DISCRETE SELECTIVITY AND DISJOINT LOCAL π-BASES

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Abstract. We answer affirmatively two questions of Gruenhage and Tkachuk from [3]. The first result is that every compact space of countable tightness has a countable disjoint local π-base at every point. The second result is that a space X is discretely selective if it is hereditarily Lindelöf and has the property that the inequality $\pi\chi(K, X) > \omega$ holds for every compact set K of X.

1. Introduction

The definitions needed in this document are all stated in Section 2. Also, we assume that all spaces are Tychonoff.

In [3], Gruenhage and Tkachuk asked the following questions

Question 1.1 (Question 5.8)**.** *Suppose that X is a (hereditarily) Lindelöf space for which the inequality $\pi\chi(K, X) > \omega$ holds for every compact set $K \subseteq X$. Must X be discretely selective?*

Question 1.2 (Question 5.9)**.** *Is it true in ZFC that every compact space of countable tightness has a countable disjoint local π-base at every point? What about if the space is compact Frechet?*

Here, "disjoint" means pairwise disjoint. Question 1.2 is related to the well-known result of Šapirovskiĭ: If X is compact and $t(X) = \omega$, then there is a countable local π-base at every point. As shown in [3], many results involving compactness plus an additional suitable property result in obtaining a disjoint π-base. In particular, as a result of Dow in [2], Gruenhage and Tkachuk observed the forcing axiom PFA implies that every point has a countable disjoint local π-base at every point in a compact space of countable tightness. In Section 3, we prove in the affirmative Question 1.2 as a consequence of a general dichotomy result on compact spaces stated as Theorem 3.2.

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As for Question 1.1, we answer it in the affirmative if the space is hereditarily Lindelöf, whereas we show a consistent counterexample for the Lindelöf case. That is, in Section 4 we show that the discrete selectiveness of a space X follows from being hereditarily Lindelöf where the inequality $\pi\chi(K, X) > \omega$ holds for every compact set $K \subseteq X$. In Section 5, we show there are models of ZFC in which there are non-discretely selective Lindelöf spaces satisfying the inequality $\pi\chi(K, X) > \omega$ for every compact set $K \subseteq X$.

2. Notation

A space X has the *disjoint (discrete) shrinking property* if for every sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of X there are non-empty open sets $V_n \subseteq U_n$ such that the family $\{V_n : n \in \omega\}$ is disjoint (discrete). Say that a space X is *discretely selective* if for any sequence $U = \{U_n : n \in \omega\}$ of non-empty open subsets of X there exists a closed discrete set $D = \{x_n : n \in \omega\}$ such that $x_n \in U_n$ for each $n \in \omega$. We may say that a sequence $\{U_n : n \in \omega\}$ of subsets of a space X has a discrete, respectively disjoint, shrinking if there are non-empty open sets $V_n \subseteq U_n$ such that the family $\{V_n : n \in \omega\}$ is discrete, respectively disjoint. Similarly such a family $\{U_n : n \in \omega\}$ has a discrete selection if there is a closed discrete set $D = \{x_n : n \in \omega\}$ such that $x_n \in U_n$ for each $n \in \omega$.

The discrete selectiveness tells about the existence of many closed discrete subsets. In particular, countably compact infinite spaces are not discretely selective. Of course, any discrete space must be discretely selective. From the definitions, we note that for a sequence $\{U_n : n \in \omega\}$ of open subsets of a Tychonoff space X, having a discrete shrinking implies having a discrete selection, which in turn, implies having a disjoint shrinking.

We say that $\mathcal{B}$ is a *local π-base at $x \in X$* if every member of $\mathcal{B}$ is a non-empty open set and every neighbourhood of x contains some member of $\mathcal{B}$. Similarly, a family $\mathcal{N}$ of non-empty sets is a *local π-network at $x \in X$* if every neighbourhood of x contains some member of $\mathcal{N}$. The *π-character* of $x \in X$, denoted $\pi\chi(x, X)$, equals $\min\{|\mathcal{B}| : \mathcal{B}$ is a local π-base at $x\}$. The *π-character* of X, denoted $\pi\chi(X)$, is equal to $\sup\{\pi\chi(x, X) : x \in X\}$. The *tightness* of X is the least upper bound cardinal $t(X)$ such that for all $A \subseteq X$, $\overline{A} = \bigcup\{\overline{B} : B \in [A]^{\leq t(X)}\}$. Also, we say that $\mathcal{B}$ is a *outer π-base* for $A \subseteq X$ if every member of $\mathcal{B}$ is non-empy open and every neighborhood of A contains some member of $\mathcal{B}$. The *outer π-character* of $A \subseteq X$, denoted $\pi\chi(A, X)$, equals $\min\{|\mathcal{B}| : \mathcal{B}$ is an outer π-base for $A\}$. Note that $\pi\chi(x, X) = \pi\chi(\{x\}, X)$, and if $A \subseteq X$ has non-empty interior then $\pi\chi(A, X) = 1$, so in particular, $\pi\chi(X, X) = 1$.

For an ordinal λ, a sequence $\{x_\alpha : \alpha < \lambda\}$ is *free* in a space X if $\overline{\{x_\gamma : \gamma < \alpha\}} \cap \overline{\{x_\gamma : \alpha \leq \gamma \leq \lambda\}} = \emptyset$ for all $\alpha < \lambda$. Similarly, an indexed sequence, $\{Z_\alpha : \alpha < \lambda\}$, of subsets of a space X will be said to be a free sequence of sets providing, for all $\beta < \lambda$, the sets $\bigcup\{Z_\alpha : \alpha < \beta\}$ and $\bigcup\{Z_\gamma : \beta \leq \gamma < \lambda\}$ have disjoint closures in X.

3. A dichotomy in compact spaces

The main result in this section is an affirmative answer to Question 1.2. It is proven by producing an ω_1-sequence by induction using elementary submodels. It will be useful to first note the following easy result about countable disjoint local π-bases.

Proposition 3.1. *A point x in a compact Hausdorff space X has a countable disjoint local π-base if and only if there is a local π-network $\{Z_n : n \in \omega\}$ at x that is a free sequence of sets consisting of closed G_δ-sets.*

Proof. First assume that $\{U_n : n \in \omega\}$ is a disjoint local π-base at a point $x \in X$. For each n, choose any non-empty closed G_δ-set $Z_n \subset U_n$. It is easily seen seen that the family $\{Z_n : n \in \omega\}$ is a local π-network at x and is also a free sequence of G_δ-sets.

Now suppose that $\mathcal{N} = \{Z_n : n \in \omega\}$ is a free sequence of closed G_δ-subsets of X that form a local π-network at a point x. As is well-known, since X is normal, there is a family $\{U_n : n \in \omega\}$ of pairwise disjoint open sets with $Z_n \subset U_n$ for all $n \in \omega$. This is proven by a simple recursion where $Z_0 \subset U_0$ is chosen such that $\overline{U_0}$ is disjoint from the closure of the union of $\{Z_n : 0 < n \in \omega\}$. Next $Z_1 \subset U_1$ is chosen such that $\overline{U_1}$ is disjoint from each of $\overline{U_0}$ and the closure of the union of $\{Z_n : 1 < n \in \omega\}$. The induction continues for ω-many steps.

Let $A = \{n \in \omega : \text{int}(Z_n) = \emptyset\}$. For each $n \in A$, fix a family $\{U(n, m) : m \in \omega\}$ of open subsets of U_n such that Z_n equals the intersection and such that the closure of $U(n, m+1)$ is a proper subset of $U(n, m)$ for all $m \in \omega$. Now we consider the countable disjoint family of open sets $\mathcal{B} = \{\text{int}(Z_n) : n \in \omega \setminus A\} \cup \{U(n, m) \setminus \text{cl}(U(n, m+1)) : n \in A,\ m \in \omega\}$. We will check that $\mathcal{B}$ is a local π-base at x. Let W be any open neighborhood of x and choose any n such that $Z_n \subset W$. If $n \notin A$, then $\text{int}(Z_n) \subset W$. On the other hand, if $n \in A$, then $\bigcap\{\text{cl}(U(n, m)) \setminus W : m \in \omega\}$ is empty, and so there is an $m \in \omega$ such that $\text{cl}(U(n, m))$ is contained in W. Clearly then $U(n, m) \setminus \text{cl}(U(n, m+1))$ is the desired member of $\mathcal{B}$ that is contained in W. $\square$

In this next proof, we use common notation and results about elementary submodels as in [2].

Theorem 3.2. *If X is a compact space and $x \in X$, then either x has a countable disjoint local π-base or there is an free ω_1-sequence in X.*

Proof. Pick $x \in X$. By Proposition 3.1, we may assume that there is no countable local π-network at x consisting of a free sequence closed G_δ-subsets of X. In particular, $\{x\}$ itself is not a G_δ-set.

Since X is completely regular, the topology of X is generated by the continuous functions to the unit interval, $C(X,[0,1])$. Let $F = \{f \in C(X,[0,1]) : f(x) = 0\}$. Now we begin a transfinite induction: let M_0 be a countable elementary submodel of $H(\theta)$ for sufficiently large θ such that $x, X \in M_0$ (hence $C(X,[0,1])$ and F are in M_0). Declare $Z_0^0 := \bigcap\{f^{-1}(0) : f \in F \cap M_0\}$. Now, fix $\gamma < \omega_1$, and suppose we have constructed countable elementary submodels $M_{\alpha+1}$, functions $f_\alpha \in F$, and sets Z_β^α, for all $\beta \leq \alpha < \gamma$, such that

(1) Z_β^α is a non-empty closed G_δ-set;
(2) $\{Z_\beta^\xi : \beta \leq \xi \leq \alpha\}$ is descending;
(3) $Z_\alpha^\alpha = \bigcap\{f^{-1}(0) : f \in F \cap M_\alpha\}$;
(4) $f_\alpha^{-1}([0,\frac{1}{2})) \not\supseteq Z_\beta^\alpha$;
(5) $Z_\beta^{\alpha+1} = Z_\beta^\alpha \cap f_\alpha^{-1}(\{1\})$ for $\beta < \alpha$, and $Z_\alpha^{\alpha+1} = Z_\alpha^\alpha \cap f_\alpha^{-1}(0)$;
(6) $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, if α is limit; and
(7) the sequence $\langle M_\beta, f_\beta, Z_\beta^\alpha : \beta \leq \alpha\rangle$ is an element of $M_{\alpha+1}$.

Items (6) and (7) imply that we are building a $\subseteq$-continuous $\in$-chain of countable elementary submodels. By item (3), it is clear that $x \in Z_\alpha^\alpha$.

Claim 3.2.1. *For each $\xi < \alpha < \gamma$, $\bigcup\{Z_\beta^\alpha : \beta < \xi\} \subset f_\xi^{-1}(1)$ and $\bigcup\{Z_\eta^\alpha : \xi \leq \eta < \alpha\} \subset f_\xi^{-1}(0)$.*

Proof of Claim. Fix any $\xi < \alpha < \gamma$ and consider the function $f_\xi \in F$. Items (2) and (5) ensures that $\bigcup\{Z_\beta^\alpha : \beta < \xi\} \subset f_\xi^{-1}(1)$ and $Z_\xi^{\xi+1} \subset f_\xi^{-1}(0)$. By item (2), it also follows that $Z_\xi^\alpha \subset f_\xi^{-1}(0)$. For all $\xi < \eta < \alpha$, $f_\xi \in F \cap M_\eta$, and so by items (3) and (2), $Z_\eta^\alpha \subset Z_\eta^\eta \subset f_\xi^{-1}(0)$. Therefore the Claim has been proven. □

Now we perform the inductive steps. We first consider when γ is a limit ordinal. Let M_γ be the elementary submodel $\bigcup_{\alpha<\gamma} M_\alpha$ and set $Z_\gamma^\gamma = \bigcap\{f^{-1}(0) : f \in F \cap M_\gamma\}$, which we may note coincides with $\bigcap_{\alpha<\gamma} Z_\alpha^{\alpha+1}$. For all $\beta < \gamma$, set $Z_\beta^\gamma = \bigcap_{\beta\leq\alpha<\gamma} Z_\beta^\alpha$. Inductive conditions (1), (2), (3) and (6) are clearly satisfied at stage $\gamma+1$, while inductive conditions (4), (5), and (7) are unchanged from stage γ.

Now assume that $\gamma = \alpha + 1$. Choose any countable elementary submodel $M_{\alpha+1}$ so that each of $\langle M_\beta, f_\beta, Z^\alpha_\beta : \beta < \alpha\rangle$ and M_α are elements. We will choose $f_\alpha \in F \cap M_{\alpha+1}$ and so item (7) will then be satisfied. The family $\{Z^\alpha_\beta : \beta \leq \alpha\} \in M_{\alpha+1}$ has been defined and by the Claim 3.2.1 and our working assumption in the proof, the family $\{Z^\alpha_\beta : \beta < \alpha\}$ is not a local π-network at x. This is *known* in $M_{\alpha+1}$ so we may choose a function $\tilde{f} \in F \cap M_{\alpha+1}$ such that $Z^\alpha_\beta \setminus \tilde{f}^{-1}([0, \frac{1}{2})) \neq \emptyset$ for all $\beta < \alpha$. For each $\beta < \alpha$, choose $x(\alpha, \beta) \in Z^\alpha_\beta$ such that $\tilde{f}(x(\alpha, \beta)) \geq \frac{1}{2}$. By elementarity, this choice can be made such that $\{x(\alpha, \beta) : \beta < \alpha\}$ is an element of $M_{\alpha+1}$. Now choose any $x(\alpha, \alpha) \in Z^\alpha_\alpha \cap M_{\alpha+1}$ such that $x(\alpha, \alpha) \neq x$. Finally we choose any $f_\alpha \in M_{\alpha+1} \cap F$ such that $f_\alpha(x(\alpha, \beta)) = 1$ for all $\beta \leq \alpha$. Having chosen our f_α, we then define the family $\{Z^\gamma_\beta : \beta \leq \gamma\}$ as per the conditions in item (5).

Having defined the sequence, $\{Z^\gamma_\beta : \beta \leq \gamma < \omega_1\}$ for all $\gamma < \omega_1$, we note that for each $\beta < \omega_1$, $Z_\beta = \bigcap_{\beta<\gamma<\omega_1} Z^\gamma_\beta$ is non-empty by compactness. We then select a point $x_\beta \in Z_\beta$ for all $\beta < \omega_1$. We complete the proof by verifying that $\{x_\beta : \beta < \omega_1\}$ is an free ω_1-sequence. Indeed, fix any $\xi < \omega_1$ and again consider the mapping f_ξ. It follows from Claim 3.2.1 that for all $\xi < \alpha < \omega_1$, $\bigcup\{Z^\alpha_\beta : \beta < \xi\}$ and $\bigcup\{Z^\alpha_\eta : \xi \leq \eta < \alpha\}$ are completely separated by f_ξ. Therefore, by item (2), it follows that $\bigcup\{Z_\beta : \beta < \xi\}$ and $\bigcup\{Z_\eta : \xi \leq \eta < \omega_1\}$ are completely separated by f_ξ. This clearly shows that $\{x_\beta : \beta < \xi\}$ and $\{x_\eta : \xi \leq \eta < \omega_1\}$ have disjoint closures in X. □

Corollary 3.3. *If a compact space is Frechet-Urysohn or has countable tightness, then every point has a countable disjoint local π-base.*

Proof. Combine Theorem 3.2 with the fact that if a space is Frechet-Urysohn or has countable tightness, then it cannot contain free ω_1-sequences. □

4. The case of Hereditarily Lindelöf spaces

In this section we answer question 1.1 for hereditarily Lindelöf spaces. We will need the following result from [3].

Lemma 4.1 ([3])**.** *If X is a regular space for which $\pi\chi(x, X) > \omega$ for all $x \in X$, then X has the disjoint shrinking property.*

Here is the main result of this section.

Theorem 4.2. *If X is a hereditarily Lindelöf for space such that $\pi\chi(K, X) > \omega$ for every compact set $K \subseteq X$, then X is discretely selective.*

Proof. We proceed by contradiction. Assume that we have a pairwise disjoint sequence $\{U_n : n \in \omega\}$ of open sets such that no selection is closed discrete. We assume the family is pairwise disjoint by the disjoint shrinking property as per Lemma 4.1.

Let Y denote the collection of all witnesses (points in X) that $\{U_n : n \in \omega\}$ is not locally finite. We can assume Y is disjoint from the closure of every U_n, and thus, Y is closed in X. To clarify this assumption, using regularity we can consider open sets V_n such that $\overline{V_n} \subseteq U_n$, for every n, and consider the set of witnesses that $\{V_n : n \in \omega\}$ is not locally finite. Note that this set must be disjoint from the closures of each V_n.

Choose $y_0 \in Y$ and choose a neighborhood W_0 of y_0 such that $\overline{W_0}$ contains no U_n. This can be done since $\pi\chi(\{y_0\}, X) > \omega$. By recursion on $\alpha < \omega_1$ suppose that we have chosen points $y_\xi \in Y$ together with open sets W_ξ of X, $\xi < \alpha$, such that $y_\xi \in W_\xi$ and no finite union from $\{\overline{W_\xi} : \xi < \alpha\}$ covers infinitely many U_n's.

Now, re-enumerate $\{\overline{W_\xi} : \xi < \alpha\}$ as $\{W'_n : n \in \omega\}$. Choose m_0 large enough such that W'_0 does not contain U_n for any $n > m_0$. Arbitrarily choose x_i in U_i for all $i \leq m_0$. At step $n+1$, choose $m_{n+1} > m_n$ large enough so that $\bigcup_{i \leq n+1} W'_i$ does not contain any U_n, for $n > m_{n+1}$. Pick any x_i in $U_i \setminus \bigcup_{i \leq n} W'_n$ for $m_n < i \leq m_{n+1}$.

Observe that the sequence $S_\alpha = \{x_n : n < \omega\}$ with $x_n \in U_n$ is a selector that is almost disjoint from W_ξ, for all $\xi < \alpha$. Let y_α (in Y) be a limit point of the selection S_α (this is possible since we are assuming S_α is not closed discrete).

Now, again by the fact that $\pi\chi(\{y_\alpha\}, X) > \omega$, we may choose W_α a neighborhood of y_α such that $\overline{W_\alpha}$ does not contain any non-empty set of the form $U_n \setminus \bigcup_{\xi \in F} \overline{W_\xi}$ where $F \in [\alpha]^{<\omega}$. This preserves the fact that no finite union of $\{\overline{W_\xi} : \xi \leq \alpha\}$ will cover infinitely many U_n's: if F is a finite subset of α, then the union of $\overline{W_\xi}$'s for $\xi \in F \cup \{\alpha\}$ will not contain any U_n that is not already contained in the union over just F.

To conclude, note that the collection $\{y_\alpha : \alpha < \omega_1\}$ has open cover $\{W_\alpha : \alpha < \omega_1\}$ with no countable subcover: if $\{W_\xi : \xi < \alpha\}$ was a subcover, then y_α is in W_ξ for some $\xi < \alpha$ which is not possible since S_α is almost disjoint from W_ξ. This contradicts the fact that X is hereditarily Lindelöf. □

5. Some Lindelöf examples

In this section we prove that, at least consistently, hereditarily Lindelöf can not be replaced by Lindelöf in Theorem 4.2. At the end of the section we include a Theorem that shows that our approach in constructing the examples for the next Theorem does not work if CH is assumed.

Theorem 5.1. *It is consistent with $\mathfrak{c} \geq \omega_2$ that there is a Lindelöf space X that is not discretely selective and in which $\pi\chi(K, X) > \omega$ for every compact set $K \subseteq X$.*

Clearly to prove the Theorem we must provide a construction of an Example. We provide two such examples in Example 5.6 and Example 5.7 below. We first introduce the additional set-theoretic assumptions that will be required.

Let κ be the value of $2^{\aleph_1}$ in the ground model V. We force with the standard poset, $\mathrm{Fn}(\kappa, 2)$, for adding κ many Cohen reals and prove there is an example in the forcing extension. So, let G_κ be a generic filter for $\mathrm{Fn}(\kappa, 2)$ and let G_{ω_2} be a generic filter for $\mathrm{Fn}(\omega_2, 2)$. Recall that for any infinite set I, $\mathrm{Fn}(I, 2)$ is the poset consisting of finite partial functions from I into 2 ordered by extension. The arguments do not require any forcing knowledge beyond those we reference in the next few Propositions from the literature. We state the Propositions for ω_2 but they also hold with κ replacing ω_2.

Proposition 5.2 ([1, Lemma 3.3])**.** *If (X, τ) is a Lindelöf space and if G is Fn$(I, 2)$-generic, then X remains Lindelöf in $V[G]$ with respect to the topology generated by having τ as a base.*

Proposition 5.3 ([4])**.** *For each $\omega \leq \lambda < \omega_2$, the filter $G_\lambda = G_{\omega_2} \cap$ $\mathrm{Fn}(\lambda, 2)$ is a generic filter for $\mathrm{Fn}(\lambda, 2)$, and the final model $V[G_\kappa]$ is equal to the forcing extension by $\mathrm{Fn}(\omega_2 \setminus \lambda, 2)$ over the forcing extension $V[G_\lambda]$.*

Therefore if Y_λ is the space 2^{ω_1} in the model $V[G_\lambda]$, then Y_λ is Lindelöf in the final model $V[G_{\omega_2}]$.

This next result is an easy consequence of the forcing theorem and the fact that $\mathrm{Fn}(\omega_2, 2)$ is ccc.

Proposition 5.4. *Every subset $S \subset 2^{\omega_1}$ of cardinality at most $\aleph_1$ that is in $V[G_{\omega_2}]$ is an element of $V[G_\lambda]$ for some $\lambda < \omega_2$. In addition, all cardinals from V are still cardinals in $V[G_{\omega_2}]$.*

Proposition 5.5. *A compact Hausdorff space K remains compact in any forcing extension that adds reals if and only if K is scattered.*

Proof. It was proven by Kunen that compact scattered spaces remain compact in any forcing extension. We will not actually need this direction of the result so we omit the proof which is done by induction on scattering height.

For the other direction, assume that K is compact but not scattered. This implies that there is a continuous mapping from K onto the unit interval $[0,1]$. In the forcing extension, by assumption this mapping is no longer onto. It does however, still have a dense range. Therefore K is not compact in the extension. □

We present two constructions of a space X as in Theorem 5.1. The first will be in the forcing extension $V[G_\kappa]$ and the second in the forcing extension $V[G_{\omega_2}]$. Of course if $2^{\aleph_1} = \aleph_2$, then these are the same extension. The second result is more general but the reason for including the first simpler example is that it will better motivate the open question asking if Luzin's Hypothesis, $2^{\aleph_0} = 2^{\aleph_1}$, implies there is an example X as in Theorem 5.1.

The space $2^{\leq\omega}$ is the standard Cantor tree where, for each $\sigma \in 2^{<\omega}$, $\{\sigma\}$ is clopen, and $[\sigma] = \{\rho \in 2^{\leq\omega} : \sigma \subset \rho\}$ is clopen. This family of clopen sets generates the topology on $2^{\leq\omega}$.

Example 5.6. In this example we work in the forcing extension $V[G_\kappa]$ and therefore $\mathfrak{c} = 2^{\aleph_0} = 2^{\aleph_1}$ holds in this model. We let Y denote the space 2^{ω_1} from the ground model V and we note that Y is Lindelöf of cardinality at most $\mathfrak{c}$. Fix an enumeration $\{f_{\xi+1} : \xi \in \mathfrak{c}\}$ (i.e. indexed by non-limit ordinals) of all functions f from $2^{<\omega}$ into Y. Also fix an enumeration, $\{A_\nu : \omega_1 \leq \nu \in \lim(\mathfrak{c})\}$, of $[\mathfrak{c}]^{\omega_1}$, and assume without loss of generality that $A_\nu \subset \nu$ for all $\omega_1 \leq \nu \in \lim(\mathfrak{c})$.

Let B be a Bernstein subset of 2^ω. By recursion on $\xi \in \mathfrak{c}$, we choose $r_\xi \in B \setminus \{r_\alpha : \alpha < \xi\}$ and $x_\xi \in 2^{\omega_1}$. For $\nu \in \lim(\mathfrak{c})$, consider the set $\{(r_\xi, x_\xi) : \xi \in A_\nu\} \subset 2^\omega \times 2^{\omega_1}$. Choose any $r_\nu \in B$ that is a complete accumulation point of the set $\{r_\xi : \xi \in A_\nu\}$ and then choose $x_\nu \in \bigcap_{n\in\omega} \mathrm{cl}(\{x_\xi : \xi \in A_\nu \text{ and } r_\nu \restriction n \subset r_\xi\})$. It is easily seen that (r_ν, x_ν) is a complete accumulation point of $\{(r_\xi, x_\xi) : \xi \in A_\nu\}$. This step is designed to, and succeeds to, ensure that X is Lindelöf since X has weight $\aleph_1$.

For $\xi \in \mathfrak{c}$, let $r_{\xi+1}$ be any point in $B \setminus \{r_\alpha : \alpha \leq \xi\}$ and let $x_{\xi+1}$ be any point in 2^{ω_1} such that

$$(\forall n \in \omega)\ x_{\xi+1} \in \mathrm{cl}(\{f_{\xi+1}(\sigma) : \sigma \in \{r_{\xi+1} \restriction k : n < k \in \omega\}) \ .$$

The example X is the subspace of $2^{\leq\omega} \times 2^{\omega_1}$

$$X = (2^{<\omega} \times Y) \cup \{(r_\xi, x_\xi) : \xi < \mathfrak{c}\} \ .$$

For each $\sigma \in 2^{<\omega}$, the set $U_\sigma = \{\sigma\} \times Y$ is clopen and it is immediate by the construction that this sequence witnesses that the space is not discretely selective.

We now let K be any compact subset of X and we verify that $\pi\chi(K, X) > \omega$. We will use the two coordinate projection maps $p_1 : 2^{\leq\omega} \times 2^{\omega_1} \mapsto 2^{\leq\omega}$ and $p_2 : 2^{\leq\omega} \times 2^{\omega_1} \mapsto 2^{\omega_1}$. Certainly $p_1(K)$ is a compact subset of $2^{<\omega} \cup \{r_\xi : \xi < \mathfrak{c}\}$ and $p_2(K)$ is a compact subset of 2^{ω_1}.

Since $\{r_\xi : \xi \in \mathfrak{c}\}$ is a Bernstein set, it follows that $p_1(K)$ is countable. For each $\sigma \in p_1(K) \cap 2^{<\omega}$, $K_\sigma = K \cap (\{\sigma\} \times Y)$ is also compact, and we verify that it is scattered. Choose any $S \subset K_\sigma$ of cardinality $\aleph_1$ that is dense in K_σ. Similar to Proposition 5.4, there is a $\lambda < \kappa$ such that S is an element of $V[G_\lambda]$. The closure of S in $V[G_\lambda]$ is equal to the closure of S in $V[G_\kappa]$ because Y is the set of points in V. By Proposition 5.5, K is scattered. It thus follows that the compact space $p_2(K)$ is a countable union of compact scattered subsets. This implies, by the Baire category theorem, that $p_2(K)$ itself is a compact scattered subset of 2^{ω_1}. Now let $\{\tau_n : n \in \omega\}$ be any countable subset of $\mathrm{Fn}(\omega_1, 2)$. For each n, $[\tau_n] = \{x \in 2^{\omega_1} : \tau_n \subset x\}$ is a standard basic clopen subset of 2^{ω_1}. It suffices to prove the Claim that the restrictions to X of

$$P = \{\{\sigma\} \times [\tau_n], [\sigma] \times [\tau_n] : \sigma \in 2^{<\omega}, n \in \omega\}$$

is not an external local π-base for K in X. Choose any $\delta < \omega_1$ such that, for all n, $\mathrm{dom}(\tau_n) \subset \delta$. Let us take another projection: $(p_2(K))_\delta = \{x \in 2^{\omega_1 \setminus \delta} : x \in p_2(K)\}$. Since $(p_2(K))_\delta$ is also a compact scattered subset of $2^{\omega_1 \setminus \delta}$, it is nowhere dense and so we may choose a $\tau \in \mathrm{Fn}(\omega_1 \setminus \delta, 2)$ such that $[\tau] \cap p_2(K)$ is empty. Since Y is dense in 2^{ω_1}, it follows that $\{\sigma\} \times (Y \cap [\tau_n \cup \tau])$ is not empty for all $\sigma \in 2^{<\omega}$ and $n \in \omega$. Since $X \setminus \left(2^{\leq\omega} \times [\tau]\right)$ is an open set containing K but no member of P, this proves the claim.

Let us remark that the set Y in this, and the next example, can be replaced by any countable dense subset of 2^{ω_1} since we only needed that Y be Lindelof and that compact subsets are scattered.

Example 5.7. For this example we are working in the extension $V[G_{\omega_2}]$ with no assumptions on the value of $2^{\aleph_1}$ in V, and therefore in $V[G_{\omega_2}]$. This time we choose the set $\{r_\xi : \xi \in \omega_2\} \subset 2^\omega$ such that r_ξ is the Cohen generic element of 2^ω given naturally by G_{ω_2} as follows. Note that, since G_{ω_2} is a maximal filter on $\mathrm{Fn}(\omega_2, 2)$, its union $g = \bigcup\{p \in \mathrm{Fn}(\omega_2, 2) : p \in G\}$ is a total function from ω_2 onto 2. We let r_ξ be defined by $r_\xi(n) = g(\xi + n)$ for $n \in \omega$. We will need, as is shown in [5, Page 49], that every nowhere dense subset of $\{r_\xi : \xi \in \omega_2\}$ is countable.

Also, it follows easily from Proposition 5.3, that $[\sigma]\cap\{r_\xi : \xi\in\omega_2\}$ has cardinality $\aleph_2$ for all $\sigma\in 2^{<\omega}$. Note that since every nowhere dense subset of $\{r_\xi : \xi\in\omega_2\}$ is countable, it is immediate that every compact subset of $\{r_\xi : \xi\in\omega_2\}$ is nowhere dense and therefore also countable. For each $\xi\in\omega_2$, let Y_ξ be the points of 2^{ω_2} that are in the model $V[G_\xi]$ as discussed above. Also note that Y_ξ is a Lindelöf subset of 2^{ω_1} in $V[G_{\omega_2}]$. We again let Y be the points of 2^{ω_1} that are in V. Here is our space X:

$$X = (2^{<\omega}\times Y)\cup\bigcup\{\{r_\xi\}\times Y_\xi : \xi\in\omega_2\}\ .$$

The same proof as for Y in Example 1, shows that for all $\alpha\in\lim(\omega_2)$, compact subsets of Y_α are scattered.

Claim 5.7.1. *X is Lindelöf.*

Proof of Claim. Since X has weight $\aleph_1$, it suffices to prove that if $S=\{(r_{\xi_\alpha}, y_\alpha) : \alpha\in\omega_1\}$ is a subset of X such that $\xi_\alpha\neq\xi_\beta$ for all $\alpha<\beta<\omega_1$, then S has a complete accumulation point in X. Since nowhere dense subsets of $\{r_{\xi_\alpha} : \alpha\in\omega_1\}$ are countable, for each $\delta<\omega_1$, there is a $\sigma_\delta\in 2^{<\omega}$ such that $\{r_{\xi_\alpha} : \delta<\alpha\in\omega_1\}$ contains a dense subset of $[\sigma_\delta]$. Therefore, there is a $\sigma\in 2^{<\omega}$ such that every point in $[\sigma]$ is a complete accumulation point of $\{r_{\xi_\alpha} : \delta<\alpha\in\omega_1\}$. Recall that $[\sigma]\cap\{r_\lambda : \lambda<\omega_2\}$ has cardinality $\aleph_2$. Therefore by Proposition 5.4, we may choose $\lambda<\omega_2$ such that r_λ is a complete accumulation point of $\{r_{\xi_\alpha} : \alpha\in\omega_1\}$ and that $\{y_\alpha : \alpha<\omega_1\}\in V[G_\lambda]$. Working in $V[G_{\lambda+\omega}]$, there is an $x_\lambda\in\bigcap_{n\in\omega}\mathrm{cl}(\{y_{\xi_\alpha} : r_\lambda\restriction n\subset r_{\xi_\alpha}\})$. It follows that $(r_\lambda, x_\lambda)\in X$ is a complete accumulation point of S. □

The verifications that $\pi\chi(K,X)>\omega$ for all compact $K\subset X$ and that X is not discretely selective are sufficiently similar to those in Example 5.6 that they can be omitted.

In this final result we prove that CH implies that the ideas used in Examples 5.6 and 5.7 will not be sufficient.

Theorem 5.8 (CH)**.** *If X is a Lindelöf subspace of $2^{\leq\omega}\times 2^{\omega_1}$ such that $X\cap(2^{<\omega}\times 2^{\omega_1})$ is dense in $2^{<\omega}\times 2^{\omega_1}$. If some infinite subsequence of $\{X\cap(\{\sigma\}\times 2^{\omega_1}) : \sigma\in 2^{<\omega}\}$ is a witness to X failing to be discretely selective, then there is a compact $K\subset X$ with $\pi\chi(K,X)=\omega$.*

Proof. For each $\sigma\in 2^{<\omega}$, let $U_\sigma = X\cap(\{\sigma\}\times 2^{\omega_1})$. Let $\{\sigma_n : n\in\omega\}\subset 2^{<\omega}\}$ be a witness to the failure of discrete selectivity. In other words, for every sequence $\{(\sigma_n, x_n)\in U_{\sigma_n} : n\in\omega\}$ there is a limit point $(r,x)\in X$.

For each $r\in 2^\omega$, take $Y_r\subset 2^{\omega_1}$ such that $X\cap(\{r\}\times 2^{\omega_1})) = \{r\}\times Y_r$. Fix an enumeration $\{r_\alpha : \alpha\in\omega_1\}$ of 2^ω. For any $\delta\in\omega_1$ and $\rho\in 2^\delta$, let

$[\rho]$ denote the compact G_δ-subset of 2^{ω_1} consisting of all $x \in 2^{\omega_1}$ that extend ρ.

For each $r \in 2^\omega$, Y_r is a Lindelöf subset of 2^{ω_1}. Therefore if there is any $\rho \in 2^{<\omega_1}$, such that Y_r is G_δ-dense in the set $[\rho]$, then $Y_r \cap [\rho]$ would be pseudocompact, and thus would contain $[\rho]$. For any such $r \in 2^\omega$ and $\rho \in 2^{<\omega_1}$, the compact set $K = \{r\} \times [\rho]$ would be an example with $\pi\chi(K, X) = \omega$.

Therefore we will work towards a contradiction by assuming that for all $r \in 2^\omega$ and $\rho \in 2^{<\omega_1}$, there is a $\rho \subset \rho' \in 2^{<\omega_1}$ such that $Y_r \cap [\rho']$ is empty. Recursively construct a strictly increasing sequence $\{\rho_\alpha : \alpha < \omega_1\} \subset 2^{<\omega_1}$ such that $Y_{r_\alpha} \cap [\rho_\alpha]$ empty for all $\alpha \in \omega_1$.

Let $x = \bigcup\{\rho_\alpha : \alpha \in \omega_1\}$ and note that $x \in 2^{\omega_1}$ and that $x \notin Y_r$ for all $r \in 2^\omega$. Notice that $\bigcup\{Y_r : r \in 2^\omega\}$ is equal to $p_2(X \cap (2^\omega \times 2^{\omega_1}))$ where p_2 is the second coordinate projection. Since X is Lindelöf, so is this projection. Since x has character ω_1 and is not in this Lindelöf projection there must be a $\delta < \omega_1$ such that $[\rho] \cap \bigcup\{Y_r : r \in \omega_1\} = \emptyset$ where $\rho = x \restriction \delta$. Let $\{\xi_k : k \in \omega\}$ be an enumeration of δ and for each $n \in \omega$, choose x_n in the clopen set $[\rho \restriction \{\xi_k : k \leq n\}]$ such that $(\sigma_n, x_n) \in U_{\sigma_n}$. By assumption there is a point $(r, \bar{x}) \in X$ that is a limit of the sequence $\{(\sigma_n, x_n) : n \in \omega\}$. It follows easily that $\bar{x} \in [\rho \restriction \{\xi_k : k \leq n\}]$ for all n. In other words $\bar{x} \in Y_r \cap [\rho]$ and this is our contradiction. □

We have answered Question 1.1 consistently in the negative for Lindelöf spaces. Clearly the question remains open in ZFC. As suggested above, we also raise the question of whether Luzin's Hypothesis implies a negative answer for Question 1.1 for Lindelöf spaces.

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