rangle$ that is a $V[G_{\mu_0}]$-sealing extension of $\langle a_\alpha, x_\alpha : \alpha < \delta+1\rangle$.

For the remainder of this section we prove the main Lemma.

Lemma 4.3. *There is an $(F, \delta+3)$-gap extending $\langle a_\alpha, x_\alpha : \alpha < \delta+2\rangle$ that F-avoids all $V[G_{\mu_0}]$-names.*

We are working to find a $\mu_1 < \mu_2 \in C(F)$ and an extending $(F_{\mu_2}, \delta+3)$-gap that is F-*avoiding* for all $P_{\langle a_\alpha, x_\alpha : \alpha<\delta+1\rangle}$-names $\dot{Y}$ in $V[G_{\mu_0}]$. We may assume that we have chosen $\mu_1 < \lambda_0 \in C(F)$ and $a_{\delta+2} \in V[G_{\lambda_0}] \setminus V[G_{\mu_1}]$ so that neither $a_{\delta+2} \setminus a_{\delta+1}$ nor $\omega \setminus a_{\delta+2}$ are elements of $\mathrm{triv}(F)$. Let $\{y_\xi : \xi < \lambda_0\}$ be an enumeration of $[\omega]^{\aleph_0} \cap V[G_{\lambda_0}]$.

Let $J_\delta \subset \omega$ be the infinite subset of ω in item (4) of Definition 3.9 with respect to $\langle a_\alpha, x_\alpha : \alpha < \delta+1\rangle$ being $V[G_{\mu_0}]$-sealing. One of the purposes of sealing is to ensure that the value of $\dot{Y} \cap k_\ell$ (for a relevant $\dot{Y}$) is determined by $p^{\downarrow}s$ for every $s \subset k_\ell$ and $p \in P_{\langle a_\alpha, x_\alpha : \alpha<\delta+2\rangle}$ such that $\delta+1 \in L_p$ and $n_p = k_\ell$.

Let $\{\lambda_\zeta : \zeta < \kappa\}$ be a cofinal subset of $C(F) \cap S^\kappa_1$ such that, for all $0 < \zeta < \kappa$, there is a strictly increasing sequence of successor ordinals $\{\mu^\zeta_\xi : \xi < \lambda_0\} \subset C(F)$ that is cofinal in λ_ζ. Assume also that $\lambda_\eta < \mu^\zeta_0$ for all $\eta < \zeta < \kappa$.

For each $0 < \zeta < \kappa$ and each $\xi < \lambda_0$, choose a coordinate $\beta(\zeta, \xi) \in \mu^\zeta_{\xi+1} \setminus \mu^\zeta_\xi$ such that $\dot{Q}_{\beta(\zeta,\xi)}$ is the $P_{\beta(\zeta,\xi)}$-name of the Cohen poset $\mathcal{C}_\omega$. Let $\dot{c}_{\beta(\zeta,\xi)}$ be the canonical name for the Cohen subset of ω added by $\dot{Q}_{\beta(\zeta,\xi)}$. Let $\dot{y}^\zeta_\xi$ be the name $y_\xi \cap \dot{c}_{\beta(\zeta,\xi)}$. Let $\mathcal{Y}(\zeta)$ be the canonical P_{λ_ζ}-name of the indexed family $\{\dot{y}^\zeta_\xi : \xi < \lambda_0\}$. Finally, choose a coordinate $\bar{\lambda}_\zeta \in \mu^{\zeta+1}_0 \setminus \lambda_\zeta$ so that $\dot{Q}_{\bar{\lambda}_\zeta}$ is the $P_{\bar{\lambda}_\zeta}$-name of the poset $Q(\mathcal{Y}(\zeta))$ from Definition 4.2. We might as well let $\dot{x}_{\bar{\lambda}_\zeta}$ denote the canonical name for the subset added. I.e. a condition $p \in P_\kappa$ forces $j \in \dot{x}_{\bar{\lambda}_\zeta}$ so long as $p \restriction \bar{\lambda}_\zeta$ forces that j is an element of the first coordinate of $p(\bar{\lambda}_\zeta)$.

We will let $\hat{x}_\zeta$ denote the canonical name for $\dot{x}_{\bar{\lambda}_\zeta} \cap (a_{\delta+2} \setminus a_{\delta+1})$ and this allows us to consider if $x_{\delta+1} \cup \hat{x}_\zeta$ will be a suitable choice for $x_{\delta+2}$.

Claim 4.3.1. *Let $0 < \zeta < \kappa$ and $\xi < \lambda_0$. Suppose that $h_\xi \in V[G_{\lambda_0}]$ is a 1-to-1 function from $F_{\lambda_0}(y_\xi)$ into ω such that $h_\xi(F_{\lambda_0}(y_\xi))$ is disjoint from y_ξ. Then 1 forces that $x^\zeta_\xi = (h_\xi \circ F)(\dot{y}^\zeta_\xi)$ is not in the ideal generated by the family $\mathcal{Y}(\zeta)$.*

We prove this Claim. Since we are assuming that F is forced to induce an automorphism on $\mathcal{P}(\omega)/\text{fin}$, it follows that the function H_ξ defined by $H_\xi(x) = (h_\xi \circ F)(x)$ induces an isomorphism from $\mathcal{P}(y_\xi)/\text{fin}$ to $\mathcal{P}(h_\xi(F_{\lambda_0}(y_\xi)))/\text{fin}$. In addition, for all $\lambda_0 \leq \mu \in C(F)$, $H_\xi \restriction V[G_\mu] = (h_\xi \circ F_\mu)$ also induces such an automorphism. It is clear that $\dot{y}^\zeta_\xi$ is forced to meet every infinite subset of y_ξ that is an element of $V[G_{\mu^\zeta_\xi}]$. It follows therefore that x^ζ_ξ will meet every infinite subset of $F(y_\xi)$ that is an element of $V[G_{\mu^\zeta_\xi}]$. Similarly, for all $\eta < \lambda_0$, y^ζ_η is forced to not contain any infinite subset of ω from the model $V[G_{\mu^\zeta_\eta}]$. Now consider any elements $\bar{y}_1, \bar{y}_2$ of the ideal generated by $\mathcal{Y}(\zeta)$ where $\bar{y}_1$ is a finite union from $\{y^\zeta_\eta : \eta < \xi\}$ and $\bar{y}_2$ is the union of $\{y^\zeta_\eta : \eta \in \Gamma\}$ for a finite $\Gamma \subset \{y^\zeta_\eta : \xi < \eta\}$. By the above properties, we have that $z_1 = F_{\lambda_0}(y_\xi) \setminus \bar{y}_1$ is forced to be an infinite element of $V[G_{\mu^\zeta_\xi}]$. Then, by recursion on $\eta \in \Gamma$, we have that the infinite set $z_1 \setminus \bigcup\{y^\zeta_\gamma : \gamma \in \Gamma \cap \eta\} \in V[G_{\mu^\zeta_\eta}]$ is not contained in y^ζ_η.

Now we assume, towards a contradiction, that, in $V[G_\kappa]$, for every subset $x \subset a_{\delta+2} \setminus a_{\delta+1}$, the choice $x_{\delta+2} = x_{\delta+1} \cup x$ fails, in $V[G_\kappa]$, to satisfy the forcing statement

$$1 \Vdash_{P_{\langle a_\alpha, x_\alpha : \alpha < \delta+3\rangle}} \dot{Y} \cap F(a_{\delta+2} \setminus a_{\delta+1}) \neq^* F(x)$$

for some $P_{\langle a_\alpha, x_\alpha : \alpha < \delta\rangle}$-name $\dot{Y}$ in $V[G_{\mu_0}]$.

Given a P_κ-name $\dot{x}$ that is forced to be a subset of $a_{\delta+2} \setminus a_{\delta+1}$, let $\vec{a}_{\dot{x}}$ denote the P_κ-name of the sequence $\langle a_\alpha, x_\alpha : \alpha < \delta + 3\rangle$ where $x_{\delta+2} = x_{\delta+1} \cup \dot{x}$. In particular then, for all $0 < \zeta < \kappa$, there is a condition $p_\zeta \in P_\kappa$, an integer m_ζ, and a $P_{\langle a_\alpha, x_\alpha : \alpha < \delta\rangle}$-name $\dot{Y}_\zeta$ in $V[G_{\mu_0}]$ such that $p_\zeta \restriction \lambda_0 \in G_{\lambda_0}$ and there is a condition r_ζ in $P_{\vec{a}_{\hat{x}_\zeta}}$ such that p_ζ forces that

$$r_\zeta \Vdash_{P_{\vec{a}_{\hat{x}_\zeta}}} \left(\dot{Y}_\zeta \cap F(a_{\delta+2} \setminus a_{\delta+1})\right) \Delta \dot{F}(\hat{x}_\zeta) \quad \subset m_\zeta \ .$$

For each $0 < \zeta < \kappa$, we assume that for each $\xi < \mu$ such that $\beta(\zeta, \xi) \in dom(p_\zeta)$ (i.e. the coordinate determining the value of $\dot{y}^\zeta_\xi$), we have that $p_\zeta \restriction \beta(\zeta, \xi)$ forces a value on $p_\zeta(\beta(\zeta, \xi)) = (s^\zeta_\xi, n^\zeta_\xi) \in \mathcal{C}_\omega$. Similarly, we assume that $p_\zeta \restriction \bar{\lambda}_\zeta$ forces a value on $p_\zeta(\bar{\lambda}_\zeta) = (s^\zeta, L_\zeta) \in [\omega]^{<\aleph_0} \times [\mu]^{<\aleph_0}$. There is no loss to assume also that $\{\beta(\zeta, \xi) : \xi \in L_\zeta\} \subset dom(p_\zeta)$. Finally, by possibly increasing m_ζ and extending p_ζ, we can assume that $s^\zeta \subset m_\zeta$ and $n^\zeta_\xi = m_\zeta$ for all $\xi \in L_\zeta$.

We may pass to a cofinal subset $\Gamma \subset \kappa$ of such ζ so that the sequence $\{dom(p_\zeta) : \zeta \in \Gamma\}$ is a Δ-system that satisfies, in addition, that $dom(p_\eta) \subset \mu_0^\zeta$ for all $\eta < \zeta \in \Gamma$. Naturally we can also assume that there is a single name $\dot{Y}_1$ and integer m so that $\dot{Y}_\zeta = \dot{Y}_1$ and $m_\zeta = m$ for all $\zeta \in \Gamma$. Next, we can assume that there is an integer n_0, a set $s_0 \subset n_0$, and a finite $L_0 \subset \delta + 2$ such that $\{\delta, \delta+1, \delta+2\} \subset L_0$ and r_ζ is equal to $(s_0 \cup (x_{\delta+1} \setminus n_0) \cup (\hat{x}_\zeta \setminus n_0), n_0, L_0)$.

Now we may choose $\zeta_1 < \zeta_2$ from Γ so that $s^{\zeta_1} = s^{\zeta_2}$, $L_{\zeta_1} = L_{\zeta_2}$, and for all $\xi \in L_{\zeta_1}$, $s_\xi^{\zeta_1} = s_\xi^{\zeta_2}$.

Now let $\hat{x}^1 = \hat{x}_{\zeta_1}$, $\hat{x}^2 = \hat{x}_{\zeta_2}$ and let $\hat{x}^3$ be the canonical name for $\hat{x}^1 \cup \hat{x}^2$. Finally choose a condition $q_3 \in P_\kappa$ extending p_{ζ_1} and p_{ζ_2} so that there is an integer $\bar{m}$, a $P_{\langle a_\alpha, x_\alpha : \alpha < \delta\rangle}$-name $\dot{Y}_3$ in $V[G_{\mu_0}]$ and a condition r_3 in $P_{\vec{a}_{\hat{x}^3}}$ so that q_3 forces that

$$r_3 \Vdash_{P_{\vec{a}_{\hat{x}^3}}} \Big(\dot{Y}_3 \cap F(a_{\delta+2} \setminus a_{\delta+1})\Big) \Delta \dot{F}(\hat{x}^3) \quad \subset \bar{m}$$

and

$$\Big(\dot{F}(\hat{x}^1) \cup \dot{F}(\hat{x}^2)\Big) \Delta \dot{F}(\hat{x}^3) \subset \bar{m} \ .$$

Again, there is no loss to assuming that $\{\delta, \delta+1, \delta+2\} \subset L_{r_3}$ and let $n_3 = n_{r_3}$ and $s_3 = x_{r_3} \cap n_3$, and $x_{r_3} = s_3 \cup (x_{\delta+1} \setminus n_3) \cup (\hat{x}^3 \setminus n_3)$.

We continue our analysis in the model $V[G_{\lambda_0}]$. Consider the condition q_3. We again assume that, for $i = 1, 2$, $q_3 \restriction \bar{\lambda}_{\zeta_i}$ forces a value $q_3(\bar{\lambda}_{\zeta_i}) = (\bar{s}_i, \bar{L}_i) \in Q(\mathcal{Y}(\zeta_i))$. Also, that $\{\beta(\zeta_i, \xi) : \xi \in \bar{L}_i\} \subset dom(q_3)$ for $i = 1, 2$, and, as with p_{ζ_i}, for each $\xi \in \bar{L}_i$, $q_3 \restriction \beta(\zeta_i, \xi)$ forces a value on $q_3(\beta(\zeta_i, \xi))$ that has the form $(s, n(\beta(\zeta_i, \xi))$. By possibly again increasing the value of $\bar{m}$, we can assume that $(\bar{s}_1 \cup \bar{s}_2) \subset \bar{m}$ and $n(\beta(\zeta_i, \xi)) \leq \bar{m}$ for all $\xi \in \bar{L}_i$ and $i = 1, 2$. Set $\bar{s}_3 = \bar{s}_1 \cup \bar{s}_2$.

Consider the sequence $\mathcal{D} = \langle D_k : k \in \omega\rangle \in V[G_{\mu_0}]$ where $d \in D_k \subset P_{\langle a_\alpha, x_\alpha : \alpha < \delta\rangle}$ providing d forces a value on each of $\dot{Y}_1 \cap k$ and $\dot{Y}_3 \cap k$. By possibly increasing $\bar{m}$, we can assume that for all $\bar{m} < \ell \in J_\delta$ there are integers k_ℓ^0, k_ℓ^1 such that for all $s \subset \ell$, the condition $(s \cup (x_{\delta+1} \setminus \ell), k_\ell^0, \{\delta{+}1\} \cup \{\alpha_i^{\delta+1} : i < k_\ell^1\})$ forces a value on each of $\dot{Y}_1 \cap \ell$ and $\dot{Y}_3 \cap \ell$. For simplicity assume that $\bar{m} < \min(J_\delta)$.

Claim 4.3.2. *for any finite $s_1, s_2 \subset (\omega \setminus a_\delta)$ such that $\bar{m} \leq \min(s_1 \cup s_2)$ and $\max(s_1 \cup s_2) \leq \ell \in J_\delta$, there is a $q_{s_1,s_2} \leq q_3$ forcing that*

(1) $\hat{x}^1 \cap \ell = \bar{s}_1 \cup s_1$,
(2) $\hat{x}^2 \cap \ell = \bar{s}_2 \cup s_2$,
(3) $\hat{x}^3 \cap \ell = \bar{s}_3 \cup (s_1 \cup s_2)$.

Let B_δ denote the set $F(a_{\delta+1} \setminus a_\delta)$ (which is in the model $V[G_{\lambda_0}]$). For each $i = 1, 2, 3$, $\ell \in J_\delta$ and $s \subset \ell$, let $H_s^i(\ell)$ denote the value forced on $\dot{Y}_1 \cap B_\delta \cap (\ell \setminus \bar{m})$ (for $i = 1, 2$) and $\dot{Y}_3 \cap B_\delta \cap (\ell \setminus \bar{m})$ for $i = 3$ by the condition $((\bar{s}_i \cup s) \cup (x_{\delta+1} \setminus \ell), k^0_\ell, \{\delta{+}1\} \cup \{\alpha^{\delta+1}_i : i < k^1_\ell\})$, $i = 1,2,3$ respectively. Notice that this is simply a property of the names $\dot{Y}_1$ and $\dot{Y}_3$ and does not depend on the properties of $\dot{F}$. However, using the connections to $\dot{F}$ we have this next critical claim.

Claim 4.3.3. *For all $\ell \in L_\delta$ and $s_1, s_2 \subset (a_{\delta+2} \setminus a_{\delta+1}) \cap (\ell \setminus \bar{m})$,*
$$H^1_{s_1}(\ell) \cup H^2_{s_2}(\ell) = H^3_{s_1 \cup s_2}(\ell).$$

This is because, in addition to the conditions on $H^1_{s_1}(\ell), H^2_{s_2}(\ell), H^3_{s_1\cup s_2}(\ell)$ in connection to $\dot{Y}_1$ and $\dot{Y}_3$ mentioned above, q_{s_1,s_2} forces that

(1) $\dot{F}(\hat{x}^1) \cap B_\delta \cap (\ell \setminus \bar{m}) = H^1_{s_1}(\ell)$,
(2) $\dot{F}(\hat{x}^2) \cap B_\delta \cap (\ell \setminus \bar{m}) = H^2_{s_2}(\ell)$,
(3) $\dot{F}(\hat{x}^3) \cap B_\delta \cap (\ell \setminus \bar{m}) = H^3_{s_3}(\ell)$,
(4) $\Big(\dot{F}(\hat{x}^1) \cup \dot{F}(\hat{x}^2)\Big) \cap B_\delta \setminus \bar{m}$ is equal to $\dot{F}(\hat{x}^3) \cap B_\delta \setminus \bar{m}$.

Claim 4.3.4. *For all $\ell \in J_\delta$, each of $H^1_\emptyset(\ell)$ and $H^2_\emptyset(\ell)$ are forced by q_3 to be subsets of $\dot{F}(\hat{x}^3)$.*

In the forcing extension $V[G_\kappa]$, we have that for each $s = \hat{x}^3 \cap \ell$ ($\ell \in J_\delta$), $H^3_s(\ell) = F(\hat{x}^3) \cap B_\delta \cap (\ell \setminus \bar{m})$. We also have that
$$H^3_s(\ell) = H^1_s(\ell) \cup H^2_\emptyset(\ell) = H^1_\emptyset(\ell) \cup H^2_s(\ell).$$

It will be convenient to simply assume that $\bar{m}$ was chosen large enough so that $H^1_\emptyset(\ell) \cup H^2_\emptyset(\ell)$ is a subset of $\bar{m}$ for all $\ell \in L_\delta$. In fact, by the definition of $H^i_s(\ell)$, this means that $H^1_\emptyset(\ell)$ and $H^2_\emptyset(\ell)$ are empty for all $\ell \in L_\delta$.

Claim 4.3.5. *It is forced by 1 that $\hat{x}^1$ contains no infinite set from $V[G_{\lambda_0}]$ and that $\hat{x}^2$ contains no infinite set from $V[G_{\mu_0^{\zeta_1}}]$. Similarly, therefore, $\dot{F}(\hat{x}^1)$ is forced by 1 to contain no infinite set from $V[G_{\lambda_0}]$ and $\dot{F}(\hat{x}^2)$ contains no infinite set from $V[G_{\mu_0^{\zeta_1}}]$. Finally, we also have that 1 forces that $\dot{F}(\hat{x}^3)$ contains no infinite set from $V[G_{\lambda_0}]$.*

Claim 4.3.6. *For all $\ell \in J_\delta$ and $s \subset (\omega \setminus a_{\delta+1}) \cap (\ell \setminus \bar{m})$, the value of $H^1_s(\ell)$ is equal to the value of $H^1_{s\cap a_{\delta+2}}(\ell)$.*

Let $s' = s \cap (a_{\delta+2} \setminus a_{\delta+1})$ and consider a generic filter G_κ with $q_{s',s'} \in G_\kappa$. Then consider the forcing statement
$$(s_0 \cup (x_{\delta+1} \setminus n_0) \cup (\hat{x}^1 \setminus n_0), n_0, L_0) \Vdash \dot{Y}_1 \cap B_\delta \setminus \bar{m} = F(\hat{x}^1) \setminus \bar{m}$$
in the model $V[G_\kappa]$. By the assumption that $p_{\zeta_1} \Vdash r_{\zeta_1} \in P_{\langle a_\alpha, x_\alpha : \alpha < \delta+3\rangle}$, we have that $s' \subset \hat{x}^1 \setminus n_0$ and $s \setminus s'$ is disjoint from $s_0 \cup x_{\delta+1} \cup \hat{x}^1$. Moreover, $(s_0 \cup (s \setminus s') \cup (x_{\delta+1} \setminus n_0) \cup (\hat{x}^1 \setminus n_0), n_0, L_0)$ is an extension

of $(s_0 \cup (x_{\delta+1} \setminus n_0) \cup (\hat{x}^1 \setminus n_0), n_0, L_0)$ because $a_\alpha \setminus a_{\delta+1} \subset n_0$ for all $\alpha \in L_0$. This shows that the value of $H^1_s(\ell)$ is indeed the same as that of $H^1_{s'}(\ell)$.

Let us also note that if r is any extension of $(s_0 \cup (x_{\delta+1} \setminus n_0) \cup (\hat{x}^1 \setminus n_0), n_0, L_0)$ in $P_{\langle a_\alpha, x_\alpha : \alpha < \delta+3\rangle}$, then $x_r \setminus (x_{\delta+1} \cup \hat{x}^1)$ is contained in $\omega \setminus a_{\delta+1}$.

Claim 4.3.7. *For all finite $s \subset (a_{\delta+2} \setminus a_{\delta+1})$, the sequence $\{H^1_s(\ell) : s \subset \ell \in J_\delta\}$ is eventually constant.*

It follows from the fact that $r \uparrow \ell_2$ is an extension of r if $n_r \leq \ell_1$ that for $s \subset \ell_1 < \ell_2$, $H^1_s(\ell_1) \subset H^1_s(\ell_2)$. Just as was the case with $s = \emptyset$, it follows that $q_{s,\emptyset}$ forces that $H^1_s(\ell)$ is a subset of $F(\hat{x}^3)$ for all $s \subset \ell \in J_\delta$. Since $\bigcup\{H^1_s(\ell) : s \subset \ell \in J_\delta\}$ is an element of $V[G_{\lambda_0}]$ and is a subset of $F(\hat{x}^3)$, it follows that this union must be finite.

Now that we know that $\{H^1_s(\ell) : s \subset \ell \in L_\delta\}$ is eventually constant for all $s \subset a_{\delta+2} \setminus (a_{\delta+1} \cup \bar{m})$, there is no loss to assuming that $H^1_s(\ell) = H^1_s(\ell')$ for all $s \subset \ell < \ell'$ with $\ell, \ell' \in J_\delta$. Clearly, by symmetry, we can also arrange to having the same conditions holding for $H^2_s(\ell)$ for such s and $\ell \in J_\delta$. For these reasons we shall henceforth let H^1_s and H^2_s denote the value of $H^1_s(\ell)$ and $H^2_s(\ell)$ respectively where ℓ is the minimum in J_δ with $s \subset \ell$. Similarly, the value of H^3_s can simply be defined as $H^1_s \cup H^2_s$.

Claim 4.3.8. *For all finite $s \subset (a_{\delta+2} \setminus a_{\delta+1})$, $H^1_s = H^2_s = H^3_s$.*

Recall that for all $s \subset \ell \in L_\delta$, we have seen that $H^3_s(\ell) = H^1_s(\ell) \cup H^2_\emptyset(\ell)$ and $H^3_s(\ell)$ is also equal to $H^1_\emptyset(\ell) \cup H^2_s(\ell)$. Of course with our updated assumption on $H^1_\emptyset(\ell)$ and $H^2_\emptyset(\ell)$ being empty, this shows that $H^1_s = H^3_s = H^2_s$ as claimed.

Claim 4.3.9. *For all finite $s, t \subset (a_{\delta+2} \setminus a_{\delta+1})$, $H^1_s \cup H^1_t = H^1_{s\cup t}$.*

We have that $H^1_t = H^2_t$ and $H^1_s \cup H^2_t = H^3_{s\cup t} = H^1_{s\cup t}$.

For each $j \in (a_{\delta+2} \setminus a_{\delta+1})$, let $H^1_j = H^1_{\{j\}}$. It is now immediate, but worth noting, that

Claim 4.3.10. *For all finite $s \subset (a_{\delta+2} \setminus a_{\delta+1})$, $H^1_s = \bigcup_{j\in s} H^1_j$.*

The claim follows easily by induction on $|s|$.

Claim 4.3.11. *In $V[G_\kappa]$ with $q_3 \in G_\kappa$,*
$F(\hat{x}^1) \setminus \bar{m}$ is equal to $\bigcup\{H^1_j : j \in \hat{x}^1 \setminus \bar{m}\}$.

This unlikely Claim follows from the forcing assertion

$$(s_0 \cup (x_{\delta+1} \setminus n_0) \cup (\hat{x}^1 \setminus n_0), n_0, L_0) \Vdash \dot{Y}_1 \cap B_\delta \setminus \bar{m} = F(\hat{x}^1)$$

and the fact that, for all $\ell \in J_\delta$, the condition $(s_0 \cup (x_{\delta+1} \setminus n_0) \cup (\hat{x}^1 \cap (\ell \setminus n_0)), k^0_\ell, \{\delta+1, \delta+2\} \cup \{\alpha^{\delta+1}_i : i < k^1_\ell\})$ decides the value of $\dot{Y}_1 \cap B_\delta \cap (\ell \setminus \bar{m})$

Let $\bar{a} = a_{\delta+2} \setminus a_{\delta+1}$.

Claim 4.3.12. $B_\delta \subset^* \bigcup\{j \in \bar{a} : H^1_j\}$

Otherwise, choose any infinite $z \subset \bar{a}$ in $V[G_{\lambda_0}]$, so that $F(z) =^* B_\delta \setminus \bigcup\{H^1_j : j \in \bar{a}\}$. It is routine to check that $\hat{x}^1$ is forced to meet z in an infinite set and therefore $F(\hat{x}^1)$ must meet $F(z)$ in an infinite set. This however contradicts Claim 4.3.11.

Claim 4.3.13. *For each $\tilde{m} \in \omega$, the set $z_{\tilde{m}} = \{j \in \bar{a} : H^1_j \subset \bar{m}\}$ is finite.*

Assume otherwise and recursively choose an infinite sequence of pairs $\{j_i, k_i : i \in \omega\}$ so that $k_i \in F(z_{\tilde{m}}) \cap H^1_{j_i} \setminus \bigcup\{H^1_{j_n} : n < i\}$, with $j_i \in \bar{a}$. This we may do since $F(z_{\tilde{m}}) \subset^* B_\delta$. Choose $\xi < \mu$ so that $F(y_\xi) = \{k_i : i \in \omega\}$. Consider the infinite set $y^{\zeta_1}_\xi \subset y_\xi$. Set $x^{\zeta_1}_\xi$ to be the set $\{j_i : k_i \in F(y^{\zeta_1}_\xi)\}$. We note that $y_\xi \subset^* z_{\tilde{m}}$ and $\{j_i : i \in \omega\} \cap z_{\tilde{m}}$ is empty. Therefore $x^{\zeta_1}_\xi$ and $y^{\zeta_1}_\xi$ are disjoint. By Claim 4.3.1, $x^{\zeta_1}_\xi$ is not contained in any finite union from the family $\mathcal{Y}(\zeta_1)$. This implies that $\hat{x}^1 \cap x^{\zeta_1}_\xi$ is infinite and yet $\hat{x}^1 \cap y^{\zeta_1}_\xi$ is finite. This is a contradiction since $F(\hat{x}^1) \cap F(y^{\zeta_1}_\xi)$ is not finite because $F(\hat{x}^1)$ contains $H^1_{j_i}$ (and therefore k_i) for all infinitely many $j_i \in \hat{x}^1$.

Claim 4.3.14. *There is $\tilde{m} \in \omega$ such that the elements of the family $\{H^1_j \setminus \bar{m} : j \in \bar{a}\}$ are pairwise disjoint.*

If this Claim fails to hold then we may recursively choose a sequence $\{i, j^1_i, j^2_i : i \in I\}$ for some infinite set $I \subset \bar{a}$ so that for each $i < i' \in I$

(1) $j^1_i \neq j^2_i$ and $\max(\{i, j^1_i, j^2_i\}) < \min(\{i', j^1_{i'}, j^2_{i'}\})$
(2) $i \in H^1_{j^1_i} \cap H^1_{j^2_i}$.

Since $F(\{j^1_i : i \in I\})$ and $F(\{j^2_i : i \in I\})$ are almost disjoint, we may suppose that $I_2 = I \setminus F(\{j^1_i : i \in I\})$ is infinite. Choose $\xi < \mu$ so $F(y_\xi) =^* I_2$. It follows that $y_\xi \cap \{j^1_i : i \in I\}$ is finite. Now given $y^{\zeta_1}_\xi$ we let $x^{\zeta_1}_\xi = \{j^1_i : i \in F(y^{\zeta_1}_\xi)\}$. We again have by Claim 4.3.1, that $x^{\zeta_1}_\xi$ is not in the ideal generated by $\mathcal{Y}(\zeta_1)$. While $\hat{x}^1$ is almost disjoint from $y^{\zeta_1}_\xi$, we have that $F(\hat{x}^1)$ contains the infinite set $\{i \in F(y^{\zeta}_\xi) : j^1_i \in \hat{x} \cap x^{\zeta_1}_\xi\}$.

Claim 4.3.15. *There is a $\tilde{m} \in \omega$ such that the family $\{H^1_j \setminus \tilde{m} : \tilde{m} < j \in \bar{a}\}$ are disjoint singleton sets.*

The failure of this Claim, together with the conclusion of Claim 4.3.13, would have that there is a strictly increasing sequence $\{j_k : k \in \omega\}$ so that, for each $k \in \omega$, $H^1_{j_k}$ contains a pair $\{i^1_k, i^2_k\}$ with $\max(\bigcup\{H^1_{j_\ell} : \ell < k\}) < \min(\{i^1_k, i^2_k\})$. Given that the pair $F^{-1}(\{i^1_k : k \in \omega\})$ and $F^{-1}(\{i^2_k : k \in \omega\})$ are almost disjoint, we may assume that $\{j_k : k \in \omega\} \setminus F^{-1}(\{i^1_k : k \in \omega\})$ is infinite. Let $I_2 = \{k : j_k \notin F^{-1}(\{i^1_k : k \in \omega\})\}$ and choose y_ξ so that $F(y_\xi) =^* \{i^1_k : k \in I_2\}$. We may choose y_ξ to be disjoint from $x_\xi = \{j_k : k \in I_2\}$. Now, with $y^{\zeta_1}_\xi \subset y_\xi$ as given above, set $x^{\zeta_1}_\xi = \{j_k : i^1_k \in F(y^{\zeta_1}_\xi)\}$. Again, Claim 4.3.1 ensures that $x^{\zeta_1}_\xi$ is not in the ideal generated by $\mathcal{Y}(\zeta_1)$. Again this leads to a contradiction from the facts that $\hat{x}^1 \cap x^{\zeta_1}_\xi$ is infinite while $F(\hat{x}^1) \cap F(y^{\zeta_1}_\xi)$ is finite. This is because each $j_k \in \hat{x}^1 \cap x^{\zeta_1}_\xi$ gives rise to another value $i^1_k \in F(\hat{x}^1) \cap F(y^{\zeta_1}_\xi)$.

Now that we have established the above claims, we choose $\tilde{m}$ as in Claim 4.3.15 and let $h_{\bar{a}}$ be a bijection between a cofinite subset of $\bar{a}$ and $B_\delta \setminus \tilde{m}$ such that $h_{\bar{a}}(j) \in H^1_j \setminus \tilde{m}$ for all $j \in dom(h_{\bar{a}})$. We complete the proof of Lemma 4.3 by obtaining a final contradiction. Since $\bar{a} \notin \mathrm{triv}(F)$, we can choose in $V[G_{\lambda_0}]$, by Proposition 2.1, an infinite $z \subset dom(h_{\bar{a}})$ such that $h_{\bar{a}}(z) \cap F(z)$ is finite. Notice that $h_{\bar{a}}(x) = \bigcup\{H^1_j \setminus \tilde{m} : j \in x\}$ for all $x \subset dom(h_{\bar{a}})$.

Choose $\xi < \mu$ so that $F(y_\xi) =^* h_{\bar{a}}(z)$ and note that we may assume that $y_\xi \cap z$ is empty. Let $x^{\zeta_1}_\xi = \{j \in z : F(y^{\zeta_1}_\xi) \cap H^1_j \setminus \tilde{m} \neq \emptyset\}$. Since $x^{\zeta_1}_\xi \subset z$ we have that $x^{\zeta_1}_\xi$ and $y^{\zeta_1}_\xi$ are disjoint. As in the previous cases, $x^{\zeta_1}_\xi$ is not in the ideal generated by $\mathcal{Y}(\zeta_1)$. It follows that $F(\hat{x}^1) \supset H^1_j$ for infinitely many $j \in x^{\zeta_1}_\xi$, which in turn implies that $F(\hat{x}^1) \cap F(y^{\zeta_1}_\xi)$ is infinite. This contradicts that $\hat{x}^1 \cap y^{\zeta_1}_\xi$ is finite.

5. Completing the proof

Theorem 5.1. *Assume that $\kappa > \omega_1$ is a regular cardinal, GCH holds below κ, and that there is a $\diamondsuit(S^\kappa_1)$-sequence. Then there is a finite support ccc iteration sequence $\langle P_\xi, \dot{Q}_\eta : \xi \leq \kappa, \eta < \kappa\rangle$ satisfying that in the forcing extension by P_κ, all automorphisms of $\mathcal{P}(\omega)/fin$ are trivial.*

Proof. Fix the $\diamondsuit(S^\kappa_1)$-sequence $\{D_\lambda : \lambda \in S^\kappa_1\}$. Assume that $\langle P_\xi, \dot{Q}_\eta : \xi \leq \kappa, \eta < \kappa\rangle$ satisfies all the conditions detailed in Lemma 2.14. We supplement the requirements on the iteration as follows. Consider any $\lambda \in S^\kappa_1$. If D_λ is a P_λ-name, rename it $\dot{F}_\lambda$, that is forced by 1 to induce a non-trivial automorphism on $\mathcal{P}(\omega)/\mathrm{fin}$, then choose, if it is forced by 1 to be possible, an (F_λ, ω_1)-gap $\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle$ in $V[G_\lambda]$ such that

(1) the poset $P_{\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle}$ is ccc, and

(2) the resulting ω_1-gap $\langle F_\lambda(a_\alpha), F(x_\alpha) : \alpha < \omega_1\rangle$ is not split in the forcing extension by $P_{\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle}$.

In such a case, then $\dot{Q}_\lambda$ is the P_λ-name of the poset $P_{\langle a_\alpha, x_\alpha:\alpha<\omega_1\rangle}$, and $\dot{Q}_{\lambda+1}$ is the $P_{\lambda+1}$-name of the poset $Q(\langle F(a_\alpha), F(x_\alpha) : \alpha < \omega_1\rangle)$ as identified in Proposition 3.4.

Now we verify that with this completed definition of $\langle P_\xi, \dot{Q}_\eta : \xi \leq \kappa, \eta < \kappa\rangle$ that the forcing extension has the desired property. We recall that, as per Lemma 2.14, P_κ forces that ω^ω-cohere and P-cohere will hold.

Let $F = \{s_\xi : \xi \in \kappa\}$ be a sequence of pairs from $\mathcal{P}(\omega, P_\kappa)$ such that F, when viewed as a P_κ-name of a function from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$ is forced, by 1 to induce a non-trivial automorphism on $\mathcal{P}(\omega)$/fin. For simplicity we assume that every element of $\mathcal{P}(\omega, P_\kappa)$ appears as the first coordinate of s_ξ for some $\xi < \kappa$.

Let $\varphi \in \kappa^\kappa$ be the function satisfying that $s_\xi = d_{\varphi(\xi)}$ and let $A = \{\varphi(\xi) : \xi \in \kappa\}$. Choose a sequence $\{M_\mu : \mu < \kappa\}$ of elementary submodels of $H(\kappa^+)$ such that each M_μ has cardinality less than κ, $F, A, \varphi \in M_0$, $M_{\mu+1}^\omega \subset M_{\mu+1}$ for all μ, and $M_\lambda = \bigcup\{M_\mu : \mu < \lambda\}$ for all limit λ. Let $C(F) = \{\mu < \kappa : M_\mu \cap \kappa = \mu\}$. Note that, for $\mu < \kappa$ with $\mathrm{cf}(\mu) \neq \omega$, $F_\mu = F \cap M_\mu$ is a function from $\mathcal{P}(\omega, P_\mu)$ to $\mathcal{P}(\omega, P_\mu)$. We recall that, as per Lemma 2.14, P_κ forces that ω^ω-cohere and P-cohere will hold, and so by elementarity, P_μ, for $\mathrm{cf}(\mu) > \omega$, forces that F_μ satisfies that $\mathrm{triv}(F_\mu) \cap \mathcal{P}(A)$ is not ccc over fin for any $A \in \mathrm{triv}(F_\mu)^+$.

Fix any $\lambda \in C(F) \cap S_1^\kappa$ such that A_λ is equal to $A \cap \lambda$. Naturally it follows from elementarity that D_λ is equal to $\{s_\xi : \xi \in \lambda\} = F_\lambda = F \restriction \mathcal{P}(\omega, P_\lambda)$.

First pass to the extension $V[G_\lambda]$ and assume that it was possible to choose an (F_λ, ω_1)-gap as described in (1) and (2) above. There is a canonical $P_{\lambda+1}$-name $\dot{Y}_\lambda$ for the subset of ω that splits the gap $\langle a_\alpha, x_\alpha : \alpha < \omega_1\rangle$. Choose the $\zeta \in \kappa$ such that $\dot{Y}_\lambda$ is the first coordinate of the pair s_ζ and let $\dot{Z}_\lambda$ denote the second coordinate of s_ζ. Now pass to the extension $V[G_\kappa]$. By the assumption on $\dot{Q}_{\lambda+1}$, the set $Z_\lambda = \mathrm{val}_{G_\kappa}(\dot{Z}_\lambda)$ does not (and can not) split the gap $\{F(a_\alpha), F(x_\alpha) : \alpha < \omega_1\}$. That is, there is an $\alpha < \omega_1$ such that $Z_\lambda \cap F(a_\alpha) \neq^* F(x_\alpha)$. However, $Y_\lambda = \mathrm{val}_{G_\kappa}(\dot{Y}_\lambda)$ does satisfy that $Y_\lambda \cap a_\alpha =^* x_\alpha$. This implies that $F(Y_\lambda) = Z_\lambda$ fails to satisfy the requirements of an automorphism of $\mathcal{P}(\omega)$/fin.

Now we prove that there is indeed such a required (F_λ, ω_1)-gap as above. Fix any strictly increasing sequence $\{\mu_\alpha : \alpha < \omega_1\} \subset C(F) \cap \lambda$

consisting of successor ordinals. By recursion on $\alpha \in \omega_1$ we choose a pair $a_\alpha, x_\alpha \in [\omega]^{\aleph_0} \cap V[G_{\mu_\alpha}]$. We $x_0 \subset a_0 \subset \omega$ in $V[G_{\mu_0}]$ simply so that $\omega \setminus a_0 \in \text{triv}(F)^+$. Recall that the conclusions of Lemma 2.4 and Lemma 2.9 hold in $V[G_\lambda]$ for F_λ.

The inductive assumptions are, for each limit $\delta < \omega_1$ and integer $n \in \omega$ are:

(1) $\langle a_\alpha, x_\alpha : \alpha < \delta + n\rangle$ is an $(F_\lambda, \delta + n)$-gap in $V[G_{\mu_{\delta+n}}]$,
(2) $\langle a_\alpha, x_\alpha : \alpha < \delta + 1\rangle$ is a $V[G_{\mu_\delta}]$-sealing extension of $\langle a_\alpha, x_\alpha : \alpha < \delta\rangle$,
(3) $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n + 2\rangle$ F_λ-avoids all $V[G_{\mu_{\delta+2n}}]$-names,
(4) $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n + 3\rangle$ is a $V[G_{\mu_{\delta+2n+2}}]$-sealing extension of $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n + 2\rangle$.

Assume that δ is a limit and that we have chosen the sequence $\langle a_\alpha, x_\alpha : \alpha < \delta\rangle$ satisfying the inductive hypotheses. It follows from Lemma 4.1 that a pair a_δ, x_δ exists in $V[G_{\mu_\delta}]$ as required.

Now assume that $n \in \omega$ and that we have chosen the sequence $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n + 1\rangle$ satisfying the inductive hypotheses. It follows from Lemma 4.3, and elementarity, that there is a pair $a_{\delta+2n+1}, x_{\delta_2 n+1}$ in $V[G_{\mu_{\delta+2n+2}}]$ so that $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n + 2\rangle$ is an $(F_\lambda, \delta + 2n + 2)$-gap that F-avoids all $V[G_{\mu_{\delta+2n}}]$-names. Similarly, it follows again from Lemma 4.1 that there is a pair $a_{\delta+2n+2}, x_{\delta+2n+2}$ in $V[G_{\mu_{\delta+2n+3}}]$ such that $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n + 3\rangle$ is a $V[G_{\mu_{\delta+2n+2}}]$-sealing extension of $\langle a_\alpha, x_\alpha : \alpha < \delta + 2n + 2\rangle$.

This completes the proof that the recursion can be completed. It follows from Lemma 3.11 that $P_{\langle a_\alpha, x_\alpha : \alpha \in \omega_1\rangle}$ is ccc. It follows from Lemma 3.12 that $P_{\langle a_\alpha, x_\alpha : \alpha \in \omega_1\rangle}$ forces that $\langle F_\lambda(a_\alpha), F_\lambda(x_\alpha) : \alpha < \omega_1\rangle$ is an ω_1-gap that is not split.

This completes the proof. □

References

[1] James E. Baumgartner, *Applications of the proper forcing axiom*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 913–959. MR776640

[2] David Chodounský, *Strong-Q-sequences and small $\mathfrak{d}$*, Topology Appl. **159** (2012), no. 13, 2942–2946, DOI 10.1016/j.topol.2012.05.012. MR2944766

[3] Ilijas Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. **148** (2000), no. 702, xvi+177, DOI 10.1090/memo/0702. MR1711328

[4] Martin Goldstern, *Tools for your forcing construction*, Set theory of the reals (Ramat Gan, 1991), Israel Math. Conf. Proc., vol. 6, Bar-Ilan Univ., Ramat Gan, 1993, pp. 305–360. MR1234283

[5] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs. MR597342

[6] Walter Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419. MR80902

[7] Saharon Shelah, *Proper forcing*, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin-New York, 1982. MR675955

[8] Saharon Shelah and Juris Steprāns, *PFA implies all automorphisms are trivial*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1220–1225, DOI 10.2307/2047617. MR935111

[9] ———, *Martin's axiom is consistent with the existence of nowhere trivial automorphisms*, Proc. Amer. Math. Soc. **130** (2002), no. 7, 2097–2106, DOI 10.1090/S0002-9939-01-06280-3. MR1896046

[10] Boban Velicković, *Definable automorphisms of* $\mathcal{P}(\omega)$/fin, Proc. Amer. Math. Soc. **96** (1986), no. 1, 130–135, DOI 10.2307/2045667. MR813825

[11] ———, OCA *and automorphisms of* $\mathcal{P}(\omega)$/fin, Topology Appl. **49** (1993), no. 1, 1–13, DOI 10.1016/0166-8641(93)90127-Y. MR1202874

[12] ———, *Forcing axioms and stationary sets*, Adv. Math. **94** (1992), no. 2, 256–284, DOI 10.1016/0001-8708(92)90038-M. MR1174395

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