MAXIMAL REALCOMPACT SPACES AND MEASURABLE CARDINALS

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Abstract. Comfort and Hager investigate the notion of a maximal realcompact space and ask about the relationship to the first measurable cardinal $\mathfrak{m}$. A space is said to be a $P(\mathfrak{m})$ space if the intersection of fewer than $\mathfrak{m}$ open sets is again open. They ask if each realcompact $P(\mathfrak{m})$ space is maximal realcompact. We establish that this question is undecidable.

1. Introduction

A (Tychonoff) space X is realcompact if there is an index set I so that X can be embedded into the product $\mathbb{R}^I$ as a closed subset. It is immediate that if $A \subseteq X$ is closed in a realcompact space X, then A is also realcompact. The category theoretic properties of the class of realcompact spaces (closed hereditary and closed under arbitrary products) ensure that for each realcompact space X (with topology τ), there is a largest topology $\sigma \supseteq \tau$, such that (X, σ) (denoted μX) is still realcompact (see [CH]). It is useful to also recall that a space is realcompact if every countably complete Z-ultrafilter is fixed (see [GJ76, Ch.8]). For the reader's convenience we formulate the main idea in the following proposition.

Proposition 1.1. *Let (X, τ) be realcompact and let $\mathcal{S}$ denote the collection of all topologies σ on X which contain τ and satisfy that (X, σ) is realcompact. The topology on X induced by the identity mapping to the diagonal $\Delta X \subseteq \Pi\{(X, \sigma) : \sigma \in \mathcal{S}\}$ will be a maximal realcompact topology on X.*

Definition 1.2. Let RC denote the class of realcompact spaces and let M(RC) denote the class of maximal realcompact spaces.

It is well known that $\mathfrak{m}$, the first measurable cardinal, is also the smallest cardinal κ with the property that the discrete space of cardinality κ is not realcompact. A discrete space X is not realcompact precisely when there is a countably complete (set) ultrafilter over X. A filter $\mathcal{F}$ on a set X is said to be κ-complete if the intersection of fewer than κ members of $\mathcal{F}$ is again in $\mathcal{F}$. Any countably complete (set) ultrafilter over X will be $\mathfrak{m}$-complete by the minimality of $\mathfrak{m}$.

Definition 1.3. If (X, τ) is a space and κ is a cardinal, let τ_κ denote the topology on X generated by the base $\{\bigcap \mathcal{W} : W \subseteq \tau\ ,\ |\mathcal{W}| < \kappa\}$. Let $P(\kappa)$ denote the class of spaces (X, τ) such that $\tau_\kappa = \tau$.

1991 *Mathematics Subject Classification.* 54D60.

Key words and phrases. realcompact spaces, strongly compact cardinals.

Supported by NSF grant DMS-0103985. This paper benefitted from a very well informed referee.

The following very interesting result can be found in [CR85] and see [HM]. It shows that $M(RC) \subseteq RC \cap P(\mathfrak{m})$.

Proposition 1.4. *If (X, τ) is realcompact, then so is $(X, \tau_{\mathfrak{m}})$.*

Of course if $\kappa > \mathfrak{m}$, then (X, τ_κ) is realcompact if and only if $|X| < \mathfrak{m}$. In fact, just as in [GJ76, p.120], we have the following stronger result.

Lemma 1.5. *If (X, τ) is realcompact and $\mathcal{U} \subseteq \mathcal{P}(X)$ is a countably complete (set) ultrafilter over X, then there is an $x \in X$ which is the $\mathcal{U}$-limit, i.e. $\{x\} = \bigcap\{\overline{U} : U \in \mathcal{U}\}$.*

The converse is false of course; for example, the ordinal space ω_1 is not realcompact but every countably complete (set) ultrafilter on ω_1 is principal (because the cardinal ω_1 is not measurable).

Proof. We may assume that X is a closed subspace of $\mathbb{R}^I$ for some index set I. For each $i \in I$, let π_i denote the projection map from $\mathbb{R}^I$ onto $\mathbb{R}$. In addition, let $\mathcal{U}_i$ denote the filter of subsets of $\mathbb{R}$ generated by

$$\{\pi_i[U] : U \in \mathcal{U}\} \ .$$

Since $\mathcal{U}$ is countably complete, so is $\mathcal{U}_i$ for each $i \in I$. In addition, since $\mathcal{U}$ is an ultrafilter on X, it follows that $\mathcal{U}_i$ is an ultrafilter over $\mathbb{R}$. By the countable completeness of $\mathcal{U}_i$, there must be an integer n such that the set $\mathbb{R} \setminus [-n, n]$ is not in $\mathcal{U}_i$. Therefore the compact set $[-n, n]$ will be a member of $\mathcal{U}_i$. By the compactness, there is a real $r_i \in \bigcap\{\overline{\pi_i[U]} : U \in \mathcal{U}\}$. Furthermore, for each $\epsilon > 0$, there is a $U \in \mathcal{U}$ such that $\pi_i[U] \subseteq (r_i - \epsilon, r_i + \epsilon)$. It follows then that for each finite $I' \subseteq I$ and $\epsilon > 0$ there is a $U \in \mathcal{U}$, such that $\pi_i[U] \subseteq (r_i - \epsilon, r_i + \epsilon)$ for each $i \in I'$. By the definition of the product topology, we have that the point $\langle r_i : i \in I\rangle \in \mathbb{R}^I$ is in $\overline{U}$ for each $U \in \mathcal{U}$. Since X is closed, this point is the x we seek. □

The question from [CH, 2.5(b)] that we wish to address is "Is M(RC)$\supseteq P(\mathfrak{m})\cap$RC valid?" The answer seems, to us, quite surprising and relies on a very deep result of Magidor [Mag76] concerning *strongly compact* cardinals. They are also sometimes called simply *compact* although this is now much less common.

Definition 1.6. ([Jec78, §33]) A cardinal $\kappa > \omega$ is a *compact cardinal* if, for every set S, every κ-complete filter over S can be extended to a κ-complete ultrafilter over S.

The interested reader is referred to [Kan03] for a comprehensive treatment of large cardinals.

In the remainder of the paper we will establish the following answer on the Comfort and Hager question.

Theorem 1.7. *M(RC) = $P(\mathfrak{m})\cap$RC if and only if $\mathfrak{m}$ is a compact cardinal.*

It certainly makes this theorem more interesting to know that Magidor has established [Mag76] that it is consistent (from a supercompact cardinal) that $\mathfrak{m}$ is a strongly compact cardinal. It is considerably easier to establish from just a measurable cardinal that it is consistent that $\mathfrak{m}$ is not a strongly compact cardinal (an even stronger result was established by Vopěnka and Hrbáček [VH66] or see [Jec78, Thm. 79]). By results of Mitchell [Mit74], there are models in which $\mathfrak{m}$ is not strongly compact and there is a proper class of measurable cardinals.

The following results are standard facts from Gillman and Jerison [GJ76]. The extension of X, υX introduced in the next result is known as the Hewitt realcompactification (see [GJ76, p.118]).

Lemma 1.8. *If (X, τ) is Tychonoff then there is a subset $\upsilon X \subseteq \beta X$ such that υX is the minimal realcompact subset of βX which contains X. A point $p \in \beta X$ is a member of υX iff for each continuous $f : \beta X \to \mathbb{R}$, there is an $x \in X$ such that $f(x) = f(p)$ (i.e. $f(p) \in f[X]$).*

Lemma 1.9. *If X is discrete, then $p \in \upsilon X$ iff $\{A \subseteq X : p \in cl_{\beta X}(A)\}$ is a countably complete ultrafilter over X.*

Lemma 1.10. *If $\mathfrak{m}$ is not a strongly compact cardinal (the most likely case) then there is an $X \in P(\mathfrak{m}) \cap$ RC which is not in M(RC).*

Proof. Let S be a set and let $\mathcal{F}$ be an $\mathfrak{m}$-complete (free) filter over S which does not extend to an $\mathfrak{m}$-complete ultrafilter over S. We work in βS where S is given the discrete topology. Let K denote the closed set $\bigcap\{cl_{\beta S}(F) : F \in \mathcal{F}\}$. Our space X will simply be the quotient space of $\upsilon S \cup K$ obtained by collapsing K to a single point. It is easily seen to follow from Lemmas 1.8 and 1.9 that $\upsilon S \cup K$ is in RC (and follows from [GJ76, 8.16]). Furthermore, by [GJ76, 8.16], X being the union of the realcompact space υS with the compact space the collapsed point K, is also realcompact. Next we must check that $X \in P(\mathfrak{m})$. By Lemma 1.9, the space υS is itself in $P(\mathfrak{m})$. The fact that X is in $P(\mathfrak{m})$ as well follows from the fact that $\mathfrak{F}$ is $\mathfrak{m}$-complete. Finally, the fact that X is not in M(RC) follows from the fact that we can enlarge the topology by making the singleton F isolated. To see that the resulting space is RC, we simply have to check that υS is disjoint from K in the original space βS. Of course this is because of the hypothesis that $\mathcal{F}$ does not extend to an $\mathfrak{m}$-complete ultrafilter. □

Remark: It is actually the case that in each model in which $\mathfrak{m}$ exists and is not strongly compact, there is a very natural example of a space X as in Lemma 1.10. Ketonen [Ket73] (or see [Kan03]) has shown that in each such model there is a regular cardinal $\kappa > \mathfrak{m}$ such that there is no *uniform* ultrafilter on κ which is $\mathfrak{m}$-complete (a filter on κ is uniform if each element of the filter has cardinality κ). Then the space X is $\upsilon(\kappa) \cup \{\infty\}$ where κ has the discrete topology, $\upsilon(\kappa)$ is the Hewitt realcompactification of κ (consisting of all the fixed and countably complete ultrafilters on κ) and the single additional point ∞. The neighborhoods of ∞ are the complements of the closures of bounded subsets of κ. This is a realcompact $P_{\mathfrak{m}}$ topology on X. There is also a stronger such topology, namely let ∞ now be an isolated point.

Lemma 1.11. *If $\mathfrak{m}$ is a strongly compact cardinal, then M(RC) is equal to $P(\mathfrak{m}) \cap$ RC.*

Proof. Let (X, τ) be a member of $P(\mathfrak{m}) \cap$ RC, i.e. a realcompact space for which $\tau_{\mathfrak{m}} = \tau$. We show that (X, τ) is maximal realcompact. Assume that $\sigma \supseteq \tau$ is a topology and that $A \subseteq X$ is a closed set in (X, σ) which is not closed in (X, τ). Let x be a point of X which is in the τ-closure of A but which is not in A. Let $\mathcal{U}_x$ denote the collection of members of τ which contain x (the neighborhood base of x in (X, τ)). Since $(X, \tau) \in P(\mathfrak{m})$ and x is in the closure of A, it follows that $\mathcal{U}_{x,A} = \{U \cap A : U \in \mathcal{U}_x\}$ generates a $\mathfrak{m}$-complete filter over X. Let $\mathfrak{U}$ be an

$\mathfrak{m}$-complete ultrafilter over X which extends $\mathcal{U}_{x,A}$. By Lemma 1.5, there is a point $z \in X$ such that $\{z\} = \bigcap\{\overline{U} : U \in \mathfrak{U}\}$ where the closure is taken in (X, σ). Since (X, τ) is Hausdorff and $\sigma \supseteq \tau$, it of course follows that z must actually be x. This contradicts the assumption that x is not in the closure of A in (X, σ). □

For the author's interest we have assembled the following related facts about the cardinal $\mathfrak{m}$ which show that it can be very far from being strongly compact. If κ is any measurable cardinal and $\mathcal{U}$ is a κ-complete ultrafilter on κ, then using the concept of relative constructibility, there is a smallest model $L[\mathcal{U}]$ (all sets constructible from $\mathcal{U}$) in which κ is measurable. In $L[\mathcal{U}]$, κ is $\mathfrak{m}$ because not only is κ the smallest measurable cardinal, Solovay showed it is the only measurable cardinal (see [Kun70, 5.11]). Silver [Sil71] showed that GCH holds if $V = L[\mathcal{U}]$. Now, suppose $\mathcal{V}$ is any uniform countably complete ultrafilter on a cardinal λ, hence $\lambda \geq \mathfrak{m}$. But then λ must be $\mathfrak{m}$, since $\lambda > \mathfrak{m}$ would yield a contradiction by the method of [VH66] (or, see [Kun70, §10]). In particular, there is no uniform $\mathfrak{m}$-complete ultrafilter on $\mathfrak{m}^+$ as in the remark following Lemma 1.10. Also, by [Kun70, 7.6], $\mathcal{V}$ is equivalent via a bijection to some finite power $\mathcal{U}^n$ of $\mathcal{U}$ on the set $\mathfrak{m}^n$. Therefore, in $L[\mathcal{U}]$, there are only $\mathfrak{m}^+$ many $\mathfrak{m}$-complete ultrafilters on $\mathfrak{m}$, since there are only that many bijections. Consider the usual Tychonoff product $2^{(2^{\mathfrak{m}})}$ with topology τ. Since $\mathfrak{m}$ is strongly inaccessible, this space with the $\tau_{\mathfrak{m}}$ topology has a dense set of cardinality $\mathfrak{m}$. Then, analogous to the proof that there are $2^{\mathfrak{c}}$ ultrafilters on N, it is follows that there are $2^{2^{\mathfrak{m}}}$ $\mathfrak{m}$-complete filters on $\mathfrak{m}$ that pairwise do not extend to a common ultrafilter. It immediately follows that there are $\mathfrak{m}$-complete filters on $\mathfrak{m}$ itself which do not extend to $\mathfrak{m}$-complete ultrafilters.

Most of the above facts about $L[\mathcal{U}]$ are also in Kanamori's text [Kan03].

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