ON THE BOUNDING, SPLITTING, AND DISTRIBUTIVITY NUMBERS OF $\mathcal{P}(\mathbb{N})$; AN APPLICATION OF LONG-LOW ITERATIONS

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Abstract. The cardinal invariants $\mathfrak{t}, \mathfrak{h}, \mathfrak{b}, \mathfrak{s}$ of $\mathcal{P}(\mathbb{N})$ are known to satisfy that $\omega_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \min\{\mathfrak{b}, \mathfrak{s}\}$ and that $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{s} < \mathfrak{b}$ are both consistent. We prove the consistency of each of the following inequalities $\omega_1 < \mathfrak{t} < \mathfrak{h} < \mathfrak{b} = \mathfrak{s}$ and $\omega_1 < \mathfrak{t} < \mathfrak{h} = \mathfrak{b} < \mathfrak{s}$. The key method is to construct a forcing poset with finite support matrix iterations of ccc posets following [3].

1. Introduction

The cardinal invariants of the continuum discussed in this article are very well known (see [7, van Douwen, p111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. A set A is a pseudointersection of a family $\mathcal{Y} \subset [\omega]^\omega$ if $A \setminus Y$ is finite for all $Y \in \mathcal{Y}$. The tower number $\mathfrak{t}$ is the minimum cardinal for which there is a mod finite descending family $\mathcal{Y} \subset [\omega]^\omega$ with no infinite pseudointersection. Of course the pseudointersection number $\mathfrak{p}$ is the minimum cardinal for which there is a family $\mathcal{Y} \subset [\omega]^\omega$ with all finite intersections infinite (the finite intersection property) and with no infinite pseudointersection ($\mathfrak{p} = \mathfrak{t}$ is shown in [12]). A family $\mathcal{I} \subset \mathcal{P}(\omega)$ is an ideal if it is closed under finite unions and mod finite subsets. An ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is dense if every $Y \in [\omega]^\omega$ contains an infinite member of $\mathcal{I}$. The distributivity number (degree) $\mathfrak{h}$ is the minimum number of dense ideals whose intersection is simply the Fréchet ideal $[\omega]^{<\aleph_0}$. A set $S \subset \omega$ is *unsplit* by a family $\mathcal{Y} \subset [\omega]^\omega$ if S is mod finite contained in one member of $\{Y, \omega \setminus Y\}$ for each $Y \in \mathcal{Y}$. The splitting number $\mathfrak{s}$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (i.e. every $S \in [\omega]^\omega$ is *split* by some

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member of $\mathcal{Y}$). The bounding number $\mathfrak{b}$ can easily be defined in these same terms, but it is best defined by the mod finite ordering, $<^*$, on the family of functions ω^ω. The cardinal $\mathfrak{b}$ is the minimum cardinal for which there is a $<^*$-unbounded family $B \subset \omega^\omega$ with $|B| = \mathfrak{b}$.

The finite support iteration of the standard Hechler poset was shown in [2] to produce models of $\mathfrak{s} < \mathfrak{b}$. It is shown in [13] that the poset we will call $\mathcal{Q}_{Bould}$ can be used to obtain a model of $\mathfrak{b} = \omega_1 < \mathfrak{s} = \omega_2$. It is shown in [8] that one can use Cohen forcing to select ccc subposets of $\mathcal{Q}_{Bould}$ to obtain a model of $\mathfrak{b} = \kappa < \mathfrak{s} = \kappa^+$. We utilize this method as well. This result was improved in [4] to show that the gap between $\mathfrak{b}$ and $\mathfrak{s}$ can be made arbitrarily large. The paper [4] also nicely expands on the method of matrix iterated forcing first introduced in [3]. The papers [3] and [4] are able to use ccc versions of the well known Mathias forcing in their iterations in place of the ccc subposets of $\mathcal{Q}_{Bould}$. In [4], the authors develop an elegant iteration property designed to preserve the maximality of a single almost disjoint family through the matrix iteration. We seem to continue to need the possibly harder to handle subposets of $\mathcal{Q}_{Bould}$ in order to control our desired value of $\mathfrak{h}$. However we refer the reader to [5] where these matrix iteration methods are again used and in which it is shown that $\mathcal{Q}_{Bould}$ can be expressed as the iteration of a σ-closed poset with a Mathias type poset.

2. Matrix Iterations

The terminology "matrix iterations" is used in [4], see also forthcoming preprint (F1222) from the second author.

Definition 2.1. *We will say that an object* $\underline{\mathbf{P}}$ *is a pre-matrix iteration if there is an infinite cardinal* κ *and an ordinal* γ *(thence a pre-*(γ,κ)*-matrix iteration) such that* $\underline{\mathbf{P}} = \langle\langle \mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i} : \alpha \leq \gamma, i \leq \kappa\rangle, \langle \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i} : \alpha < \gamma, i \leq \kappa\rangle\rangle$ *where, for each* $(\alpha, i) \in \gamma \times \kappa{+}1$ *and each* $j < i$*,*

1. $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,j}$ *is a complete suborder of the poset* $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i}$ *(i.e.* $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,j} \lessdot \mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i}$*),*
2. $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i}$ *is a* $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i}$*-name of a poset,* $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha+1,i}$ *is equal to* $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i} * \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i}$*,*
3. *for limit* $\delta \leq \gamma$*,* $\mathbb{P}^{\underline{\mathbf{P}}}_{\delta,i}$ *is equal to the union of the family* $\{\mathbb{P}^{\underline{\mathbf{P}}}_{\beta,i} : \beta < \delta\}$
4. $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,\kappa}$ *is the union of the chain* $\{\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,j} : j < \kappa\}$*.*

When the context makes it clear, we omit the superscript $\underline{\mathbf{P}}$ when discussing a pre-matrix iteration. Throughout the paper, κ will be a fixed regular cardinal and, for simpler notation, whenever we discuss a pre-matrix iteration $\underline{\mathbf{P}}$ we assume that it is a pre-(γ,κ)-matrix iteration for some ordinal γ.

Let us recall that a poset $(P,<_P)$ is a complete suborder of a poset $(Q,<_Q)$ providing $P\subset Q$, $<_P\,\subset\,<_Q$, and each maximal antichain A of $(P,<_P)$ is also a maximal antichain of $(Q,<_Q)$. Note that it follows that incomparable members of $(P,<_P)$ are still incomparable in $(Q,<_Q)$, i.e. $p_1\perp_P p_2$ implies $p_1\perp_Q p_2$. An element p of P is a reduction of $q\in Q$ if $r\not\perp_Q q$ for each $r<_P p$. If $P\subset Q$, $<_P\subset<_Q$, $\perp_P\,\subset\,\perp_Q$, and each element of Q has a reduction in P, then $P<\!\circ\, Q$. The reason is that if $A\subset P$ is a maximal antichain and $p\in P$ is a reduction of $q\in Q$, then there is an $a\in A$ and an $r<p,a$ in P, such that $r\not\perp_Q q$.

Definition 2.2. *A matrix-iteration (or (κ,γ)-matrix iteration for a cardinal κ and ordinal γ), is pre-matrix system $\underline{\mathbf{P}}$, where, for each $(\alpha,i)\in\gamma\times\kappa$ the poset $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i}$ is ccc.*

When constructing a matrix-iteration by induction, it will be necessary to have the notation and language for extension. We will use, for an ordinal γ, $\underline{\mathbf{P}}^\gamma$ to indicate that $\underline{\mathbf{P}}^\gamma$ is a pre-(γ,κ)-matrix iteration.

Definition 2.3. *A pre-matrix iteration $\underline{\mathbf{P}}^\gamma$ is an extension of $\underline{\mathbf{P}}^\delta$ providing $\delta\leq\gamma$, and, for each $\alpha\leq\delta$ and $i\leq\kappa$, $\mathbb{P}^{\underline{\mathbf{P}}^\delta}_{\alpha,i}=\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\alpha,i}$. We can use $\underline{\mathbf{P}}^\gamma\restriction\delta$ to denote the unique pre-(δ,κ)-matrix iteration extended by $\underline{\mathbf{P}}^\gamma$.*

If, for each $i\leq\kappa$, $\dot{\mathbb{Q}}_{\gamma,i}$ is a $\mathbb{P}^{\underline{\mathbf{P}}}_{\gamma,i}$-name of a poset, then we let $\underline{\mathbf{P}}^\frown\langle\dot{\mathbb{Q}}_{\gamma,i}: i\leq\kappa\rangle$ denote the $(\gamma+1,\kappa)$-matrix $\langle\langle\mathbb{P}_{\alpha,i}:\alpha\leq\gamma+1\rangle,\langle\dot{\mathbb{Q}}_{\alpha,i}:\alpha<\gamma+1,i\leq\kappa\rangle\rangle$, where for $i\leq\kappa$, $\mathbb{P}_{\gamma,i}=\mathbb{P}^{\underline{\mathbf{P}}}_{\gamma,i}$, $\mathbb{P}_{\gamma+1,i}=\mathbb{P}^{\underline{\mathbf{P}}}_{\gamma,i}*\dot{\mathbb{Q}}_{\gamma,i}$, and for $\alpha<\gamma$, $(\mathbb{P}_{\alpha,i},\dot{\mathbb{Q}}_{\alpha,i})=(\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i},\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i})$.

The following, from [4, Lemma 3.10], shows that extension at limit steps is canonical.

Lemma 2.4. *If γ is a limit and if $\{\underline{\mathbf{P}}^\delta:\delta<\gamma\}$ is a sequence of pre-matrix iterations satisfying that for $\beta<\delta<\gamma$, $\underline{\mathbf{P}}^\delta\restriction\beta=\underline{\mathbf{P}}^\beta$, then there is a unique pre-matrix iteration $\underline{\mathbf{P}}^\gamma$ such that $\underline{\mathbf{P}}^\gamma\restriction\delta=\underline{\mathbf{P}}^\delta$ for all $\delta<\gamma$.*

Proof. For each $\delta<\gamma$ and $i<\kappa$, we define $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\delta,i}$ to be $\mathbb{P}^{\underline{\mathbf{P}}^\delta}_{\delta,i}$ and $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^\gamma}_{\delta,i}$ to be $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^{\delta+1}}_{\delta,i}$. It follows that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}^\gamma}_{\delta,i}$ is a $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\delta,i}$-name. Since γ is a limit, the definition of $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\gamma,i}$ is required to be $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\delta,i}:\delta<\gamma\}$ for $i<\kappa$. Similarly, the definition of $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\gamma,\kappa}$ is required to be $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\gamma,i}: i<\kappa\}$. Let us note that $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\gamma,\kappa}$ is also required to be the union of the chain $\bigcup\{\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\delta,\kappa}:\delta<\gamma\}$, and this holds by assumption on the sequence $\{\underline{\mathbf{P}}^\delta:\delta<\gamma\}$.

To prove that $\underline{\mathbf{P}}^\gamma$ is a pre-(γ,κ)-matrix it remains to prove that for $j<i\leq\kappa$, and each $q\in\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\gamma,i}$, there is a reduction p in $\mathbb{P}^{\underline{\mathbf{P}}^\gamma}_{\gamma,j}$. Since γ

is a limit, there is an $\alpha < \gamma$ such that $q \in \mathbb{P}^{\underline{\mathbf{P}}^\alpha}_{\alpha,i}$ and, by assumption, there is a reduction, p, of q in $\mathbb{P}^{\underline{\mathbf{P}}^\alpha}_{\alpha,j}$. By induction on β ($\alpha \leq \beta \leq \gamma$) we note that $q \in \mathbb{P}^{\underline{\mathbf{P}}^\beta}_{\beta,i}$ and that p is a reduction of q in $\mathbb{P}^{\underline{\mathbf{P}}^\beta}_{\beta,j}$. For limit β it is trivial, and for successor β it follows from condition (1) in the definition of pre-matrix iteration. □

We also will need the next result taken from [4, Lemma 13], which they describe as well known, for stepping diagonally in the array of posets.

Lemma 2.5. *Let $\mathbb{P}, \mathbb{Q}$ be partial orders such that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$. Let $\dot{\mathbb{A}}$ be a $\mathbb{P}$-name for a forcing notion and let $\dot{\mathbb{B}}$ be a $\mathbb{Q}$-name for a forcing notion such that $\Vdash_{\mathbb{Q}} \dot{\mathbb{A}} \subset \dot{\mathbb{B}}$, and every $\mathbb{P}$-name of a maximal antichain of $\dot{\mathbb{A}}$ is also forced by $\mathbb{Q}$ to be a maximal antichain of $\dot{\mathbb{B}}$. Then $\mathbb{P} * \dot{\mathbb{A}} \lessdot \mathbb{Q} * \dot{\mathbb{B}}$*

One forcing notion that fits well in the matrix forcing scheme is the standard Hechler forcing $\mathcal{H}$. More specifically, $(s, f) \in \mathcal{H}$ providing $s \in \bigcup_n \omega^n = \omega^{<\omega}$ is an increasing function and $(s_2, f_2) < (s_1, f_1)$ providing $s_1 \subset s_2$, $f_1 \leq f_2$ and $s_2(j) > f_1(j)$ for all $j \in \mathrm{dom}(s_2) \setminus \mathrm{dom}(s_1)$. Let $\omega^{<\omega\uparrow}$ denote the increasing functions in $\bigcup_n \omega^n$.

If we have a matrix iteration $\underline{\mathbf{P}} = \underline{\mathbf{P}}^\gamma$ as in Definition 2.2 and, for some $\alpha < \gamma$, $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i}$ is simply the $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i}$-name for $\mathcal{H}$ (including $i = \kappa$), then surprisingly the generic Hechler real $h_\alpha \in \omega^\omega$ that is added by $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,0}$ (as a subset of $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha+1,0}$) is the same real as that added by $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,\kappa}$. It is just that the posets $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i}$ ($0 < i < \kappa$) have ensured that h_α dominates even more functions.

Although this is well known we will need the method utilized in the proof. It also follows from the much more general statement in which $\mathcal{H}$ is replaced by any Souslin poset ([9]).

Proposition 2.6. *Assume that κ and γ are non-zero ordinals and that $\langle\langle \mathbb{P}_{\alpha,i} : \alpha \leq \gamma, i \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_{\alpha,i} : \alpha < \gamma, i \leq \kappa\rangle\rangle$ is a matrix iteration. If, for each $i < \kappa$, $\dot{\mathbb{Q}}_{\gamma,i}$ is the $\mathbb{P}_{\alpha,i}$-name for $\mathcal{H}$, then for each $j < i$, $\mathbb{P}_{\gamma,j} * \dot{\mathbb{Q}}_{\gamma,j} \lessdot \mathbb{P}_{\gamma,i} * \dot{\mathbb{Q}}_{\gamma,i}$.*

The proof relies on the standard (see [2]) concept of a rank function associated with Hechler names. This is also what is needed to show that forcing with Hechler does not raise the value of either $\mathfrak{p}$ or $\mathfrak{h}$. Since it is so well known, we state without proof that if $A \subset \mathcal{H}$ is a predense set and $(s_0, f_0) \in \mathcal{H}$, then there is function $\mathrm{rk} = \mathrm{rk}_A$ from $\omega^{<\omega\uparrow}$ into ω_1 satisfying the following. For each $s \in \omega^{<\omega\uparrow}$, if there is some $f \in \omega^\omega$ such that (s, f) is below some member of A, then

$\mathrm{rk}(s) = 0$, and if $0 < \mathrm{rk}(s)$, then there is an integer $\ell_s \in \omega$ and an (any) $f \in \omega^\omega$, such that for all $0 < k \in \omega$, there is an s_k such that $\mathrm{rk}(s_k) < \mathrm{rk}(s)$, $(s_k, f+k) < (s, f+k)$, and $\mathrm{dom}(s_k) \setminus \mathrm{dom}(s)$ has cardinality ℓ_s. Note that for every increasing function $g \in \omega^\omega$, there is a k such that $(s_k, f+g) < (s, g)$ since there is a k such that $g(|s| + \ell_s) < k$. Now we prove the Lemma.

Proof. Let $j < i \leq \kappa$ and assume that $\dot{A}$ is a $\mathbb{P}_{\alpha,j}$-name of a predense subset of $\mathcal{H}$. Let G_i be a generic filter for $\mathbb{P}_{\alpha,i}$ and let $G_j = G_i \cap \mathbb{P}_{\alpha,j}$. Within $V[G_j]$ let A be the downwards closure of the predense set $\mathrm{val}_{G_j}(\dot{A})$ and choose a rank function rk_A for A. Now we prove that A (and thus $\mathrm{val}_{G_j}(\dot{A})$) is a predense subset of $\mathcal{H}$ in $V[G_i]$. We prove, by induction on $\mathrm{rk}_A(s)$, that for any $(s, f) \in \mathcal{H}$ there is an extension which is below some member of A. If $\mathrm{rk}_A(s) = 0$, then there is some $f_s \in \omega^\omega$ such that $(s, f_s) \in A$. Clearly $(s, f + f_s)$ is an extension of (s, f) that is below some member of A. Now suppose that $\mathrm{rk}_A(s) > 0$ and let $f_s \in \omega^\omega$ be chosen to witness this value assigned to $\mathrm{rk}_A(s)$. Let k be greater than $f(i)$ for each $i < \mathrm{dom}(s) + \ell_s$ and choose s_k so that $\mathrm{rk}_A(s_k) < \mathrm{rk}_A(s)$ and so that $(s_k, f_s + k) < (s, f_s + k)$. It is easily checked that $(s_k, f) < (s, f)$ and, since $\mathrm{rk}_A(s_k) < \mathrm{rk}_A(s)$, we have an extension below some member of A. By Lemma 2.5, this completes the proof. $\square$

Let us also recall the following important fact about $\mathcal{H}$.

Proposition 2.7 ([2])**.** *Suppose that $\dot{Y}$ is an $\mathcal{H}$-name of an infinite subset of ω and $\dot{Y}$ is in M for some countable elementary submodel of $H(\mathfrak{c}^+)$. Then if some condition forces that $\dot{Y} \subset a$, then a must contain some infinite subset of M.*

Proof. Suppose that there is an extension (s, f) that forces that $\dot{Y} \subset a$. Within M we can define, for $s' \in \omega^{<\omega\uparrow}$, $L_{s'} = \{n : (\forall g \in \omega^\omega)\ (s', g) \not\Vdash n \notin \dot{Y}\}$. It follows that if $(s', f) \leq (s, f)$, then $(s', f) \Vdash L_{s'} \subset a$. Therefore we may as well assume that L_s is finite.

Choose $m \in \omega$ so that $L_s \subset m$. Since $\dot{Y}$ is forced to be infinite the set $A \in M$ is a dense set of conditions in $\mathcal{H}$ where $(s', f') \in A$ providing $L_{s'} \setminus m$ is not empty. By definition, $\mathrm{rk}_A(s) > 0$. By finite induction on rank, there is an s' with $\mathrm{rk}_A(s') = 1$ such that $(s', f) < (s, f)$. Let $\ell_{s'}$ denote the integer witnessing that $\mathrm{rk}_A(s') = 1$ (and so $L_{s'} \subset m$). Working in M, we choose for each k, an s_k with $(s_k, f+k) < (s', f+k)$, $\mathrm{rk}_A(s_k) = 0$, and $|\mathrm{dom}(s_k) \setminus \mathrm{dom}(s')| = \ell_{s'}$. For each k, choose any $n_k \in L_{s_k} \setminus m$. Since $(s_k, f+k) < (s, f)$, we have that $n_k \in a$. Since $\{n_k : k \in \omega\}$ is in M and is contained in a, we just need to prove that it is infinite. The reason is that if there is an n such that $\{k : n_k = n\}$ is infinite, then $n \in L_{s'}$. To see this, let $g \in \omega^\omega$ and choose k so that $n_k = n$ and $g(i) < k$ for all $i \in \mathrm{dom}(s_k)$. Since $(s_k, g) \le (s', g)$, and $(s_k, g) \nVdash n \notin \dot{Y}$, we have that $(s', g) \nVdash n \notin \dot{Y}$. □

We will use the Hechler poset when we want to add a dominating real, but we also need to be able to iterate without adding a dominating real. This next lemma is specialized for the limit steps of a matrix iteration.

Lemma 2.8. *Assume that κ and γ are non-zero limit ordinals and that $\langle\langle \mathbb{P}_{\alpha,i} : \alpha \le \gamma, i \le \kappa\rangle, \langle \dot{\mathbb{Q}}_{\alpha,i} : \alpha < \gamma, i \le \kappa\rangle\rangle$ is a matrix iteration. Suppose that $\dot{h}$ is a $\mathbb{P}_{\alpha,i}$-name, for some $\alpha < \gamma$, such that for some $j < i$, $\dot{h}$ is forced by $\mathbb{P}_{\gamma,i}$ to be dominated by some function in the extension by $\mathbb{P}_{\gamma,j}$, then there is a $\delta < \gamma$ such that $\dot{h}$ is also forced by $\mathbb{P}_{\delta,i}$ to be dominated by some function in the extension by $\mathbb{P}_{\delta,j}$.*

Proof. Choose any $p \in \mathbb{P}_{\gamma,i}$ and $\mathbb{P}_{\gamma,j}$-name $\dot{f}$ such that $p \Vdash \dot{h} < \dot{f}$. Let $\alpha \le \delta < \gamma$ be chosen so that $p \in \mathbb{P}_{\delta,i}$. For each $n \in \omega$, let $D_n \subset \mathbb{P}_{\delta,j}$ be a maximal antichain such that for each $d \in D_n$, there are $q(n,d) \in \mathbb{P}_{\gamma,j}$ and $m(n,d) \in \omega$ such that $d = q(n,d) \restriction \delta$ and $q(n,d) \Vdash \dot{f}(n) = m(n,d)$. Clearly there is a $\mathbb{P}_{\delta,j}$-name $\dot{g}$ such that for each n and $d \in D_n$, $d \Vdash \dot{g}(n) = m(n,d)$. We claim that $p \Vdash_{\mathbb{P}_{\delta,i}} \dot{h} < \dot{g}$. To see this, assume $p_1 < p$ (in $\mathbb{P}_{\delta,i}$) and $n \in \omega$ are such that $p_1 \Vdash \dot{g}(n) \le \dot{h}(n)$. Since D_n is predense in $\mathbb{P}_{\delta,i}$, there is a $d \in D_n$ that is compatible with p_1. However, we also then have that $q(n,d)$ is compatible with p_1 since the sequence $\langle \mathbb{P}_{\beta,i}, \dot{\mathbb{Q}}_{\beta,i} : \beta < \gamma$ is just a finite support iteration (conditions (2),(3) of pre-matrix iterations). This shows that p fails to force that $\dot{h} < \dot{f}$ because $q(n,d)$ forces that $\dot{g}(n) = \dot{f}(n)$. □

We outline here the plan for the $\mathfrak{t} = \kappa = \mathfrak{h} < \mathfrak{b} = \lambda = \mathfrak{s}$ case. The other case is similar. The poset $\mathbb{P}_0$ will be the trivial poset, and for each $i < \kappa$, the poset $\dot{\mathbb{Q}}_{i,i}$ will simply be the $\mathbb{P}_{i,i}$-name for a countable Cohen poset, and $\mathbb{P}_{i+1,\kappa}$ will equal $\mathbb{P}_{i+1,i}$. For cofinally many $\alpha \in \lambda$, there will be an $i = i_\alpha < \kappa$ such that $\dot{\mathbb{Q}}_{\alpha,i}$ will provide a pseudointersection to a mod finite chain from the forcing extension by $\mathbb{P}_{\alpha,i}$. By bookkeeping arrangements, this will ensure that $\mathfrak{p} \ge \kappa$ holds in the final $\mathbb{P}_{\lambda,\kappa}$ forcing extension model. This of course guarantees that $\mathfrak{h}$ is at least κ in the final model, but we must take care to not let it get larger. This is accomplished as follows. For each $i < \kappa$ and $\gamma < \lambda$ we let $\mathcal{I}_{\gamma,i}$ denote the ideal generated by the family of sets added by such $\mathbb{Q}_{\alpha,i}$ (where $\kappa < \alpha < \gamma$ and $i_\alpha = i$). We show that $\langle \mathcal{I}_{\lambda,i} : i < \kappa\rangle$ is a descending sequence of dense ideals and the plan is to ensure that the intersection of this family of ideals is $[\omega]^{<\aleph_0}$. This will show that $\mathfrak{h}$ is no larger than κ. Arranging that $\mathfrak{b} = \lambda$ is easily done by ensuring that $\dot{\mathbb{Q}}_{\alpha,\kappa}$ is standard Hechler forcing for cofinally many $\alpha \in \lambda$. It is well known that Hechler forcing will not add a set which is a member of each $\mathcal{I}_{\lambda,i}$. The role of the matrix structure of the forcing comes into focus when we construct a sequence $\{\dot{\mathbb{Q}}_{\alpha,i} : i \leq \kappa\}$ designed to add an unsplit real. The poset $\dot{\mathbb{Q}}_{\alpha,\kappa}$ will be forced to be a special ccc subposet of the poset $\mathcal{Q}_{Bould}$ introduced in [13] as mentioned above. Such ccc subposets were introduced and studied in [8]. Our game plan to utilize the matrix structure is that the construction of $\dot{\mathbb{Q}}_{\alpha,i}$ must ensure that $\mathbb{P}_{\alpha,i} * \dot{\mathbb{Q}}_{\alpha,i}$ will force that, for all $j < i$, no $\mathbb{P}_{\alpha,j} * \dot{\mathbb{Q}}_{\alpha,j}$-name will be in the ideal generated by $\mathcal{I}_{\alpha,i}$. In the next section we introduce the poset $\mathcal{Q}_{Bould}$ and the special class of ccc subposets that can be used as a sequence to extend a matrix iteration system analogous to Lemma 2.6 for Hechler.

3. CCC SUBPOSETS OF $\mathcal{Q}_{Bould}$

The proper poset $\mathcal{Q}_{Bould}$ is introduced in [13] to establish the consistency of $\mathfrak{b} < \mathfrak{s} = \mathfrak{a}$. Let us note the important properties of $\mathcal{Q}_{Bould}$ shown to hold in [13]. The first is that it adds an unsplit real.

Proposition 3.1. *If $\dot{X}$ is the generic subset of ω added by $\mathcal{Q}_{Bould}$, then the set $\{A \subset \omega : A \in V$ and $|\dot{X} \setminus A| < \omega\}$ is a free ultrafilter over $V \cap \mathcal{P}(\omega)$.*

The second is that the forcing does not add a dominating real and does not add a pseudointersection to any ground model tower.

Proposition 3.2. *If $\dot{f}$ is a $\mathcal{Q}_{Bould}$-name of a strictly increasing function in ω^ω and if $\dot{X}$ is the generic subset of ω added by $\mathcal{Q}_{Bould}$, then there is a (ground model) $h \in \omega^\omega$ so that, for every infinite set $A \subset \omega$ in the ground model, the set $\{n \in A : \dot{f}(n) < h(n)\}$ will be forced to be infinite.*

We adopt the elegant representation of this poset from [1]. Also many of the technical details for constructing ccc subposets of this poset, sharing the above mentioned properties, are similar to the results in [8]. The main tool is the properties of logarithmic measures.

Definition 3.3. *A function h is a logarithmic measure on a set $S \subset \omega$ if h is a function from $[S]^{<\aleph_0}$ into ω with the property that whenever $\ell \geq 0$ and $h(a \cup b) \geq \ell + 1$, then either $h(a) \geq \ell$ or $h(b) \geq \ell$. A pair $(s, h) \in \mathcal{L}_n$ if $s \in [\omega]^{<\aleph_0}$ and h is a logarithmic measure on s with $h(s) \geq n$. The elements e of $[\omega]^{<\aleph_0}$ such that $h(e) > 0$ are called the positive sets.*

Note that if $(s,h) \in \mathcal{L}_n$ and $\emptyset \neq e \subset s$, then $(e, h \restriction [e]^{<\aleph_0}) \in \mathcal{L}_{h(e)}$.

Definition 3.4. *The poset $\mathcal{Q}_{Bould}$ consists of all pairs (u,T) where*

1. *$u \in [\omega]^{<\aleph_0}$,*
2. *$T = \langle t_\ell : \ell \in \omega\rangle$ is a sequence of members of $\mathcal{L}_1$ where for each ℓ, $t_\ell = (s_\ell, h_\ell)$, $\max(s_\ell) < \min(s_{\ell+1})$, and the sequence $\{h_\ell(s_\ell) : \ell \in \omega\}$ is monotone increasing and unbounded.*

For each $t \in \mathcal{L}_1$, let $int(t) = s$ where $t = (s,h)$. For each $(u,T) \in \mathcal{Q}_{Bould}$, let $\ell_{u,T}$ be the minimal ℓ such that $\max(u) < \min(int(t_\ell))$ and let $int(u,T) = \bigcup\{int(t_\ell) : \ell_{u,T} < \ell\}$.

The extension relation is defined by $(u_2, \langle t^2_\ell : \ell \in \omega\rangle) \le (u_1, \langle t^1_\ell : \ell \in \omega\rangle)$ providing

1. *$u_2 \supset u_1$ and $u_2 \setminus u_1$ is contained in $int(u_1, T_1)$,*
2. *$int(u_2, T_2) \subset int(u_1, T_1)$*
3. *there is a sequence of finite subsets of ω, $\langle B_k : k \in \omega\rangle$, such that for each $k \ge \ell_{u_2,T_2}$, $\max(B_k) < \min(B_{k+1})$ and $int(t^2_k) \subset \bigcup\{int(t^1_\ell : \ell \in B_k\}$,*
4. *for every $k \ge \ell_{u_2,T_2}$ and every h^2_k-positive $e \subset int(t^2_k)$ there is a $j \in B_k$ such that $e \cap int(t^1_j)$ is h^1_j-positive.*

We shall say that a pair (s,h) in $\mathcal{L}_1$ is built from a condition $(u, \langle t_k : k \in \omega\rangle) \in \mathcal{Q}_{Bould}$ providing there is a finite $B \subset \omega$ such that $\max(u) < \min(\bigcup\{int(t^1_\ell : \ell \in B\}))$, $s \subset \bigcup\{int(t^1_\ell : \ell \in B\})$ and for every h-positive $e \subset s$ there is a $j \in B$ such that $e \cap int(t_j)$ is t_j-positive.

Definition 3.5. *For $q \in \mathcal{Q}_{Bould}$, let $q = (u_q, T_q)$. Say that $q_1, q_2 \in \mathcal{Q}_{Bould}$ are mod finite equivalent if there are ℓ_1, ℓ_2 such that the sequences $\{t^1_j : \ell_1 < j\} \subset T_{q_1}$ and $\{t^2_k : \ell_2 < k\} \subset T_{q_2}$ are the same. Then we say that $Q \subset \mathcal{Q}_{Bould}$ is closed under finite changes if for each $q_1 \in Q$ any $q_2 \in \mathcal{Q}_{Bould}$ which is mod finite equivalent to q_1 is also in Q. Say that $q \in \mathcal{Q}_{Bould}$ is a pure condition if u_q is the empty set. Say that $Q \subset \mathcal{Q}_{Bould}$ is directed mod finite if it is closed under finite changes and every finite set of pure conditions from Q has a lower bound in Q.*

Proposition 3.6. *If $Q \subset \mathcal{Q}_{Bould}$ is directed mod finite then Q is σ-centered.*

Definition 3.7. *Say that Q is $\aleph_1$-directed mod finite if it is directed mod finite and if for each countable predense set $\{(u_n, T_n) : n \in \omega\} \subset Q$, there is a pure condition $(\emptyset, T) = (\emptyset, \langle t_\ell : \ell \in \omega\rangle) \in Q$ such that, for each $0 < \ell \in \omega$ and for each $w \subset \max(int(t_{\ell-1}))$, and for each h_ℓ-positive $e \subset int(t_\ell)$, there is an $n < \min int(t_{\ell+1})$ and a $w_e \subset e$ such that $(w \cup w_e, T) < (u_n, T_k)$ for each $k \le \max(n,\ell)$. When this condition holds, let us say that T is a mod finite meet of the family $\{(u_n, T_n) : n \in \omega\}$.*

Lemma 3.8. *Suppose that* $\{(u_n, T_n) : n \in \omega\}$ *is a subset of* $\mathcal{Q}_{Bould}$ *and let* T *be a mod finite meet. Then, for each* $w \in [\omega]^{<\aleph_0}$, $\{(u_n, T_n) : n \in \omega\}$ *is predense below* (w, T) *in all of* $\mathcal{Q}_{Bould}$.

Proof. Let (w, T') be an arbitrary member of $\mathcal{Q}_{Bould}$ that is compatible with $(w, T) = (w, \{t_\ell : \ell \in \omega\})$ in $\mathcal{Q}_{Bould}$. By extending (w, T'), we may assume that $(w, T') < (w, \{t_\ell : \ell > \ell_w\})$, where $w \subset \min(int(t_{\ell_w}))$. Choose any T'-positive e so that $(w \cup e, T') < (w, T')$ and $\max(w) < \min(e)$. Therefore there is an $\ell > \ell_w$ such that $h_\ell(e \cap int(t_\ell)) > 0$, and so, by Definition 3.7, there is an $n < \min(int(t_{\ell+1}))$ and a $w_e \subset e$ so that $(w \cup w_e, T) < (u_n, T_n)$ and we have that $(w \cup w_e, T') < (w, T) < (u_n, T_n)$. □

Definition 3.9. $\mathbb{Q}_{207}$ *is the set of* $Q \subset \mathcal{Q}_{Bould}$ *that are* $\aleph_1$*-directed mod finite.*

Proposition 3.10. *Assume that* κ *and* γ *are non-zero ordinals and that* $\langle\langle \mathbb{P}_{\alpha,i} : \alpha \leq \gamma, i \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_{\alpha,i} : \alpha < \gamma, i \leq \kappa\rangle\rangle$ *is a matrix iteration. If, for each* $i < \kappa$, $\dot{\mathbb{Q}}_{\gamma,i}$ *is a* $\mathbb{P}_{\alpha,i}$*-name for a member of* $\mathbb{Q}_{207}$ *containing* $\dot{\mathbb{Q}}_{\gamma,j}$ *for all* $j < i$*, then for each* $j < i \leq \kappa$, $\mathbb{P}_{\gamma,j} * \dot{\mathbb{Q}}_{\gamma,j} <\!\circ\, \mathbb{P}_{\gamma,i} * \dot{\mathbb{Q}}_{\gamma,i}$.

Proof. By Lemma 2.5, it suffices to prove that each $\mathbb{P}_{\gamma,j}$-name of a predense subset of $\dot{\mathbb{Q}}_{\gamma,j}$ is forced by $\mathbb{P}_{\gamma,i}$ to be predense in $\dot{\mathbb{Q}}_{\gamma,i}$. Since $\dot{\mathbb{Q}}_{\gamma,j}$ is forced to be a subset of $\dot{\mathbb{Q}}_{\gamma,i}$, it is immediate that $[\omega]^{<\aleph_0} \times \{T\}$ is a predense subset of $\dot{\mathbb{Q}}_{\gamma,i}$ for each $(\emptyset, T) \in \dot{\mathbb{Q}}_{\gamma,j}$. Now the Proposition follows by Lemma 3.8. □

We now establish notation that will be useful when preserving a function is (forced to be) unbounded.

Definition 3.11. *For a condition* $q = (u, \{t_\ell : \ell \in \omega\})$ *in* $\mathcal{Q}_{Bould}$ *and a pair of functions* g, f *in* ω^ω *with* $f \not<^* g$*, let* $q[g, f] = (u, \{t_\ell : g(\ell) < f(\ell)\})$*. Similarly, for an infinite set* $L \subset \omega$*, let* $q[L] = (u, \{t_\ell : \ell \in L\})$.

The application of this definition (see Corollary 3.13) will use this next lemma.

Lemma 3.12. *If* $Q \in \mathbb{Q}_{207}$ *and if* $\dot{f}$ *is a* Q*-name such that* $\Vdash \dot{f} \in \omega^\omega$*, then there is a pair* $((\emptyset, \{t^{\dot{f}}_\ell : \ell \in \omega\}), F_{\dot{f}}) \in Q \times \omega^\omega$ *such that for each* $\ell > 0$, $w \subset \max(int(t^{\dot{f}}_{\ell-1}))$ *and* $t^{\dot{f}}_\ell$*-positive* $e \subset int(t^{\dot{f}}_\ell)$*, there is a* $w_e \subset e$ *such that* $(w \cup w_e, \{t^{\dot{f}}_k : \ell < k \in \omega\})$ *forces that* $\dot{f}(\ell) < F_{\dot{f}}(\ell)$.

Proof. For each k, there is a pre-dense set $\{(u^k_n, T^k_n) : n \in \omega\} \subset Q$ satisfying that, for each n, k, (u^k_n, T^k_n) forces a value on $\dot{f} \restriction k{+}1$. Since $Q \in \mathbb{Q}_{207}$, there is, for each k, a condition $(\emptyset, \{t^k_\ell : \ell \in \omega\}) \in Q$ such that, for each ℓ, for any $w \subset \max int(t^k_{\ell-1})$ and any t^k_ℓ-positive e, there is a $w_e \subset e$, such that $(w \cup w_e, \{t^k_m : \ell < m \in \omega\})$ forces a value on $\dot f \restriction k+1$. Let $\{w_k : k \in \omega\}$ be an enumeration of $[\omega]^{<\aleph_0}$. For each k, let $T_k = \{t^k_\ell : \ell \in \omega\}$, and choose $T = \{t_\ell : \ell \in \omega\}$ to be a mod finite meet of the family $\{(w_k, T_k) : k \in \omega\}$. That is, in this case, for each $0 < \ell$ and each h_ℓ-positive $e \subset int(t_\ell)$, we have that $(e, T) < (\emptyset, T_k)$ for each $k \leq \ell$. Of course it follows that, for all $w \subset \max int(t_\ell)$, and $t_{\ell+1}$-positive e $(w \cup e, T) < (w, T_\ell)$. Also, since both t_ℓ and $t_{\ell+1}$ are built from T_ℓ, we have that there are $m < m'$ such that t_ℓ is built from $\{t^\ell_j : j \leq m\}$ and $e \cap int(t^\ell_{m'})$ is $t^\ell_{m'}$-positive. Therefore, there is a $w_e \subset e \cap int(t^\ell_{m'})$ such that $(w \cup w_e, \{t^\ell_k : m' < k \in \omega\})$ forces a value on $\dot f \restriction \ell+1$. It follows also that $(w \cup w_e, \{t_k : \ell+1 < k \in \omega\})$ forces the same value on $\dot f \restriction \ell+1$. To finish the proof, simply define $F_{\dot f}(\ell)$ large enough so that, for each $w \subset \max int(t_{\ell+1})$ such that $(w, \{t_k : \ell+1 < k \in \omega\})$ forces a value on $\dot f(\ell)$, this value is less than $F_{\dot f}(\ell)$. The Lemma is now proven by using (and re-indexing) the condition $(\emptyset, \{t_{\ell+1} : \ell \in \omega\})$. $\square$

Corollary 3.13. *If $\dot f$, $q = (\emptyset, \{t^{\dot f}_\ell : \ell \in \omega\})$ and $F_{\dot f}$ are as in Lemma 3.12, then for any $h \in \omega^\omega$ with $h \not<^* F_{\dot f}$, $q[F_{\dot f}, h] \Vdash_Q h \not<^* \dot f$ for any $Q \in \mathbb{Q}_{207}$ with $q[F_{\dot f}, h] \in Q$.*

4. Constructing the posets

Let $\aleph_1 < \kappa < \lambda$ be regular cardinals and assume that $\theta^\kappa < \lambda$ for all $\theta < \lambda$.

Theorem 4.1. *There is a poset $\mathbb{P}_{\lambda,\kappa}$ as in Definition 2.2 so that in the forcing extension, we have $\omega_1 < \kappa = \mathfrak{p} = \mathfrak{h} = \mathfrak{b} < \mathfrak{s} = \lambda = 2^\omega$.*

Theorem 4.2. *There is a poset $\mathbb{P}_{\lambda,\kappa}$ as in Definition 2.2 so that in the forcing extension, we have $\omega_1 < \kappa = \mathfrak{p} = \mathfrak{h} < \mathfrak{b} = \mathfrak{s} = \lambda = 2^\omega$.*

The proof of the above two results must wait until the final section.

If the ground model is chosen to satisfy GCH, then there is a simple and well known further step which can be used to adjust the value of $\mathfrak{t}$ to any regular uncountable cardinal less than κ. It relies on Easton's lemma that if P is a ccc poset (in fact if P satisfies the τ-cc) and if Q is a τ-closed poset (sometimes called $<\tau$-closed [10, 15.19]) for some uncountable regular cardinal, then in the forcing extension by P, the poset Q is still τ-distributive and so further forcing by Q will not add any sequences of ordinals of length less than τ. See also [6] for a more general situation.

Corollary 4.3. *If GCH holds in the ground model and if τ, κ, λ are regular cardinals such that $\omega_1 \leq \tau < \kappa \leq \lambda$, then there is a cardinal preserving forcing extension in which $\mathfrak{t} = \tau$, $\mathfrak{h} = \kappa$, $\mathfrak{s} = \lambda = \mathfrak{c}$, and $\mathfrak{b}$ can be either κ or λ as desired.*

Proof. First let $\mathbb{P}_{\kappa,\lambda}$ be the ccc poset as in Theorem 4.1, so that $\mathfrak{t} = \mathfrak{h} = \mathfrak{b} = \kappa$ and $\mathfrak{s} = \lambda = 2^\omega$. Still in the ground model of GCH, let Q denote the forcing poset $\tau^{<\tau}$. Our desired model is the forcing extension by $\mathbb{P}_{\kappa,\lambda} \times Q$. Note that in the extension by $\mathbb{P}_{\kappa,\lambda}$ the poset Q is now τ-distributive and has cardinality τ. It follows that it, and thus the product $\mathbb{P}_{\kappa,\lambda} \times Q$, is cardinal preserving. Furthermore, this further forcing by Q will add no new subsets of ω. It is easily seen that this means that the values of $\mathfrak{b}$ and $\mathfrak{s}$ are preserved. Since $\mathfrak{t} = \mathfrak{p} = \kappa$ in the extension by $\mathbb{P}_{\kappa,\lambda}$, the poset $([\omega]^\omega, \supset^*)$ is κ-closed, and so forcing by Q will preserve that it is κ-distributive. This shows that $\mathfrak{h} = \kappa$ in the extension by $\mathbb{P}_{\kappa,\lambda} \times Q$. Finally, to see that $\mathfrak{t} = \tau$ holds in this extension, we simply note that, in the extension by $\mathbb{P}_{\kappa,\lambda}$, the poset $\mathbb{Q}$ can be order-theoretically embedded in $([\omega]^\omega, \supset^*)$ and so the generic for Q will introduce a mod finite descending τ-sequence with no pseudointersection. $\square$

The recursive constructions of the posets for the two main theorems is quite involved and will require additional notational conventions and devices. Recall that, for a poset $\mathbb{P}$, a $\mathbb{P}$-name $\dot{Y}$ for a subset of ω is a *nice* name (or canonical name) if, for each $n \in \omega$, there is an antichain $Y_n \subset \mathbb{P}$ so that $\dot{Y} = \{(p, \check{n}) : n \in \omega, p \in Y_n\}$ where $\check{n}$ is the canonical $\mathbb{P}$-name for the integer n. In order to ensure that $\mathfrak{p} \geq \kappa$ we will need bookkeeping so as to ensure we eventually consider all $<\kappa$-sized sets of nice names of infinite subsets of ω. For this we will fix $\prec$, a well-ordering of $H(\lambda)$ in order type λ. With this well-ordering we have, for any ccc poset $\mathbb{P} \in H(\lambda)$, an implicit enumeration of all the nice $\mathbb{P}$-names of subsets of ω, as well as all the $<\kappa$-sized sets of such names. We will use the letter $\mathcal{Y}$ when denoting either a $<\kappa$-sized set of subsets of ω, or, for a poset $\mathbb{P}$, a $<\kappa$-sized set of $\mathbb{P}$-names of subsets of ω. For an infinite $Y \subset \omega$, let $f_Y \in \omega^\omega$ denote the strictly increasing enumeration function.

We introduce notation for the standard σ-centered poset which forces a pseudointersection of a family $\mathcal{Y}$ of subsets of ω which has the strong finite intersection property (all finite intersections are infinite).

Definition 4.4. *Let $\mathcal{F}_\omega$ denote the cofinite filter $\{\omega \setminus u : u \in [\omega]^{<\aleph_0}\}$. For any family $\mathcal{Y} \subset [\omega]^{\aleph_0}$ with the strong finite intersection property,*

$Q[\mathcal{Y}]$ is the standard poset for adding a pseudointersection. That is, we let $\langle\mathcal{Y}\rangle$ denote the closure of the family $\mathcal{F}_\omega \cup \mathcal{Y}$ under finite intersections, and $Q[\mathcal{Y}]$ is the set $[\omega]^{<\aleph_0} \times \langle\mathcal{Y}\rangle$ ordered by $(u_1, Y_1) < (u_0, Y_0)$ providing $Y_1 \subset Y_0$ and $u_0 \subset u_1 \subset u_0 \cup (Y_0 \setminus \max(u_0))$.

We set $\dot{A}_{Q[\mathcal{Y}]}$ to be any nice name $\dot{A}$ where for each $(u, Y) \in Q[\mathcal{Y}]$ and $n \in \omega$, (u, Y) forces that $n \in \dot{A}$ if and only if $n \in u$.

We adopt the convention that $Q[\emptyset]$ is the trivial poset. Of course $Q[\{\omega\}]$ just adds a Cohen real. Let us also note that if $\mathcal{Y}$ is not empty, then $\dot{A}_{Q[\mathcal{Y}]}$ is forced to not contain any infinite ground model set.

As part of our bookkeeping let us note that the set $\{\kappa\cdot\eta : 0 < \eta < \lambda\}$ is the cub set of ordinals equalling the closure of all ordinals of cofinality equal to κ. Each ordinal $0 < \alpha \in \lambda$ has a representation as $\alpha = \kappa\cdot\eta + i$ for some $0 \leq i < \kappa$, and for $\gamma \leq \lambda$, we will let $\Gamma_{\gamma,i} = \{\kappa\cdot\eta + i : 0 < \eta \text{ and } \kappa\cdot\eta + i < \gamma\}$ and $\Gamma_i = \Gamma_{\lambda,i}$.

Definition 4.5. *For $\gamma < \lambda$, let $\underline{\mathbf{Q}}_0^\gamma$ denote the set of all (γ, κ)-matrix iterations $\underline{\mathbf{P}}$ in $H(\lambda)$ such that for each $\alpha < \gamma$, $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,\kappa}$ forces that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,\kappa}$ adds a new real, and, for each $i < \kappa$, $\mathbb{P}^{\underline{\mathbf{P}}}_{0,i}$ is the trivial poset.*

For each $\underline{\mathbf{P}} \in \underline{\mathbf{Q}}_0^\gamma$ and each η with $\kappa\cdot\eta \leq \gamma$, we let $\mathcal{Y}[\underline{\mathbf{P}}, \kappa\cdot\eta]$ denote the $\prec$-minimal family $\mathcal{Y}$ of cardinality less than κ satisfying that

1. *each member of $\mathcal{Y}$ is a nice $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,\kappa}$-name of a subset of ω,*
2. *the condition 1 forces over the poset $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,\kappa}$ that the family $\mathcal{Y}$ has the strong finite intersection property, and*
3. *there is a condition $p \in \mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,\kappa}$ that forces over the poset $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,\kappa}$ that $\mathcal{Y}$ has no infinite pseudointersection.*

Further, let $i^{\underline{\mathbf{P}}}_{\kappa\cdot\eta}$ denote the minimal $i < \kappa$ such that each member of $\mathcal{Y}[\underline{\mathbf{P}}, \kappa\cdot\eta]$ is a $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$-name. We let $\underline{\mathbf{Q}}_0^\lambda$ denote all those (λ, κ)-matrix iterations $\underline{\mathbf{P}}$ such that, for all $\gamma < \lambda$, $\underline{\mathbf{P}}\restriction\gamma \in \underline{\mathbf{Q}}_0^\gamma$.

Clearly the definition of $\mathcal{Y}[\underline{\mathbf{P}}, \kappa\cdot\eta]$ is just using the well-ordering $\prec$ to select *the next* family needing a pseudointersection if we are using the iteration sequence $\underline{\mathbf{P}}$. The requirement that each $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,\kappa}$ adds a new real in the definition of $\underline{\mathbf{Q}}_0^\gamma$ ensures that such a (non-empty) family $\mathcal{Y}[\underline{\mathbf{P}}, \kappa\cdot\eta]$ exists. Before continuing with $\mathcal{Y}[\underline{\mathbf{P}}, \kappa\cdot\eta]$ we make precise how we will be defining our sequence of ideals that will be used to witness $\mathfrak{h} \leq \kappa$.

In the next definition, we reserve a special role for ordinals of the form $\kappa\cdot\eta$ and $\kappa\cdot\eta + 1$ for all $0 < \eta < \lambda$.

Definition 4.6. *We define enhanced structures* $\underline{\mathbf{Q}}_1^\gamma$ *where* $\underline{\mathbf{q}} = (\underline{\mathbf{P}}_{\underline{\mathbf{q}}}, \mathbb{A}_{\underline{\mathbf{q}}})$ *is in* $\underline{\mathbf{Q}}_1^\gamma$ *if* $\underline{\mathbf{P}}_{\underline{\mathbf{q}}} \in \underline{\mathbf{Q}}_0^\gamma$ *and,* $\mathbb{A}_{\underline{\mathbf{q}}}$ *is a sequence* $\{\dot{A}_\alpha^{\underline{\mathbf{q}}} : \alpha < \gamma\}$ *where, for each* $\alpha < \gamma$, $\dot{A}_\alpha^{\underline{\mathbf{q}}}$ *is a nice* $\mathbb{P}^{\underline{\mathbf{P}}_{\underline{\mathbf{q}}}}_{\alpha+1,\kappa}$*-name of an infinite subset of* ω.

A structure $\underline{\mathbf{q}} = (\underline{\mathbf{P}}, \{\dot{A}_\alpha : \alpha < \gamma\})$ *is in* $\underline{\mathbf{Q}}_2^\gamma$ *providing* $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_1^\gamma$ *and for all* $i < \kappa$ *and* $\kappa \le \kappa\cdot\eta + i < \gamma$*, it is forced by* $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+i+1,\kappa}$ *that* $\dot{A}_{\kappa\cdot\eta+i}$ *is mod finite contained in every member of* $\{\dot{A}_{\kappa\cdot\eta+j} : 1 < j < i\}$.

For $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_2^\gamma$ *and* $0 < i < \kappa$*, we let* $\mathcal{I}(\underline{\mathbf{q}}, i)$ *denote the* $\mathbb{P}^{\underline{\mathbf{P}}_{\underline{\mathbf{q}}}}_{\gamma,\kappa}$*-name of the ideal generated by the family* $\{\dot{A}^{\underline{\mathbf{q}}}_{\kappa\cdot\eta+i} : \kappa \le \kappa\cdot\eta + i < \gamma\}$*. Let us note that* $\mathbb{P}^{\underline{\mathbf{P}}_{\underline{\mathbf{q}}}}_{\gamma,\kappa}$ *forces that* $\mathcal{I}(\underline{\mathbf{q}}, j)$ *contains* $\mathcal{I}(\underline{\mathbf{q}}, i)$ *for all* $1 < j < i < \kappa$.

For $\underline{\mathbf{q}} = (\underline{\mathbf{P}}, \mathbb{A}) \in \underline{\mathbf{Q}}_1^\gamma$ and $(\alpha, i) \in \gamma{+}1 \times \kappa$, we may write $\mathbb{P}^{\underline{\mathbf{q}}}_{\alpha,i}$ rather than $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,i}$. The analogous convention holds for $\mathbb{Q}^{\underline{\mathbf{q}}}_{\alpha,i}$ for $\alpha < \gamma$. Similar to the notion of extension of matrix iterations, we formulate the obvious definition of extension for members of $\bigcup_{\gamma\le\lambda} \underline{\mathbf{Q}}_1^\gamma$.

Definition 4.7. *We say that* $\underline{\mathbf{q}} = (\underline{\mathbf{P}}_{\underline{\mathbf{q}}}, \{\dot{A}_\alpha^{\underline{\mathbf{q}}} : \alpha < \gamma\}) \in \underline{\mathbf{Q}}_1^\gamma$ *is an extension of* $(\underline{\mathbf{P}}, \{\dot{A}_\alpha : \alpha < \delta\}) \in \underline{\mathbf{Q}}_1^\delta$ *providing* $\delta \le \gamma$, $\underline{\mathbf{P}} = \underline{\mathbf{P}}_{\underline{\mathbf{q}}} \restriction \delta$*, and, for all* $\alpha < \delta$, $\dot{A}_\alpha = \dot{A}_\alpha^{\underline{\mathbf{q}}}$.

Next we merge the requirements towards achieving $\mathfrak{p} \ge \kappa$ while maintaining the descending sequence of ideals $\mathcal{I}(\underline{\mathbf{q}}, i)$ $(i < \kappa)$. For each $i < \kappa$, we will have that $\mathbb{P}_{i+1,i}$ simply adds a Cohen real not added by $\mathbb{P}_{i+1,j}$ for any $j < i$.

Definition 4.8. *A structure* $\underline{\mathbf{q}} = (\underline{\mathbf{P}}, \{\dot{A}_\alpha : \alpha < \gamma\})$ *is in the set* $\underline{\mathbf{Q}}_2^\gamma(\prec)$ *providing* $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_1^\gamma$ *and, for all* $i, j < \kappa$*, and* $\alpha = i < \gamma$ *or* $\kappa < \kappa\cdot\eta + 1 < \alpha = \kappa\cdot\eta + i < \gamma$*, then it is forced by* $\mathbb{P}^{\underline{\mathbf{P}}}_{\alpha,j}$ *that*

1. *if* $j < i = \alpha$*, then* $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,j}$ *is* $Q[\emptyset]$,
2. *if* $j > i$*, then* $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,j} = \dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i}$,
3. *if* $i = \alpha$*, then* $\mathcal{Y}_\alpha^{\underline{\mathbf{q}}} = \{\omega\}$,
4. *if* $i^{\underline{\mathbf{P}}}_{\kappa\cdot\eta} \ne i < \alpha$*, then* $\mathcal{Y}_\alpha^{\underline{\mathbf{q}}} = \{\omega\} \cup \{\dot{A}_\beta : \kappa\cdot\eta + 1 < \beta < \alpha\}$,
5. *if* $i^{\underline{\mathbf{P}}}_{\kappa\cdot\eta} = i < \alpha$*, then* $\mathcal{Y}_\alpha^{\underline{\mathbf{q}}} = \mathcal{Y}[\underline{\mathbf{P}}, \kappa\cdot\eta] \cup \{\dot{A}_\beta : \kappa\cdot\eta + 1 < \beta < \alpha\}$,
6. $\mathcal{Y}_\alpha^{\underline{\mathbf{q}}}$ *has the strong finite intersection property,*
7. $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\alpha,i} = Q[\mathcal{Y}_\alpha^{\underline{\mathbf{q}}}]$*, and*
8. $\dot{A}_\alpha$ *is equal to* $\dot{A}_{Q[\mathcal{Y}_\alpha^{\underline{\mathbf{q}}}]}$.

Lemma 4.9. *For each* $\gamma \le \lambda$, $\underline{\mathbf{Q}}_2^\gamma(\prec)$ *is non-empty, and for each* $\underline{\mathbf{q}} = (\underline{\mathbf{P}}_{\underline{\mathbf{q}}}, \mathbb{A}^{\underline{\mathbf{q}}})$ *in* $\underline{\mathbf{Q}}_2^\gamma(\prec)$*, if* $\gamma < \lambda$*, there is an extension in* $\underline{\mathbf{Q}}_2^{\gamma+1}(\prec)$*. Furthermore, if* $\gamma \notin \{\kappa\cdot\eta : 0 < \eta < \lambda\}$*, then the extension is unique.*

Proof. It is immediate that if $\gamma \le \lambda$ is a limit, and if $\langle \underline{\mathbf{q}}_\alpha : \alpha \in \gamma\rangle$ is a sequence satisfying that, for each $\beta < \alpha < \gamma$, $\underline{\mathbf{q}}_\alpha \in \underline{\mathbf{Q}}_2^\alpha(\prec)$ is an extension of $\underline{\mathfrak{q}}_\beta \in \underline{\mathbf{Q}}_2^\beta(\prec)$, then there is a common extension $\underline{\mathfrak{q}}$ in $\underline{\mathbf{Q}}_2^\gamma(\prec)$.

Now we assume that $\gamma < \lambda$ and that $\underline{\mathfrak{q}} \in \underline{\mathbf{Q}}_2^\gamma(\prec)$ and prove there is an extension in $\underline{\mathbf{Q}}_2^{\gamma+1}(\prec)$. Let $\underline{\mathbf{P}} = \underline{\mathbf{P}}_{\mathfrak{q}}$ and, for $\alpha < \gamma$, let $\dot{A}_\alpha$ denote $\dot{A}_\alpha^{\mathfrak{q}}$. If $\gamma = 0$, then we can define $\mathbb{Q}_{0,0} = \mathcal{H}$, and, by recursion, for $i < \kappa$. $\mathbb{Q}_{0,i}$ to be the poset $(\bigcup_{j<i} \mathbb{Q}_{0,j}) * \mathcal{H}$ (or any other small ccc iteration). We also set $\mathbb{Q}_{0,\kappa}$ to be the poset $\bigcup_{i<\kappa} \mathbb{Q}_{0,i}$. Technically, $\dot{\mathbb{Q}}_{0,i}$ should be the $\mathbb{P}_{0,i}$-name of $\mathbb{Q}_{0,i}$, where $\mathbb{P}_{0,i} = \mathbb{P}_{0,i}^{\mathbf{P}}$ is the trivial poset. We also let $\mathbb{P}_{1,i} = \mathbb{P}_{0,i} * \dot{\mathbb{Q}}_{0,i}$ for each $i \leq \kappa$. It is evident that $\underline{\mathbf{P}}' = \langle \langle \mathbb{P}_{\alpha,i} : \alpha \leq 1, i \leq \kappa \rangle, \langle \dot{\mathbb{Q}}_{\alpha,i} : \alpha < 1, i \leq \kappa \rangle \rangle$ is in $\underline{\mathbf{Q}}_0^1$. We can define $\dot{A}_0$ to be a nice name for the range of the first Hechler real and then we have that $(\underline{\mathbf{P}}', \{\dot{A}_0\}) \in \underline{\mathbf{Q}}_2^1$. It follows vacuously that $(\underline{\mathbf{P}}', \{\dot{A}_0\})$ is also in $\underline{\mathbf{Q}}_2^1(\prec)$.

The case for γ having the form $\kappa \cdot \eta$, or $\kappa \cdot \eta + 1$ with $\eta > 0$, is proven in exactly the same way. The chain $\{\dot{\mathbb{Q}}_{\gamma,i} : i \leq \kappa\}$ can be chosen arbitrarily just so long as it is non-trivial and $\underline{\mathbf{P}}^\frown \langle \dot{\mathbb{Q}}_{\gamma,i} : i \leq \kappa \rangle$ is in $\underline{\mathbf{Q}}_0^{\gamma+1}$. One can set $\dot{A}_\gamma$ to be any $\mathbb{P}_{\gamma,\kappa} * \dot{\mathbb{Q}}_{\gamma,\kappa}$-name of an infinite subset of ω, and then we have our required extension in $\underline{\mathbf{Q}}_2^{\gamma+1}(\prec)$.

Now suppose that $\gamma = \kappa \cdot \eta + i$ for some $1 < i < \kappa$. We have that $\mathcal{Y}[\underline{\mathbf{P}}, \kappa \cdot \eta]$ is forced by $\mathbb{P}_{\kappa\cdot\eta,\kappa}^{\mathbf{P}}$ to have the strong finite intersection property. To define our desired extension $\underline{\mathfrak{q}}'$ of $\underline{\mathfrak{q}}$ we first suppose that $i_{\kappa\cdot\eta}^{\mathbf{P}}$ is not equal to i. It should be evident from Definition 4.8 that the family $\mathcal{Y}_\gamma = \{\omega\} \cup \{\dot{A}_\beta^{\mathfrak{q}} : \kappa \cdot \eta < \beta < \gamma\}$ is forced by $\mathbb{P}_{\gamma,\kappa}^{\mathfrak{q}}$ to have the strong finite intersection property; in fact the set is forced to be descending mod finite. Now we define $\dot{\mathbb{Q}}_{\gamma,j} = Q[\emptyset]$ for $j < i$, and $\dot{\mathbb{Q}}_{\gamma,j} = Q[\mathcal{Y}_\gamma]$ for $i \leq j \leq \kappa$. We set $\underline{\mathbf{P}}_{\mathfrak{q}'} = \underline{\mathbf{P}}^\frown \langle \dot{\mathbb{Q}}_{\gamma,j} : j \leq \kappa \rangle$. It follows immediately from Lemma 2.5 that $\mathbb{P}_{\gamma+1,j}^{\mathfrak{q}'} <\!\circ\, \mathbb{P}_{\gamma+1,\bar{\imath}}^{\mathfrak{q}'}$ for all $j < \bar{\imath} \leq \kappa$. Therefore $\underline{\mathbf{P}}_{\mathfrak{q}'} \in \underline{\mathbf{Q}}_0^{\gamma+1}$, and we set $\dot{A}_{\gamma+1}^{\mathfrak{q}'}$ to be $\dot{A}_{Q[\mathcal{Y}_\gamma]}$ as in Definition 4.8. It should be clear that $\underline{\mathfrak{q}}' = \langle \underline{\mathbf{P}}^{\mathfrak{q}'}, \langle \mathbb{A}^{\mathfrak{q}} \cup \{\dot{A}_{\gamma+1}^{\mathfrak{q}'}\} \rangle \rangle$ is the required extension of $\underline{\mathfrak{q}}$ in $\underline{\mathbf{Q}}_2^{\gamma+1}(\prec)$

Now we establish the Lemma for the case that i is equal to $i_{\kappa\cdot\eta}^{\mathbf{P}}$. According to the definition of members of $\underline{\mathbf{Q}}_2^\gamma(\prec)$ (Definition 4.8), using a routine genericity argument and, by induction on β with $\kappa \cdot \eta \leq \beta < \min(\gamma, \kappa \cdot \eta + i_{\kappa\cdot\eta}^{\mathbf{P}})$, we have that $\mathbb{P}_{\beta,\kappa}^{\mathfrak{q}}$ forces that the family $\mathcal{Y}[\underline{\mathbf{P}}, \kappa \cdot \eta] \cup \{\dot{A}_{\alpha+1}^{\mathfrak{q}} : \alpha < \beta\}$ has the strong finite intersection property. Now the only change from the above case where $i \neq i_{\kappa\cdot\eta}^{\mathbf{P}}$, is that we now necessarily have that $\mathcal{Y}_\gamma^{\mathfrak{q}'} = \mathcal{Y}[\underline{\mathbf{P}}, \kappa \cdot \eta] \cup \{\dot{A}_{\alpha+1}^{\mathfrak{q}} : \alpha < \gamma\}$. The construction of the extension $\underline{\mathfrak{q}}'$ proceeds as above. □

Lemma 4.10. *Assume that* $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_2^\lambda(\prec)$*. Then*

(1) $\mathbb{P}^{\mathbf{q}}_{\lambda,\kappa}$ *forces that* $\mathfrak{p} \geq \kappa$,

(2) *the family* $\{\mathcal{I}(\underline{\mathbf{q}}, i) : 1 < i < \kappa\}$ *is forced to be a descending sequence of ideals that are dense in* $([\omega]^{\aleph_0}, \subset)$,

(3) *if* $1 < i < \kappa$ *and* $\dot{Y}$ *is* $\mathbb{P}^{\mathbf{q}}_{\gamma,\kappa}$*-name that is forced by some* $p \in \mathbb{P}^{\mathbf{q}}_{\lambda,\kappa}$ *to be a member of* $\mathcal{I}(\underline{\mathbf{q}}, i)$*, then* $p \restriction \gamma$ *forces that* $\dot{Y}$ *is in the ideal generated by* $\{\dot{A}^{\mathbf{q}}_{\kappa\cdot\eta+i} : \kappa \cdot \eta + i < \gamma\}$.

Proof. Fix any $\eta < \lambda$ and consider the family $\mathcal{Y}[\underline{\mathbf{P}}^{\mathbf{Q}}, \kappa\cdot\eta]$. It is immediate from Definition 4.8 that every condition from $\mathbb{P}^{\mathbf{q}}_{\kappa\cdot(\eta+1),\kappa}$ forces that $\mathcal{Y}[\underline{\mathbf{P}}^{\mathbf{q}}, \kappa \cdot \eta]$ has an infinite pseudointersection. Therefore, by condition (iii) of Definition 4.5, $\mathcal{Y}[\underline{\mathbf{P}}^{\mathbf{q}}, \kappa \cdot \eta']$ is not equal to $\mathcal{Y}[\underline{\mathbf{P}}^{\mathbf{q}}, \kappa \cdot \eta]$ for all $\eta' > \eta$. It now follows easily that $\mathbb{P}^{\mathbf{q}}_{\lambda,\kappa}$ forces that $\mathfrak{p} \geq \kappa$.

If $1 < j < i < \kappa$ and $0 < \eta < \lambda$, then it is forced that $\dot{A}^{\mathbf{q}}_{\kappa\cdot\eta+i}$ is mod finite contained in $\dot{A}^{\mathbf{q}}_{\kappa\cdot\eta+j}$. Therefore the ideal $\mathcal{I}(\underline{\mathbf{q}}, j)$ is contained in the ideal $\mathcal{I}(\underline{\mathbf{q}}, i)$. To see that $\mathcal{I}(\underline{\mathbf{q}}, j)$ is forced to be a dense ideal, let $\dot{Y}$ be a $\mathbb{P}^{\mathbf{q}}_{\gamma,\kappa}$-name of an infinite subset of ω. Choose any $\eta \in \lambda \setminus \gamma$ so that $i^{\mathbf{P}}_{\kappa\cdot\eta}$ is greater than j. Exactly as in Lemma 4.9, it follows, by induction on $1 < j \leq i$ that $\dot{A}^{\mathbf{q}}_{\kappa\cdot\eta+j}$ is forced to meet $\dot{Y}$ in an infinite set.

Finally, we fix a $\mathbb{P}^{\mathbf{q}}_{\gamma,\kappa}$-name $\dot{Y}$ for a subset of ω and suppose some $p \in \mathbb{P}^{\mathbf{q}}_{\gamma,\kappa}$ forces that $\dot{Y}$ is not in the ideal generated by the collection $\{\dot{A}^{\mathbf{q}}_{\kappa\cdot\eta+i} : \kappa\cdot\eta+i < \gamma\}$. We prove that there is no extension of p forcing that $\dot{Y}$ is in the ideal $\mathcal{I}(\underline{\mathbf{q}}, i)$. To do so, suppose that H is any finite subset of λ, and let $H_0 = \{\eta \in H : \kappa\cdot\eta+i < \gamma\}$. By extending p in $\mathbb{P}^{\mathbf{q}}_{\gamma,\kappa}$ we can assume that p forces that $\dot{Y}_\gamma = \dot{Y} \setminus \bigcup\{\dot{A}^{\mathbf{q}}_{\kappa\cdot\eta+i} : \eta \in H_0\}$ is infinite. Then, by induction on $\eta \in H \setminus H_0$, $\mathbb{P}^{\mathbf{q}}_{\kappa\cdot\eta+i,\kappa}$ forces that $\dot{\mathbb{Q}}^{\mathbf{q}}_{\kappa\cdot\eta+i,\kappa}$ is $Q[\mathcal{Y}]$ for some non-empty family with strong finite intersection property, and so it follows from Definition 4.4, that p forces that $\dot{Y}_\gamma = \dot{Y} \setminus \bigcup\{\dot{A}^{\mathbf{q}}_{\kappa\cdot\eta+i} : \eta \in H\}$ is infinite. The third conclusion of the Lemma is thus proven. □

Now we isolate the future demands on the sequence $\{\dot{\mathbb{Q}}_{\kappa\cdot\eta+1,i} : i \leq \kappa\}$ for each $\eta < \lambda$. These posets must be chosen so as to add an unsplit real, but we must also ensure that they do not add any infinite set which is a member of $\mathcal{I}(i)$ for each $1 < i < \kappa$. First of all, it will be necessary to add many Cohen reals so we include it in our definition. For an index set J, we let $\mathcal{C}_J$ denote the standard poset, $Fn(J, 2)$ ([11, p204]), for adding Cohen reals.

First we define the systems that will also add dominating reals.

Definition 4.11. *A structure* $\underline{\mathbf{q}} = (\underline{\mathbf{P}}, \{\dot{A}_\alpha : \alpha < \gamma\})$ *is in the set* $\underline{\mathbf{Q}}_3^\gamma(\prec)$ *providing* $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_2^\gamma(\prec)$ *and, for all* $i < \kappa$*, and* $\kappa \leq \kappa \cdot \eta < \gamma$,

(1) it is forced by $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta,i}$ that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta,i}$ is the poset $\mathcal{C}_{i+1\times 2^\omega}$,
(2) and if $\kappa\cdot\eta+1<\gamma$ with η a successor, then it is forced by $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$ that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$ is $\mathcal{H}$.
(3) if $\kappa\cdot\eta+1<\gamma$ with η a limit, then it is forced by $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$ that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$ is in $\mathbb{Q}_{207}$,
(4) if $\kappa\cdot\eta+1<\gamma$ with η a limit, then $\dot{A}_{\kappa\cdot\eta+1}$ is the $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+2,i}$-name of the generic real added by $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$, and, it is forced that $\dot{A}_{\kappa\cdot\eta+1}$ is not split by any $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta,i}$-name.

This next result is similar to the results of Sections 4 and 5 in [8]. It suffices for the proof of Lemma 4.13. To improve the flow of the paper, we defer the proof.

Lemma 4.12. *If $Q\subset\mathcal{Q}_{Bould}$ is closed under finite changes and if every finite set of pure conditions has a lower bound in $\mathcal{Q}_{Bould}$, then in the forcing extension by $\mathcal{C}_{2^\omega}$, Q has an extension Q_1 in $\mathbb{Q}_{207}$.*

Proof. See Lemma 6.5. □

And the following Lemma is an immediate consequence of Lemma 4.12.

Lemma 4.13. *If $\gamma\leq\lambda$ is a limit and if $\{\underline{\mathbf{q}}_\xi:\xi<\gamma\}$ is a sequence with $\underline{\mathbf{q}}_\xi\in\underline{\mathbf{Q}}^\xi_3(\prec)$ and $\underline{\mathbf{q}}_\xi=\underline{\mathbf{q}}_\zeta\restriction\xi$ for all $\xi\leq\zeta<\gamma$, then there is an extension $\underline{\mathbf{q}}_\gamma\in\underline{\mathbf{Q}}^\gamma_3(\prec)$. If $\underline{\mathbf{q}}=(\underline{\mathbf{P}}^{\mathbf{q}},\mathbb{A}^{\mathbf{q}})$ is in $\underline{\mathbf{Q}}^\gamma_3(\prec)$, for any $\gamma<\lambda$, then there is an extension in $\underline{\mathbf{Q}}^{\gamma+1}_3(\prec)$. Therefore, for each $\gamma\leq\lambda$, $\underline{\mathbf{Q}}^\gamma_3(\prec)$ is non-empty.*

Next we introduce the structures to preserve the value of $\mathfrak{b}$. Let us remind the reader that $q[f,g]$ is defined in Definition 3.11.

Definition 4.14. *A structure $\underline{\mathbf{q}}=(\underline{\mathbf{P}},\{\dot{A}_\alpha:\alpha<\gamma\})$ is in the set $\underline{\mathbf{Q}}^\gamma_4(\prec)$ providing $\underline{\mathbf{q}}\in\underline{\mathbf{Q}}^\gamma_2(\prec)$ and, for all $j<i<\kappa$, and $\kappa\leq\kappa\cdot\eta<\gamma$,*
(1) it is forced by $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta,j}$ that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta,j}$ is the poset $\mathcal{C}_{j+1\times 2^\omega}$, and
(2) if $\kappa<\kappa\cdot\eta+1<\gamma$, it is forced, by $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,j}$, that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,j}$ is in $\mathbb{Q}_{207}$,
(3) again if $\kappa<\kappa\cdot\eta+1<\gamma$ and i is a limit, then
  (a) $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$ forces that $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$ is in $\mathbb{Q}_{207}$,
  (b) $\dot{A}_{\kappa\cdot\eta+1}$ is the $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+2,i}$-name of the generic real added by $\dot{\mathbb{Q}}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta+1,i}$, and, it is forced that $\dot{A}_{\kappa\cdot\eta+1}$ is not split by any $\mathbb{P}^{\underline{\mathbf{P}}}_{\kappa\cdot\eta,i}$-name,

(c) $\mathbb{P}^{\mathbf{P}}_{\kappa\cdot\eta+1,i}$ *forces that, for each* $\mathbb{P}^{\mathbf{P}}_{\kappa\cdot\eta+2,j}$*-name* $\dot g\in\omega^\omega$ *and each* $q=(u_q,\{t^q_\ell:\ell\in\omega\})\in\dot{\mathbb{Q}}^{\mathbf{P}}_{\kappa+1,j}$*, the condition* $q[\dot g, f_{\dot A_i}]$ *is in* $\dot{\mathbb{Q}}^{\mathbf{P}}_{\kappa+1,i}$*.*

For convenience, we let $\underline{\mathbf{Q}}^\gamma_5(\prec)$ *be* $\underline{\mathbf{Q}}^\gamma_3(\prec)\cup\underline{\mathbf{Q}}^\gamma_4(\prec)$*.*

Lemma 4.15. *Let* $\underline{\mathbf{q}}$ *be a member of* $\underline{\mathbf{Q}}^\gamma_4(\prec)$*. Then for each* $j<i$ *and each* $i<\alpha\le\gamma$*, the poset* $\mathbb{P}_{\alpha,i+1}$ *forces that* $f_{\dot A_i}$ *is not dominated by any* $\mathbb{P}_{\alpha,j}$*-name* $\dot g$*.*

Proof. Since $\underline{\mathbf{q}}\in\underline{\mathbf{P}}^\gamma_2(\prec)$, the name $\dot A_i$ is added by the poset $\mathbb{P}_{i+1,i}$ and is the name $\dot A_{Q[\{\omega\}]}$. Moreover, by Lemma 2.5, we have that $\mathbb{P}_{i+1,j}*\mathbb{P}_{\gamma+1,j}/\mathbb{P}_{i+1,j}$ is completely embedded in $\mathbb{P}_{i+1,i}*\mathbb{P}_{\gamma+1,j}/\mathbb{P}_{i+1,j}$ for all $j<i$. It follows easily that $\mathbb{P}_{i+1,i}*\mathbb{P}_{\gamma+1,j}/\mathbb{P}_{i+1,j}$ forces that $f_{\dot A_i}$ is unbounded.

We proceed by induction on $\alpha\le\gamma$, so assume that $\dot g$ is a $\mathbb{P}_{\alpha,j}$-name and no $\mathbb{P}_{\beta,j}$-name dominates $f_{\dot A_i}$ for any $\beta<\alpha$. Let $p\in\mathbb{P}_{\alpha,i}$ be arbitrary, and it suffices to show that p does not force that $\dot f(n)<\dot g(n)$ for all n. If α is a limit, then choose $\beta<\alpha$ such that $p\in\mathbb{P}_{\beta,i}$. Define the $\mathbb{P}_{\beta,j}$-name, g^+ where $q\in\mathbb{P}_{\beta,i}$ forces that $g^+(n)=k$ if k is the minimal value for which there is an extension $p(q,n)\in\mathbb{P}_{\alpha,j}$ such that $p(q,n)\restriction\beta=q$ and $p(q,n)$ forces that $\dot g(n)=k$. By the inductive assumption, we may choose some $q\in\mathbb{P}_{\beta,j}$ and $p'<p$ in $\mathbb{P}_{\beta,i}$ such that $p'<q$, $p'\Vdash g^+(n)<\dot f(n)$ and $q\Vdash g^+(n)=k$. Since $p(q,n)\restriction\beta=q$ is compatible with p', we have that $p(q,n)$ is compatible with p', and this contradicts that p forces that $\dot f(n)<\dot g(n)$.

Therefore α must be a successor, say $\alpha=\beta+1$. Choose $\eta>0$ and $\bar\imath<\kappa$ such that $\beta=\kappa\cdot\eta+\bar\imath$. We consider the cases based on the value of $\bar\imath$. We know that $\bar\imath\le i$ since, otherwise, $\dot{\mathbb{Q}}_{\beta,j}$ is the trivial poset and $\dot g$ would actually be a $\mathbb{P}_{\beta,j}$-name. The most interesting case is when $\bar\imath=1$. In this case, we have, by the assumption on α, that $\mathbb{P}_{\beta,i}$ forces that $f_{\dot A_i}$ is unbounded. In Definition 4.14, we have $q[\dot g,f_{\dot A_i}]$ is in $\dot{\mathbb{Q}}_{\beta,i}$, where $q[\dot g,f_{\dot A_i}]$ is defined in Definition 3.11. By Corollary 3.13, we have $\mathbb{P}_{\beta+1,i}$ preserves that $f_{\dot A_i}$ remains unbounded. The case when $\bar\imath=0$ can be handled almost exactly as in the case when $1<\bar\imath<i$ and so we skip it.

Finally, suppose that $1<\bar\imath<i$, $p(\beta)\in\dot{\mathbb{Q}}_{\beta,j}$. In this case we have that $\dot{\mathbb{Q}}_{\beta,i}$ is equal to $\dot{\mathbb{Q}}_{\beta,j}$. We proceed similarly as in the case when α is a limit. We define a $\mathbb{P}_{\beta,j}$-name g^+. For each n, let $D_n\subset\mathbb{P}_{\alpha,j}$ be a dense open below p set of conditions that force a value on $\dot g(n)$. By extending any given $d\in D_n$ finitely many times, there are $d\in D_n$, such that there is a k such that $d\restriction\beta\Vdash\dot g(n)\ge k$ and there is some $d'\in D_n$ with $d\restriction\beta\le d'\restriction\beta$, $d'\Vdash\dot g(n)=k$. Then, we define the

$\mathbb{P}_{\beta,i}$-name g^+ so that for each such d, $d \restriction \beta$ forces that $g^+(n) = k$. Choose any $p' < p \restriction \beta$ in $\mathbb{P}_{\beta,i}$ and n such that $p' \Vdash g^+(n) < f_{\dot{A}_i}(n)$. We may suppose there is a $d \in D_n$ such that $p' < d \restriction \beta$ and $d \restriction \beta$ forces that $g^+(n) = k$. By simply changing our choice of $d \in D_n$, we may suppose also that d forces that $\dot{g}(n) = k$. This completes the proof of the Claim since we now have that $p' * d(\beta)$ is below p and forces that $\dot{g}(n) < f_{\dot{A}_i}(n)$. □

Again, in order to improve the flow of the main ideas, we will defer the proof of following lemma until later.

Lemma 4.16. *For each $\gamma \leq \lambda$, $\underline{\mathbf{Q}}_4^\gamma(\prec)$ is non-empty, and, for each $\underline{\mathfrak{q}}$ in $\underline{\mathbf{Q}}_4^\gamma(\prec)$, if $\gamma < \lambda$, there is an extension in $\underline{\mathbf{Q}}_4^{\gamma+1}(\prec)$.*

Proof. See Lemma 6.8. □

5. The preservation result for $\mathfrak{h}$

We introduce an iterable condition for $\underline{\mathfrak{q}} \in \underline{\mathbf{Q}}_5^\gamma(\prec)$ that will ensure no infinite set is a member of all of the $\mathcal{I}(\underline{\mathfrak{q}}, i)$'s.

Definition 5.1. *Suppose that $\underline{\mathfrak{q}} = (\underline{\mathbf{P}}, \{\dot{A}_\alpha : \alpha \in \gamma\})$ is in $\underline{\mathbf{Q}}_5^\gamma(\prec)$. For each $i \in \kappa$ and $p \in \mathbb{P}_{\gamma,i}$, say that p is determined if for each $\alpha \in \mathrm{dom}(p)$, and $j < \kappa$ such that $\alpha = j$ or $\kappa \leq \kappa \cdot \eta + j = \alpha$, $p \restriction \alpha$ forces:*

1. *if $j \leq i$, and $\alpha < \kappa$ or $1 < j$, then there is a finite $u = u(p, \alpha) \subset \omega$ so that $p(\alpha) = (u, \dot{Y})$ for some $\dot{Y} \in \dot{\mathcal{Y}}_\alpha^{\mathfrak{q}}$,*
2. *if $j = 0$, then there is an $r \in \mathcal{C}_{i+1\times 2^\omega}$ such that $p(\alpha) = r$,*
3. *if $j = 1$ and $\dot{\mathbb{Q}}_{\alpha,i}^{\mathfrak{q}} = \mathcal{H}$, then there is an $u = u(p, \alpha) \in \omega^{<\omega\uparrow}$, such that $p(\alpha) = (u, \dot{g}) \in \mathcal{H}$ for some $\mathbb{P}_{\alpha,i}^{\mathfrak{q}}$-name $\dot{g}$,*
4. *if $j = 1$ and $\dot{\mathbb{Q}}_{\alpha,i}^{\mathfrak{q}} \in \mathbb{Q}_{207}$ then there is a finite $u = u(p, \alpha) \subset \omega$ and an $t \in \mathcal{L}_1$ such that such that $u \subset \min int(t)$, $p(\alpha) = (u, \dot{T})$ for some $\mathbb{P}_{\alpha,i}^{\mathfrak{q}}$-name $\dot{T}$, and $p \restriction \alpha \Vdash t \in \dot{T}$.*

and for each determined condition p and $\alpha \in \mathrm{dom}(p)$, let $\pi_0(p(\alpha))$ denote the finite set $u(p, \alpha)$ as in the definition. In the case where α is of the form $\kappa \cdot \eta + 1 > \kappa$, we will let $\pi_1(p(\alpha))$ denote the indicated $\dot{Y}$, $\dot{g}$, or $\dot{T}$ as appropriate.

For the remainder of this section, let $\underline{\mathfrak{q}}$ denote any member of $\underline{\mathbf{Q}}_5^\gamma(\prec)$ for any $\gamma \leq \lambda$. We suppress using the superscript $\underline{\mathfrak{q}}$ and so, for $\alpha < \gamma$ and $i \leq \kappa$, we use $\mathbb{P}_{\alpha,i}, \dot{\mathbb{Q}}_{\alpha,i}$ rather than $\mathbb{P}_{\alpha,i}^{\mathfrak{q}}, \dot{\mathbb{Q}}_{\alpha,i}^{\mathfrak{q}}$. Also α will always be an ordinal less than or equal γ, and $1 < i$ will be an ordinal less than κ. Recall that we let $\Gamma_{\alpha,j}$ denote the set $\{\kappa \cdot \eta + j : \kappa \leq \kappa \cdot \eta + j < \gamma\}$. Recall also that for each $\xi < \gamma$, $\dot{A}_\xi^{\mathfrak{q}}$ is a $\mathbb{P}_{\xi+1,\kappa}$-name usually added by

$\dot{\mathbb{Q}}_{\xi,j}$ for some $j < \kappa$. In particular, for each $\xi \in \Gamma_{\alpha,i}$, $\dot{A}^{\mathfrak{a}}_{\xi}$ is determined by the values of $p(\xi)$ for $p \in \mathbb{P}_{\xi+1,i}$ (or in $\mathbb{P}_{\gamma,i}$).

It is sufficient and notationally easier to focus on successor levels.

Definition 5.2. *For determined conditions $p_0, p_1, p_2 \in \mathbb{P}_{\alpha,i+1}$, we use the notation $p_1\Delta_{\alpha,i}p_2 = p_0$ to denote the relation that p_1, p_2 are each extensions of p_0 and that for each $\xi \in \alpha \setminus \Gamma_{\alpha,i+1}$, $\pi_0(p_1(\xi)) = \pi_0(p_2(\xi))$ and, for any $\xi \in \mathrm{dom}(p_0) \cap \Gamma_{\alpha,i+1}$,*

$$\pi_0(p_1(\xi)) \cap \pi_0(p_2(\xi)) = \pi_0(p_0(\xi)).$$

Say that $p_1, \dots p_n$ is a p_0-fan (wrt $\mathbb{P}_{\alpha,i}$ if needed) if $p_j\Delta_{\alpha,i}p_k = p_0$, for any $1 \leq j < k \leq n$.

Here is the new condition.

Definition 5.3. *Say that $\mathbb{P}_{\alpha,i+1}$ is an i-pure extension of $\mathbb{P}_{\alpha,i}$ if for any dense $D \subset \mathbb{P}_{\alpha,i+1}$ and any p_0-fan, $p_1, \dots, p_n$, there are extensions $\bar{p}_j \leq p_j$ $(1 \leq j \leq n)$ in D so that $\bar{p}_1, \dots, \bar{p}_n$ is also a p_0-fan.*

Here is the reason.

Lemma 5.4. *If $\mathbb{P}_{\alpha,i+1}$ is an i-pure extension of $\mathbb{P}_{\alpha,i}$, then any $\mathbb{P}_{\alpha,i}$-name $\dot{Y}$ of a subset of ω that is forced by any $p \in \mathbb{P}_{\alpha,i+1}$ to be a member of $\mathcal{I}_{\alpha,i+1}$, is actually forced to be finite.*

In order to prove this, we first prove

Lemma 5.5. *Suppose that $\mathbb{P}_{\alpha,i+1}$ is an i-pure extension of $\mathbb{P}_{\alpha,i}$ and let p_1, p_2 be a p_0-fan (wrt $\mathbb{P}_{\alpha,i}$) in $\mathbb{P}_{\alpha,i+1}$. Then for any determined condition $d \in \mathbb{P}_{\alpha,i}$, if p_1 is below d, then there is a p_0-fan $\bar{p}_1, \bar{p}_2$ extending p_1, p_2 such that $\bar{p}_2$ is also below d.*

Proof. We prove this statement by induction on α. Let β be the maximum element of $\mathrm{dom}(p_1)$. If $\beta \in \Gamma_{\alpha,i_\beta}$ for some $i_\beta > i$, then $d(\beta) \in \dot{\mathbb{Q}}_{\beta,i}$ is simply the maximal element of $\dot{\mathbb{Q}}_{\beta,i+1}$. We can simply apply the induction hypothesis to $p_1 \restriction \beta, p_2 \restriction \beta$.

Similarly if β is equal to $\kappa \cdot \eta$ for some $\eta > 0$, then $p_1(\beta)$ is a member of the Cohen poset and is equal to $p_2(\beta)$. Simply choose $\bar{p}_1, \bar{p}_2$ a $p_0 \restriction \beta$-fan extending $p_1 \restriction \beta, p_2 \restriction \beta$ so that each are below $d \restriction \beta$. Then we have that $\bar{p}_1 \cup \{(\beta, p_1(\beta))\}, \bar{p}_2 \cup \{(\beta, p_2(\beta))\}$ is as required.

Now suppose that $1 < \beta < i$ or $\kappa + 1 < \beta = \kappa \cdot \eta + i_\beta$ and $i_\beta \leq i$. Then $d(\beta)$ has the form $(u_\beta, \dot{Y}_\beta)$ for some $\mathbb{P}_{\beta,i}$-name. We also have that $p_1{\restriction}\beta \Vdash w_\beta = \pi_0(p_1(\beta)) \setminus u_\beta \subset \dot{Y}_\beta$ and that $\pi_0(p_2(\beta)) = \pi_0(p_1(\beta))$. Let $D \subset \mathbb{P}_{\beta,i}$ be the dense set of determined conditions $\bar{d}$ which determine the value of $w_\beta \setminus \dot{Y}_\beta$. Suppose further that every member of D is either below $d \restriction \beta$ or is incompatible with $d \restriction \beta$. Choose any $p_0 \restriction \beta$-fan $\bar{p}_1, \bar{p}_2$ in $\mathbb{P}_{\beta,i+1}$ such that $\bar{p}_1$ is below some $\bar{d} \in D$. By the induction hypothesis, we may assume that $\bar{p}_2$ is also below $\bar{d}$. It is immediate that the pair $\bar{p}_1 \cup \{(\beta, p_1(\beta))\}, \bar{p}_2 \cup \{(\beta, p_2(\beta))\}$ is p_0-fan. It also now follows that $\bar{p}_2$ forces that $\bar{p}_2 \Vdash \pi_0(p_2(\beta)) \setminus d(\beta) \subset \dot{Y}_\beta$. The desired p_0-fan extension of p_1, p_2 is then $\bar{p}_1 \cup \{(\beta, p_1(\beta))\}, \bar{p}_2 \cup \{(\beta, (u_\beta \cup w_\beta, \dot{Y}_\beta \cap \pi_1(p_2(\beta)))\}$ since $\bar{p}_2$ forces that $(u_\beta \cup w_\beta, \dot{Y}_\beta \cap \pi_1(p_2(\beta)))$ is below $d(\beta)$.

If $\dot{\mathbb{Q}}_{\beta,i+1}$ is Hechler forcing, then let $d(\beta) = (s, \dot{g}) \in \dot{\mathbb{Q}}_{\beta,i}$. Also let $w_\beta = \pi_0(p_1(\beta))$. We again have that $p_1 \restriction \beta$ forces that $w_\beta(\ell) > \dot{g}(\ell)$ for each $\ell \in \mathrm{dom}(w_\beta) \setminus \mathrm{dom}(s)$ and again, by induction, there is a $p_0 \restriction \beta$-fan $\bar{p}_1, \bar{p}_2$ extending $p_1 \restriction \beta, p_2 \restriction \beta$ such that $\bar{p}_2$ also forces that $w_\beta(\ell) > \dot{g}(\ell)$ for each $\ell \in \mathrm{dom}(w_\beta) \setminus \mathrm{dom}(s)$. The definition of $\bar{p}_2(\beta)$ will again be $(w_\beta, \dot{g}')$ for some $\mathbb{P}_{\beta,i+1}$-name so that $\bar{p}_2 \restriction \beta$ forces that $\dot{g}'$ is above $\dot{g}$ and that $(w_\beta, \dot{g}') \leq p_j(\beta)$. The argument for the case that $\dot{\mathbb{Q}}_{\beta,i+1}$ is in $\mathbb{Q}_{207}$ is exactly similar, and will be left to the reader. □

Corollary 5.6. *Suppose that $\mathbb{P}_{\alpha,i+1}$ is an i-pure extension of $\mathbb{P}_{\alpha,i}$ and let p_1, p_2 be a p_0-fan (wrt $\mathbb{P}_{\alpha,i}$) in $\mathbb{P}_{\alpha,i+1}$. Then for any $\mathbb{P}_{\alpha,i}$-name $\dot{Y}$ and integer k, if p_1 forces that $k \in \dot{Y}$, then so does p_2.*

Proof of Lemma 5.4. Assume that there are $\eta_1, \ldots, \eta_{n-1} \in \Gamma_{\alpha,i}$ and $m \in \omega$ such that $p \Vdash \dot{Y} \setminus \left(\dot{A}_{\eta_1} \cup \cdots \cup \dot{A}_{\eta_{n-1}}\right) \subset m$. We may assume that $\{\eta_1, \ldots, \eta_{n-1}\} \subset \mathrm{dom}(p)$. Also, by increasing m we can assume that $\pi_0(p(\beta)) \subset m$ for all $\beta \in \mathrm{dom}(p) \cap \Gamma_{\alpha,i+1}$. Let D be the dense set of conditions d in $\mathbb{P}_{\alpha,i+1}$ for which there is a pair (d', v) where d' is a determined condition in $\mathbb{P}_{\alpha,i}$, $d \leq d'$, $v \in \omega \setminus m$ and $d' \Vdash v \in \dot{Y}$. Of course this means that d (and d') force that $v \in \dot{Y} \setminus m$. We may view $\{p_0, p_1, \ldots, p_n\}$ as a kind of trivial p_0-fan where $p_k = p_0$ for $1 \leq k \leq n$. Using that $\mathbb{P}_{\alpha,i+1}$ is an i-pure extension, choose a p_0-fan $\{p_0, \bar{p}_1, \ldots, \bar{p}_n\}$ so that each of the $\bar{p}_k$ are in D. Let v be the value such that $\bar{p}_1 \Vdash v \in \dot{Y}$. By assumption, we have (by a trivial re-indexing) that $\bar{p}_1 \Vdash v \in \dot{A}_{\eta_1}$. Of course this means that $v \in \pi_0(\bar{p}_1(\eta_1))$ and so $v \notin \pi_0(\bar{p}_k(\eta_1))$ for each $2 \leq k \leq n$. Similarly, we may rearrange the indexing so that for each k with $1 \leq k \leq n-1$, $v \in \pi_0(\bar{p}_k(\eta_k))$ and $v \notin \pi_0(\bar{p}_n(\eta_k))$. If necessary, we can extend $\bar{p}_n$ so as to ensure that $\bar{p}_n \Vdash v \notin \dot{A}_{\eta_1} \cup \cdots \cup \dot{A}_{\eta_n}$. But now, by Lemma 5.5, we may also assume that $\bar{p}_n$ forces that $v \in \dot{Y}$. This is our desired contradiction, since the pair $\bar{p}_n, v$ witness that p_0 did not force that $\dot{Y}$ is contained in $m \cup \dot{A}_{\eta_1} \cup \cdots \cup \dot{A}_{\eta_{n-1}}$. □

And now we record the main purpose of this notion.

Corollary 5.7. *If* $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_5^\lambda(\prec)$ *and, for each* $i < \kappa$, $\mathbb{P}^{\underline{\mathbf{q}}}_{\lambda,i+1}$ *is an* i*-pure extension of* $\mathbb{P}^{\underline{\mathbf{q}}}_{\lambda,i}$*, then the intersection of the family* $\{\mathcal{I}(\underline{\mathbf{q}}, i) : i < \kappa\}$ *of ideals, is just the ideal of finite sets.*

Next we prove that, except possibly for members of $\mathbb{Q}_{207}$, all the posets in our iteration sequences for $\underline{\mathbf{q}}$ will preserve the property of being i-pure. The members of $\mathbb{Q}_{207}$ have to be specially chosen to do so and this is done in Lemma 6.7.

Lemma 5.8. *Let* $0 < \alpha \in \gamma$ *and* $1 < i \in \kappa$*, and assume that for all* $\beta < \alpha$, $\mathbb{P}_{\beta,i+1}$ *is an* i*-pure extension of* $\mathbb{P}_{\beta,i}$*. If* α *is a limit, or if* $\alpha = \beta + 1$ *and* $\mathbb{P}_{\beta,i}$ *forces that* $\dot{\mathbb{Q}}_{\beta,i}$ *is not in* $\mathbb{Q}_{207}$*, then* $\mathbb{P}_{\alpha,i+1}$ *is an* i*-pure extension of* $\mathbb{P}_{\alpha,i}$*.*

Proof. We proceed by cases.

Case 1: α is a limit. Let $p_1, \ldots, p_n$ be a p_0-fan. Choose $\beta_0 < \alpha$ so that $\{p_0, \ldots, p_n\} \subset \mathbb{P}_{\beta_0,i+1}$. Let $D \subset \mathbb{P}_{\alpha,i+1}$ be a dense set of determined conditions, and set $D_{\beta_0} = \{d \restriction \beta_0 : d \in D\}$. Since D_{β_0} is a dense subset of $\mathbb{P}_{\beta_0,i+1}$, we can apply the induction hypotheses and thereby assume that $\{p_1, \ldots, p_n\}$ are in D_{β_0}. Choose any $d_n \in D$ so that $d_n \restriction \beta_0 = p_n$. Choose $\beta_1 < \alpha$ so that $d_n \in \mathbb{P}_{\beta_1,i+1}$. Let q denote the function $d_n \restriction [\beta_0, \beta_1)$. Now let us consider the p_0-fan $\{p_1 \cup q, p_2 \cup q, \ldots, p_{n-1} \cup q, d_n = p_n \cup q\}$. By induction on n, we can choose a $\mathbb{P}_{\alpha,i+1}$-fan $\{\bar{p}_1, \ldots, \bar{p}_{n-1}\} \subset D$ suitably extending the elements $\{p_1 \cup q, p_2 \cup q, \ldots, p_{n-1} \cup q\}$. We finish by setting $\bar{p}_n = d_n \cup (\bar{p}_1 \restriction [\beta_1, \alpha))$.

Case 2: $\alpha = \beta + 1$ for some $\beta \in \Gamma_{\alpha,i_\beta}$ with $1 < i_\beta \leq i$. In this case $\mathbb{P}_{\alpha,i+1} = \mathbb{P}_{\beta,i+1} * Q[\dot{\mathcal{Y}}_\beta]$ where $\dot{\mathcal{Y}}_\beta$ is a $\mathbb{P}_{\beta,i}$-name of a filter on ω. Let $p_1, \ldots, p_n$ be a p_0-fan and let D be a dense subset of $\mathbb{P}_{\beta,i+1} * Q[\dot{\mathcal{Y}}_\beta]$. This case is similar to the previous case but critically relies on the fact that $\mathcal{Y}_\beta$ is a $\mathbb{P}_{\beta,i}$-name rather than a $\mathbb{P}_{\beta,i+1}$-name. We again assume that the elements of D are determined conditions. Let us note that $\pi_0(p_j(\beta)) = \pi_0(p_k(\beta))$ for $1 \leq j \leq k \leq n$. If needed, we can extend each $\pi_1(p_k(\beta))$ $(1 \leq k \leq n)$ so that there is a $\mathbb{P}_{\beta,i}$-name $\dot{Y}_n$ that is forced by each p_k to equal $\pi_1(p_k(\beta))$. As in the previous case, there is a $p_0 \restriction \beta$-fan $\{\bar{p}_1, \ldots, \bar{p}_n\}$ suitably extending $\{p_1 \restriction \beta, \ldots, p_n \restriction \beta\}$ so that there is a $d \in D$ with $\bar{p}_n = d \restriction \beta$ and $d \restriction \beta \Vdash \pi_0(d(\beta)) \setminus \pi_0(p_n(\beta)) \subset \dot{Y}_n$. Fix any $\ell \in \pi_0(d(\beta)) \setminus \pi_0(p_n(\beta))$. By Lemma 5.5, we have that $\bar{p}_k \Vdash \ell \in \dot{Y}_1$ for each $1 \leq k \leq n$. Recall that $\pi_0(p_k(\beta)) = \pi_0(p_n(\beta))$ for each $1 \leq k \leq n$. It then follows that $\bar{p}_k \cup \{(\beta, d(\beta))\} \leq p_k$. The p_0-fan $\{\bar{p}_1 \cup \{(\beta, d(\beta))\}, \bar{p}_2 \cup \{(\beta, d(\beta))\}, \ldots, \bar{p}_n \cup \{(\beta, d(\beta))\}\}$ satisfies that the final entry is in D. By induction on n, we extend to the required p_0-fan of members of D.

Case 3: $\alpha = \beta + 1$ for some $\beta \in \Gamma_{\alpha,i_\beta}$ with $1 < i_\beta = i + 1$. In this case $\mathbb{P}_{\alpha,i+1} = \mathbb{P}_{\beta,i+1} * Q[\dot{\mathcal{Y}}_\beta]$ where $\dot{\mathcal{Y}}_\beta$ is a $\mathbb{P}_{\beta,i+1}$-name of a filter on ω. Let $p_1, \ldots, p_n$ be a p_0-fan and let D be a dense subset of (determined conditions of) $\mathbb{P}_{\beta,i+1} * Q[\dot{\mathcal{Y}}_\beta]$. The key step here is to keep things disjoint in the final coordinate. Choose any m_1 large enough so that $\pi_0(p_k(\beta)) \subset k_1$ for $1 < k \leq n$. By strengthening each $\pi_1(p_k(\beta))$ we can assume that $p_k \restriction \beta \Vdash \pi_1(p_k(\beta)) \cap m_1$ is empty for each $1 < k \leq n$. Proceeding as in Case 1, we can choose a $p_0 \restriction \beta$-fan $\{\bar{p}_1, \ldots, \bar{p}_n\} \subset \mathbb{P}_{\beta,i+1}$ so that there is some $d_1 \in D$ with $d_1 \leq p_1$ and $\bar{p}_1 \leq d_1 \restriction \beta$. Next, choose m_2 large enough so that $\pi_0(d_1(\beta)) \subset m_2$ and similarly strengthen $\pi_1(p_k(\beta))$ for $1 < k \leq n$ so that $\bar{p}_k \Vdash \pi_1(p_k(\beta))$ is disjoint from m_2. Now, by extending, we may assume that $\{\bar{p}_1, \ldots, \bar{p}_n\}$ is a $p_0 \restriction \beta$-fan such that there is a $d_2 \in D$ with $\bar{p}_2 < d_2 \restriction \beta$ and $\bar{p}_2 \Vdash d_2(\beta) < p_2(\beta)$. Let use note that $\pi_0(d_1(\beta)) \cap \pi_0(d_2(\beta)) = \pi_0(p_1(\beta)) \cap \pi_0(p_2(\beta)) = \pi_0(p_0(\beta))$. Continuing this recursive construction for n steps we obtain the p_0-fan

$$\{\bar{p}_1 \cup \{(\beta, d_1(\beta))\}, \bar{p}_2 \cup \{(\beta, d_2(\beta))\}, \ldots, \bar{p}_n \cup \{(\beta, d_n(\beta))\}\}$$

as required.

Case 4: $\alpha = \beta + 1$ for some successor β with $i_\beta > i$. This case is vacuous since $\mathbb{P}_{\beta,i+1}$ forces that $\dot{\mathbb{Q}}_{\beta,i+1}$ is the trivial poset.

Case 5: $\alpha = \beta + 1$ where β is a limit with cofinality not equal to κ. In this case, $\mathbb{P}_{\beta,i+1}$ forces that $\dot{\mathbb{Q}}_{\beta,i+1}$ is the Hechler poset. Let $p_1, \ldots, p_n$ be a p_0-fan and let D be a dense set of determined conditions from $\mathbb{P}_{\alpha,i+1}$. We may assume (arrange) that $p_j(\beta) = (s_0, \dot{g}_0)$ for each $0 \leq j \leq n$. Let σ denote the $\mathbb{P}_{\beta,i+1}$-name of the subset of $\omega^{<\omega}$ where $p \Vdash s \in \sigma$ providing there is a $d \in D$ such that $p < d \restriction \beta$ and $p \Vdash (s, \dot{h}) < d(\beta)$ for some $\mathbb{P}_{\beta,i+1}$-name $\dot{h}$. There is also a $\mathbb{P}_{\beta,i+1}$-name for the rank function $\mathrm{rk}_\sigma : \omega^{<\omega} \to \omega_1$. Given any increasing $s \in \omega^{<\omega}$, we will say that p forces that an integer x_0 witnesses the value of $\mathrm{rk}_\sigma(s)$ providing that for each n there is an increasing function $s_n \in \omega^{x_0}$ so that $s_n(|s|) > n$ and $\mathrm{rk}_\sigma(s_n) < \mathrm{rk}_\sigma(s)$.

By induction, we may assume that $p_j \restriction \beta$ forces a value on each of $\mathrm{rk}_\sigma(s_0)$ and $\dot{x}_0$ (a witness to $\mathrm{rk}_\sigma(s_0)$) for each $1 \leq j \leq n$. We can thus, further assume that p_j also forces a value on $\dot{g}_0 \restriction \dot{x}_0$ for each $1 \leq j \leq n$. Fix a value m_0 large enough so that p_j forces that $\dot{g}_0 \restriction \dot{x}_0 \in (m_0)^{\dot{x}_0}$ for each $1 \leq j \leq n$. Let D_0 denote the dense set of $d \in \mathbb{P}_{\beta,i+1}$ which force a value on $\dot{s}_1$ witnessing the value of $\mathrm{rk}_\sigma(s_0)$ in the sense that d forces that $\dot{s}_1(|s_0|) > m_0$ and $(\dot{s}_1, \dot{h}) < (s_0, \dot{g}_0)$. Again by induction, we can assume that each $p_j \restriction \beta$ is in D_0 and we let s_1 be the value forced on $\dot{s}_1$ by p_1. We can repeat this step finitely many times, rather we assume that by induction on the value forced on $\mathrm{rk}_\sigma(s_0)$ by p_1, we simply assume that we have that for some $\dot{g}_1$ $(1 \Vdash \dot{g}_0 \leq \dot{g}_1)$, $p_1 \restriction \beta \cup \{\langle \beta, (s_1, \dot{g}_1)\rangle\} \in D$. We continue this same process but next focus on the values that p_2 forces on $\mathrm{rk}_\sigma(s_1)$ and the $\mathbb{P}_{\beta,i+1}$-name $\dot{x}_1$ witnessing this rank. In the next step, there is an s_2 extending s_1, a $\mathbb{P}_{\beta,i+1}$-name $\dot{g}_2$, and a $p_0 \restriction \beta$-fan $\{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n\}$ satisfying that $\bar{p}_j \cup \{\langle \beta, (s_2, \dot{g}_2)\rangle\} < p_j$ for $1 \leq j \leq n$ and such that each of $\bar{p}_1 \cup \{\langle \beta, (s_2, \dot{g}_2)\rangle\}$ and $\bar{p}_2 \cup \{\langle \beta, (s_2, \dot{g}_2)\rangle\}$ are in D. Continuing this induction for n steps completes the proof in this case.

This completes the proof of the Lemma □

6. Building members of $\mathcal{Q}_{Bould}$ for pure extensions

A family $P \subset [\omega]^{<\aleph_0}$ naturally induces a logarithmic measure.

Definition 6.1. *Let $P \subset [\omega]^{<\aleph_0}$ and define the relation $h(s) \geq \ell$ for $s \in [\omega]^{<\aleph_0}$ by induction on $|s|$ and ℓ as follows:*

(1) $h(e) \geq 0$ for all $e \in [\omega]^{<\aleph_0}$,
(2) $h(e) > 0$ if e contains some non-empty element of P,
(3) for $\ell > 0$, $h(e) \geq \ell + 1$ if and only if, $|e| > 1$ and whenever $e_1, e_2 \subset e$ are such that $e = e_1 \cup e_2$ then $h(e_1) \geq \ell$ or $h(e_2) \geq \ell$.

The definition of $h(e)$ is the maximum ℓ such that $h(e) \geq \ell$.

Here is the main tool for constructing members of $\mathcal{L}_n$ for arbitrarily large n.

Lemma 6.2 ([1, Lemma 4.7])**.** *Let $P \subset [\omega]^{<\aleph_0}$ be an upward closed family of non-empty sets and let h be the associated logarithmic measure. Assume that whenever ω is partitioned into finitely many sets $\mathcal{A}$, there is some $A \in \mathcal{A}$ such that $P \cap [A]^{<\aleph_0}$ is non-empty. Then, for any finite partition $\mathcal{A}$ of ω, and any integer n, there is an $A \in \mathcal{A}$ and an $e \subset A$ such that $h(e) \geq n$.*

Definition 6.3. *Let $\tau = \langle (\emptyset, T_n) : n \in \omega\rangle$ be any descending sequence of pure conditions. Define the poset $\mathbb{P}_\tau$ to be the set of finite tuples $\bar{r} = \langle r_i : i < m\rangle$ of members of $\mathcal{L}_1$ such that there is an extension $R_{\bar{r}} = \langle r_i : i \in \omega\rangle$ so that $(\emptyset, R_{\bar{r}})$ is a pure condition and, for each $i < m$, $(int(r_i), R_{\bar{r}}) < (\emptyset, T_{\max int(r_{i-1})})$. The ordering on $\mathbb{P}_\tau$ is simply end extension.*

The next result is based on the result [8, Corollary 3.8].

Lemma 6.4. *Suppose that $Q_0 \subset \mathcal{Q}_{Bould}$ is closed under finite changes and satisfies that every finite subset of pure conditions has a lower bound in $\mathcal{Q}_{Bould}$. Let $\tau = \langle (\emptyset, T_n) : n \in \omega\rangle$ be any descending sequence of pure conditions from Q_0. In the forcing extension by $\mathbb{P}_\tau$, there is an* *extension $Q_1 \subset \mathcal{Q}_{Bould}$ of Q_0 which is directed mod finite and satisfies that for every ground model set $A \subset \omega$, there is a $q \in Q_1$ such that $int(q) \subset A$ or $int(q) \cap A = \emptyset$.*

Proof. Let Q_0 and τ be as in the Lemma, and let $\mathcal{U}$ be a maximal directed family of infinite subsets of ω satisfying that for all $U \in \mathcal{U}$ and finitely many $\{q_1, \ldots, q_n\} \subset Q_0$, there is a $q \in \mathcal{Q}_{Bould}$ satisfying that $int(q) \subset U$ and $q < q_i$ for each $1 \leq i \leq n$. The fact that there is such a maximal $\mathcal{U}$ is an easy consequence of Zorn's Lemma, and, it follows from Proposition 3.1, that $\mathcal{U}$ is an ultrafilter over $\mathcal{P}(\omega)$. Now it also easy to see that for each $\vec{q} = \{q_1, \ldots, q_n\} \subset Q_0$ and $U \in \mathcal{U}$, the set $D(\vec{q}, U) \subset \mathbb{P}_\tau$ is dense where $\langle r_k : k < \ell\rangle$ is in $D(\vec{q}, U)$ providing that for some $n \leq k < \ell$, $int(r_k) \subset U$ and r_k is built from q_i for each $1 \leq i \leq n$. Let $\{\langle r_i : i < m\rangle : m \in \omega\}$ be a generic filter for $\mathbb{P}_\tau$, and set $r_\tau = \langle r_i : i \in \omega\rangle$. In the forcing extension by $\mathbb{P}_\tau$, let $L(\vec{q}, U) = \{\ell \in \omega : int(r_\ell) \subset U$ and r_ℓ is built from $q_i (1 \leq i \leq n)\}$. Let $Q_1 = Q_0 \cup \{(u, \{r_\ell : \ell \in L(\vec{q}, U)\}) : u \in [\omega]^{<\aleph_0}, \vec{q} \in [Q_0]^{<\aleph_0}, U \in \mathcal{U}\}$. We omit the routine verification that Q_1 is directed mod finite. □

From earlier in the paper we need to prove:

Theorem 6.5. *If $Q \subset \mathcal{Q}_{Bould}$ is closed under finite changes and if every finite set of pure conditions has a lower bound in $\mathcal{Q}_{Bould}$, then in the forcing extension by $\mathcal{C}_{2^\omega}$, Q has an extension Q_1 in $\mathbb{Q}_{207}$.*

It follows from a length 2^ω recursion applying the following Lemma.

Lemma 6.6. *If $Q \subset \mathcal{Q}_{Bould}$ is directed mod finite and $\{(u_n, T_n) : n \in \omega\} \subset Q$ is a predense subset of Q then in the forcing extension by $\mathcal{C}_\omega$ there is a directed mod finite $Q_1 \supset Q$ and a condition $(\emptyset, T) = (\emptyset, \langle t_\ell : \ell \in \omega\rangle) \in Q_1$ such that either $\{(u_n, T_n) : n \in \omega\}$ is not predense in Q_1 or T is the mod finite meet of the family $\{(u_n, T_n) : n \in \omega\} \subset Q$.*

Proof. Let $P \subset [\omega]^{<\aleph_0}$ be the set of $w \in [\omega]^{<\aleph_0}$ such that there is an $n \in \omega$ such that $u_n \subset w$. For each $w \in [\omega]^{<\aleph_0}$, let $P_w = \{w_1 \in [\omega]^{<\aleph_0} : w \cup (w_1 \setminus \max(w)) \in P\}$.

Claim 1. Let $\{A_1, \ldots, A_n\}$ be subsets of ω. Either there is an i, $1 \leq i \leq k$ such that for all $q \in Q$, there is no bound on the values of h_q on the set $[A_i \cap int(q)]^{<\aleph_0}$, or there is a $q \in Q$ such that values of h_q are bounded on the set $[(A_1 \cup \cdots \cup A_n) \cap int(q)]^{<\aleph_0}$.

Proof of Claim. Assume that for each $1 \leq i \leq n$ there a $q_i \in Q$ so that there is a bound, L_i, on the values of the members of $[A_i \cap int(q_i)]^{<\aleph_0}$. Let L be larger than $\Sigma_{i=1}^k L_i$ and choose any q which is below each of the

q_i. If $L+n$ is not a bound of h_q on the set $[(A_1 \cup \cdots \cup A_n) \cap int(q)]^{<\aleph_0}$, then choose any $t \in T_q$ so that $h_t((A_1 \cup \cdots \cup A_n) \cap int(t)) > L + n$. Recursively choose a descending sequence $int(t) = e_0 \supset e_1 \supset \cdots \supset e_n$ so that $h_t(e_i) > L + n - i$ and $e_{i+1} \in \{e_i \cap A_i, e_i \setminus A_i\}$. Since h_t is a logarithmic measure, this is possible. Since $h_t(e_n) > L$, there must be an i such that $e_n \cap A_i$ is non-empty. Of course this means that $e_n \subset A_i$ and contradicts that $L_i \leq L$ was the bound on such measures. □

Now we break into two cases. In the first case, assume there is an $A_1 \subset \omega$ so that for each $q \in Q$, there is no bound of h_q on $[int(q) \cap A_1]^{<\aleph_0}$ and yet there is a $w \in [\omega]^{<\aleph_0}$ and a $q_w = (w, T_{q_w}) \in Q$ such that there is a bound L on the values of h_{q_w} on the set $P_w \cap [int(T_{q_w}) \cap A_1]^{<\aleph_0}$.

Let $T_{q_w} = \langle t^w_\ell : \ell \in \omega \rangle$ and for each ℓ, $t^w_\ell = (s^w_\ell, h^w_\ell)$. Fix ℓ_w large enough so that for all $\ell > \ell_w$ and $e \subset s^w_\ell \cap A_1$, the value of $h^w_\ell(e) < L < h^w_\ell(s^w_\ell)$. Define the new sequence $T = \langle t_\ell : \ell \in \omega \rangle$ so that for all ℓ, $t_\ell = (s_\ell, h_\ell)$ where $s_\ell = s^w_{\ell+\ell_w} \cap A_1$ and for $e \subset s_\ell$, $h_\ell(e) = h^w_{\ell+\ell_w}(e) - L$. It is routine to show that any finitely many pure conditions from the family $Q \cup \{(\emptyset, T)\}$ have a lower bound in $\mathcal{Q}_{Bould}$. Let τ be the ω-sequence consisting of $(\emptyset, T)$ repeated in each coordinate. In the forcing extension by $\mathcal{C}_\omega$ there is a generic sequence r_τ for the poset $\mathbb{P}_\tau$ from Definition 6.3. Thus, by Lemma 6.4, we have an extension Q_1 of $Q \cup \{r_\tau\}$ that is directed mod finite. To finish the proof in this case, we show that $\{(u_n, T_n) : n \in \omega\}$ is not predense in Q_1. Fix any n and we show, by contradiction, that (w, T) is not compatible with (u_n, T_n) in $\mathcal{Q}_{Bould}$. Assume that $(w \cup w_1, T') \in \mathcal{Q}_{Bould}$ is below (w, T) and (u_n, T_n). By the definition of extension, $w \cup w_1$ is an end-extension of both w and u_n. In other words, $w_1 \subset \omega \setminus \max w$ and either $u_n \subset w$ or $w \subset u_n \subset w \cup w_1$. Choose any $\ell \in \omega$ so that there is an $e \subset s_\ell$ so that $h_\ell(e) > 0$ and $e \subset int((w \cup w_1, T'))$. Since $(w \cup w_1, T') < (u_n, T_n)$, we have that $e \subset int((u_n, T_n))$. This implies that $(w \cup w_1 \cup e, T') < (u_n, T_n)$. In other words, $w_1 \cup e \in P_w$. By definition of h_ℓ, we have that $h_q(w_1 \cup e) \geq h_q(e) > L$. Finally, $w_1 \cup e \subset int((w, T)) \subset A_1$, which provides our desired contradiction.

In the other case we have an A_1 such that there is no bound of h_q on the set $[int(T_q) \cap A_1]^{<\aleph_0}$, and for every $A_2 \subset A_1$, if for each $q \in Q$ there is no bound of h_q on the set $[int(T_q) \cap A_2]^{<\aleph_0}$, then also, for each $q \in Q$ and each $w \in [\omega]^{<\aleph_0}$, there is no bound of h_q on the set $P_w \cap [int(T_q) \cap A_2]^{<\aleph_0}$. We define a new poset similar to the $\mathbb{P}_\tau$ above. Choose any descending sequence $\{(\emptyset, R_n) : n \in \omega\}$ contained in Q so that, for each n, $(u_n, R_n) < (u_n, T_n)$.

Claim 2. For each m and each $q \in Q$, there is an $(s,h) \in \mathcal{L}_m$, such that $s \subset A_1$, (s,h) is built from $(u_q \cup \{m\}, T_q)$, and for each h-positive $e \subset s$, $e \in P_w$ for all $w \subset m$.

Proof of Claim. Let $m \in \omega$ and $q \in Q$. Define the family $P_{q,m} \subset [\omega]^{<\aleph_0}$ to be all sets e such that e has positive q-measure and $e \in P_w$ for each $w \subset m$. Let $h_{q,m}$ be the logarthmic measure associated with $P_{q,m}$ as in Definition 6.1. We need to get $s \in P_{q,m}$ to have $h_{q,m}(s) > m$.

By Lemma 6.2, it suffices to show that if $\{A_2, \cdots, A_n\}$ is any partition of A_1, there is an $e \in P_{q,m}$ which is contained in one of the A_j $(1 < j \leq n)$.

By assumption, there is a $2 \leq j \leq n$ such that there is no bound on the q-measure of sets from $[A_j]^{<\aleph_0}$. By the working assumption in this case, there is also no bound on the q-measures of $P_w \cap [A_j]^{<\aleph_0}$ for each $w \subset m$. For each $w \subset m$, choose $e_w \in P_w \cap [A_j]^{<\aleph_0}$ such that the q-measure of e_w is positive. Let $e = \bigcup\{e_w : w \subset m\}$. By definition of the q-measure, we certainly have that e has positive q-measure. Evidently, $e \in P_w$ for each $w \subset m$. By Lemma 6.2, there is an s with $h(s) > m$. □

Now we define our poset $\mathbb{P}_{\tau,2}$ where τ is the sequence $\{(\emptyset, R_n) : n \in \omega\}$. Elements of $\mathbb{P}_{\tau,2}$ are those $\langle r_i : i < n\rangle \in \mathbb{P}_\tau$ that also satisfy that, for each i and each r_{i+1}-positive set e, e is a member of P_w for each $w \subset \max(int(r_i))$; moreover, there is an $n < \max int(r_i)$ witnessing that $e \in P_w$ in that there is a $w_e \subset e$ such that $(w \cup w_e, T_n) < (u_n, T_n)$.

By definition, if $\langle r_0\rangle \in \mathbb{P}_\tau$, then $\langle r_0\rangle \in \mathbb{P}_{\tau,2}$. Moreover, if $\langle r_i : i < n\rangle \in \mathbb{P}_{\tau,2}$, then by applying Claim 2 to the pair $m = \max(int(r_{n-1}))$ and $q = (\emptyset, R_m)$, to obtain (s,h) as in Claim 2, and then choose $\bar{m}$ sufficiently large, we let $r = (s \cup \{\bar{m}\}, h)$ to show that every element of $\mathbb{P}_{\tau,2}$ has a proper extension.

Again let $r_\tau = \langle r_i : i \in \omega\rangle$ be a generic sequence for $\mathbb{P}_{\tau,2}$. We check that r_τ is a mod finite meet of the sequence (u_n, T_n). Let $\{j_i : i \in \omega\}$ be the strictly increasing sequence where $j_i = \max(int(r_i))$ for each $i \in \omega$. Notice then that, for each n, $\langle r_i : i < n\rangle$ is in $\mathbb{P}_\tau$, and, for each $i \geq n$, r_i is built from $(\{j_i\}, R_{j_i})$. It follows that $(u_n, r_\tau) < (u_n, T_n)$ in $\mathcal{Q}_{Bould}$. Now consider any $w \in [\omega]^{<\aleph_0}$ and choose ℓ_w such that $w \subset \max(int(r_{\ell_w}))$. Now, for each $\ell > \ell_w$ and each r_ℓ-positive $e \subset int(r_\ell)$, $e \in P_w$, and there is an $n < j_\ell$ and a $w_e \subset e$ such that $(w \cup w_e, T_n) < (u_n, T_n)$. Since $n < j_\ell$ we have that $(w \cup w_e, r_\tau) < (w \cup w_e, T_n)$. This completes the verification that r_τ is the mod finite meet of the family $\{(u_n, T_n) : n \in \omega\}$.

The same density argument, using Claim 2 as in Lemma 6.4, shows that there is $Q_1 \subset Q_{Bould}$ which contains $Q \cup \{(\emptyset, r_\tau)\}$ and is directed mod finite. □

Now we return to discussing a member $\underline{\mathbf{q}}$ of $\underline{\mathbf{Q}}_5^\gamma(\prec)$ and prove a stronger result than Lemma 4.13 because we will be preserving i-pure for some $i < \kappa$. Most of the proof involves a generalization of Lemma 6.6 to incorporate the extra requirement demanded of i-pure extensions.

Lemma 6.7. *Suppose that* $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_3^{\gamma+1}(\prec)$ *where* $\gamma = \kappa \cdot \eta$ *for some limit* $0 < \eta < \lambda$. *Note that any extension of* $\underline{\mathbf{q}}$ *in* $\underline{\mathbf{Q}}_3^{\gamma+2}(\prec)$ *must be by members of* $\mathbb{Q}_{207}$. *If we have that* $\mathbb{P}_{\gamma,i+1}$ *is an* i*-pure extension of* $\mathbb{P}_{\gamma,i}$ *for* $i < \kappa$, *then there is such an extension of* $\underline{\mathbf{q}}$ *in* $\underline{\mathbf{Q}}_3^{\gamma+2}(\prec)$.

Proof. We proceed to construct $\{\dot{Q}_i : i < \kappa\}$ by recursion on $i < \kappa$ satisfying that

(1) $\dot{Q}_i$ is a $\mathbb{P}_{\gamma+1,i}$-name of a member of $\mathbb{Q}_{207}$,
(2) for $j < i$, $\dot{Q}_i$ is forced to contain $\dot{Q}_j$,
(3) for each $j < i$, $\mathbb{P}_{\gamma+1,j+1} * \dot{Q}_{j+1}$ is a j-pure extension of $\mathbb{P}_{\gamma+1,j} * \dot{Q}_j$,
(4) for each limit $j < i$ and each $\mathbb{P}_{\gamma,j}$-name of a subset of ω, $\dot{A}$, there is a $q \in \dot{Q}_j$ such that $int(q)$ is forced to be contained in $\dot{A}$ or in $\omega \setminus \dot{A}$.

If i is a limit, then we work in the extension by $\mathbb{P}_{\gamma,i} * \mathcal{C}_{i\times 2^\omega}$. We have the poset $\mathcal{C}_{\{i\}\times 2^\omega}$ at our disposal. We apply Lemma 6.4 to the poset $\bigcup\{\dot{Q}_j : j < i\}$ to obtain a poset Q_1 as in Lemma 6.4 and then apply Lemma 6.5, to choose the necessary extension $\dot{Q}_i$ in $\mathbb{Q}_{207}$.

Now suppose we have selected the sequence $\{\dot{Q}_j : j \leq i\}$ and we now prove we can find $\dot{Q}_i$. We are seeking a $\mathbb{P}_{\gamma,i+1} * \mathcal{C}_{i+2\times 2^\omega}$-name for $\dot{Q}_{i+1}$ which must contain the $\mathbb{P}_{\gamma,i} \times \mathcal{C}_{i+1\times 2^\omega}$-name $\dot{Q}_i$. The main task is to ensure that $\mathbb{P}_{\gamma+2,i+1}$ is an i-pure extension and this requires that we generically extend fans as in Definition 5.3. The trick will be to maintain a stronger hypothesis.

For any $\rho \leq 2^\omega$, let $\bar{\mathcal{C}}_\rho$ denote the poset $\mathcal{C}_{(i+1\times 2^\omega)\cup(\{i+1\}\times\rho)}$. By induction on $\rho < 2^\omega$, we will define a $\bar{\mathcal{C}}_\rho$-name of a poset $\dot{Q}_{i,\rho}$. We start with $\dot{Q}_{i,0} = \dot{Q}_i$. We will then let $\dot{Q}_{i+1}$ equal $\dot{Q}_{i,2^\omega}$. The purpose of $\dot{Q}_{i,\rho+\omega}$ will be to introduce a mod finite meet of a $\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$-name of a predense set $\{(\dot{u}_n, \dot{T}_n) : n \in \omega\} \subset \dot{Q}_{i,\rho}$. For each $m \in \omega$, $\dot{Q}_{i,\rho+m}$ will simply equal $\dot{Q}_{i,\rho}$.

Note that each $p \in \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}$ can be regarded as a member of $\mathbb{P}_{\gamma+2,i+1}$, and so we will continue with the notational conventions concerning $\pi_0(p(\beta))$ and $\pi_1(p(\beta))$ for $\beta \leq \gamma+1$. We will always assume that $p\restriction\gamma+1$ is determined.

With this notation, even though we have only partially defined $\dot{\mathbb{Q}}_{\gamma,i+1}$, it is still clear what it means when we say that $\{p_0, p_1, \cdots, p_n\} \subset \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}$ is a fan with respect to $\mathbb{P}_{\gamma+1,i}$, i.e. $\{p_1\restriction\gamma+1, \cdots, p_n\restriction\gamma+1\}$ is a $p_0\restriction\gamma+1$-fan, and, for each $1 \leq j \leq n$, $\pi_0(p_j(\gamma+1))$ is equal to $\pi_0(p_j(\gamma+1))$.

We require that the following inductive condition, IH(ρ), is maintained. If $\{p_0^k, p_1^k, \cdots, p_{n_k}^k : k < \ell\}$ is a sequence of fans (from $\mathbb{P}_{\gamma+1,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}$), and if $p \in \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$, then for each integer m there is a $\bar{p} < p$ and an $r \in \mathcal{L}_m$, such that for each $k < \ell$ for which there is a $1 \leq j_k \leq n_k$ with $p < p_{j_k}^k\restriction_{\gamma+1}$, there is a p_0^k-fan $\{\bar{p}_1^k, \ldots, \bar{p}_{n_k}^k\}$ (from $\mathbb{P}_{\gamma+1,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}$) extending $\{p_1^k, \ldots, p_{n_k}^k\}$ such that $\bar{p}$ extends a member of $\{\bar{p}_1^k\restriction_{\gamma+1}, \ldots, \bar{p}_{n_k}^k\restriction_{\gamma+1}\}$ and for all $1 \leq j \leq n_k$, $\bar{p}_j^k\restriction_{\gamma+1}$ forces that r is built from $\pi_1(p_j^k(\gamma+1))$.

We leave it as a trivial exercise that IH(0) holds. Also, if IH(ξ) holds for all $\xi < \rho$, then IH(ρ) holds where we let $\dot{Q}_{i,\rho}$ be defined as $\bigcup\{\dot{Q}_{i,\xi} : \xi < \rho\}$. In particular, if IH($\rho$) holds, then so does IH($\rho+\omega$) if we set $\dot{Q}_{i,\rho+\omega}$ to be equal to $\dot{Q}_{i,\rho}$.

Now we examine our predense set $\{(\dot{u}_n, \dot{T}_n) : n \in \omega\}$ to see if we can introduce an element which is a mod finite meet, or if we must add an element so that it is no longer predense. In fact, we may assume that there is a predense set $D = D_\rho \subset \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}$ such that, if $G \subset \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$ is any generic filter, then $A_D = \{(u,T) : (\exists d \in D)\ d\restriction_{\gamma+1} \in G \ \text{ and } \ (u,T) = (\pi_0(d(\gamma+1)), \pi_1(d(\gamma+1)))\}$ is our predense set $\{(u_n, T_n) : n \in \omega\}$ under consideration. For each generic filter G for $\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$, we will split into two cases based on whether or not we can preserve the pre-density of A_D while maintaining the induction hypothesis IH($\rho+\omega$). In particular, when we can not, we will call this Case 1.

Case 1: there is a finite set $\mathcal{P} \subset [\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}]^{<\aleph_0}$ fans, and a pair $w \in [\omega]^{<\aleph_0}$ and $m \in \omega$, such that there is **no** expansion set $\bar{\mathcal{P}}$ of fans for which there is an $r_w \in \mathcal{L}_m$ such that

(1) for each $\{p_0, p_1, \ldots, p_n\} \in \mathcal{P}$ with $p_1\restriction\gamma+1 \in G$, there is a tuple $\{p_0, \bar{p}_1, \ldots, \bar{p}_n\}$ in $\bar{\mathcal{P}}$ with $\bar{p}_1\restriction\gamma+1 \in G$ and for each $1 \leq j \leq n$, $\bar{p}_j \leq p_j$ and $\bar{p}_j\restriction_{\gamma+1}$ forces that r_w is built from $\pi_1(p_j(\gamma+1))$,
(2) for each r_w-positive $e \subset int(r_w)$, there is a $d \in D$ and a $w_e \subset e$ such that $\bar{p}_1\restriction\gamma+1 < d \restriction \gamma+1$ and forces that $(w \cup w_e, \pi_1(d(\gamma+1))$ is an extension of $d(\gamma+1)$ in $\dot{Q}_{i,\rho}$.

Let D be a member of a countable elementary submodel $M \prec H(\theta)$ and recall that for a generic filter G, we have that $M[G]$ is an elementary submodel of $H(\theta)$ in $V[G]$. Therefore, if there is a set of fans $\mathcal{P}$ as in Case 1, then there is such a set in M. There is a maximal antichain $D_1 \subset \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$ (in M) satisfying that each $d \in D_1$ decides if Case 1 holds or if it fails. Let $D_{1,1}$ denote the set of $d \in D_1$ which force that Case 1 holds. We define a $\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$-name of a countable poset $\mathbb{P}(D, M)$ (analogous to the poset $\mathbb{P}_\tau$ of Lemma 6.6) that will force the existence of a sequence $\langle r_\ell : \ell \in \omega\rangle$ so that $(\emptyset, \langle r_\ell : \ell \in \omega\rangle)$ will be in $\dot{Q}_{i,\rho+\omega}$ and will be the mod finite meet of A_D if Case 1 fails, and will have an extension in $\dot{Q}_{i,\rho+\omega}$ that is incompatible with each member of A_D if Case 1 holds.

We work in $V[G]$. If Case 1 holds then a condition in $\mathbb{P}(D, M)$ is a pair $(\langle r_i : i < \ell\rangle, \mathcal{P})$ such that

(1) for each $i < \ell$, $r_i \in \mathcal{L}_i$,
(2) $\mathcal{P} \in M$ is a finite set of fans $\langle p_0, p_1, \ldots, p_n\rangle$ such that $p_1 \in G$, and
(3) a condition $(\langle r_i : i < \bar{\ell}\rangle, \bar{\mathcal{P}})$ is an extension of $(\langle r_i : i < \ell\rangle, \mathcal{P})$ providing $\ell \leq \bar{\ell}$, $\mathcal{P} \subset \bar{\mathcal{P}}$, and for each fan $\langle p_0, p_1, \ldots, p_n\rangle$ in $\mathcal{P}$, there is a fan $\langle p_0, \bar{p}_1, \ldots, \bar{p}_n\rangle$ in $\bar{\mathcal{P}}$ satisfying that, for each $\ell \leq i < \bar{\ell}$ and each $1 \leq j \leq n$, it is forced by $\bar{p}_j \restriction_{\gamma+1}$ that r_i is built from $\pi_1(p_j(\gamma + 1))$.

It is fairly routine to verify that $\mathbb{P}(D, M)$ is transitive. What will happen in Case 1 is that the side conditions $\mathcal{P}$ and the requirement on extension, will ensure that D is not dense.

If Case 1 fails, then a condition in $\mathbb{P}(D, M)$ is simply a tuple $\langle r_j : j < \ell\rangle$ such that $r_0 \in \mathcal{L}_1$ and

(1) for each $0 < j < \ell$, $r_j \in \mathcal{L}_j$ and for each $n < \max(int(r_{j-1}))$, r_j is built from T_n,
(2) for each $0 < j < \ell$ and each $w \subset \max(int(r_{j-1}))$ and each e that is r_j-positive, there is an $n < \max int(r_j)$ and a $w_e \subset e$ such that $(w \cup w_e, T_n) < (u_n, T_n)$, and
(3) of course $\langle r_j : j < \bar{\ell}\rangle$ is an extension of $\langle r_j : j < \ell\rangle$ providing $\ell \leq \bar{\ell}$.

We obtain a generic for $\mathbb{P}(D, M)$, regardless of whether it is Case 1 or not, by choosing any canonical isomorphism between $\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho * \mathbb{P}(D, M)$ and $\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_{\rho+\omega}$. Recall also, from Lemma 6.4, that for any finite $\vec{q} \in [\dot{Q}_{i,\rho}]^{<\aleph_0}$, $L(\vec{q}, \omega)$ is the set of $\ell > 0$ such that r_ℓ is built from $(int(r_{\ell-1}), T_q)$ for each $q \in \vec{q}$. For each $\vec{q} \in [Q_{i,\rho}]^{<\aleph_0}$ and $u \in [\omega]^{<\aleph_0}$ let $r(u, \vec{q})$ denote the element $(u, \{r_\ell : \ell \in L(\vec{q}, \omega)\})$ of $Q_{i,\rho+\omega}$. The definition of $\dot{Q}_{i,\rho+\omega}$, as in Lemma 6.4, is the set $Q_{i,\rho} \cup \{r(u,\vec{q}) : u \in [\omega]^{<\aleph_0}, \vec{q} \in [Q_{i,\rho}]^{<\aleph_0}\}$. The set $\{r(u,\vec{q}) : u \in [\omega]^{<\aleph_0}, \vec{q} \in [Q_{i,\rho}]^{<\aleph_0}\}$ is dense in $Q_{i,\rho+\omega}$. It should be clear that if Case 1 fails, then the generic sequence $\{r_\ell : \ell \in \omega\}$ is a mod finite meet of the family $\{(u_n, T_n) : n \in \omega\}$.

We check that IH$(\rho+\omega)$ holds by letting $\{\{p_0^k, p_1^k, \ldots, p_{n_k}^k\} : k < \ell\}$ be any sequence of $\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_{\rho+\omega} * \dot{Q}_{i,\rho+\omega}$ fans, and let p be any element of $\mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_{\rho+\omega}$. For each $k < \ell$ and $1 \le j \le n_k$, we can assume that $p_j^k(\gamma+1)$ is equal to $r(u_j^k, \vec{q}_j^{\,k})$ for some $\vec{q}_j^{\,k} \in [\dot{Q}_{i,\rho}]^{<\aleph_0}$. Let us recall that $p_j^k(\gamma) = p_{j'}^k(\gamma)$ for $1 \le j, j' \le n_k$. By extending p we can assume that for each $k < \ell$, $p(\gamma) \le p_1^k(\gamma)$ and that there is some $d_1 \in D_1$ such that $p{\restriction}_{\gamma+1} < d_1$. Rather than working with the isomorphism between $\mathbb{P}_{\beta,i+1} * \bar{\mathcal{C}}_\rho * \mathbb{P}(D,M)$ and $\mathbb{P}_{\beta,i+1} * \bar{\mathcal{C}}_{\rho+\omega}$, it will be easier to identify these posets. For any $\psi \in \bar{\mathcal{C}}_{\rho+\omega}$, let $\psi^\rho * \psi^{\rho+\omega}$ be its representation in $\bar{\mathcal{C}}_\rho * \mathbb{P}(D,M)$. Thus we can let a pair $\langle r_j : j < \ell_p\rangle$ and $\mathcal{P}$ from M such that, with $\psi = p(\gamma)$, we have that $\psi^{\rho+\omega}$ is either $\langle r_j : j < \ell_p\rangle$ (if $d_1 \notin D_{1,1}$) or $(\langle r_j : j < \ell_p\rangle, \mathcal{P})$ (if $d_1 \in D_{1,1}$). For each $k < \ell$ and $0 \le j \le n_k$, let $\tilde{p}_j^k = p_j^k{\restriction}\gamma+1$.

First assume that $d_1 \in D_{1,1}$, and by possibly expanding it, we can assume that $\mathcal{P}$ is a witness to the fact that d_1 forces that Case 1 holds, and we can asssume that $\mathcal{P}$ contains the list $M \cap \{\{\tilde{p}_0^k, \tilde{p}_1^k, \ldots, \tilde{p}_{n_k}^k\} : k < \ell\}$.

Now apply IH(ρ) to the collection $\mathcal{P} \cup \{\{\tilde{p}_0^k, \tilde{p}_1^k, \ldots, \tilde{p}_{n_k}^k\} : k < \ell\}$ and $p{\restriction}_{\gamma+1}$ to select $\bar{p} < p{\restriction}_{\gamma+1}$ and the collection $\bar{\mathcal{P}} \supset \{\{\bar{p}_0^k, \bar{p}_1^k, \ldots, \bar{p}_{n_k}^k\} : k < \ell\}$ for a suitably large m. Note that $\bar{\mathcal{P}} \subset \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$. By choosing m large enough (considerably greater than ℓ_p), we can assume that the hypothesized $r \in \mathcal{L}_m$ (but we call it r_{ℓ_p}) satisfies that $\max(int(r_{\ell_p-1})) < \min(int(r_{\ell_p}))$. Applying elementarity, we can also assume that for each $k < \ell$ such that $\{\tilde{p}_0^k, \tilde{p}_1^k, \ldots, \tilde{p}_{n_k}^k\}$ is in M, we also have that $\{\bar{p}_0^k, \bar{p}_1^k, \ldots, \bar{p}_{n_k}^k\}$ is in M.

Now each condition $\bar{p}_j^k{\restriction}_{\gamma+1} * (\vec{r}, \bar{\mathcal{P}} \cap M)$, where $\vec{r} = \langle r_i : i \le \ell_p\rangle$, we claim, forces that $\ell_p \in L(\vec{q}_j^{\,k}, \omega)$ and therefore that r_{ℓ_p} is built from $\pi_1(q)$ for each $q \in \vec{q}_j^{\,k}$ with $k < \ell$ and $1 \le j \le n_k$. Let, for lack of better notation, $\tilde{r}$ denote the pair $(\vec{r}, \bar{\mathcal{P}} \cap M) \in \mathbb{P}(D,M)$. The required sequence for the witness to this instance of IH$(\beta, \rho+\omega)$ is then the condition $\bar{p} * \tilde{r} < p$, the sequence

$$\{\{p_0^k, \bar{p}_1^k{\restriction}_{\gamma+1} * \tilde{r} * r(u_1^k, L(\vec{q}_1^{\,k}))\ , \ldots,\ \bar{p}_{n_k}^k{\restriction}_{\beta,\rho} * \tilde{r} * (u_{n_k}^k, L(\vec{q}_{n_k}^{\,k}))\} : k < \ell\}$$

and the condition r_{ℓ_p}.

If $d_1 \in D_1 \setminus D_{1,1}$ then we proceed similarly except that r_{ℓ_p} will have to satisfy condition (2) in the definition of $\mathbb{P}(D, M)$. The fact that we can do this is by the assumption that Case 1 fails to hold, and by an argument as in Claim 2 of Lemma 6.6.

We finish the proof of the Lemma by proving that (following suitable enumerating procedures) the property IH(2^ω) will ensure that $\dot{Q}_{i+1} = \dot{Q}_{i,2^\omega}$ is forced to be a member of $\mathbb{Q}_{207}$ and that $\mathbb{P}_{\gamma+2,i+1}$ is then an i-pure extension of $\mathbb{P}_{\gamma+1,i}$. The argument that $\dot{Q}_{i+1}$ is in $\mathbb{Q}_{207}$ is simply that if D is any countable predense subset of $\mathbb{P}_{\gamma,i+1} * \mathbb{Q}_{i+1}$ that we have to worry about, then there is a $\rho < 2^\omega$ such that $D = D_\rho \subset \mathcal{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}$ and is such that 1 forces that Case 1 did not hold. At stage $\rho + \omega$ in the recursive construction we added a suitable mod finite meet.

Now we show that $\mathbb{P}_{\gamma+2,i+1}$ is an i-pure extension of $\mathbb{P}_{\gamma+2,i}$. Fix any dense set $D \subset \mathbb{P}_{\gamma+2,i+1}$ and any p_0-fan $p_1, \ldots, p_n$ as in definition 5.3. Choose any $\rho < 2^\omega$ large enough so that D contains the downward closure of the predense set D_ρ and $\{p_0, p_1, \ldots, p_n\} \subset \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho * \dot{Q}_{i,\rho}$. Since D is predense in $\mathbb{P}_{\gamma+2,i}$, Case 1 must have failed at stage ρ. Choose any generic filter $G \subset \mathbb{P}_{\gamma,i+1} * \bar{\mathcal{C}}_\rho$ with $p_1 \restriction \gamma + 1 \in G$. Let $w = \pi_0(p_1(\gamma + 1))$. Recall that $\pi_0(p_j(\gamma + 1)) = \pi_0(p_1(\gamma))$ for each $1 \leq j \leq n$. Since each p_j is determined, we may choose m_0 so that, for each $1 \leq j \leq n$, $p_j \restriction \gamma + 1$ forces that $\max int(t) < m_0$ for any $t \in \pi_1(p_j(\gamma + 1))$ such that $\max(w) \not< \min int(t)$.

Since Case 1 fails, we can an $r_0 \in \mathcal{L}_{m_0}$ with $m_0 < \min int(r_0)$ and a p_0-fan $\{p_0, \bar{p}_1, \ldots, \bar{p}_n\}$ such that for each $1 \leq j \leq n$, $\bar{p}_j < p_j$ and $\bar{p}_j \restriction \gamma + 1$ (which is in $\mathbb{P}_{\gamma,i} * \bar{\mathcal{C}}_\rho$) forces that r_0 is built from $\pi_1(p_j(\gamma + 1))$, and so that there is a $w_0 \subset int(r_{\ell_0})$ and a $d_1 \in D$ such that $\bar{p}_1 \restriction \gamma + 1 * \ (w \cup w_0, T)$ is less than d_1. Redefine $\bar{p}_1$ to be any common extension of the current $\bar{p}_1$ and d_1 so that $\bar{p}_1 \restriction \gamma + 1$ is unchanged, and $\pi_0(\bar{p}_1(\gamma + 1)) = w \cup w_0$. Similarly, for each $1 < j \leq n$, let $\bar{p}_j$ be the extension $\bar{p}_j \restriction \gamma + 1 * (w \cup w_0, \pi_1(\bar{p}_j(\gamma + 1))$.

We no longer need the generic filter G, and we can summarize this step by saying that, given the p_0-fan $\{p_1, \ldots, p_n\}$, we have found an extension p_0-fan $\{\bar{p}_1, \ldots, \bar{p}_n\}$ such that $\bar{p}_1 \in D$. Now repeat this step with the p_0-fan $\{p_0, p^1_2, p^1_3, \ldots, p^1_n, p^1_1\}$ where, for each $1 \leq j \leq n$, $p^1_j = \bar{p}_j$. By this we mean that we can next choose an extension $\langle r_i : i \leq \ell_0 + 1\rangle$ of $\langle r_i : i \leq \ell_0\rangle$ so that there is again a p_0-fan $\{p_0, \bar{p}^1_2, \ldots, \bar{p}^1_n, \bar{p}^1_1\}$ such that there is some $d_2 \in D$ so that $\bar{p}^1_2 < d_2$. After n-repetitions, we have our required p_0-fan witnessing that $\mathbb{P}_{\gamma+2,i+1}$ is an i-pure extension. □

Now we need a similar lemma for $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_4^{\gamma+1}(\prec)$.

Lemma 6.8. *Suppose that* $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_4^{\gamma+1}(\prec)$ *where* $\gamma = \kappa \cdot \eta$ *for some* $0 < \eta < \lambda$. *If we have that* $\mathbb{P}_{\gamma,i+1}$ *is an* i*-pure extension of* $\mathbb{P}_{\beta,i}$ *for* $i < \kappa$, *then there is such an extension of* $\underline{\mathbf{q}}$ *in* $\underline{\mathbf{Q}}_4^{\gamma+2}(\prec)$.

Proof. Note that any extension of $\underline{\mathbf{q}}$ in $\underline{\mathbf{Q}}_4^{\gamma+2}(\prec)$ must be by members of $\mathbb{Q}_{207}$. We proceed to construct $\{\dot{Q}_i : i < \kappa\}$ by recursion on $i < \kappa$ satisfying that

(1) $\dot{Q}_i$ is a $\mathbb{P}_{\gamma+1,i}$-name of a member of $\mathbb{Q}_{207}$,
(2) for $j < i$, $\dot{Q}_i$ is forced to contain $\dot{Q}_j$,
(3) for each $j < i$, $\mathbb{P}_{\gamma+1,j+1} * \dot{Q}_{j+1}$ is a j-pure extension of $\mathbb{P}_{\gamma+1,j} * \dot{Q}_j$,
(4) for each limit $j < i$ and each $\mathbb{P}_{\gamma,j}$-name $\dot{A}$ of a subset of ω, there is a $q \in \dot{Q}_j$ such that $int(q)$ is forced to be contained in $\dot{A}$ or in $\omega \setminus \dot{A}$,
(5) for each limit $j < i$ and each $\mathbb{P}_{\gamma,j}$-name $\dot{g}$ of a member of ω^ω and each $q = (u_q, \{t_\ell^q : \ell \in \omega\}) \in \dot{Q}_j$, the condition $q[\dot{g}, f_{\dot{A}_i}]$ is in $\dot{Q}_i$.

The proof for the construction of $\dot{Q}_{i+1}$ is exactly the same as in Lemma 6.7 and can be skipped.

If i is a limit, then we work in the extension by $\mathbb{P}_{\gamma,i} * \mathcal{C}_{i\times 2^\omega}$. Let $G_{\gamma,i}$ be any $\mathbb{P}_{\gamma,i}$-generic filter, and, in $V[G_{\gamma,i}]$, let H_i be a $\mathcal{C}_{i\times 2^\omega}$-generic filter. For each $j < i$, let $G_j * H_j$ denote the generic filter $(G_{\gamma,i} * H_i) \cap \mathbb{P}_{\gamma+1,j} = (G_{\gamma,i} * H_i) \cap (\mathbb{P}_{\gamma,j} * \mathcal{C}_{j+1\times 2^\omega}$. We recall from Lemma 4.15 that $f_{\dot{A}_i}$ is forced by $\mathbb{P}_{\gamma+1,i}$ to be unbounded. For each $j < i$, let F_j denote the set $\omega^\omega \cap V[G_j * H_j]$. Let us note that $\tilde{Q}_0$ is closed under finite changes and every finite set of pure conditions has a lower bound in $\mathcal{Q}_{Bould}$, where

$$\tilde{Q}_0 = \{q, q[g, f_{\dot{A}_i}] : j < i,\ q \in Q_j \text{ and } g \in F_j\}\ .$$

We have the poset $\mathcal{C}_{\{i\}\times 2^\omega}$ at our disposal, and we choose a partition I, J of 2^ω into sets of cardinality 2^ω. We now apply Lemma 6.4 in the further extension by forcing with $\mathcal{C}_{\{i\}\times I}$ so as to obtain $\tilde{Q}_1 \supset \tilde{Q}_0$ (closed under finite changes and having a directed set of pure conditions) which satisfies that, for all $A \subset \omega$ in the extension by $V[G_{\gamma,i}]$, there is a $q \in \tilde{Q}_1$ such that $int(q) \subset A$ or $int(q) \cap A = \emptyset$. To finish, we then pass to the full extension by further forcing with $\mathcal{C}_{\{i\}\times J}$ and apply Theorem 6.5 to choose the necessary extension $\dot{Q}_i$ in $\mathbb{Q}_{207}$. □

7. Final proofs and open questions

Proof of Theorem 4.1. We first check that there is a $\underline{\mathbf{q}}$ in $\underline{\mathbf{Q}}_3^\lambda(\prec)$ satisfying that, for all $i < \kappa$, $\underline{\mathbf{P}}_{i+1}^\lambda$ is an i-pure extension of $\underline{\mathbf{P}}_i^\lambda$. Indeed, if not, then we may suppose that $\gamma \leq \lambda$, and we have a maximal sequence $\{\underline{\mathbf{q}}^\xi : \xi < \gamma\}$ such that, for all $\xi \leq \zeta < \gamma$, $\underline{\mathbf{q}}^\xi \in \underline{\mathbf{Q}}_3^\xi(\prec)$, $\underline{\mathbf{q}}_\zeta$ is an extension of $\underline{\mathbf{q}}_\xi$, and, for all $i < \kappa$, $\underline{\mathbf{P}}^{\mathbf{q}_\xi}_{\xi,i+1}$ is an i-pure extension of $\underline{\mathbf{P}}^{\mathbf{q}_\xi}_{\xi,i}$. If γ is a limit, then there is a unique common extension $\underline{\mathbf{q}}_\gamma$, which, by Lemma 4.13, is in $\underline{\mathbf{Q}}_3^\gamma(\prec)$. Also, by Lemma 5.8, $\underline{\mathbf{P}}^{\mathbf{q}_\gamma}_{\gamma,i+1}$ is an i-pure extension of $\underline{\mathbf{P}}^{\mathbf{q}_\gamma}_{\gamma,i}$ for all $i < \kappa$. Therefore γ must be a successor, but then one of Lemma 5.8 or Lemma 6.7 implies that we have an non-trivial extension.

So we may choose $(\langle\langle\mathbb{P}_{\alpha,i} : \alpha \leq \lambda, i \leq \kappa\rangle, \langle\dot{\mathbb{Q}}_{\alpha,i} : \alpha < \lambda, i \leq \kappa\rangle\rangle, \{\dot{A}_\alpha : \alpha \in \lambda\})$ in $\underline{\mathbf{Q}}_3^\lambda(\prec)$ such that $\mathbb{P}_{\lambda,i+1}$ is an i-pure extension of $\mathbb{P}_{\lambda,i}$ for all $i < \kappa$. Properties (2) and (4) of the definition of $\underline{\mathbf{Q}}_3^\lambda(\prec)$ (Definition 4.11) imply that $\mathfrak{b} = \mathfrak{s} = \lambda$. Finally, we have that $\mathfrak{p} = \mathfrak{h} = \kappa$ by Lemma 4.10 and Corollary 5.7. □

Proof of Theorem 4.2. It follows as in the previous proof that there is a $\underline{\mathbf{q}} \in \underline{\mathbf{Q}}_4^\lambda(\prec)$ (using Lemma 6.8 in place of Lemma 6.7). The only change is that we use Lemma 4.15 to deduce that $\mathfrak{b} = \kappa$. □

We close with two open problems.

(1) Is it consistent to have $\omega_1 < \mathfrak{h} < \mathfrak{b} < \mathfrak{s}$?
(2) Is it consistent to have $\omega_1 < \mathfrak{h} < \mathfrak{s} < \mathfrak{b}$?

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