SOME SPECIAL REGULAR OPEN SUBSETS OF ω^*

Abstract. We prove the Main Lemma and Theorem 4.11 of [3] (i.e. Bezhanishvili and Harding) in ZFC.

For any family $\mathcal{A} \subset [\omega]^{\aleph_0}$, $\mathcal{A}^\perp$ denotes the set of $b \in [\omega]^{\aleph_0}$ that are almost disjoint from each $a \in \mathcal{A}$. $\mathcal{A}^+$ denotes the set of $X \subset \omega$ that are not in the ideal generated by $\mathcal{A} \cup \mathcal{A}^\perp$. In particular, if $\mathcal{A}$ is an adf (almost disjoint family), then $\mathcal{A}^+$ denotes the set of $X \subset \omega$ that meet infinitely many members of $\mathcal{A}$ in an infinite set. If U is an open subset of ω^* and $\mathcal{A}_\mathcal{U}$ is the ideal of those infinite $a \subset \omega$ satisfying that $a^* \subset U$, then X being in $\mathcal{A}^+$ is equivalent to X^* meeting the boundary of $\overline{U}$.

Lemma 1. *For any $\mathcal{A} \subset [\omega]^{\aleph_0}$, $\left(\mathcal{A}^\perp\right)^+ \subset \mathcal{A}^+$.*

Proof. Since $\mathcal{A} \subset \left(\mathcal{A}^\perp\right)^\perp$, if X is not in the ideal generated by $\mathcal{A}^\perp \cup \left(\mathcal{A}^\perp\right)^\perp$, then X is not in the ideal generated by $\mathcal{A}^\perp \cup \mathcal{A} = \mathcal{A} \cup \mathcal{A}^\perp$. This proves the Lemma. ☐

Definition 2. *A family $\mathcal{A} \subset [\omega]^{\aleph_0}$ is completely separable if for all $X \in \mathcal{A}^+$, there is an $a \in \mathcal{A}$ such that $a \subset^* X$.*

Proposition 3 ([2])**.** *There is an infinite completely separable adf.*

Lemma 4. *For any $m \in \omega$, there are $\mathcal{B}_i$ $(i \le m)$ such that for all $i \neq j \le m$,*

1. *$\mathcal{B}_i$ is an infinite completely separable adf,*
2. *$\mathcal{B}_i \subset \mathcal{B}_j^\perp$,*
3. *$\mathcal{B}_i^+ = \mathcal{B}_j^+$ for $i, j \le m$.*

Proof. Let $\mathcal{A}$ be a completely separable adf as in Proposition 3 and let $\{a_\alpha : \alpha \in \mathfrak{c}\}$ be an enumeration of $\mathcal{A}$. It is shown in [2, 4.9] that each infinite completely separable adf has cardinality $\mathfrak{c}$ and that $\{a \in \mathcal{A} : a \subset^* X\}$ has cardinality $\mathfrak{c}$ for all $X \in \mathcal{A}^+$. Let $\{X_\xi : \xi \in \mathfrak{c}\}$ be an enumeration of $\mathcal{A}^+$ so that each $X \in \mathcal{A}^+$ is listed infinitely many times. By induction on $\xi \in \mathfrak{c}$, choose $H_\xi \in [\mathfrak{c} \setminus \bigcup_{\eta<\xi} H_\eta]^{m+1}$ so that $a_\alpha \subset^* X_\xi$ for each $\alpha \in H_\xi$. Choose pairwise disjoint subsets of $\mathfrak{c}$, $\{J_i : i < m\}$, so that $|J_i \cap H_\xi| = 1$ for all $i < m$ and $\xi < \mathfrak{c}$. For $i < m$, set $\mathcal{B}_i = \{a_\alpha : \alpha \in J_i\}$ and let $\mathcal{B}_m = \{a_\alpha : \alpha \in \mathfrak{c} \setminus \bigcup_{i<m} J_i\}$. Clearly each $X \in \mathcal{A}^+$ contains mod finite infinitely many elements of $\mathcal{B}_i$ for each $i \le m$. It thus follows that each of $\{\mathcal{B}_i : i < m\}$ is completely separable and that $\mathcal{B}_i^+ = \mathcal{A}^+$ for each $i \le m$. Since $\mathcal{A}$ is an adf and the famliy $\{J_i : i < m\}$ are pairwise disjoint, we also have that $\mathcal{B}_i \subset \mathcal{B}_j^\perp$ for $i \neq j \le m$. ☐

Definition 5. *$\mathcal{B} \prec^+ \mathcal{A}$ if*

1. *for each $b \in \mathcal{B}$, there is an $a \in \mathcal{A}$ with $b \subset^* a$ (or $\mathcal{B} \prec \mathcal{A}$),*
2. *for each $X \in \mathcal{A}^+$, there is an $a \in \mathcal{A}$ with $X \cap a \in \mathcal{B}^+$.*

Lemma 6. *For each completely separable adf $\mathcal{A}$ and each $m < \omega$, there is a family $\{\mathcal{B}_i : i \le m\}$ such that, for each $i \neq j \le m$,*

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(1) $\mathcal{B}_i \prec^+ \mathcal{A}$ *and is an infinite completely separable adf,*
(2) $\mathcal{B}_i \subset \mathcal{B}_j^\perp$,
(3) $\mathcal{B}_i^+ = \mathcal{B}_j^+$ *and* $\mathcal{A} \subset \mathcal{B}_i^+$,
(4) $\mathcal{B}_m^+ = \left((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{m-1})^\perp\right)^+$.

Proof. For each $a \in \mathcal{A}$, choose $\{\mathcal{B}_i(a) : i < m\}$ as in Lemma 4 so that $\mathcal{B}_i(a) \subset [a]^{\aleph_0}$ for each $i \leq m$. Set $\mathcal{B} = \bigcup\{\mathcal{B}_i(a) : a \in \mathcal{A}\}$. We verify each item.

(1) It is clear that $\mathcal{B}_i \prec \mathcal{A}$. Similarly, if $X \in \mathcal{A}^+$, then there is an $a \in \mathcal{A}$ such that $a \subset X$, hence it follows that $X \in \mathcal{B}_i^+$.
(2) It is obvious that $\mathcal{B}_i \cap \mathcal{B}_j$ is empty.
(3) Suppose that $X \in \mathcal{B}_i^+$. If there is an $a \in \mathcal{A}$ such that $X \cap a \in \mathcal{B}_i(a)^+$, then $X \in \mathcal{B}_j^+$. Otherwise $X \in \mathcal{A}^+$, and so there is an $a \in \mathcal{A}$ such that $a \subset X$. Of course this ensures that $X \in \mathcal{B}_j(a)^+$.
(4) It is immediate from (1) - (3) that $\mathcal{B}_m \subset (\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{m-1})^\perp$ and this implies that $\mathcal{B}_m^+ \subset \left((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{m-1})^\perp\right)^+$. Now assume that $X \notin \mathcal{B}_m^+$. By (3), there is a B in the ideal generated by $\bigcup_{i<m} \mathcal{B}_i$ such that $X \setminus B$ is in $\mathcal{B}_i^\perp$ for each $i \leq m$. Therefore, $X \setminus B$ is in $(\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{m-1})^\perp$ and so X is in the ideal generated by $(\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{m-1}) \cup (\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{m-1})^\perp$. Equivalently, X is not in $((\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{m-1}))^+$. By Lemma 1, this completes the proof of (4).

□

Definition 7. *If* $\mathcal{A}$ *is an adf, let* $\{\mathcal{B}_i : i \leq m\} \prec_m^+ \mathcal{A}$ *denote the relations as in Lemma 6.*

Using an easy inductive process and Lemma 6 we have the following.

Corollary 8. *Let* $0 < m, n \in \omega$ *and let* T *be the maximum subtree of* $(m+1)^{<n}$ *satisfying that* $t \in T$ *is maximal if* $t(k) = m$ *for some (unique)* $k \in dom(t)$ *(of course* k *is the maximum element of* $dom(t)$*). Then there is a sequence* $\{\mathcal{B}_t : t \in T\}$ *satisfying*

(1) $\mathcal{B}_\varnothing = \{\omega\}$,
(2) *if* $t \in T$ *is not maximal, then* $\{\mathcal{B}_{t^\frown i} : i \leq m\} \prec_m^+ \mathcal{B}_t$.

Following [3], for a finite tree $T \subset \omega^{<\omega}$ the topology τ_T is defined by simply saying that a set $U \subset T$ is open if for each $t \in U$, $t^\uparrow = \{s : t \subseteq s \in T\}$ is a subset of U. Thus each maximal node is isolated and the closure of any node equals the set of all nodes below it.

Lemma 9. *If* t *is a maximal node of a finite tree* T *and if* T^{-t} *is the subtree of* T *obtained by removing* t*, then* $f : (T, \tau_T) \mapsto (T^{-t}, \tau_{T^{-t}})$ *is open and continuous (and onto) if* $f(s) = s$ *for* $s \in T^{-t}$ *and* $f(t) = x$ *is any maximal node of* T^{-t} *that is above the immediate predecessor of* t.

Proof. We first prove that f is continuous. Let $U \in \tau_{T^{-t}}$ and consider any $s \in f^{-1}(U)$. We must show that $s^\uparrow \subset f^{-1}(U)$. Note that $U \subset f^{-1}(U)$. Since each of t and x are maximal, we may assume that $s \notin \{t, x\}$. If s is not below the immediate predecessor of t, then $s^\uparrow$ (in T) is contained in U and therefore in $f^{-1}(U)$. If s is below the immediate predecessor of t, then both x and each point of $s^\uparrow$ (in T^{-t})

are in U. This implies that $s^\uparrow$ (in T) is contained in U and completes the proof that f is continuous.

Now assume that U is an open subset of T. It is immediate that $U \cap T^{-t}$ is an open subset of T^{-t}. Since each of U and $U \cup \{x\}$ are open in T^{-t}, it follows that $f(U)$ is open in T^{-t}. $\square$

Proposition 10. *If $f : X \mapsto Y$ and $g : Y \mapsto Z$ are open, continuous, and onto maps, then $g \circ f : X \mapsto Z$ is also open, continuous, and onto.*

Corollary 11. *There is an open, continous, and onto mapping from the tree topology (T, τ_T) of Corollary 8 to $m^{<n}$ with the subspace topology.*

Theorem 12. *For each $m, n \in \omega$, there is an open, continuous, and onto mapping from ω^* to $m^{<n}$ such that the preimage of every point is locally compact and has no isolated points.*

Proof. By Lemma 9 and Proposition 10 it suffices to prove the Theorem for values of $m > 0$. Similarly, it suffices to prove that for each T as in Lemma 8, there is an open, continuous mapping f from ω^* onto T also with the stated property on point pre-images. Let $\{\mathcal{B}_t : t \in T\}$ be the family as stated in Lemma 8. For each maximal $t \in m^{<n}$, let $U_t = \bigcup\{b^* : b \in \mathcal{B}_t\}$ and set $f(U_t) = t$. For each non-maximal $t \in T$, let

$$U_{t^\frown m} = \bigcup\{b^* : b \in \left(\mathcal{B}_{t^\frown 0} \cup \cdots \cup \mathcal{B}_{t^\frown (m-1)}\right)^\perp\} \text{ and set } \ f(U_{t^\frown m}) = t^\frown m \ .$$

Define $U_\varnothing = \omega^*$ and for non-maximal $\varnothing \neq t \in T$, $U_t = \bigcup\{b^* : b \in \mathcal{B}_t\}$. We set $f(U_t \setminus \bigcup\{U_s : t \subsetneq s \in T\}) = t$. For each $b \in \mathcal{B}_t$, $f^{-1}(t) \cap b^*$ is closed so it follows that $f^{-1}(t)$ is locally compact.

Claim 1. For each $t \in T$, $t^\uparrow = f(U_t)$ and

$$t \in f(X^*) \quad \text{iff} \quad X \in \mathcal{B}^+_{t^\frown 0}$$

if t is non-maximal and $X \subset \omega$.

The statement of the claim clearly holds for each maximal $t \in T$. We prove the claim by reverse induction on $dom(t)$. We note that, by definition, $f(U_t) \subset t^\uparrow$ for all $t \in T$. Fix any non-maximal $t \in T$. To show that $f(U_t) = t^\uparrow$, it suffices, proceeding by induction, to show that $t \in f(U_t)$. Choose any $b \in \mathcal{B}_t$ and note that $b \in \mathcal{B}^+_{t^\frown i}$ for each $i \leq m$. It follows that $\overline{b^* \cap U_{t^\frown 0}}$ is non-compact and disjoint from $U_{t^\frown i}$ for all $0 < i \leq m$. Since $\overline{b^* \cap U_{t^\frown 0}} \subset b^* \subset U_t$, we have that $\overline{b^* \cap U_{t^\frown 0}} \setminus U_{t^\frown 0}$ is a non-empty subset of $U_t \setminus \bigcup_{i \leq m} U_{t^\frown i}$ which is mapped to t. Since $\mathcal{B}_{t^\frown 0}$ is completely separable, it therefore follows that $t \in f(X^*)$ for each $X \in \mathcal{B}^+_{t^\frown 0}$. Now assume that $x \in X^*$ (for some $X \subset \omega$) and that $f(x) = t$. Choose the unique $b \in \mathcal{B}_t$ so that $x \in b^*$. Since $x \notin U_{t^\frown m}$, we have that $X \cap b$ is not in $\left(\bigcup_{i<m} \mathcal{B}_{t^\frown i}\right)^\perp$. Additionally, $X \cap b$ is not in the ideal generated by $\bigcup_{i<m} \mathcal{B}_{t^\frown i}$ since $x \notin \bigcup_{i<m} U_{t^\frown i}$. Therefore we have, as needed, that $X \in \left(\bigcup_{i<m} \mathcal{B}_{t^\frown i}\right)^+ = \mathcal{B}^+_{t^\frown 0}$.

It follows from Claim 1 that f is continuous (i.e. $f^{-1}(t^\uparrow)$ is open for each $t \in T$) and onto. We finish by proving that f is open. Choose any infinite $X \subset \omega$ and let $t \in f(X^*)$. We must prove that $t^\uparrow \subset f(X^*)$. Again, we can proceed by reverse induction on $dom(t)$. By Claim 1, $X \in \mathcal{B}^+_{t^\frown 0}$, and therefore by the assumptions of Corollary 8, $X \in \mathcal{B}^+_{t^\frown i}$ for all $i \leq m$. Each $\mathcal{B}_{t^\frown i}$ is completely separable, hence there are $\{b_i : i < m\} \subset [X]^{\aleph_0}$ such that $b_i \in \mathcal{B}_{t^\frown i}$ for each $i \leq m$. Since $b^*_m \subset U_{t^\frown m}$

and $t^\frown m$ is maximal in T, $t^\frown m \in f(X^*)$. Fix any $i < m$, and note that, by (3) of Lemma 6, $b_i \in \mathcal{B}^+_{t^\frown i^\frown 0}$. Therefore, by Claim 1, $t^\frown i \in f(X^*)$ and, by the induction hypothesis, $(t^\frown i)^\uparrow \subset f(X^*)$.

Finally we prove that $f^{-1}(t)$ has no isolated points. By Claim 1, it suffices to show that $X^* \cap f^{-1}(t)$ is not a single point for any $X \in \mathcal{B}^+_{t^\frown 0}$. Choose any infinite $\{b_n : n \in \omega\} \subset \mathcal{B}_{t^\frown 0}$ such that $X \cap b_n$ is infinite for each n. Let $Y = \bigcup\{b_{2n} \setminus \bigcup_{k<n} b_{2k+1} : n \in \omega\}$. Note that, for each $n \in \omega$, $X \cap Y \cap b_{2n}$ is infinite and $(X \setminus Y) \cap b_{2n+1}$ is infinite. Therefore, $X \cap Y$ and $X \setminus Y$ are both in $\mathcal{B}^+_{t^\frown 0}$. Since $(X \cap Y)^*$ and $(X \setminus Y)^*$ are disjoint, the proof is complete. □

References

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