AUTOHOMEOMORPHISMS OF PRE-IMAGES OF $\mathbb{N}^*$

ALAN DOW

Abstract. In the study of the Stone-Čech remainder of the real line a detailed study of the Stone-Čech remainder of the space $\mathbb{N}\times[0,1]$, which we denote as $\mathbb{M}$, has often been utilized. Of course the real line can be covered by two closed sets that are each homeomorphic to $\mathbb{M}$. It is known that an autohomeomorphism of $\mathbb{M}^*$ induces an autohomeomorphism of $\mathbb{N}^*$. We prove that it is consistent with there being non-trivial autohomeomorphism of $\mathbb{N}^*$ that those induced by autohomeomorphisms of $\mathbb{M}^*$ are trivial.

1. Introduction

We consider the remainder $\mathbb{M}^* = \beta(\mathbb{N}\times[0,1])\setminus(\mathbb{N}\times[0,1]) = \beta\mathbb{M}\setminus\mathbb{M}$ and the projection $\pi:\mathbb{M}^*\to\mathbb{N}^*$ satisfying that $\pi((a\times[0,1])^*) = a^*$ for all infinite $a\subset\mathbb{N}$.

For each $u\in\mathbb{N}^*$, we let $\mathbb{I}_u$ denote the pre-image $\pi^{-1}(u)$ of u. Each $\mathbb{I}_u$ is a continuum, and it is an example of a standard subcontinuum as per [7]. Perhaps it is appropriate to refer to each of these (more special) $\mathbb{I}_u$ as primary standard subcontinuum.

It was shown in [4] that every autohomeomorphism Ψ of $\mathbb{M}^*$ induces an autohomeomorphism H_Ψ of $\mathbb{N}^*$ satisfying that $\Psi(\mathbb{I}_u) = \mathbb{I}_{H_\Psi(u)}$ (i.e. $H_\Psi\circ\pi = \pi\circ\Psi$). The question addressed in this article, raised in [4], is whether every autohomeomorphism H of $\mathbb{N}^*$ is induced by some autohomeomorphism of $\mathbb{M}^*$ in this manner. It is evident that this question has an affirmative answer if all autohomemorphisms of $\mathbb{N}^*$ are trivial. We prove in Theorem 5.1 below that it is consistent that a negative answer is also consistent. The affirmative answer under CH will appear in another paper. The main reference for results in this section is [7].

Within any compact space X and a sequence $\{A_n : n\in\mathbb{N}\}$ of subsets, it is common to let $u-\lim\{A_n\}_n$ denote the usual set of u-limits, $\bigcap\{\overline{\left(\bigcup_{n\in a} A_n\right)} : a\in u\}$.

In particular, if $\{[a_m,b_m] : m\in\mathbb{N}\}$ is a sequence of pairwise disjoint connected subintervals of $\mathbb{N}\times I$, we let $[a_m,b_m]_u$ denote $u-\lim\{[a_m,b_m]\}_m$. If there is some $U\in u$ satisfying that the sequence $\{[a_m,b_m] : m\in U\}$ is locally finite in $\mathbb{N}\times I$, then $[a_m,b_m]_u$ is also an example of a standard subcontinuum of $\mathbb{M}^*$.

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Say that a sequence $\{[a_m, b_m] : m \in \mathbb{N}\}$ is a standard sequence if it is a locally finite set of pairwise disjoint non-trivial connected intervals in $\mathbb{N} \times I$. A standard sequence will be called rational if all the end-points are rational numbers.

Definition 1.1. *If $\mathcal{A} = \{[a_m, b_m] : m \in \mathbb{N}\}$ is a standard sequence, let $\mathbb{M}^*_{\mathcal{A}}$ denote the set $cl_{\beta\mathbb{M}}(\bigcup \mathcal{A}) \setminus \mathbb{M}$.*

Lemma 1.2. *If $\mathcal{A}$ is a standard sequence, then $\bigcup \mathcal{A}$ is homeomorphic to $\mathbb{M}$, and $\mathbb{M}^*_{\mathcal{A}}$ is homeomorphic to $\mathbb{M}^*$.*

Proposition 1.3. *If K and L are disjoint compact subsets of $\mathbb{M}^*$, then there are standard sequences $\mathcal{A} = \{[a_m, b_m] : m \in \mathbb{N}\}$ and $\mathcal{C} = \{[c_m, d_m] : m \in \mathbb{N}\}$ and disjoint sets N_K, N_L of $\mathbb{N}$ such that*

1. *for all $m \in \mathbb{N}$, $a_m < c_m < d_m < b_m$,*
2. *K is contained in the $\beta\mathbb{M}$-closure of $\bigcup\{[c_m, d_m] : m \in N_K\}$*
3. *L is contained in the $\beta\mathbb{M}$-closure of $\bigcup\{[c_m, d_m] : m \in N_L\}$.*

A point x of $[a_m, b_m]_u$ is a cut-point if for every $f \in \mathbb{N}^{\mathbb{N}}$, there is a sequence $a_m < c_m < d_m < b_m$ $(m \in \mathbb{N})$ such that

1. x is a cut-point of $[c_m, d_m]_u$,
2. $\{m \in \mathbb{N} : d_m - c_m < 1/f(m)\} \in u$.

Say that a standard sequence $\{[c_m, d_m] : m \in \mathbb{N}\}$ is f-thin if $d_m - c_m < 1/f(m)$ for all $m \in \mathbb{N}$.

Certainly, if $a_m < x_m < b_m$ (for all $m \in \mathbb{N}$), then $x_u = u-\lim\{x_m : m \in \mathbb{N}\}$ is a (standard) cut-point of $[a_m, b_m]_u$.

The standard cut-points of $[a_m, b_m]_u$ are naturally linearly ordered as in the ultraproduct. The closure of any subinterval of the standard cut-points is said to be a subinterval of $[a_m, b_m]_u$.

If $\{[a_m, b_m] : m \in \mathbb{N}\}$ is a standard sequence then for every selection $a_m \leq x_m \leq b_m$, the set $\{x_u : u \in \mathbb{N}^*\}$ maps homeomorphically to $\mathbb{N}^*$ by the map x_u being sent to u. Say that a sequence $\{x_m : m \in \mathbb{N}\}$ is a selector sequence (for $\{[a_m, b_m] : m \in \mathbb{N}\}$) if $a_m < x_m < b_m$ for all $m \in \mathbb{N}$.

Proposition 1.4. *If a subset L of $\mathbb{M}^*$ is homeomorphic to an interval of any standard subcontinuum of any standard sequence then it is an actual subinterval of a standard subinterval of a standard sequence.*

I believe this next result is new. For any continuous $g : \mathbb{M} \to I$ let g^* denote the natural extension of g mapping $\mathbb{M}^*$ to I.

Theorem 1.5. *Suppose that $\Psi : \mathbb{M}^* \to \mathbb{M}^*$ is a homeomorphism and let $\{[a_m, b_m] : m \in \mathbb{N}\}$ be a standard sequence in $\mathbb{M}$. Then, for any selector sequence $\{x_m : m \in \mathbb{N}\}$ for $\{[a_m, b_m] : m \in \mathbb{N}\}$, there is a standard sequence $\{[c_j, d_j] : j \in \mathbb{N}\}$ and a homeomorphism $h : \mathbb{N}^* \to \mathbb{N}^*$ such that for each $u \in \mathbb{N}^*$,*

(1) $\Psi(x_u) \in [c_j, d_j]_{h(u)}$,
(2) $[c_j, d_j]_{h(u)}$ *is a subinterval of* $\Psi([a_m, b_m]_u)$,
(3) *the mapping* $\Psi(x_u)$ *to* $h(u)$ *is a homeomorphism from* $\Psi(\{x_m : m \in \mathbb{N}\}^*) = \{\Psi(x_u) : u \in \mathbb{N}^*\}$ *to* $\mathbb{N}^*$.

Proof. Choose a continuous function $g : \mathbb{M} \to I$ satisfying that, for every $m \in \mathbb{N}$, $g(x_m) = 1$ and $g(r) = 0$ for all $r \notin \bigcup\{(a_m, b_m) : m \in \mathbb{N}\}$.

Choose a continuous function $g_1 : \mathbb{M} \to I$ satisfying that $g_1^* = g^* \circ \Psi^{-1}$. For each $u \in \mathbb{N}^*$, let $y_u = \Psi(x_u)$. Observe that $g_1^*(y_u) = 1$ for all $u \in \mathbb{N}^*$.

Let $\{[c_j, d_j] : j \in \mathbb{N}\}$ enumerate all maximal connected intervals in $\mathbb{M}$ that satisfy that $(c_j, d_j) \cap g_1^{-1}(\frac{1}{2}, 1]$ is dense in $[c_j, d_j]$.

Consider any $u \in \mathbb{N}^*$ and the subcontinuum $\tilde{I}_u = \Psi(I_u)$. There is a standard interval subsequence $\{[\tilde{a}_m, \tilde{b}_m] : m \in \mathbb{N}\}$ of $\{[a_m, b_m] : m \in \mathbb{N}\}$ that has $\{x_m : m \in \mathbb{N}\}$ as a selector and is contained in $g^{-1}(\frac{3}{4}, 1])$. The point y_u is in $\Psi([\tilde{a}_m, \tilde{b}_m]_u)$ which is a subinterval of $\tilde{I}_u$. Also $\Psi([\tilde{a}_m, \tilde{b}_m]_u)$ is contained in the interior of $g_1^{-1}(\frac{1}{2}, 1]$. Now choose any standard sequence $\{[r_n, s_n] : n \in \mathbb{N}\}$ so that $g_1\left(\bigcup\{[r_n, s_n] : n \in \mathbb{N}\}\right)^* \subset (\frac{1}{2}, 1]$ and so that $\Psi([\tilde{a}_m, \tilde{b}_m]_u)$ is a subinterval of $[r_n, s_n]_v$ for some (unique) ultrafilter $v \in \mathbb{N}^*$.

For each $n \in \mathbb{N}$, choose the unique $j_n \in \mathbb{N}$ so that $[r_n, s_n] \subset (c_{j_n}, d_{j_n})$. Since $\{[r_n, s_n] : n \in \mathbb{N}\}$ is locally finite, the sequence $\{[c_{j_n}, d_{j_n}] : n \in \mathbb{N}\}$ is also locally finite. Hence $\{[c_{j_n}, d_{j_n}] : n \in \mathbb{N}\}$ is a standard sequence. The mapping $n \mapsto j_n$ is finite-to-one and now let w be the finite-to-one image of v. Consider the standard subcontinuum $[c_j, d_j]_w$. We check that $[c_j, d_j]_w$ is an interval in $\tilde{I}_u$, and that $h(u) = w$ is the map that satisfies the statement of the Lemma.

The continuum $[c_j, d_j]_w$ is contained in the component $\tilde{I}_u$ of $\Psi\left((\bigcup\{[a_m, b_m] : m \in \mathbb{N}\})^*\right)$ and contains the interval $\Psi([\tilde{a}_m, \tilde{b}_m]_u)$. It follows from the results in [Hart92], that $[r_n, s_n]_v$ is an interval in $[c_j, d_j]_w$. Choose any continuous function $g_2 : \mathbb{M} \to I$ satisfying that $g_2^*(y_u) = 1$ and $g_2^*(\tilde{I}_u \setminus [r_n, s_n]_v) = 0$.

Set $L = \{n : g_2([a_n, b_n]) \setminus [0, \frac{1}{4}] \neq \emptyset\}$ and $\tilde{L} = \{j_n : n \in L\}$. Since $g_2^*(y_u) = 1$, it follows that $L \in v$ and $\tilde{L} \in w$. Let the function sending each n to j_n be denoted by ρ. Thus $\rho^*(v) = w$. For every $\tilde{v} \in \mathbb{N}^*$ such that $\rho^*(\tilde{v}) = w$, we have that $[r_n, s_n]_{\tilde{v}}$ is a subset of $[c_j, d_j]_w \subset \tilde{I}_u$. Therefore v is the unique ultrafilter satisfying that $\rho^*(v) = w$ and $L \in v$. Therefore, by removing a finite set from L, we can assume that ρ is 1-to-1 on L.

Next, we note that y_u is in the interior of $(\bigcup\{[r_n, s_n] : n \in L)^*$ (easily checked by the results in [Hart92]) and so there is a $U \in u$ such that $\Psi(\{x_m : m \in U\}^*) \subset (\bigcup\{[r_n, s_n] : n \in L)^*$.

We now have our desired mapping defined on U^*. For each $\tilde{u} \in U^*$, there is a unique $v_{\tilde{u}} \in L^*$ so that $y_{\tilde{u}} \in [r_n, s_n]_{v_{\tilde{u}}}$. The continuous projection mapping on $(\bigcup\{[r_n, s_n] : n \in L)^*$ to L^* agrees with the mapping sending $y_{\tilde{u}}$ to $v_{\tilde{u}}$. The mapping sending $\tilde{u}$ to $y_{\tilde{u}}$ is induced by the continuous mapping $\Psi \restriction \{x_m : m \in U\}^*$. Therefore the mapping $h(\tilde{u})$ sent to $\rho^*(v_{\tilde{u}})$ is continuous (and 1-to-1). $\square$

2. $\mathbb{N}^*$ cut-sets

Definition 2.1. *Say that a subset K of $\mathbb{M}^*$ is an $\mathbb{N}^*$ cut-set (for the standard sequence) if there is a standard sequence $\mathcal{A} = \{[a_m, b_m] : m \in \mathbb{N}\}$ such that*

1. *K is a subset $\mathbb{M}^*_{\mathcal{A}}$,*
2. *$K \cap [a_m, b_m]_u$ is a cut-point of $[a_m, b_m]_u$ for every $u \in \mathbb{N}^*$,*
3. *the mapping from $(\bigcup\{[a_m, b_m] : m \in \mathbb{N}\})^*$ where $[a_m, b_m]$ is sent to m, to $\mathbb{N}^*$ is 1-to-1 on K.*

An $\mathbb{N}^*$ cut-set K is trivial if K equals D^* for some closed discrete $D \subset \mathbb{M}$.

A concept called non-trivial *maximal nice filters* on spaces of the form $\mathbb{N} \times X$ (with X compact) was introduced and studied in [2] where it was shown that PFA implies these do not exist if X is metrizable. The neighborhood filters in $\beta\mathbb{M}$ of a $\mathbb{N}^*$ cut-set result in maximal nice filters.

Let us a recall that a set K is said to be a P_κ-set in a space X if every $G_{<\kappa}$ of X that contains K has K in its interior. It is well-known that D^* is a $P_{\mathfrak{b}}$-set for every closed subset D of $\mathbb{M}$. In this next result we prove that an $\mathbb{N}^*$ cut-set has a neighborhood base resembling that of a cut-point. This result isn't strictly needed since the $\mathbb{N}^*$ cut-sets that we intend to consider are those of the form $\Psi(D^*)$ when D^* is a trivial $\mathbb{N}^*$ cut-set.

Proposition 2.2. *Every $\mathbb{N}^*$ cut-set K of $\mathbb{M}^*$ is a $P_{\mathfrak{b}}$-set in $\mathbb{M}^*$. Also, if the standard sequence $\mathcal{A} = \{[a_m, b_m] : m \in \mathbb{N}\}$ witnessing that K is an $\mathbb{N}^*$ cut-set, then there is a family of rational standard sequences $\{\mathcal{C}_f : f \in \mathbb{N}^{\mathbb{N}}\}$ such that, for each $f \in \mathbb{N}^{\mathbb{N}}$,*

$\mathcal{C}_f = \{[c^f_m, d^f_m] \subset [a_m, b_m] : m \in \mathbb{N}\}$ is f-thin

and the family $\{(\bigcup \mathcal{C}_f)^* : f \in \mathbb{N}^{\mathbb{N}}\}$ is a neighborhood base for K in $\mathbb{M}^*_{\mathcal{A}}$. Note also that the family $\{(\bigcup \mathcal{C}_f)^* : f \in \mathbb{N}^{\mathbb{N}}\}$ is $<\mathfrak{b}$-directed.

Proof. Let $\mathcal{A} = \{[a_m, b_m] : m \in \mathbb{N}\}$ be a standard sequence witnessing that K is an $\mathbb{N}^*$ cut-set. Let U be an open subset of $\beta\mathbb{M}_{\mathcal{A}}$ that contains K. Select, by Lemma 1.3, a pair $\tilde{\mathcal{A}} = \{[\tilde{a}_m, \tilde{b}_m] : m \in \mathbb{N}\}$ and $\tilde{\mathcal{C}} = \{[\tilde{c}_m, \tilde{d}_m] : m \in \mathbb{N}\}$ so that there are disjoint subsets N_K, N_U of $\mathbb{N}$ so that

1. for all $m \in \mathbb{N}$, $\tilde{a}_m < \tilde{c}_m < \tilde{d}_m < \tilde{b}_m$,
2. for all $m \in \mathbb{N}$ there is a k such that $[\tilde{a}_m, \tilde{b}_m] \subset [a_k, b_k]$,
3. K is contained in the $\beta\mathbb{M}$-closure of $\bigcup\{[\tilde{c}_m, \tilde{d}_m] : m \in N_K\}$,
4. $\mathbb{M}^*_{\mathcal{A}} \setminus U_n$ is contained in the $\beta\mathbb{M}$-closure of $\bigcup\{[\tilde{c}_m, \tilde{d}_m] : m \in N_{U_n}\}$.

Let ψ denote the function on $\mathbb{M}_{\mathcal{A}}$ onto $\mathbb{N}$ obtained by sending every point of $[a_m, b_m]$ to m. We may let ψ^* then denote the continuous extension of ψ to $\mathbb{M}^*_{\mathcal{A}}$ onto $\mathbb{N}^*$.

For each $x \in K$ (a cut-point) with $\psi^*(x) = u \in \mathbb{N}^*$, there is a set $L_x \subset N_K$ such that $\psi \restriction L_x$ is 1-to-1 and x is in the $\beta\mathbb{M}$-closure of the union of the standard sequence $\tilde{\mathcal{A}}_{L_x} = \{[\tilde{a}_m, \tilde{b}_m] : m \in L_x\}$. The closure of the $\tilde{\mathcal{A}}_{L_x}$ meets K in a clopen set and so, by possibly shrinking L_x we can ensure that x is still in the closure and that this clopen subset of K is equal to the intersection of K with the closure of the union of the set $\{[a_j, b_j] : j \in \psi[L_x]\}$ (i.e. all the $\mathcal{A}$-intervals that are hit by $\tilde{\mathcal{A}}_{L_x}$). Since there is a finite cover of K by such clopen sets (associated with the closures of these $\tilde{\mathcal{A}}_{L_x}$'s), this shows that there is a selector function $\sigma : \mathbb{N} \to \mathbb{N}$ so that

1. for each $m \in \mathbb{N}$, $[\tilde{a}_{\sigma(m)}, \tilde{b}_{\sigma(m)}]$ is a subset of $[a_m, b_m]$,
2. the closure of the union of the intervals in $\{[\tilde{a}_{\sigma(m)}, \tilde{b}_{\sigma(m)}] : m \in \mathbb{N}\}$ contains K in its interior,
3. the closure of the union of the intervals in $\{[\tilde{a}_{\sigma(m)}, \tilde{b}_{\sigma(m)}] : m \in \mathbb{N}\}$ is contained in U_n.

Choose any function $f \in \mathbb{N}^{\mathbb{N}}$ so that for all but finitely many m, each of the distances $\{|\tilde{a}_{\sigma(m)} - \tilde{c}_{\sigma(m)}|, |\tilde{d}_{\sigma(m)} - \tilde{c}_{\sigma(m)}|, \{|\tilde{d}_{\sigma(m)} - \tilde{b}_{\sigma(m)}|\}$ are greater than $5/f(m)$. For every $x \in K$, applying the definition of cut-point, choose an f-thin standard sequence $\mathcal{C}_x = \{[c^x_m, d^x_m] : m \in \mathbb{N}\}$ so that $x \in [c^x_m, d^x_m]_u$ where $u = \psi^*(x)$. Again, we can choose $L_x \subset \mathbb{N}$ so that $L_x \in u$ and so that for all $y \in K$ with $L_x \in \psi^*(y)$, we also have that y is in the $\beta\mathbb{M}$ closure of the union of $\mathcal{C}_x$. In fact L_x is simply any set such that $L^*_x = \psi^*(K \cap \mathrm{cl}_{\beta\mathbb{M}}(\bigcup\{[c^x_m, d^x_m] : m \in \mathbb{N}\}))$. If needed, we can make $<1/f(n)$ changes to any of the c^x_m's and d^x_m's so as to ensure every $y \in K$ with $L_x \in \psi^*(y)$, is in the interior of $\mathrm{cl}_{\beta\mathbb{M}}(\bigcup\{[c^x_m, d^x_m] : m \in \mathbb{N}\})$. Now consider the sequence $\{[\tilde{a}_{\sigma(m)}, \tilde{b}_{\sigma(m)}] : m \in \mathbb{N}\}$. It follows that, for all but finitely many $m \in L_x$, the interval $[c^x_m, d^x_m]$ meets the interval $[\tilde{c}_{\sigma(m)}, \tilde{d}_{\sigma(m)}]$. By the choice of f, we then have that, for all but finitely many $m \in L_x$, the interval $[c^x_m, d^x_m]$ is contained in $[\tilde{a}_{\sigma(m)}, \tilde{b}_{\sigma(m)}]$. This shows that $(\bigcup\{[c^x_m, d^x_m] : m \in \mathbb{N}\})^*$ is contained in U and is a neighborhood of a relatively clopen set of points of K. By the compactness of K there is an f-thin standard sequence $\mathcal{C}_f = \{[c^f_m, d^f_m] \subset [a_m, b_m] : m \in \mathbb{N}\}$ such that $(\bigcup \mathcal{C}_f)^*$ is contained in U.

Now we prove that the family is $<\mathfrak{b}$-directed. Now let $F \subset \mathbb{N}^{\mathbb{N}}$ be any set with $|F| < \mathfrak{b}$. For each $f \in F$, choose also $g_f \in \mathbb{N}^{\mathbb{N}}$ so that $f < g_f$ and for all $m \in \mathbb{N}$, $[c^m_{g_f}, d^m_{g_f}] \subset (c^m_f, d^m_f)$ (using that the closure of $D = \{c^m_f, d^m_f : m \in \mathbb{N}\}$ in $\beta\mathbb{M}$ is disjoint from K. For each $f \in F$, let $h_f \in \mathbb{N}^{\mathbb{N}}$ satisfy that, for all $m \in \mathbb{N}$, $5/h_f(m)$ is less than each of the values $c^m_{g_f} - c^m_f$, $d^m_f - d^m_{g_f}$, and $d^m_{g_f} - c^m_{g_f}$. Choose $\bar{f} \in \mathbb{N}^{\mathbb{N}}$ so that for all $f \in F$, $h_f <^* \bar{f}$. Fix any $f \in F$, and note that there is some m_f so that, for all $m > m_f$, we have that $[c^m_{\bar{f}}, d^m_{\bar{f}}] \cap [c^m_{g_f}, d^m_{g_f}]$ is not empty and that $h_f(m) < f(m)$. Consider any $m > m_f$ and the required condition: $c^m_{g_f} \leq d^m_{\bar{f}}$ or $c^m_{\bar{f}} \leq d^m_{g_f}$. In the first case we then have

$$c^m_f < c^m_{g_f} - \frac{1}{\bar{f}(m)} < d^m_{\bar{f}} - \frac{1}{\bar{f}(m)} < c^m_{\bar{f}} < d^m_{\bar{f}} < c^m_{\bar{f}} + \frac{1}{\bar{f}(m)} < c^m_{g_f} + \frac{1}{g_f(m)} < d^m_{g_f} < d^m_f$$

which implies that $[c^m_{\bar{f}}, d^m_{\bar{f}}] \subset (c^m_f, d^m_f)$. The case when $c^m_f \leq d^m_{g_f}$ also implies that $[c^m_{\bar{f}}, d^m_{\bar{f}}] \subset (c^m_f, d^m_f)$ by a similar argument. ☐

Definition 2.3. *Say that an indexed family $\{\mathcal{C}_f : f \in \mathbb{N}^{\mathbb{N}}\}$ is an $\mathbb{N}^*$ cut-set of $\mathbb{M}^*$*

1. *every $\mathcal{C}_f$ is a rational standard sequence indexed as $\{[c^m_f, d^m_f] : m \in \mathbb{N}\}$,*
2. *the sequence $\mathcal{C}_f$ is f-thin,*

(3) *the family is countably directed in the sense that, for each family* $\{f_n : n \in \omega\} \subset \mathbb{N}^{\mathbb{N}}$, *there is an* $f \in \mathbb{N}$ *such that, for all* $n \in \omega$, $f <^* f_n$ *and* $\{m : [c_f^m, d_f^m] \not\subset (c_{f_n}^m, d_{f_n}^m)\}$ *is finite.*

Given any $\mathcal{C}_f$ *and subset* I *of* $\mathbb{N}$, *let* $\mathcal{C}_f \restriction I = \{[c_f^m, d_f^m] : m \in I\}$. *For an* $\mathbb{N}^*$ *cut-set family* $\mathfrak{C} = \{\mathcal{C}_f : f \in \mathbb{N}^{\mathbb{N}}\}$, *and a subset* $I \subset \mathbb{N}$, *let* $\mathfrak{C} \restriction I$ *denote the (re-indexed)* $\mathbb{N}^*$ *cut-set* $\{\mathcal{C}_f \restriction I : f \in \mathbb{N}^{\mathbb{N}}\}$. *Let Triv*$(\mathfrak{C})$ *denote the ideal of subsets* I *of* $\mathbb{N}$ *satisfying that the family* $\mathfrak{C} \restriction I$ *is trivial.*

3. The effects of two consequences of PFA

In this section we work with two assumptions $(\dagger_1^+)$ and $(\dagger_2^+)$, each of which is a consequence of PFA. In fact, $(\dagger_2^+)$ is a consequence of the open graph axiom (formerly OCA). Let H be an autohomeomorphism of $\mathbb{N}^*$. For a subset a of $\mathbb{N}$, H is said to be trivial on a^* if (by possibly removing a finite subset of a) there is a bijection h_a from a into $\mathbb{N}$ such that $H(c^*) = (h_a(c))^*$ for all $c \subset a$. The family, $\mathrm{Triv}(H) = \{a \subset \mathbb{N} : H \text{ is trivial on } a^*\}$, is an ideal on $\mathcal{P}(\mathbb{N})$. An ideal $\mathcal{I}$ on a countable set D is said to be ccc over fin, if given any uncountable family of pairwise almost disjoint subsets of D, all but countably many of them are in $\mathcal{I}$.

Thus one says that an autohomeomorphism H of $\mathbb{N}^*$ (or of D^* for any countable discrete set D) is trivial modulo ccc over fin, if the ideal $\mathrm{Triv}(H)$ is ccc over fin. Similarly H is somewhere trivial if there is some infinite set $I \in \mathrm{Triv}(H)$.

Definition 3.1. *The statement* $(\dagger_1^+)$ *is the assertion that every autohomeomorphism of* $\mathbb{N}^*$ *is trivial. The statement* $(\dagger_1)$ *is the statement that every such autohomeomorphism is trivial modulo ccc over fin. While we are at it, let* $(\dagger_1^-)$ *be the statement that every autohomeomorphism of* $\mathbb{N}^*$ *is somewhere trivial.*

The statement $(\dagger_2^+)$ *is the assertion that every* $\mathbb{N}^*$ *cut-set of* $\mathbb{M}^*$ *is trivial, and again* $(\dagger_2)$ *is the statement that every* $\mathbb{N}^*$ *cut-set of* $\mathbb{M}^*$ *is trivial on an ideal that is ccc over fin.*

Lemma 3.2. $(\dagger_2^+)$ *holds if* $\mathfrak{c} = \aleph_2$ *and the open graph axiom holds.*

Proof. Let $\{\mathcal{C}_f : f \in \mathbb{N}^{\mathbb{N}}\}$ be an $\mathbb{N}^*$ cut-set of $\mathbb{M}^*$. Using that the open graph axiom implies that $\mathfrak{b} = \aleph_2$ and we are assuming $\mathfrak{c} = \aleph_2$, we can recursively choose a dominating family $\{f_\gamma : \gamma \in \omega_2\} \subset \mathbb{N}^{\mathbb{N}}$ so that, by possibly making finite modifcations to each, the family $\{\mathcal{C}_{f_\gamma} : \gamma \in \omega_2\}$ is directed as in Proposition 2.2 and is cofinal in the family $\{\mathcal{C}_f : f \in \mathbb{N}^{\mathbb{N}}\}$. Let F denote the family $\{f_\gamma : \gamma \in \omega_2\}$ without the indexing.

Let R be the set of pairs $(f, f') \in [F]^2$ such that there is an $m \in \mathbb{N}$ such that $[c_f^m, d_f^m] \cap [c_{f'}^m, d_{f'}^m]$ is empty. If we identify each $f \in F$ with the corresponding sequence $\mathcal{C}_f$ as an element of $(\mathbb{Q}^2)^{\mathbb{N}}$ viewed as a product of discrete spaces, then the relation R is open in the resulting metric space. Assume that $\{f_\alpha : \alpha \in \omega_1\} \subset F$ satisfies that $(f_\alpha, f_\beta) \in R$ for all $\alpha \neq \beta \in \omega_1$. We may choose an $f \in F$ so that, for all $\alpha < \omega_1$, there is an $m_\alpha \in \mathbb{N}$ so that $[c_f^m, d_f^m] \subset (c_{f_\alpha}^m, d_{f_\alpha}^m)$ for all $m > m_\alpha$. Fix an $\bar{m} \in \mathbb{N}$ and an uncountable $\Lambda \subset \omega_1$ so that $m_\alpha = \bar{m}$ for all $\alpha \in \Lambda$. Also choose uncountable $\Lambda_1 \subset \Lambda$ so that, for all $\alpha, \beta \in \Lambda_1$

and $m \leq \bar{m}$, $[c^m_{f_\alpha}, d^m_{f_\alpha}] = [c^m_{f_\beta}, d^m_{f_\beta}]$. We now have a contradiction since, for all $\alpha \neq \beta \in \Lambda_1$, $[c^m_{f_\alpha}, d^m_{f_\alpha}] \cap [c^m_{f_\beta}, d^m_{f_\beta}]$ is not empty (contradicting the pair (f_α, f_β) is supposed to be in R).

By the OGA, it follows there must be a cover of F by a countable collection $\{F_n : n \in \omega\}$ satisfying that $[F_n]^2$ is disjoint from R for all $n \in \omega$. Choose any n so that F_n is a $<^*$-dominating subfamily of $\mathbb{N}^\mathbb{N}$. By our construction, this also ensures that $\{\mathcal{C}_f : f \in F_n\}$ is cofinal in the original family $\{\mathcal{C}_f : f \in \mathbb{N}^\mathbb{N}\}$ in the sense of Proposition 2.2. Fix any $m \in \mathbb{N}$ and observe that the family $\{[c^m_f, d^m_f] : f \in F_n\}$ is linked. Of course this means we can choose a sequence $\{x_m : m \in \mathbb{N}\}$ satisfying that $x_m \in [c^m_f, d^m_f]$ for all $f \in F_n$ and this contradicts that there should be some $f \in F_n$ satisfying that the $\beta\mathbb{M}$-closure of $\{x_m\}_{m\in\mathbb{N}}$ should be disjoint from the $\beta\mathbb{M}$-closure of $\bigcup \mathcal{C}_f$.

□

Theorem 3.3 (Assume ($\dagger_1$) and ($\dagger_2$)). *Every autohomeomorphism Ψ on $\mathbb{M}^*$ induces a trivial autohomeomorphism on $\mathbb{N}^*$.*

Proof. Assume that Ψ is an autohomeomorphism of $\mathbb{M}$. For simplicity assume that $\Psi(\,(\mathbb{N}\times \{0\})^*\,) = (\mathbb{N} \times \{0\})^*$. Let H denote the autohomeomorphism on $\mathbb{N}^*$ induced by Ψ, hence $H \circ \pi = \pi \circ \Psi$.

Start with the continuous function $g : \mathbb{M} \to [0, 1]$ satisfying that, for each $n \in \mathbb{N}$,

(1) $g^{-1}(0) \cap (\{n\} \times [0, 1]) = \{n\} \times \{\frac{i}{n} : 0 \leq i \leq n\}$,
(2) $g^{-1}(1) \cap (\{n\} \times [0, 1]) = \{n\} \times \{\frac{2i+1}{2n} : 0 \leq i < n\}$,
(3) $g \restriction \big(\{n\} \times [\frac{i}{2n}, \frac{i+1}{2n}]\big)$ is linear for all $0 \leq i < 2n$.

Simply the graph of $g \restriction (\{n\} \times [0, 1])$ oscillates 0 to 1 and back to 0 exactly n times. Let $D = \{(n, q) : n \in \mathbb{N}, q \in \{\frac{i}{2n} : 0 < i < 2n\}$. Of course D is a closed discrete subset of $\mathbb{M}$. For an infinite subset J of $\mathbb{N}$, let $D_J = D \cap (J \times [0, 1])$.

Let g_1 be a continuous map on $\mathbb{M}$ satisfying that $g_1^* = g^* \circ \Psi^{-1}$. For each $n \in \mathbb{N}$, let us count, call it L_n, the number of times that g_1 oscillates as follows. Fix any $n \in \mathbb{N}$ and set $t^n_0 = 0$ and define (if it exists) $t^n_1 = \inf\{r : g_1(n, r) = \frac{2}{3}\}$ (we think of g_1 as "on its way towards 1"). Next, we define t^n_2 to be $\inf\{r \in [t^n_1, 1] : g_1(n, r) = \frac{1}{3}\}$ (i.e. "g_1 is on its way back towards 0"). Continue recursively defining this increasing sequence $t^n_0, t^n_1, \ldots, t^n_{L_n}$ so that $g_1(n, t^n_i) = \frac{1}{3}$ for even $i \leq L_n$ and $g_1(n, t^n_i) = \frac{2}{3}$ for odd $i \leq L_n$. Also, $g_1 \restriction \big(\{n\} \times [t^n_i, t^n_{i+1}]\big)$ is either contained in $[0, \frac{2}{3}]$ or in $[\frac{1}{3}, 1]$. For each $0 \leq i < L_n$ also choose $s^n_i \in [t^n_i, t^n_{i+1}]$ so that, if i is even, $g_1((n, s^n_i)) = \min(g_1(\{n\} \times [t^n_i, t^n_{i+1}])$, and, if i is odd, $g_1((n, s^n_i)) = \max(g_1(\{n\} \times [t^n_i, t^n_{i+1}])$. It follows from the fact that $g_1^* = g^* \circ \Psi$, that if $0 \leq i_n < L_n$ ($n \in \mathbb{N}$) is a sequence with each i_n even, then $\limsup\{g_1((n, s^n_{i_n})) : n \in \mathbb{N}\} = 0$. Similarly the limit would be 1 for a sequence of odd values for i_n. It also follows similarly, that the sequence $\{L_n : n \in \mathbb{N}\}$ is, mod finite, a diverging sequence of even numbers. For convenience (by possibly redefining finitely many) we assume L_n is even for all $n \in \mathbb{N}$.

Let σ be the function in ${}^\mathbb{N}\mathbb{N}$ defined by $\sigma(m) = L_m/2$ (it will help to change letters). Let us assume that H is not trivial (meaning $\mathbb{N}$ is not in $\mathrm{Triv}(H)$). We break the rest of the proof into cases based on properties of σ. In each case we prove that there is an infinite $a \in \mathrm{Triv}(H)$ satisfying that $L_{h_a(n)} \neq n$ for all $n \in a$. Appealing to symmetry we complete the proof in the case that $L_{h_a(n)} < n$ for all $n \in a$. Note that $(\dagger_1)$ implies that every infinite $b \subset \mathbb{N}$ has an infinite subset that is in $\mathrm{Triv}(H)$.

First case is when σ is, mod finite, a permutation on $\mathbb{N}$. Since $\mathbb{N} \notin \mathrm{Triv}(H)$, the almost permutation σ^{-1} does not induce H. There is an infinite $b \subset \mathbb{N}$ such that $(\sigma^{-1}(b))^*$ and $H(b^*)$ are distinct. By taking complements if needed, we can assume that $(\sigma^{-1}(b))^*$ is not contained in $H(b^*)$. Then choose an infinite $J \subset b$ so that $(\sigma^{-1}(J))^*$ is disjoint from $H(b^*) \supset H(J^*)$. Now by $(\dagger_1)$ we can assume that $J \in \mathrm{Triv}(H)$ and choose the injection $h_J : J \to \mathbb{N}$ witnessing that $a \in \mathrm{Triv}(H)$. It follows that, for every $n \in J$, $L_{h_J(n)} \neq n$.

Next case is that there is some infinite $b \subset \mathbb{N}$ such that b is disjoint from $\{L_m : m \in \mathbb{N}\}$. Again choose an infinite $J \subset b$ in $\mathrm{Triv}(H)$ and we again have $L_{h_J(n)} \neq n$ for all $n \in J$.

The final case is that σ is not 1-to-1. Choose any disjoint pair b_0, b_1 of infinite subset of $\mathbb{N}$ satisfying that σ is 1-to-1 on each while $\sigma[b_0] = \sigma[b_1]$. First choose an infinite $a_0 \subset b_0$ with $a_0 \in \mathrm{Triv}(H)$. Next choose $a_1 \subset b_1$ so that $a_1 \in \mathrm{Triv}(H)$ and $\sigma[a_1] \subset \sigma[a_0]$. Shrink a_0 so that $\sigma[a_0] = \sigma[a_1]$. We may choose $J \in \mathrm{Triv}(H)$ to be a subset of one of a_0, a_1 so that again $L_{h_a(n)} \neq n$ for all $n \in a$.

Now we continue the proof by assuming that J is an infinite set in $\mathrm{Triv}(H)$ satisfying that $L_{h_J(n)} < n$ for all $n \in J$.

Choose a standard sequence (indexed by D) $\{[a_d, b_d] : d \in D\}$ so that $d \in (a_d, b_d)$ for all $d \in D$. Of course D is a selector set for the standard sequence $\mathcal{A} = \{[a_d, b_d] : d \in D\}$ and D^* is an $\mathbb{N}^*$ cut-set for $\mathcal{A}$. We pass to the subset $D_J = D \cap (J \times [0,1])$ and $\mathcal{A}_J = \{[a_d, b_d] : d \in D_J\}$. More generally, for $I \subset J$, let $\mathcal{A}_I = \{[a_d, b_d] : d \in D_I\}$.

By considering an uncountable almost disjoint family of infinite subsets of J and using Theorem 1.5 we have that $(\dagger_2)$ implies that the $\mathbb{N}^*$ cut-set $\Psi(D_J^*)$ is trivial on an ideal that is ccc over fin, there is an infinite $I \subset J$ so that $\Psi(D_I^*) = K$ is a trivial $\mathbb{N}^*$ cut-set with respect to the homeomorphic copy of $\mathbb{M}^*$ we get from $(H(I) \times [0,1])^*$. Therefore we may choose a closed discrete $E \subset H(I) \times [0,1]$ so that $E^* = K$. Notice that $\Psi \restriction D_I^*$ is an homeomorphism from D_I^* to E^*. Again, by using $(\dagger_1)$ and by possibly further shrinking I in the same manner that we shrunk J to obtain I, we can assume that $\Psi \restriction (D_I^* \cap \mathbb{M}_I^*)$ is trivial. That means there is a (mod finite) bijection $f : D_I \to E \cap \mathbb{M}_{H(I)}$ inducing Ψ on $D_I^* \cap \mathbb{M}_I^*$. We omit the easy verification that, by removing a finite set from I, we have that for $d_1 < d_2 \in D \cap (\{n\} \times [0,1])$ $(n \in I)$, $f(d_1) < f(d_2) \in E \cap (\{h_I(n)\} \times [0,1])$.

Fix any $n \in I$, and for each $0 \leq i \leq 2n$, let $e_{i,n} = f((n, \frac{i}{2n}))$ (noting that $(n, \frac{i}{2n}) \in D_I$).

Claim 1. For all but finitely many $n \in b$,

(1) $g_1(e_{i,n}) < \frac{1}{3}$ for all even $i \leq 2n$, and
(2) $g_1(e_{i,n}) > \frac{2}{3}$ for all odd $i < 2n$.

Indeed, g^* will send every point of $\left(\{(n, \frac{2i}{2n}) : n \in b, i \leq n\}\right)^*$ to 0, and so g_1^* must send every point of $\left(\{f((n, \frac{2i}{2n})) = e_{2i,n} : n \in b, i \leq n\}\right)^*$ to 0. The analogous property holds for the set of $e_{2i+1,n}$ $(n \in b,\ i < n)$. It thus follows that $g_1 \restriction (h_a(n) \times [0,1])$ must oscillate at least n times, contradicting that the oscillation number $L_{h_a(n)}$ is less than n. □

4. PFA

In this section, in working towards our main result, we give an alternate proof of Vignati's ([19]) theorem that PFA implies that every autohomeomorphism of $\mathbb{M}^*$ is trivial. We will work with $(\dagger_1^+)$ and $(\dagger_2^+)$ and we also assume the principle defined next.

Definition 4.1. *Say that $\mathcal{H}$ is an ω^ω-family if $\mathcal{H} = \{h_f : f \in \omega^\omega\}$ is a family of functions satisfying simply that $\mathrm{dom}(h_f) = \{(n,m) \in \omega^2 : m < f(n)\}$. We then say that such a family $\mathcal{H}$ is coherent, if whenever $f <^* g$ are in ω^ω, then $\{(n,m) \in \mathrm{dom}(h_f) \cap \mathrm{dom}(h_g) : h_f((n,m)) \neq h_g((n,m))\}$ is finite.*

Say that the principle ω^ω-cohere holds if each ω^ω-family $\mathcal{H}$ that is coherent, there is a function h with domain $\omega \times \omega$ such that $\mathcal{H} \cup \{h\}$ is also coherent.

The principle ω^ω-cohere (not so named) is a well-known consequence of OCA due to Todorcevic (see [6, 2.2.7]). It is well-known (essentially due to Hausdorff) that the principle ω^ω-cohere implies that $\mathfrak{b} > \omega_1$. Todorcevic also proved the following Proposition.

Proposition 4.2. *If $\omega_1 < \mathfrak{b}$, then for any coherent ω^ω-family $\mathcal{H}$, there is a countable set that mod finite contains the range of each $h \in \mathcal{H}$.*

Proof. Assume there is no such countable set. Recursively choose $\{f_\alpha : \alpha < \omega_1\} \subset \omega^\omega$, so that the range of h_{f_α} is not, mod finite, contained in the union of the ranges of the family $\{h_{f_\beta} : \beta < \alpha\}$. Choose any $f \in \omega^\omega$ satisfying that $f_\alpha <^* f$ for all $\alpha < \omega_1$. Let L denote the range of h_f and choose $\delta < \omega_1$ large enough so that any point of L that is in the range of any h_{f_α} $(\alpha < \omega_1)$ is already in the range of some h_{f_α} with $\alpha < \delta$. We now have a contradiction to the coherence assumption that guarantees that h_{f_δ} is, mod finite, contained in h_f. □

Theorem 4.3. *If $(\dagger_1^+), (\dagger_2^+)$ and the principle ω^ω-cohere hold, then every autohomeomorphism of $\mathbb{M}^*$ is trivial.*

Proof. Let Ψ be an autohomeomorphism of $\mathbb{M}^*$. Since we are assume $(\dagger_1^+)$, there is no loss to assume that $\pi = \pi \circ \Psi$ (i.e. Ψ is a lifting of the identity map on $\mathbb{N}^*$). Fix an enumeration, $\{q_\ell : \ell \in \omega\}$, of the rationals in $[0,1]$. For every $f \in \omega^{\mathbb{N}}$, let $D_f = \{(n, q_\ell) : n \in \mathbb{N}, \ell < f(n)\}$. Before proceeding we note that the collection $\bigcup\{(D_f)^* : f \in \omega^{\mathbb{N}}\}$ is a dense subset of $\mathbb{M}^*$. By $(\dagger_2)$, we may fix, for each $f \in \omega^{\mathbb{N}}$, a countable set $E_f \subset \mathbb{M}$, satisfying that $(E_f)^* = \Psi((D_f)^*)$. Then by $(\dagger_1^+)$, we may fix a lifting $\sigma_f : D_f \to E_f$ that induces the homeomorphism $\Psi \restriction (D_f)^*$.

Loosely identifying $\mathbb{N}$ and $\{q_\ell : \ell < \omega\}$ with ω, it is apparent that $\mathcal{H} = \{\sigma_f : f \in \omega^{\mathbb{N}}\}$ can be regarded as an ω^ω-family. Since σ_f and σ_g (for $f <^* g$) induce the same mapping on $(D_f)^*$ (i.e. the mapping Ψ), it follows that $\mathcal{H}$ is a coherent family. By the principle ω^ω-cohere, we may choose a function $\sigma : \mathbb{N} \times \{q_\ell : \ell \in \omega\} \to \mathbb{M}$ satisfying that $\sigma_f \subset^* \sigma$ for all $f \in \omega^{\mathbb{N}}$. It follows that σ is, mod finite, 1-to-1 and $\sigma \restriction (\{n\} \times I)$ is an order-preserving

preserving function into $\{n\}\times I$ because any failure would violate $\sigma_f \subset^* \sigma$ for some suitably large f. Similarly, it is easily shown that the range of σ is dense in $\{n\} \times I$ for all but finitely many $n \in \mathbb{N}$.

The final thing to show is that, for all but finitely many $n \in \mathbb{N}$, $\sigma \restriction (\{n\} \times [0,1])$ is the restriction of a homeomorphism $g_n : \{n\} \times [0,1] \to \{n\} \times [0,1]$, and that the resulting (almost homeomorphism) $g = \bigcup_n g_n$ from $\mathbb{M}$ to $\mathbb{M}$ satisfies that the Stone-Čech extension βg contains Ψ. By continuity and density, it suffices to show that for any sequence $R = \{(n, r_n) : n \in \mathbb{N}\} \subset \mathbb{M}$, for all but finitely many $n \in \mathbb{N}$, g_n is continuous at (n, r_n) and that $\beta g \restriction R^* = \Psi \restriction R^*$. This we do now. Given such a set R (an $\mathbb{N}^*$ cut-set), we can choose a sequence $S = \{(n, s_n) : n \in \mathbb{N}\} \subset \mathbb{M}$ satisfying that $\Psi(R^*) = S^*$. Suppose there is an infinite set $a \subset \mathbb{N}$ satisfying that the oscillation of σ on each neighborhood of (n, r_n) is greater than some $\epsilon_n > 0$. Apply the continuity of Ψ so as to choose a standard sequence $\{\{n\} \times [c_n, d_n] : n \in a\}$ satisfying that $c_n < r_n < d_n$ for all $n \in a$ and so that Ψ sends the set $\left(\bigcup_{n\in a}\{n\} \times [c_n, d_n]\right)^*$ into the set $\left(\bigcup_{n\in a}\{n\} \times (s_n - \epsilon_n/4, s_n + \epsilon_n/4)\right)^*$. Next choose two functions $\rho_1, \rho_2 \in \omega^{\mathbb{N}}$ satisfying that, for all $n \in a$, $\{q_{\rho_1(n)}, q_{\rho_2(n)}\} \subset (c_n, d_n)$ and so that $|\sigma((n, q_{\rho_1(n)})) - \sigma((n, q_{\rho_2(n)}))| > \epsilon_n$. Choose any $f \in \omega^{\mathbb{N}}$ large enough so that $\rho_1(n) + \rho_2(n) < f(n)$ for all $n \in \mathbb{N}$. Clearly there is an infinite set $b \subset a$ such that, by symmetry, $\sigma((n, q_{\rho_1(n)})) \notin [s_n - \epsilon_n/4, s_n + \epsilon_n/4]$ for all $n \in b$. But now the set $R_1 = \{(n, q_{\rho_1(n)}) : n \in b\}$ is a subset of $R_2 = D_f \cap \left(\bigcup_{n\in a}\{n\} \times [c_n, d_n]\right)$. Since $\Psi \restriction R_2^*$ is induced by σ, we have a contradiction since $(\sigma(R_1))^*$ and $\Psi(R_2^*)$ are disjoint. $\square$

5. Non-trivial autohomeomorphisms

In this section we return to the question raised in [4]: "Given an autohomeomorphism $H : \mathbb{N}^* \to \mathbb{N}^*$, does there exist an autohomeomorphism $\Psi : (\mathbb{N} \times [0,1])^* \to (\mathbb{N} \times [0,1])^*$ satisfying $H \circ \pi = \pi \circ \Psi$?" We prove that it is consistent to have $(\dagger_1)$ and $(\dagger_2)$ holding in a model in which there is a non-trivial autohomeomorphism of $\mathbb{N}^*$ (i.e. $(\dagger_1^+)$ fails). Once we succeed, then the following is a consequence of Theorem 3.3.

Theorem 5.1. *There is a model in which there are non-trivial autohomeomorphisms of $\mathbb{N}^*$ and every automorphism on $\mathbb{N}^*$ induced by an autohomeomorphism of $\mathbb{M}^*$ is trivial.*

The model was introduced by Velickovic ([17]) and we will use the further analysis of the forcing initiated in [12]. We do not know if $(\dagger_2^+)$ holds in this model. It would be interesting to have a formulation of a suitable weakening of OCA that can be shown to hold in this model.

First we define the partial order $\mathbb{P}_2$ from [17]

Definition 5.2. *The partial order $\mathbb{P}_2$ is defined to consist of all 1-to-1 functions f where*

(1) $\operatorname{dom}(f) = \operatorname{range}(f) \subset \mathbb{N}$,
(2) *for all* $i \in \operatorname{dom}(f)$ *and* $n \in \omega$, $f(i) \in [2^n, 2^{n+1})$ *if and only if* $i \in [2^n, 2^{n+1})$
(3) $\limsup_{n\to\omega} |[2^n, 2^{n+1}) \setminus \operatorname{dom}(f)| = \omega$
(4) *for all* $i \in \operatorname{dom}(f)$, $i = f^2(i) \neq f(i)$.

The ordering on $\mathbb{P}_2$ is $\subseteq^*$.

It is shown in [17], see also [12], that $\mathbb{P}_2$ is σ-directed closed. The following partial order was introduced in [12] as a great tool to uncover the forcing preservation properties of $\mathbb{P}_2$, such as the fact that $\mathbb{P}_2$ is $\aleph_2$-distributive (and so introduces no new ω_1-sequences of subsets of $\mathbb{N}$).

Definition 5.3 ([12, 2.2])**.** *Given $\{p_\xi : \xi \in \mu\}$, define $\mathbb{P}_2(\{p_\xi : \xi \in \mu\})$ to be the partial order consisting of all $q \in \mathbb{P}_2$ such that there is some $\xi \in \mu$ such that $q =^* p_\xi$. The ordering on $\mathbb{P}_2(\{p_\xi : \xi \in \mu\})$ is $p \leq q$ if $p \supseteq q$ (rather than $p \supseteq^* q$ for $\mathbb{P}_2$).*

Lemma 5.4 ([14])**.** *In the forcing extension, $V[H]$, by $2^{<\omega_1}$, there is a maximal $\subset^*$-descending sequence $\{p_\xi : \xi \in \omega_1\} \subset \mathbb{P}_2$ which is $\mathbb{P}_2$-generic over V and for which $\mathbb{P}_2(\{p_\xi : \xi \in \omega_1\})$ is ccc, ω^ω-bounding, and preserves that $\mathbb{R} \cap V$ is not meager.*

Proposition 5.5. *If $\xi \in \omega_1$ and $\{p_\xi : \xi \in \mu\}$ is a descending sequence in $\mathbb{P}_2$, then $\mathbb{P}_2(\{p_\xi : \xi \in \mu\})$ is a countable atomless poset.*

Lemma 5.6 ([12, 2.4])**.** *Given $\eta \in \omega_1$, a $\subset^*$-descending sequence $\{p_\xi : \xi \in \eta\} \subset \mathbb{P}_2$, and a countable elementary submodel $\mathfrak{U} \prec (H(\aleph_2), \in)$, such that $\{p_\xi : \xi \in \eta\} \in \mathfrak{U}$, then there is a $p \in \mathbb{P}_2$ which is $\mathfrak{U}$-generic for $\mathbb{P}_2(\{p_\xi : \xi \in \eta\})$. Moreover, for any extension $\{p_\xi : \xi \in \mu\} \subset \mathbb{P}_2$ (again $\subset^*$-descending) such that $\eta < \mu$ and $p_\eta = p$, every $D \in \mathfrak{U}$ is predense in $\mathbb{P}_2(\{p_\xi : \xi \in \mu\})$ provided it is dense in $\mathbb{P}_2(\{p_\xi : \xi \in \eta\})$.*

Almost all of the work we have to do is to establish additional preservation results for the poset $\mathbb{P}_2(\{p_\xi : \xi \in \omega_1\})$. The sequence $\{p_\xi : \xi \in \omega_1\}$ (chosen in the forcing extension by $2^{<\omega_1}$) will always be assumed to be $\mathbb{P}_2$-generic over the PFA model. Following [12], let $\mathcal{F}$ be the filter on $\mathbb{P}_2$ generated by the sequence $\{p_\xi : \xi \in \omega_1\}$. Note that $V[\mathcal{F}]$ is a generic extension of V meaning that $\{p_\xi : \xi \in \omega_1\}$ selects an element of every maximal antichain of $\mathbb{P}_2$ that is an element of V but it is chosen within the model of CH and also introduces an ω_1 sequence cofinal in ω_2. We may assume, as per [12], that many statements about $\mathbb{P}_2$-names of Borel subsets of $\mathbb{M}$ that are forced by $\mathbb{P}_2$ to hold will hold in $V[\mathcal{F}]$. In addition, there are no new Borel subsets of $\mathbb{M}$.

Once these are established, we are able to apply the standard PFA type methodology as demonstrated in [12,14]. The technique is to construct a $\mathbb{P}_2(\{p_\xi : \xi \in \omega_1\})$-name $\dot{Q}$ of a proper poset and to then invoke PFA in the ground model so as to select a filter meeting a given choice of ω_1-many dense subsets of $2^{<\omega_1} * \mathbb{P}_2(\{p_\xi : \xi \in \omega_1\}) * \dot{Q}$. Using the proof that $\mathbb{P}_2$ is ω_1-distributive, there is then a condition $p \in \mathbb{P}_2$ that shows that simply forcing with $\mathbb{P}_2$ yields the desired conclusion from meeting those ω_1-many dense sets.

For example, it is shown in [14] that $\mathbb{P}$ forces that $\mathrm{Triv}(H)$ is a dense P-ideal (i.e. $(\dagger_1^-)$ holds). However, since this is weaker than $(\dagger_1)$ we refer to the following theorem to assert that $\mathbb{P}$ forces that $(\dagger_1)$ holds.

Theorem 5.7 (4.17 of [3])**.** *In the extension obtained by forcing over a model of PFA, if Φ is a homomorphism from $\mathcal{P}(\mathbb{N})/fin$ onto $\mathcal{P}(\mathbb{N})/fin$, then Triv$(\Phi)$ is a ccc over fin ideal.*

Now we adapt the construction from [12] of $\{p_\xi : \xi \in \omega_1\}$ so that we also have that $(\dagger_2)$ holds. The approach also incorporates ideas from Velickovic's proof that OCA and $\mathbb{MA}(\omega_1)$ implies $(\dagger_1)$.

Theorem 5.8. *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_2$, every $\mathbb{N}^*$ cut-set of $\mathbb{M}^*$ is trivial on an ideal that is ccc over fin.*

Proof. Let $\{\dot{\mathcal{C}}_f : f \in \mathbb{N}^\mathbb{N}\}$ be a family of $\mathbb{P}_2$-names that is forced (by $\mathbf{1}$) to be an $\mathbb{N}^*$ cut-set of $\mathbb{M}^*$. Much as in Theorem 3.2, we can assume that the family $\{\dot{\mathcal{C}}_f : f \in \mathbb{N}^\mathbb{N}\}$ is also forced to be order-preserving in the sense that $\left(\bigcup \dot{\mathcal{C}}_f\right)^*$ contains $\left(\bigcup \dot{\mathcal{C}}_g\right)^*$ whenever $f <^* g$ are in $\mathbb{N}^\mathbb{N}$.

Suppose also that the ideal of sets on which this cut-set is trivial is not ccc over fin. Since $\mathbb{P}_2$ is $\aleph_1$-distributive, we may choose a condition $p_0 \in \mathbb{P}_2$ and an almost disjoint family $\{I_\alpha : \alpha \in \omega_1\}$ of subsets of $\mathbb{N}$ such that p_0 forces that none of the I_α's are in the trivial ideal for this cut-set. By [17] (since this final model is a model of Martin's Axiom) it suffices to assume that the family $\{I_\alpha : \alpha < \omega_1\}$ is tree-like. More specifically, there is a function $\sigma : \mathbb{N} \to 2^{<\omega}$ such there is a 1-to-1 enumerated collection $\{\rho_\alpha : \alpha < \omega_1\} \subset 2^\omega$ so that for all $\alpha < \omega_1$, $\sigma(I_\alpha) \subset \{\rho_\alpha \restriction j : j \in \omega\}$.

To start the proof, let $p_0 \in G$ be a $\mathbb{P}_2$-generic filter. We consider the family F of all partial functions $f \restriction I_\alpha$ ($f \in \mathbb{N}^\mathbb{N}$ and $\alpha \in \omega_1$). We define the relation $R \subset [F]^2$ to consist of all unordered pairs $\{f, f'\}$ that satisfying

(1) if $\mathrm{dom}(f) = I_\alpha$ and $\mathrm{dom}(f') = I_\beta$, then $\alpha \neq \beta$,
(2) there is an $m \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$ such that $[c^m_f, d^m_f] \cap [c^m_{f'}, d^m_{f'}]$ is empty.

We utilize the discrete topology on $S = \{\infty\} \cup \mathbb{Q}^2$ and for any $f \in F$ we identify f with an element, s_f, of the product space $S^\mathbb{N}$, where in coordinate $m \in \mathrm{dom}(f)$, $s_f(m) = (c^m_f, d^m_f)$ and for $m \in \mathbb{N} \setminus \mathrm{dom}(f)$, $s_f(m) = \infty$. With this topology, using tree-like as in [17], the relation R is an open subset of the product space $[\{s_f : f \in F\}]^2$ (i.e. unordered pairs). Assume that Λ is an uncountable subset of ω_1 and that, for each $\xi \in \Lambda$ we set $\bar{f}_\xi = f_\xi \restriction I_\xi$. The set $\{\bar{f}_\xi : \xi \in \Lambda\}$ would be called an R-homogeneous set if it satisfied that $\{\bar{f}_\xi, \bar{f}_\eta\} \in R$ for all $\xi \neq \eta \in \Lambda$. We show that $\{\bar{f}_\xi : \xi \in \Lambda\}$ fails to be an R-homogeneous set.

Choose any $f \in \mathbb{N}^\mathbb{N}$ so that $f_\xi <^* f$ for all $\xi \in \Lambda$. Choose $\bar{m} \in \mathbb{N}$ and uncountable $\Lambda_1 \subset \Lambda$ so that $[c^m_f, d^m_f] \subset [c^m_{f_\xi}, d^m_{f_\xi}]$ for all $\bar{m} < m \in \mathbb{N}$. Choose also uncountable $\Lambda_2 \subset \Lambda_1$ so that $\{[c^m_{f_\xi}, d^m_{f_\xi}] : m \leq \bar{m}\} = \{[c^m_{f_\eta}, d^m_{f_\eta}] : m \leq \bar{m}\}$ for all $\xi, \eta \in \Lambda_2$. These reductions ensure that $\{\bar{f}_\xi, \bar{f}_\eta\}$ is not in R for any pair $\xi, \eta \in \Lambda_2$.

Suppose now we prove that there is a sequence $\{p_\xi : \xi \in \omega_1\}$ as in Lemma 5.4 that, in addition, ensures that $\mathbb{P}_2(\{p_\xi : \xi \in \omega_1\})$ forces that there is a proper poset Q that **does** force there is an uncountable R-homogeneous set. Then by our discussion in the paragraph following Lemma 5.6 we would have contradicted that $\{\dot{\mathcal{C}}_f : f \in \mathbb{N}^\mathbb{N}\}$ is forced by $\mathbb{P}_2$ to be a cut-set.

Todorcevic [15] has shown that, since there is no uncountable R-homogeneous set, the family $F \subset \mathbb{N}^\mathbb{N}$ must be covered by a countable family $\{F_n : n \in \omega\}$ of sets each with the property that $[F_n]^2$ is disjoint from R. Using our separable metrizable topology on F and the fact that R is open, this is equivalent to there being a family $\{F_n : n \in \omega\}$ of countable subsets of F satisfying that for every n, the set of pairs from the closure of F_n and that the union of the closures of the F_n's cover F. If we prove that $\{p_\xi : \xi \in \omega_1\}$ forces there is no such sequence $\{F_n : n \in \omega\}$ then we have completed the proof of the theorem. We use Lemma 5.6 to do so.

Let $\eta \in \omega_1$, $\{p_\xi : \xi \in \eta\} \subset \mathbb{P}_2$, and the countable elementary submodel $\mathfrak{U}$ be as in Lemma 5.6. We can assume further $\mathfrak{U}$ is equal to $M \cap H(\aleph_2)$ for some countable elementary submodel $M \prec H(\aleph_3)$ satisfying that each of the objects $\{p_\xi : \xi \in \eta\}$, $\{I_\alpha : \alpha \in \omega_1\}$, $\{\dot{\mathcal{C}}_f : f \in \mathbb{N}^{\mathbb{N}}\}$ are elements of M. Let us note that R itself is an element of V and that we may assume there is some $\xi < \eta$ such that p_ξ forces that R is the open set resulting from the conditions (1) and (2) above applied to the family $[F]^2$. It suffices to prove that if $\{\dot{F}_n : n \in \omega\} \in \mathfrak{U}$ is a set of $\mathbb{P}_2(\{p_\xi : \xi \in \eta\})$-names of countable subsets of F, then there is a choice of p_η (as in Lemma 5.6) that forces this sequence fails to have the covering properties mentioned in the previous paragraph.

To do so, we first consider the forcing extension by the countable poset $\mathbb{P}_2(\{p_\xi : \xi \in \eta\})$. Since the $\dot{F}_n$'s are countable names of subsets of F we can fix a $\delta \in \mathfrak{U} \cap \omega_1$ such that it is forced that every element of $\bigcup\{\dot{F}_n : n \in \omega\}$ has as its domain an element of $\{I_\beta : \beta < \delta\}$. There is a maximal antichain E_0 of $\mathbb{P}_2(\{p_\xi : \xi \in \eta\})$ (in $\mathfrak{U}$) where each $p \in E_0$ forces one of the following statements

(1) there is an $f \in F \cap \mathfrak{U}$ such that f is not in the closure of $\dot{F}_n$ for each $n \in \omega$,
(2) there is an $n \in \omega$ and a pair $\{f, f'\} \in \dot{F}_n \cap R$,
(3) each of the previous two statements (1) and (2) fail to hold.

Now consider any condition $q \in \mathbb{P}_2(\{p_\xi : \xi \in \eta\})$ that is an extension of an element of E_0 in which condition (3) holds.

Working in M we can ask if, for each $f \in \mathbb{N}^{\mathbb{N}}$, there is a condition $p_f < q$ in $\mathbb{P}_2(\{p_\xi : \xi \in \eta\})$ and an integer n_f satisfying that $f \restriction I_\delta$ is in the closure of $\dot{F}_{n_f}$. We show that the answer to this question is "no". If such a pair, p_f, n_f, exists for all $f \in \mathbb{N}^{\mathbb{N}}$, then there would be a single condition $\bar{p}$ and integer $\bar{n}$ satisfying that $\tilde{F} = \{f \in F : p_f = \bar{p}, n_f = \bar{n}\}$ is $<^*$-cofinal in $\mathbb{N}^{\mathbb{N}}$. Since $\bar{p}$ forces that $\tilde{F} \restriction I_\delta$ is a subset of the closure of $\dot{F}_{\bar{n}}$, it follows from the failure of (2) that $[\tilde{F} \restriction I_\delta]^2$ is disjoint from R. But this means that, for all $m \in I_\delta \setminus \bar{m}$, the family $\{[c^m_f, d^m_f] : f \in \tilde{F}\}$ is linked. Since a linked family of compact intervals has non-empty intersection, this, together with the cofinality of F, would imply that I_δ is in the trivial ideal.

So we have now proven there is some $f \in \mathbb{N}^{\mathbb{N}}$ such that there is no pair p_f, n_f as above. By elementarity there is such an $f \in M \cap \mathbb{N}^{\mathbb{N}} \subset \mathfrak{U}$. The failure of there being a pair p_f, n_f is equivalent to the assertion, that for each $n \in \omega$, there is a subset $E(q, f, n)$ of $\mathbb{P}_2(\{p_\xi : \xi < \eta\})$ that is dense below q and for each $e \in E(q, f, n)$, there is a basic open neighborhood of $f \restriction I_\delta$ in the topology on $S^{\mathbb{N}}$ such that e forces that $\dot{F}_n$ is disjoint from this open set.

It follows then the choice of p_η as in Lemma 5.6 will result in the family $\{\dot{F}_n : n \in \omega\}$ considered as $\mathbb{P}_2(\mathcal{F})$-names will not be the dense sets for a countable cover of F consisting of sets whose set of pairs are disjoint from R. The proof is completed, i.e. a contradiction reached, by applying the PFA methodology discussed following Lemma 5.4. □

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Department of Mathematics and Statistics, UNC Charlotte