COMPACT SPACES AND THE PSEUDORADIAL PROPERTY, II

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Abstract. There is a model of set theory in which all compact spaces of weight at most ω_2 are pseudoradial.

1. Introduction

We show that 2^{ω_2} can be pseudoradial. It is easily seen that if 2^{ω_2} is pseudoradial, then all compact spaces of weight at most ω_2 are also pseudoradial.

A well-ordered sequence $\{a_\alpha : \alpha \in \kappa\}$ converges to a point x if every neighborhood of x contains a final segment of the sequence, if $\{a_\alpha : \alpha \in \kappa\} \subset A$, we could say that x is a radial limit of A. The properties *radial* and *pseudoradial* (see [1, 3]) are natural generalizations (and stand in the same relation to each other) of the well-known properties *Frèchet-Urysohn* and *sequential* in which converging sequence is replaced by converging well-ordered sequence. A space X is radial if every point in the closure of a set is a radial limit of the set, while a space is pseudoradial if every radially closed set is closed.

Sapirovskii suggests in [5] that $[0,1]^{\omega_2}$ (equivalently 2^{ω_2}) should fail to be $\aleph_0$-pseudoradial (a property weaker than pseudoradial). It is shown in [2] that Kunen's set-theoretic principle P_1 on ω_1 implies that 2^{ω_2} is indeed $\aleph_0$-pseudoradial.

The situation for 2^{ω_1} is simpler and better understood. The analogue of P_1 for ω is the assertion that the cardinal $\mathfrak{p}$ is equal to the continuum and greater than ω_1. It is well-known that countable subsets of 2^{ω_1} are radial if $\mathfrak{p} > \omega_1$ and the space itself is pseudoradial if $\mathfrak{s} > \omega_1$. If 2^κ is pseudoradial, then $\mathfrak{s} > \kappa$ but this only serves to guarantee that countable sets that fail to be closed will not be radially closed. We study analogues for ω_1 sized subsets which we call sP_1 and wP_1 (see Definition 2.1) since they are topological versions of Kunen's P_1 principle. Although it is shown in [2] that wP_1 is equivalent to Sapirovskii's $\aleph_0$-pseudoradial for 2^{ω_2}, wP_1 is more set-theoretic and easier to deal with in isolation. The technique of this paper is based on the fact, proven in [2], that $wP_1 + \mathfrak{p} > \omega_2$ implies that 2^{ω_2} is pseudoradial.

Specifically, we start with a model in which Shelah's strengthening, referred to as GMA in [8], of Kunen's P_1 principle holds and we force with the usual finite support iteration of length ω_3 in which the factors are the usual σ-centered tower filling posets (Booth). We can choose Shelah's model [6] so that the Continuum Hypothesis holds and $2^{\aleph_1} = \aleph_3$. It is well known that $\mathfrak{p} = \omega_3 = \mathfrak{c}$ will hold in this extension. We will show that wP_1 will also hold. It was shown by Juhasz and Szentmiklossy [4] that Martin's Axiom plus $\mathfrak{c} > \omega_2$ does not imply that $[0,1]^{\omega_2}$ is pseudoradial hence we do need GMA.

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2. Elementary matrices

This section will establish properties that will be needed to apply Shelah's principle to a poset we define later. We let P denote the finite support iteration of length ω_3 in which the factors are the usual filter filling posets (Booth) with a suitable enumeration of the names of filters of cardinality at most ω_2. Conditions $p \in P$ have the form $\{\langle\gamma, \langle t_p^\gamma, A_p^\gamma\rangle\ \rangle : \gamma \in \mathrm{dom}(p)\}$ where $\mathrm{dom}(p)$ is a finite subset of ω_3, each t_p^γ is a member of $\omega^{<\omega}$ and A_p^γ is a P_γ-name of a member of $\mathcal{A}_\gamma$, which is itself a P_γ-name forced by 1 to be a filter of infinite subsets of ω. For each $p \in P$, we define p^* to be the condition $\{\langle\gamma, \langle t_p^\gamma, \check{\omega}\rangle\ \rangle : \gamma \in \mathrm{dom}(p)\}$. It will simplify notation if we also identify p^* with the obvious function into $\omega^{<\omega}$ and suppress the side conditions. We will use the notation $q <_w p$ to denote $q \leq p$ and $q^* = p^*$. Therefore, q^* is the largest condition such that $q <_w q^*$. Note that if $p^* \cup r^*$ is a function (agree on their common domain), then p and r are compatible, indeed $p \wedge r$ exists. For a condition $p \in P$ and subset A of P, we will use $p^\perp, A^\perp$ to denote the set of all conditions which are incompatible with p, respectively, each member of A. Also, $p \perp q$ denotes the relation that p and q are incompatible.

wP1 **Definition 2.1.** wP_1 is the statement that whenever $\mathcal{X} \subset \wp(\omega_1)$ and $|\mathcal{X}| < 2^{\omega_1}$, then there are a uniform filter base $\mathcal{U}$ on ω_1 so that $|\mathcal{U} \cap \{X, \omega_1 \setminus X\}| = 1$ for each $X \in \mathcal{X}$, and an uncountable set $C \subset \omega_1$ and a function $\varphi : \mathcal{U} \to \omega_1$ such that

$$\{U \cap (\beta, \gamma) : \beta \in \gamma, U \in \mathcal{U}, \text{ and } \varphi(U) < \gamma\}$$

has the finite intersection property for each $\gamma \in C$.

The statement in a sense reflects the finite intersection property of the filter $\mathcal{U}$ to countable pieces of ω_1. Therefore in order to show that wP_1 holds in the forcing extension by P, we suppose we have a family, $\mathcal{X}$, of ω_2 many P-names of uncountable subsets of ω_1. We fix a ϵ-chain $\{M_\alpha : 0 < \alpha \in \omega_2\}$ of ω-closed elementary submodels of cardinality ω_1 (recall CH holds) containing this family and so that the chain is continuous at ω_1 limits. Let $\mathcal{M}$ denote the union of this entire chain. We use this chain to factor the forcing. For convenience, we let M_0 be the empty set.

We enumerate the family of all P-names of subsets of ω_1 which are members of $\mathcal{M}$ (including $\mathcal{X}$ of course), $\{X_\gamma : \gamma \in \nu\}$, as well as $P \cap \mathcal{M} = \{p_\gamma : \gamma \in \nu\}$ in such a way that for each α in ω_2 all the $M_\alpha \cap P$-names are listed before any names that are not $M_\alpha \cap P$-names and so that for each such pair p, X, there is a γ so that $p_\gamma = p$ and $X_\gamma = X$. For each $\gamma \in \nu$, we will define a P-name F_γ so that there is a $q < p_\gamma$ such that $q \Vdash F_\gamma \in \{X_\gamma, \omega_1 \setminus X_\gamma\}$. For each $\mu \in \omega_2$, there is a minimal γ_μ such that the collection $\{X_\zeta : \zeta < \gamma_\mu\}$ enumerates all B_μ-names which are in $\mathcal{M}$. We define $\mathcal{F}_\mu$ to be the (name of the) collection $\{F_\zeta : \zeta < \gamma_\mu\}$. To start the induction, $\{X_\zeta : \zeta < \gamma_0\}$ is simply an enumeration of the canonical $B_0 = \{\emptyset\}$ names for $\mathcal{M} \cap \wp(\omega_1)$. We select any uniform ultrafilter $\mathcal{U}$ on ω_1 which is in $\mathcal{M}$ and, for $\zeta < \gamma_0$, define F_ζ to be X_ζ if (suppressing the trivial forcing) it is a member of $\mathcal{U}$ and to be $\omega_1 \setminus X_\zeta$ otherwise.

Since P is ccc, each $B_\alpha = M_\alpha \cap P$ is completely embedded in P. Given a filter G on P, we use G^α to denote $G \cap B_\alpha$, and, as is usual, for $\alpha \in \omega_3$, G_α will denote $G \cap P_\alpha$, where P_α is the set of conditions in P with support contained in α.

Balpha **Lemma 2.2.** *For any $p \in P$ and $\alpha \in \omega_2$ there is a $q \in B_\alpha$ such that $q^* = p^* \cap M_\alpha$*

and so that every extension of q in B_α is compatible with p (i.e. q is basically a projection of p).

Proof. We proceed by induction on $\gamma \in \mathrm{dom}(p) \cap M_\alpha$. Let G_γ be P_γ-generic such that $p \restriction \gamma$ and $q \restriction \gamma$ are in G_γ (where $q \restriction \gamma$ denotes the element we have defined till now). Set $A = \{r \in P \cap M_\alpha : r \perp (p \restriction \gamma + 1) \text{ and } |t^\gamma_r| \geq |t^\gamma_p|\}$. Since P is ccc, there is a countable $A' \subset A$ such that $(A')^\perp = A^\perp$; note that $A' \in M_\alpha$. Therefore, $A'/G_\gamma = \{r : r \in A' \text{ and } r \restriction \gamma \in G_\gamma\}$ is a member of $M_\alpha[G_\gamma]$. It follows that, in $V[G_\gamma]$, $p(\gamma)$ is incompatible with $r(\gamma)$ for each $r \in A'/G_\gamma$. That is, there is a $B \subset \omega$ such that the condition (t^γ_p, B) is incompatible with $r(\gamma)$ for all $r \in A'/G_\gamma$. Since the only assumption we've made on G_γ is that $p \restriction \gamma$ and $q \restriction \gamma$ are in G_γ, we have that

$$q \restriction \gamma \wedge p \restriction \gamma \Vdash \ (\exists B \in \mathcal{A}_\gamma)\ (\ (p^*(\gamma), B)\ \perp\ \{r(\gamma) : r \in A'/G_\gamma\}\ ).$$

We show that $q \restriction \gamma$ forces the same statement. Assume that $q' \in P_\gamma \cap M_\alpha$ and $q' \leq q \restriction \gamma$. It follows, by our induction hypothesis then that every extension of q' in $M_\alpha \cap P$ is compatible with both $q \restriction \gamma$ and with $p \restriction \gamma$. Therefore, in M_α, q' does not force the failure of the above statement. By elementarity, we see that M_α models that $q \restriction \gamma$ has no extension which forces the failure of the statement, hence, by the forcing lemma and elementarity, we have that $q \restriction \gamma$ forces the statement as required. The definition of $q(\gamma)$ is obtained by taking any P_γ-name in M_α of a B as in the above existential statement. □

Definition 2.3. For $p \in P$ and $\mu \in \omega_2$, let $p^{-\mu}$ denote the set of all conditions as in Lemma 2.2. That is, $q \in p^{-\mu}$, if $q \in B_\mu$, $q^* = p^* \cap M_\mu$ and r is compatible with p for each $r \leq q$ in B_μ.

Proposition 2.4. *For any $p \in P$ and $\mu < \zeta < \omega_2$, if $p' \in p^{-\zeta}$, then $(p')^{-\mu} \subset p^{-\mu}$.* fix-
In addition, there is a $q <_w p$ such that $q^{-\mu} \subset (p')^{-\mu}$

Proof. Let $p \in P$, $p' \in p^{-\zeta}$ and $q' \in (p')^{-\mu}$. Since $(q')^* = (p')^* \cap M_\mu$ and $(p')^* = p' \cap M_\zeta$, it follows that $(q')^* = p^* \cap M_\mu$. To show that $q' \in p^{-\mu}$ it remains to show that for each $r \in B_\mu$ with $r \leq q'$, we have that r is compatible with p. Given such an r we have that r is compatible with p' by definition of $(p')^{-\mu}$. Choose any $r' \in B_\mu$ which is below p' and r. By definition of $p^{-\mu}$ and the fact that $p' \in p^{-\mu}$, it follows that r' is compatible with p showing, of course, that r is compatible with p.

Now for the existence of q, simply observe that the canonical meet of p and p' will have the required property. □

Definition 2.5. If $b \in B_\alpha$ and $p \in P$, then

$$b \Vdash_{B_\alpha} p \Vdash_w \varphi$$

will be understood to mean that if $b \in G^\alpha$ (a B_α-generic filter) then there is a $q \Vdash \varphi$ such that $q^{-\alpha} \cap G^\alpha \neq \emptyset$ and $q^* \setminus M_\alpha = p^* \setminus M_\alpha$. Note that p must be compatible with b in this situation.

The following result is obvious but since the notation is new, it is worth recording.

Proposition 2.6. *If $b \Vdash_{B_\alpha} p \not\Vdash_w \varphi$, then there is a $q \leq p, b$ such that $q \Vdash \neg\varphi$.* factor

When it is clear from context, we will use $q^{-\mu} \Vdash \varphi$ to abbreviate that $b \Vdash_{B_\mu} \varphi$ for all $b \in q^{-\mu}$. We also note that the set $q^{-\mu}$ is centered.

Let $\gamma < \nu$ (from our indexing) and let λ be minimal such that X_γ is a B_λ-name. We define F_γ assuming that F_ζ has been defined for $\zeta < \gamma$. The inductive assumption is that for $\mu < \lambda$, $\mathcal{F}_\mu$ is a B_μ-name of a maximal filter on $\mathcal{M}[G^\mu] \cap \wp(\omega_1)$.

Xnotin *Case* 1. There are $A_\gamma \in [\gamma]^{<\omega}$, $p_\gamma \geq p \in B_\lambda$, and $\mu < \lambda$ such that for every $q < p$, there is $H_q \in \mathcal{F}_\mu$ such that,

$$q^{-\mu} \Vdash \text{“}q \Vdash_w \xi \notin X_\gamma \cap \bigcap\{F_\zeta : \zeta \in A_\gamma\} \text{ for all } \xi \in H_q\text{”}$$

then $\mu_\gamma = \mu$ and $\mu(F_\gamma)$ is $\max(\{\mu_\gamma\} \cup \{\mu(F_\zeta) : \zeta \in A_\gamma\})$. Note that $\mu(F_\gamma) < \lambda$. We define F_γ so that $p \Vdash F_\gamma = (\breve{\omega}_1 \setminus X_\gamma)$ and $p' \Vdash F_\gamma = \breve{\omega}_1$ for all $p' \in p^\perp$. Specifically,

$$F_\gamma = \{(q, \check{\delta}) : q \in B_\lambda, \delta \in \omega_1, q \perp p' \text{ or } (q \leq p' \text{ and } q \Vdash \check{\delta} \notin X_\gamma)\}$$

On the other hand,

Xin *Case* 2. if there are no choices as in Case 1, we set F_γ to be the name forced by p_γ to be X_γ and to be ω_1 by conditions in $p^\perp$ analogous to Case 1. In this case $\mu_\gamma = 0$ and also set $\mu(F_\gamma) = 0$ and $A_\gamma = \emptyset$.

Note that, by the failure of Case 1 and by the inductive assumption that $\mathcal{F}_\mu$ is a maximal filter over $\mathcal{M}[G^\mu]$, we have the next claim.

Claim 1. If we are in Case 2 then for each $A \in [\gamma]^{<\omega}$ and for each η there is a dense below p_γ (in B_λ) set of q such that there is $H_q \in \mathcal{F}_\eta$ with

$$q^{-\eta} \Vdash \text{“}\xi \in H_q \Rightarrow q \not\Vdash_w \xi \notin F_\gamma \cap \bigcap\{F_\zeta : \zeta \in A\}\text{”} \,.$$

Proof. Take any $p \leq p_\gamma$, and note that by the failure of Case 1, there is some $q < p$ such that for this $\mu = \eta$ there is no H_q as in Case 1. Working in the model $V[G^\eta]$, we have that $\{\xi : q \Vdash_w \xi \notin X_\gamma \cap \bigcap\{F_\zeta : \zeta \in A\}$ is not a member of $\mathcal{F}_\eta$. Since $\mathcal{F}_\eta$ is a maximal filter, it follows that H_q, the B_η-name for $\{\xi : q \not\Vdash_w \xi \notin X_\gamma \cap \bigcap\{F_\zeta : \zeta \in A\}$, is in $\mathcal{F}_\eta$. $\square$

obviouslemma **Lemma 2.7.** *If $A \in [\gamma+1]^{<\omega}$, $\xi \in \omega_1$, and $p \in P$, then*

$$p \Vdash \text{“}\xi \notin \bigcap_{\rho \in A} F_\rho\text{”} \quad \textit{iff} \quad p^{-\lambda} \Vdash \text{“}\xi \notin \bigcap_{\rho \in A} F_\rho\text{”}$$

Proof. Note that for each $\rho \in A$, F_ρ is a B_λ-name. Although X_ρ and F_ρ need not be elements of M_λ, it does follow that for each $\xi \in \omega_1$, the name for $F_\rho \cap [0, \xi]$ is a member of M_λ. Therefore the result follows directly from elementarity and Lemma 2.2. $\square$

Now we prove, by induction on γ, that the essence of the above Claim 1 also holds when we are in Case 1.

project **Lemma 2.8.** *Suppose that $A' \in [\gamma]^{<\omega}$ and that $\zeta \geq \mu(F_\rho)$ for each $\rho \in \{\gamma\} \cup A' \cup A_\gamma$. Then for each $q' \in P$, there is a $q < q'$ and $H' \in \mathcal{F}_\zeta$ such that*

$$q^{-\zeta} \Vdash \text{“}q \not\Vdash_w \xi \notin F_\gamma \cap \bigcap\{F_\eta : \eta \in A'\} \textit{ for } \xi \in H'\text{”}$$

Proof. Note that by Lemma 2.7, and by induction, we may work solely with conditions in B_λ which are below p_γ. Assume that $q' \in B_\lambda$ and also that we are in Case

1 for γ. Now apply the inductive hypothesis to $\max(A' \cup A_\gamma)$, and assume that we have an $H' \in \mathcal{F}_\zeta$ and $q < q'$ such that

$$q^{-\zeta} \Vdash \text{“}q \nVdash_w \xi \notin \bigcap\{F_\eta : \eta \in A' \cup A_\gamma\} \text{ for } \xi \in H'\text{”}.$$

Let $H_q \in \mathcal{F}_{\mu_\gamma}$ be as in Case 1. By Proposition 2.4, we may assume that there is some $p' \in (q^{-\zeta})$ such that $q^{-\mu_\gamma} = (p')^{-\mu_\gamma}$. We show that q and the canonical name for $H_q \cap H'$ are as required. Let $\dot{\xi}$ be any B_ζ-name of a member of $H_q \cap H'$ in the sense that $q^{-\zeta} \Vdash \dot{\xi} \in H_q \cap H'$. Therefore, since $q^{-\mu_\gamma} \Vdash \dot{\xi} \in H_q$ by 2.7,

$$q^{-\mu_\gamma} \Vdash q \Vdash_w \dot{\xi} \notin X_\gamma \cap \bigcap\{F_\eta : \eta \in A_\gamma\} \ .$$

Therefore we may fix $r < q$ such that $r^{-\mu_\gamma} \Vdash r <_w q$ and

$$r \Vdash \dot{\xi} \notin X_\gamma \cap \bigcap\{F_\eta : \eta \in A_\gamma\} \ . \tag{2.1}$$

Since $r^{-\zeta} \Vdash r <_w q$, it suffices to show that

$$r^{-\zeta} \Vdash r \nVdash_w \dot{\xi} \notin F_\gamma \cap \bigcap\{F_\eta : \eta \in A'\} \ .$$

Assume otherwise, hence that by further extending r (maintaining $r^{-\zeta} \Vdash r <_w q$) we can obtain that

$$r \Vdash \dot{\xi} \notin F_\gamma \cap \bigcap\{F_\eta : \eta \in A'\} \ . \tag{2.2}$$

Now we still have, by definition of H', that $r^{-\zeta} \Vdash r \nVdash_w \dot{\xi} \notin \bigcap\{F_\eta : \eta \in A_\gamma \cup A'\}$. Therefore, there is an $r' < r$ such that

$$r' \Vdash \dot{\xi} \in \bigcap\{F_\eta : \eta \in A_\gamma \cup A'\} \ .$$

Since we are in Case 1, and we have the above forcing statements 2.1 and 2.2, we have our contradiction since, seemingly, $r' \Vdash \dot{\xi} \notin X_\gamma \cup (\omega_1 \setminus X_\gamma)$. $\square$

Now we relativize lemma 2.8 to an elementary submodel. In the statement below, the restriction to $\alpha \in M$ is what allows us to overcome the complication caused by the fact that $\{B_\alpha : \alpha \in \omega_2\}$ is not a finite support iteration.

Lemma 2.9. *Suppose $M \prec H(\theta)$, $\alpha \in M$, and that $A \in M \cap M_\alpha$ is a subset of P. With $p \perp_w A$ understood to mean there is some $r <_w p$ such that $r \perp A$, we have that if $p \perp_w A$, then $(p^* \cap M) \perp_w A$.* perpw

Proof. First recall that $p \perp_w A$ is equivalent to $(p^{-\alpha}) \perp_w A$ by Lemma 2.2. Thus we can assume that $p \in B_\alpha \subset M_\alpha$.

We prove the lemma by induction on $|\operatorname{dom}(p)|$. In the first instance assume that $\gamma_0 = \max(\operatorname{dom}(p) \cap M) < \max \operatorname{dom}(p)$. The result will follow by showing that $p_1 = p \restriction (\gamma_0 + 1) \perp_w A$. Fix any $r <_w p$ such that $r \perp A$. Note that $r_1 = r \restriction (\gamma_0 + 1) <_w p_1$, so we show $r_1 \perp_w A$. Towards a contradiction, assume there is an $a \in A$ such that $r_1 \not\perp a$. Since P is ccc and $A \in M \cap M_\alpha$, we may assume that $a \in M \cap M_\alpha$. Therefore $\operatorname{dom}(a) \subset M \cap M_\alpha$, and it follows that $r \not\perp a$ – a contradiction.

Now assume that $\gamma_0 = \max \operatorname{dom}(p) \in M$ and (without loss of generality) that $p \perp A$. Set $B = \{q \in P_{\gamma_0} : (\forall \tilde{q} < q)\ \tilde{q} \in P_{\gamma_0+1} \text{ and } \tilde{q}^*(\gamma_0) = p^*(\gamma_0) \Rightarrow \tilde{q} \not\perp_w A\}$. Note that B is in $M \cap M_\alpha$ since $p^*(\gamma_0)$ is simply some member of $\omega^{<\omega}$. We check that $p \restriction \gamma_0 \perp_w B$ (in fact $p \restriction \gamma_0 \perp B$). Otherwise, there is a $b \in B$ such that $p \restriction \gamma_0 \not\perp b$

(again, we can assume $b \in M \cap M_\alpha$). Define $\tilde{b} = p \wedge b$ in the obvious sense. Check that $\tilde{b} \in P_{\gamma_0+1}$, $\tilde{b}^*(\gamma_0) = p^*(\gamma_0)$, so we must have that $\tilde{b} \not\perp_w A$. However, clearly $\tilde{b} \leq p$, hence $\tilde{b} \perp A$. Now, by our induction assumption, $((p \restriction \gamma_0)^* \cap M) \perp_w B$, so, working in $M \cap M_\alpha$, there is an $r <_w (p \restriction \gamma_0)^*$ such that there is a $\tilde{r}$ witnessing failure to be in B so that $\tilde{r} \perp_w A$. This shows that $(p^* \cap M) \perp_w A$. □

The next lemma is the key property that allows us to "weakly" replace a member F of the filter $\mathcal{F}_\lambda$ by one from $\mathcal{F}_\zeta$ for some $\zeta < \lambda$.

filterM

Lemma 2.10. *Suppose $M \prec H(\theta)$, $F \in \mathcal{F}_\lambda \cap M$ and $\zeta \in M \setminus \mu(F)$, and $p \in P$, then there is a $q < p$ and $H' \in \mathcal{F}_\zeta \cap M$ such that $q^* \setminus M \subset p^*$, $q^* \setminus M_\lambda \subset p^*$ and*

$$q^{-\zeta} \Vdash \text{“}q \not\Vdash_w \xi \notin F \text{ for } \xi \in H' \cap M\text{”}$$

Proof. Let q be chosen as in Lemma 2.8 and set $q' = q^* \cap M$, hence $q' \in M$. Observe that

$$q^{-\zeta} \text{ “} \Vdash \ q' \not\Vdash_w \xi \notin F \ \text{ for each } \xi \in H_q \text{ ” .}$$

Therefore,

$$M \models (\exists q < q')(\exists H \in \mathcal{F}_\zeta)\ (q^* \supset q') \text{ and } q^{-\zeta} \ \Vdash \ \text{“}(\forall \xi \in H)\ q' \not\Vdash_w \xi \notin F\text{ ”}$$

Thus, there are $q_1 \in M \cap B_\zeta$ and $H \in M \cap \mathcal{F}_\zeta$ such that $q_1 \Vdash$ "$(\forall \xi \in H)\ q' \not\Vdash_w \xi \notin F$ ", and $q_1^* \supset q'$. Since $q_1 \in M$ and $q_1^* \supset q'$, it follows that $q_1^* \cup q^*$ is a function; thus q and q_1 are compatible. In particular, q_1 and q have a common extension, which we denote by q_2, such that $q_2^{-\zeta} \Vdash q_2 <_w q$.

We show that q_2 and H' are as required. In fact, this is immediate from Lemma 2.9 by working in $V[G^{-\zeta}]$ and noting that $A = \{a \in P : a \Vdash \xi \in F\}$ is in $M[G^{-\zeta}] \cap M_\lambda[G^{-\zeta}]$ for each $\xi \in M \cap \omega_1$ where λ is minimal (hence in M) such that $F \in M_\lambda$. □

filterM+

Lemma 2.11. *Suppose $M \prec H(\theta)$, $F \in \mathcal{F}_\lambda \cap M$ and $\zeta \in M$, and $p \in P$, there is a $q < p$ and $H' \in \mathcal{F}_\zeta \cap M$ such that $q^* \setminus M \subset p^*$, $q^* - M_\lambda \subset p^*$ and*

$$q^{-\zeta} \Vdash \text{“}q \not\Vdash_w \xi \notin F \text{ for } \xi \in H' \cap M\text{”}$$

Proof. We proceed by induction on λ. Apply Lemma 2.10 to obtain $q_1 \leq p$, $H_1 \in \mathcal{F}_{\mu(F)} \cap M$ as in Lemma 2.10, so that

$$q_1^{-\mu(F)} \Vdash q_1 \not\Vdash_w \xi \notin F \text{ for all } \xi \in H_1 \cap M \ .$$

Apply the induction hypothesis to H_1 to obtain $q_2 < q_1$ and H_2, again as in the statement of the Lemma, so that

$$q_2^{-\zeta} \Vdash q_2 \not\Vdash_w \xi \notin H_1 \text{ for all } \ \xi \in H_2 \cap M \ .$$

It should be clear that $(q_2^* \setminus M) \cup (q_2^* \setminus M_\lambda) \subset p^*$. Now suppose that $r \leq q_2$ is such that $r^{-\zeta} \Vdash r \leq_w q_2$ and that $r \Vdash \xi \notin F$. Noting that $r^{-\mu(F)} \Vdash r <_w q_1$ it follows that $r^{-\mu(F)} \Vdash \xi \notin H_1$. This is because H_1 is a $B_{\mu(F)}$-name and no extension of $r \restriction M_{\mu(F)}$ can force $\xi \in H_1$. However, this contradicts that $r^{-\zeta} \Vdash q_2 \not\Vdash_w \xi \notin H_1$. □

Definition 2.12. A family $\mathcal{M}$ is a conforming system if each $M \in \mathcal{M}$ is an elementary submodel of some H_θ, and given $M, M' \in \mathcal{M}$, there is an $\in$-isomorphism $f : M' \to M$ such that f is the identity on $M \cap M'$ and $M \cap \mu = M' \cap \mu$ for each $\mu \in \omega_2$ such that $M \cap \mu$ and $M' \cap \mu$ are both cofinal in μ.

conform

Lemma 2.13. *If $\mathcal{M} = \{M_i : i \in n\}$ is a finite conforming system, and if $F_i \in M_i \cap \mathcal{F}$ for each i, then for each $p \in P$, there is an $A \in \mathcal{F}_0 \cap \bigcup_{i<n} M_i$ such that for each $\xi \in A \cap M_0$, there is a $q < p$ such that $q \Vdash \xi \in F_i$ for each $i < n$.*

Proof. Let λ_i be minimal such that F_i is a B_{λ_i}-name (note that $\lambda_i \in M_i$) and enumerated so that $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$. We proceed by induction on n and then on the lexicographic ordering on $\langle \lambda_{n-1}, \lambda_{n-2}, \ldots, \lambda_0 \rangle$.

For $n = 1$ we just apply Lemma 2.11. For each $i, j < n$, let $f_{i,j}$ denote the isomorphism from M_i to M_j. If $\lambda_j \in M_i$ for some $i \neq j$, then we can replace F_i by the canonical name for $F_i \cap f_{j,i}(F_j)$. If we have some $\xi \in \omega_1 \cap M_i$ and q such that $q \Vdash \xi \in f_{j,i}(F_j)$, then note that $q \Vdash \xi \in F_j$. This is simply because the set, $D(i,\xi) = \{r : r \Vdash \xi \in f_{j,i}(F_j)\}$ is determined by each of $D^*(i,\xi) = \{r^* : r \in B_{\lambda_i}$ and $r^* \Vdash \xi \in f_{j,i}(F_j)\}$ and $D^*(i,\xi) \cap M_i$, and that $f_{i,j}$ is the identity mapping on $D^*(i,\xi) \cap M_i$. Therefore we may assume that $\lambda_i \notin M_j$ for $i < j$.

The above situation will recur in other forms and it will be useful to recall some standard notation. Given an ordinal ξ and a name F, $[[\xi \in F]]$ normally denotes the unique element in the complete Boolean algebra generated by P which is the join of the open subset of P consisting of those elements that force the statement $\xi \in F$. We will instead treat $[[\xi \in F]]$ as that open subset of P. Certainly there will be a minimal ζ such that $[[\xi \in F]] \cap B_\zeta$ is predense in $[[\xi \in F]]$, and since P is ccc, ζ will not have uncountable cofinality. Let us more loosely denote this relationship by saying that $[[\xi \in F]]$ is a member of B_ζ when we really mean that $B_\zeta \cap [[\xi \in F]]$ is predense. If there are $i < j < n$ such that, for each $\xi \in M_j \cap \omega_1$, the corresponding ζ for $[[\xi \in F_j]]$ is in $M_i \cap \lambda_i$, then as above we can replace F_i by $f_{j,i}(F_j) \cap F_i$ and apply the induction hypothesis and obtain good behavior for F_j for free.

For each $i < n$, set $\tilde{\lambda}_i = \sup(M_i \cap \lambda_i)$. We first show that there cannot be $i \neq j$ such that M_i is cofinal in $\tilde{\lambda}_j$. Assume otherwise; hence, by the definition of conforming system, $M_i \cap \tilde{\lambda}_j = M_j \cap \tilde{\lambda}_j$. If $\tilde{\lambda}_j = \lambda_j$, then λ_j has countable cofinality, as does $f_{i,j}(\lambda_j) = \mu_i \in M_i$. Therefore, $f_{i,j}(M_j \cap \lambda_j)$ is cofinal in μ_i which would imply that $\mu_i = \lambda_j$. This contradicts our current assumption that $\lambda_j \notin M_i$. Therefore we have shown that λ_j has uncountable cofinality in this situation. However, we would then be in the situation of the previous paragraph since for each $\xi \in M_j \cap \omega_1$, the ζ associated with the set $[[\xi \in F_j]]$ will be a member of $M_j \cap \lambda_j \in M_i$.

Therefore we have shown that we may assume that for each j there is a $\zeta_j \in M_j \cap \lambda_j$ such that $[\zeta_j, \tilde{\lambda}_j) \cap M_i$ is empty for each $i \neq j$. Let i be such that $\tilde{\lambda}_i$ (hence also ζ_i) is maximal. It is easily checked that $\tilde{\lambda}_j < \zeta_i$ for each $j \neq i$. Let $p_1 \leq p$ be chosen according to Lemma 2.11 together with H_i so that

$$p_1^{-\zeta_i} \Vdash p_1 \nVdash_w \xi \notin F_i \text{ for each } \xi \in H_i \ .$$

Apply the induction hypothesis to the family $\{H_i\} \cup \{F_j : j \neq i, j < n-1\}$ to obtain $A \in M_0 \cap \mathcal{F}_0$ such that for each $\xi \in A \cap M_0$, there is a $q < p_1$ such that $q \Vdash \xi \in F_j$

Fix any $\xi \in A \cap M_0$ and $q < p_1$ such that $q \Vdash \xi \in F_j$ for all $j \neq i$ and $q \Vdash \xi \in H_i$. Observe that $[[\xi \in F_j]]$ is in B_{ζ_i} for each $j \neq i$ since $\tilde{\lambda}_j < \zeta_i$. Therefore

$$q^{-\zeta_i} \Vdash \xi \in F_j \cap H_i \text{ for each } j \neq i$$

Now we have that

$$q^{-\zeta_i} \Vdash_{B_{\zeta_i}} p_1 \not\Vdash \xi \notin F_i \ .$$

By Proposition 2.6, there is an $r \in P$ so that $r \leq p_1$, $r^{-\zeta_i} \leq q^{-\zeta_i}$, and $r \Vdash \xi \in F_i$. Clearly, $r \Vdash \xi \in F_j$ for all $j < n$. □

3. Property wP$_1$ and $\mathfrak{p} > \omega_1$

In this section we prove that if the ground model satisfies Shelah's generalized MA principle, then $V[G]$ is a model of wP_1.

Recall that the axiom is the following:

Definition 3.1. [7, page264] [GMA] If Q is an $\aleph_1$-complete poset such that for any $\{q_i : i \in \omega_2\} \subset Q$, there are $q_i^\dagger \leq q_i$ (for $i \in \omega_2$) and pressing down functions $f_n : \omega_2 \setminus \{0\} \to \omega_2$ such that

$$\text{if } i < j \text{ and } (\forall n)(f_n(i) = f_n(j)), \text{ then } q_i^\dagger \wedge q_j^\dagger \text{ exists}$$

then for any family of fewer than $2^{\aleph_1} = \omega_3$ dense open subsets of Q, there is a filter on Q which meets each of them.

Fix the family $\mathcal{U} = \{F_\gamma : \gamma \in \nu\}$ as in the previous section. Our poset Q is defined as follows. A condition $q \in Q$ consists of a pair $(A_q, \mathcal{M}_q)$ where $A_q \in [\omega_1]^\omega$ and $\mathcal{M}_q = \bigcup\{\mathcal{M}_{q,\alpha} : \alpha \in A\}$ and

1. each $M \in \mathcal{M}_q$ is a countable family of elementary submodels of some (fixed) suitably large H_θ,
2. for each $M \in \mathcal{M}_q$, P and $\mathcal{U}$ are in M,
3. for each $M \in \mathcal{M}_{q,\alpha}$, $M \cap \omega_1 = \alpha$
4. $\mathcal{M}_{q,\alpha}$ is a conforming system,
5. for each $\alpha < \beta \in A$ and $M \in \mathcal{M}_{q,\alpha}$, there is an $M' \in \mathcal{M}_{q,\beta}$, $M \in M'$.

The ordering on Q is $q < q'$ providing A_q is an end extension of $A_{q'}$, and $\mathcal{M}_{q'} \subset \mathcal{M}_q$.

Lemma 3.2. *The poset Q satisfies the requirements in the Axiom GMA*

Proof. Suppose we are given $\{q_i : i \in \omega_2\} \subset Q$. For each $i \in \omega_2$, let M_i be a countable elementary submodel such that $q_i, i \in M_i$. Let $\delta_i = M_i \cap \omega_1$, and set $q_i^\dagger = (A_{q_i} \cup \{\delta_i\}, \{M_i\} \cup \mathcal{M}_{q_i})$. Fix an enumeration, $\{S_\zeta : \zeta < \omega_2\}$ of the countable subsets of ω_2 (recall we are assuming CH) and let C be a cub of ω_2 so that for all $\gamma \in C$ and $\beta < \gamma$, $[\beta]^\omega \subset \{S_\zeta : \zeta < \gamma\} \subset [\gamma]^\omega$. In addition, let $\{H_\xi : \xi \in \omega_1\}$ be an enumeration of the countable subsets of $H(\omega_1)$. We are now ready to define our pressing down functions.

For each $\omega_1 \leq i \in \omega_2$, let $f_0(i) = \xi \in \omega_1$ be such that the transitive (Mostowski) collapse of M_i is equal to H_ξ. Also let $f_1(i) = \xi'$ be such that $H_{\xi'}$ is equal to the image of q_i under the collapsing function. Define $f_2(i) = \max(C \cap i)$ if $i \notin C$, $f_2(i) = \omega$ if cofinality of i is ω, otherwise set $f_2(i) = \zeta$ where $M_i \cap i = Z_\zeta$.

For $n > 2$ and $i \in \omega_2 \setminus C$, let $f_n(i) \in 2$ be any mappings just so long as for each $\gamma \in C$ and $\gamma' = \min(C \setminus (\gamma + 1))$, the mapping i to $\langle f_n(i) : 3 \leq n \in \omega\rangle$ is one-to-one on the set (γ, γ'). For $i \in C$ with countable cofinality, simply ensure that $\{f_n(i) : 3 \leq n < \omega\}$ is increasing cofinal in i.

It should be reasonably clear that the f_n's are pressing down functions. Suppose that i, j are such that $f_n(i) = f_n(j)$ for all n. By the definitions of f_n's, it easily follows that both i, j are in C and both have uncountable cofinality. Furthermore, M_i and M_j will have the same transitive collapse with q_i and q_j being sent by that collapse to the same element. By the definition of $f_2(j) = M_j \cap j = M_i \cap i$, it follows that $\{M_i, M_j\}$ is a conforming system. Suppose that $M \in \mathcal{M}_{q_i}$ and $M' \in \mathcal{M}_{q_j}$ and $\delta = M \cap \omega_1 = M' \cap \omega_1\}$. Assume $\mu \in \omega_2$ is such that M and M' are cofinal in μ. It follows immediately that $\mu \in M_i \cap M_j$, hence $\mu \in i$. Since M_i and M_j agree on i and their transitive collapses take $\mathcal{M}_{q_i,\delta}$ and $\mathcal{M}_{q_j,\delta}$ to the same set, it follows that there is an $M'' \in \mathcal{M}_{q_i,\delta}$ such that the transitive collapse of M_i sends M'' to the same set that the transitive collapse of M_j sends M'. Since M and M'' must agree on μ, it follows that M and M' also agree on μ. The rest of the details that $(A_{q_i} \cup \{\delta_i\}, \mathcal{M}_{q_i} \cup \mathcal{M}_{q_j} \cup \{M_i, M_j\})$ is the meet of $q_i^\dagger$ and $q_j^\dagger$ are straightforward.

To see that Q is $\aleph_1$-complete, suppose that $\{q_n : n \in \omega\}$ is a descending chain in Q, then simply $q = (\bigcup_n A_{q_n}, \bigcup\{\mathcal{M}_{q_n} : n \in \omega\})$ is the needed lower bound. □

Lemma 3.3. *If G is a Q-generic filter, then there is a function $\varphi : \mathcal{U} \to \omega_1$ and a cub C such that the statement of wP_1 is forced by 1 to hold in the forcing extension by P.*

Proof. Let G be a generic filter for Q (we only have to meet the following dense sets $\{D_\gamma : \gamma \in \nu\}$ where $q \in D_\gamma$ providing there is an $M \in \mathcal{M}_q$ such that $F_\gamma \in M$). For each $F_\gamma \in \mathcal{U}$, fix a minimal δ such that there is a $q \in G$ and $M \in \mathcal{M}_{q,\delta}$ such that $F_\gamma \in M$. We define $\varphi(M)$ to be this δ. The set C is the closure of the set $A = \bigcup\{A_q : q \in G\}$. It is clear that to verify the property wP_1 we need only show that the condition holds for $\delta \in A$.

Suppose then that $\delta \in A$ and that $\varphi(F_\gamma) < \delta$ for each $\gamma \in B \in [\nu]^{<\omega}$. Fix any $\beta < \delta$. By the definition of φ, there are $q_\gamma \in G$ so that $F_\gamma \in M_\gamma$ for some $M_\gamma \in \mathcal{M}_{q_\gamma}$ and such that $M_\gamma \cap \omega_1 = \varphi(F_\gamma)$. By the definition of Q and by the directedness of G, there is a single $q \in G$, and for each $\gamma \in B$ an $M'_\gamma \in \mathcal{M}_{q,\delta}$ such that $F_\gamma \in M'_\gamma$. Assume that $p \in P$ is such that $p \Vdash \bigcap\{F_\gamma : \gamma \in B\} \cap (\beta, \delta)$ is empty. Since $\beta \in M_\gamma$, we may replace F_γ by $F_\gamma \setminus \beta$ and remain in $M_\gamma \cap \mathcal{U}$. Note that the family $\{M_\gamma : \gamma \in B\}$ is a finite conforming family. Therefore we can apply Lemma 2.13 and observe that there is a $\xi \in \delta$ and a $p' \leq p$ such that $p' \Vdash \xi \in F_\gamma$ for each $\gamma \in B$. □

Theorem 3.4. *It is consistent to have $\mathfrak{p} = \omega_3$ and wP_1. It follows that it is consistent all compact spaces of weight at most ω_2 are pseudoradial.*

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