COUNTABLE π-CHARACTER, COUNTABLE COMPACTNESS AND PFA

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ABSTRACT. Balogh proved that PFA, the Proper Forcing Axiom, implies that a countably compact space with countable tightness is either compact or contains an uncountable free sequence. Eisworth established the relevance to proper forcing of strengthening the countable tightness assumption to that of hereditary countable π-character. Eisworth proved that for any countably compact space with hereditary countable π-character there is a totally proper forcing that adds an uncountable free sequence. We extend these results by showing that PFA implies that countably compact spaces are closed in spaces that have hereditary countable π-character. This gives a countably compact version of the Moore-Mrowka problem in the class of spaces with hereditary countable π-character.

1. Introduction

If a space X has countable hereditary π-character, then X has countable tightness. In the case that X is compact, Sapirovskiĭ has shown these properties are equivalent. In this paper we prove that the proper forcing axiom, PFA , implies that a regular countably compact subset of a space with hereditary countable π-character is closed. We will assume without further mention that in this article we are only dealing with regular spaces.

We briefly summarize just some of the extensive background and context to this result. Rančin [11] defined a space to be C-sequential if every non-isolated point is a limit of a convergent sequence. This was generalized in [10] to the property C-closed which asserts that every countably compact subset is closed. The question of whether a space with countable tightness is C-closed was asked by Ismail and Nyikos

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in [10]. Eisworth uncovered the importance of hereditary countable π-character in these investigations in [7] and [8]. Finally it was shown in [5] to be consistent with CH that spaces with countable hereditary π-character are C-closed and it was asked there if this statement (denoted π-$\circledast$) was a consequence of PFA . It was shown in [4] that PFA does not imply that countably compact C-closed spaces of countable tightness must be sequential (or compact) and in [6] that it is consistent to have a compact C-closed space of countable hereditary π-character that is not sequential. Hajnal and Juhasz [9] show that CH implies the existence of a countably compact space of countable tightness with no non-trivial convergent sequences (and so is not C-sequential). In this article we note there is an example (Example 3.1) of a countably compact space of countable tightness that does not have countable π-character.

Free sequences were introduced by Arhangelskii [1] and shown to be a critical tool in the study of tightness in compact spaces. A free sequence in a space X is a sequence $\{x_\alpha : \alpha \in \delta\}$ of points of X, indexed by an ordinal, in which the closure of each initial segment is disjoint from the closure of the complementary final segment. Fremlin and Nyikos ([3]) laid some of the groundwork for showing how to force the existence of uncountable free sequences in countably compact spaces, and Balogh [2] extended their work to establish that PFA implies compact spaces of countable tightness are sequential (the Moore-Mrowka problem). A key step in the proof was the new concept of algebraic free-sequence [13] which one may view as making the freeness notion first-order. This made it possible to convert the second order property of *forcing the existence of an uncountable free sequence* to establishing the existence of one under PFA through meeting only $\aleph_1$-many dense sets. This does not help in a countably compact space that already may have uncountable free sequences. The CH assumption in [5] took care of this second order problem since it can easily be shown that we may assume that ω is a dense subset of the space and so our spaces have character at most $\mathfrak{c}$. In this paper we turn to Hausdorff-Luzin gaps of infinite subsets of a dense set ω as an added component to achieve the kind of upward absoluteness needed to build an objectionable free sequence.

2. A forcing preservation theorem

A standard step in analyzing the closure of a countably compact subset of a space X (with countable tightness) has been to consider maximal filters of closed subsets of the subspace that converge to a point outside the set. A key step in [5] was to establish that in models of CH we could force such a filter that had a base of separable sets. In this section we prove that we can do this under PFA if we are working in a space with hereditary countable π-character. Towards that we establish a result of independent interest. In addition, we are able to introduce our main new idea of utilizing classical Hausdorff-Luzin gaps in this context. For any family $\mathcal{I}$ of subsets of ω, let $\mathcal{I}^\perp$ denote the ideal of sets that are almost disjoint from every element of $\mathcal{I}$.

Theorem 2.1. PFA *implies countably compact separable regular spaces with hereditary countable π-character remain countably compact and countably tight after countably closed forcing.*

Proof. Let X be a countably compact space with hereditary countable π-character. Let ω be a dense subset of X. We omit the simple argument that it suffices to prove that X has countable tightness in the forcing extension by ${}^{<\omega_1}\omega_2$. We will however initially work in this forcing extension and proceed by contradiction. Assume that Y is a subset of X with a limit z such that z is not the limit of any countable subset of Y. We may assume that Y is countably compact. Let $\mathcal{F}$ be any maximal filter of closed subsets of Y with the property that z is a limit of each $F \in \mathcal{F}$. Let $\mathcal{I}$ denote the ideal of subsets of ω satisfying that $\mathrm{cl}_X(I)$ does not contain an element of $\mathcal{F}$. By the maximality of $\mathcal{F}$, $\mathrm{cl}_X(J)$ contains an element of $\mathcal{F}$ for each $J \in \mathcal{I}^\perp$. Let $\{I_\alpha : \alpha \in \omega_1\}$ enumerate $\mathcal{I}$, and let $\{J_\alpha : \alpha \in \omega_1\}$ enumerate $\mathcal{I}^\perp$. For each $\alpha \in \omega_1$, let $\mathcal{I}_\alpha = \{I_\beta : \beta < \alpha\}$

In this forcing extension z has character ω_1 since the space is regular and separable. For each $\alpha \in \omega_1$, choose $y_\alpha \in Y$ such that $y_\alpha \in \mathrm{cl}(J_\beta)$ for all $\beta < \alpha$. Now choose disjoint open subsets W_α, U_α of X such that $y_\alpha \in W_\alpha$, $z \in U_\alpha$, and the closure of U_α is disjoint from $\{y_\beta : \beta \leq \alpha\}$.

For each α, let Z_α denote the set $\bigcup\{\mathrm{cl}_X(J) \setminus \omega : J \in \mathcal{I}_\alpha^\perp\}$. If U is a neighborhood of z then $U \cap \omega$ is not in $\mathcal{I}$ and so there are $J \in \mathcal{I}_\alpha^\perp$ contained in U. Since X is countably compact, we have that z is in the closure of Z_α. In addition, since ${}^{<\omega_1}\omega$ adds no new countable sets, $\mathcal{I}_\alpha$ and thus Z_α are elements of the ground model. This means that there is a family $\{U(\alpha, \ell) : \ell \in \omega\}$ of open subsets of X satisfying that for each neighborhood U of z, there is an ℓ such that $U(\alpha, \ell) \cap Z_\alpha$ is a non-empty subset of U. We may assume that $U(\alpha, \ell) \cap Z_\alpha$ is not empty for each ℓ. Again we have that $U(\alpha, \ell) \cap \omega$ is not in $\mathcal{I}_\alpha$, and so we may choose $J(\alpha, \ell) \subset U(\alpha, \ell) \cap \omega$ so that $J(\alpha, \ell) \in \mathcal{I}_\alpha^\perp$. We may note that if J is any set that meets $J(\alpha, \ell)$, for each $\ell \in \omega$, in an infinite set, then z is in the closure of J.

Fix any continuous increasing $\in$-chain, $\{M_\gamma : \gamma \in \omega_1\}$, of countable elementary submodels of $H(\omega_2)$ such that the sequence $\{y_\alpha, W_\alpha, \{U(\alpha, \ell) : \ell \in \omega\} : \alpha \in \omega_1\} \in M_0$ as well as $\mathcal{I}$. Let C denote the cub of $\delta \in \omega_1$

satisfying that $M_\delta \cap \omega_1 = \delta$. Define a poset $\mathbb{P}$ where $p \in \mathbb{P}$ providing there is a finite subset $C_p \subset C$ and p is a finite subset of $C_p \times \omega$. For each $p \in \mathbb{P}$ and each $\delta \in C_p$, let $W(p,\delta)$ denote the open set $W_\delta \setminus \bigcup\{\overline{U_\gamma} : \gamma \in C_p \setminus \delta\}$. We define $p < q$ providing

(1) $p \supset q$,
(2) for $\alpha \in C_q$, $p \setminus q \cap (\{\alpha\} \times \omega)$ is contained in $W(q,\alpha)$,
(3) for $\beta \in C_p \setminus C_q$ and $\alpha \in C_q \setminus \beta$, for each $\ell < |C_q|$, there is a $j \in J(\alpha,\ell)$ such that $(\alpha,\ell) \in p$.

We prove that $\mathbb{P}$ is ccc. Let $\{p_\xi : \xi \in \omega_1\}$ be a subset of $\mathbb{P}$. By passing to a subset and re-enumerating we can assume that the family $\{C_{p_\xi} : \xi \in \omega_1\}$ is a Δ-system with root R. We can assume that for all α, ξ, $p_\xi \cap (R \times \omega) = p_\alpha \cap (R \times \omega)$. Let L denote the cardinality of C_{p_0}. Let $\{p_\xi : \xi \in \omega_1\}$ be an element of a countable elementary submodel M and let $M \cap \omega_1 = \delta$. We also choose M so that $\{M_\alpha : \alpha \in \omega_1\}$ and $\{W_\alpha, \{U(\alpha,\ell) : \ell \in \omega\} : \alpha \in \omega_1\}$ are elements of M.

For each $\xi \in \omega_1$ let $C_{p_\xi} \setminus R$ be enumerated as $\{\beta^\xi_0, \ldots, \beta^\xi_{m-1}\}$ in increasing order. For each $i < m$, let $W(\xi,i)$ denote the set $W(p_\xi, \beta^\xi_i)$. For any m-tuple, $\vec{t}$ of (possibly empty) finite subsets of ω, let $E_t = \{\xi \in \omega_1 : (\forall i < m)\ t_i \subset W(\xi,i)\}$. Let T denote the set of $\vec{t}$ such that $E_{\vec{t}}$ is uncountable. Note that $E_{\vec{t}} \in M$ for all m-tuples, $\vec{t}$, of finite sets of integers. For such an m-tuple $\vec{t}$ and a pair $i < m$ and $j \in \omega$, let $\vec{t}^{\,i}_j$ denote the tuple $\langle t'_k : k < m\rangle$ where $t'_i = t_i \cup \{j\}$ and $t'_k = t_k$ for $k \neq i$.

Claim 1. For each $\vec{t} \in T$ and each $i < m$, the set $J_{\vec{t},i} = \{j : \vec{t}^{\,i}_j \in T\}$ satisfies that $\omega \setminus J_{\vec{t},i}$ is in $\mathcal{I}_\delta$.

Proof of Claim 1. Let us note that $J_{\vec{t},i}$ is also in M. Furthermore, since we are working in a model of CH, we have that $M \cap \mathcal{P}(\omega) = M_\delta \cap \mathcal{P}(\omega)$, and so, $J_{\vec{t},i} \in M_\delta$. Since $\vec{t} \in T$, we can choose $\xi \in E \setminus M$. It is immediate, for each $j \in W(p_\xi, \beta^\xi_i) = W(\xi,i)$ we have that $\xi \in E_{\vec{t}^{\,i}_j}$. Since $\xi \notin M$, it follows that each such $\vec{t}^{\,i}_j \in T$. This proves that $x_{\beta^\xi_i}$ is not a limit point of the set $\omega \setminus J_{\vec{t},i}$. Since $\omega \setminus J_{\vec{t},i}$ is in M and $x_{\beta^\xi_i}$ is in the closure of every element of $\mathcal{I}^\perp_\delta$, this shows that $\omega \setminus J_{\vec{t},i}$ is in $\mathcal{I}$. □

Now we have that for each $i,k < m$, $\ell < L$ and each $\vec{t} \in T$, $J(\beta^\delta_k, \ell) \cap J_{\vec{t},i}$ is infinite. This is because $J(\beta^\delta_k,\ell) \in \mathcal{I}^\perp_{\beta^\delta_k}$ and $\mathcal{I}^\perp_{\beta^\delta_k} \supset \mathcal{I}^\perp_\delta$. Therefore, by a simple finite recursion, we can find a $\vec{t} = \langle t_i : i < m\rangle \in T$ satisfying that $t_i \cap J(\beta^\delta_k,\ell)$ is not empty for all $k < m$ and $\ell < L$.

Finally, choose any $\xi \in E_{\vec{t}}$ and let $p = p_\xi \cup p_\delta \cup \{(\beta^\xi_i, j) : i < m \text{ and } j \in t_i\}$. It is immediate that $p < p_\xi$ since no elements of $C_p \setminus C_{p_\xi}$ are below any elements of C_{p_ξ}. The choice of each t_i has ensure that each of condition (2) and (3) in the definition of $p < p_\delta$ hold. This shows that p_ξ and p_δ are compatible and completes the proof that $\mathbb{P}$ is ccc.

For each $\delta \in C$, let $\dot{L}_\delta$ denote the $\mathbb{P}$-name satisfying that $p \Vdash j \in \dot{L}_\delta$ if and only if $(\delta, j) \in p$. We note that it is forced that $\dot{L}_\delta$ is mod finite contained in $W_\delta \setminus U_\gamma$ for all $\delta < \gamma \in C$.

Claim 2. $\mathbb{P}$ forces that for all $\delta \in C$ and integers m, ℓ, the set $\{\beta \in C \cap \delta : \dot{L}_\beta \cap J(\delta, \ell) \subset m\}$ is finite.

This is immediate by the definition of $p < q$ in $\mathbb{P}$. One may summarize Claim 2 by saying that, for each ℓ, the family $\{\dot{L}_\delta, J(\delta, \ell) : \delta \in C\}$ is forced to be a Hausdoff-Luzin family.

Now we can complete the proof of the theorem. The poset $\mathbb{Q} = {}^{<\omega_1}\omega_2 * \mathbb{P}$ is a proper poset. We choose $\mathbb{Q}$-names as follows. There is a ${}^{<\omega_1}\omega_2$-name $\dot{C}$ for C and each $\dot{L}_\delta$ will actually be a $\mathbb{Q}$-name. For each $\alpha \in \omega_1$ we have the ${}^{<\omega_1}\omega_2$- names $\dot{x}_\alpha, \dot{W}_\alpha, \dot{U}_\alpha$ and $\{\dot{J}(\alpha, \ell) : \ell \in \omega\}$ with the obvious interpretions.

There is a family $\mathcal{D}$ of ω_1-many dense subsets of $\mathbb{Q}$ satisfying that if G is $\mathcal{D}$-generic then we have a cub $C \subset \omega_1$ and the sequence $\{x_\alpha, W_\alpha, U_\alpha, L_\alpha, \{J(\alpha, \ell) : \ell \in \omega\} : \alpha \in C\}$ so that for all $\delta \in C$,

(1) U_α is an open set containing z,
(2) W_α is an open set containing x_α and $U_\alpha \cap W_\alpha = \emptyset$,
(3) $\mathrm{cl}(U_\alpha)$ is disjoint from $\{x_\beta : \beta \in C \cap \alpha\}$,
(4) for each open set U with $z \in U$, there is an ℓ such that $J(\alpha, \ell)$ is mod finite contained in $U \cap \omega$,
(5) for each $\delta < \gamma \in C$, L_δ is mod finite contained in $\omega \cap W_\delta \setminus \overline{U_\gamma}$,
(6) for each $\ell, m \in \omega$, there are only finitely many $\beta \in C \cap \delta$ such $L_\beta \cap J(\delta, \ell) \subset m$.

For a subset J of ω, let J' denote the set of limit points of J. Define A to be the union of the family $\{L'_\delta : \delta \in C\}$. For each $\delta \in C$ and $\beta \in C \cap \delta$, $L'_\beta \cap U_\delta$ is empty. Since X has countable tightness (in the ground model) z is not in the closure of A. Let U be any open set with $z \in U$ and $\mathrm{cl}(U)$ disjoint from A. Choose $\ell \in \omega$ so that $S = \{\delta \in C : J(\delta, \ell) \subset^* U\}$ is uncountable. But now, since the family $\{L_\delta, J(\delta, \ell) : \delta \in S\}$ is a Hausdorff-Luzin family, there is a $\delta \in S$ such that $U \cap L_\delta$ is infinite. Since X is countably compact, $U \cap L_\delta$ has a limit point in A. This completes the proof. $\square$

Now with the previous theorem in hand, we can remove the CH assumption from the result [5, Theorem 1]. We give an alternate proof since the notation is somewhat different and to better illustrate the role of Theorem 2.1. Since $\diamondsuit$ holds after forcing with ${}^{<\omega_1}\omega_2$ it will suffice to prove this next lemma.

Lemma 2.2. *The axiom $\diamondsuit$ implies that if X is a countably tight regular space, Y is a countably compact subset, and $z \in X \setminus Y$ is a limit point of Y, then there is a maximal filter $\mathcal{F}$ of closed subsets of Y such that $\mathcal{F}$ has a base of separable sets and z is a limit of each $F \in \mathcal{F}$.*

Proof. Choose any countable subset of Y that has z as a limit. We may pass to the closure in Y of that countable set. Therefore we may assume that the base set for Y is ω_1. Let $\{A_\alpha : \alpha \in \omega_1\}$ be a $\diamondsuit$-sequence. That is, for each $A \subset \omega_1$ and each cub $C \subset \omega_1$, there is a $\delta \in C$ such that $A \cap \delta = A_\delta$. We define our filter $\mathcal{F}$ by recursively choosing a family $\{D_\alpha : \alpha \in \omega_1\}$ of countable subsets of ω_1 and setting $\mathcal{F}$ to be the filter generated by their closures. Suppose that $\delta \in \omega_1$ and that we have constructed $\{D_\alpha : \alpha \in \delta\}$. Our inductive hypotheses are that for $\beta < \alpha < \delta$

(1) D_α is a countable subset of ω_1,
(2) $\{z\} \in \mathrm{cl}(D_\alpha) \subset \mathrm{cl}(D_\beta)$,
(3) if α is a limit and z is a limit of $A_\alpha \cap \mathrm{cl}(D_\xi)$ for all $\xi < \alpha$, then $D_\alpha \subset \mathrm{cl}(A_\alpha)$.

We let $D_n = \omega \setminus n$ for all $n \in \omega$. Now let δ be in $\omega_1 \setminus \omega$, and we construct D_δ. If there is an $\alpha < \delta$ such that z is not a limit of $A_\delta \cap \mathrm{cl}(D_\alpha)$, then let $B_\delta = \bigcup\{D_\alpha : \alpha \in \delta\}$, otherwise let $B_\delta = A_\delta$. We thus have that z is a limit of $B_\delta \cap \mathrm{cl}(D_\alpha)$ for each $\alpha \in \delta$. Moreover, if U is any neighborhood of z, we have, by the countable compactness of Y, that $Y \cap \bigcap\{\mathrm{cl}(U \cap B_\delta \cap \mathrm{cl}(D_\alpha)) : \alpha < \delta\}$ is not empty. Moreover, by the regularity of X, z is a limit point of $Y \cap \bigcap\{\mathrm{cl}(U \cap B_\delta \cap \mathrm{cl}(D_\alpha)) : \alpha < \delta\}$. By countable tightness, we can choose D_δ to be any countable subset of this intersection that has z as a limit.

It remains to prove that $\mathcal{F}$ is a maximal filter. Let A be any subset of Y that meets $\mathrm{cl}(D_\alpha)$ for all $\alpha \in \omega_1$. We prove that $\mathrm{cl}(A)$ contains $\mathrm{cl}(D_\delta)$ for some $\delta \in \omega_1$. First we show that z is a limit of all such A. Let W, U be open neighborhoods of z such that $\mathrm{cl}(W) \subset U$. We prove that $U \cap A$ is not empty. Let M be a countable elementary submodel of $H((2^{\omega_1})^+)$ such that $A, W, \{A_\alpha, D_\alpha : \alpha \in \omega_1\}$ are in M and such that $W \cap \delta = A_\delta$ where $M \cap \omega_1 = \delta$. Of course z is a limit of $W \cap D_\beta$ for each $\beta \in M$, and, by elementarity, $D_\beta \subset M$. Therefore z is a limit of $A_\delta \cap \mathrm{cl}(D_\beta)$ for each $\beta \in \delta$, and we thus have that $\mathrm{cl}(D_\delta) \subset U$. It then follows that $A \cap U$ is not empty.

Now choose such an M so that $A \cap \delta = A_\delta$. We already know that z is a limit of $A \cap \mathrm{cl}(D_\beta)$ for all $\beta \in \omega_1$. By elementarity, and countable tightness, we have that z is a limit of $A_\delta \cap \mathrm{cl}(D_\beta)$ for all $\beta \in M$. This completes the proof that A contains D_δ. □

We close this section with an example illustrating that neither Theorem 2.1 nor the conclusion of Lemma 2.2 hold for non-separable spaces.

Example 2.1. Let Y denote the set $2^{<\omega_1}$, namely the tree of all 2-valued functions with domain a countable ordinal. The topology on Y has the family of all downward closed subsets as an open base. In particular for each $f \in 2^{<\omega_1}$ and for each $\alpha \in \mathrm{dom}(f)$, the family $\{f \restriction \beta : \alpha < \beta \leq \mathrm{dom}(f)\}$ is a clopen base at f. In addition, for $\rho \in 2^{\omega_1}$, $[\rho] = \{\rho \restriction \alpha : \alpha \in \omega_1\}$ is a non-compact clopen subset of Y. Therefore Y is locally compact and first countable. In particular Y has countable hereditary π-character. Next, define X to be the space $Y \cup \{z\}$ where Y is an open subspace and a neighborhood base for z is the family $\{X \setminus \bigcup\{[\rho_i] : i < n\} : n \in \omega \text{ and } \{\rho_i : i < n\} \subset 2^{\omega_1}\}$. The point z is a limit point of every infinite antichain of $2^{<\omega_1}$. It is thus easily checked that every subspace of X has countable π-character and is sequential. On the other hand, if $\dot\rho$ is a generic function over the poset $2^{<\omega_1}$, then every (ground model) neighborhood of z meets $[\dot\rho]$ in an uncountable set, while z is not in the closure of $\{\dot\rho \restriction \alpha : \alpha < \delta\}$ for any $\delta \in \omega_1$.

3. Main Theorem

We are now ready to state and prove the main result. The proof will be to simultaneously force a free sequence as well as the Hausdorff-Luzin families playing the same role as in Theorem 2.1.

Theorem 3.1. PFA *implies that countably compact sets are closed in regular spaces with hereditary countable π-character.*

Proof. Let X be a regular space in which every subspace has countable π-character. Suppose that Y is a countably compact subset of X and that some point z is a limit point of Y. We will assume that z is not in Y and achieve a contradiction. Since X has countable tightness, we may pass to a separable subset of Y that has z in its closure, and, for convenience, asssume that the usual set ω is a dense subset of Y. For each $x \in X \setminus \{z\}$, fix an open set W_x with $x \in W_x$ and $z \notin \overline{W_x}^X$. Throughout the proof, for a set $A \subset Y$, we will use $\overline{A}$ to denote the closure in Y and, if needed, $\overline{A}^X$ to denote the closure in X.

Let G_1 denote any generic filter for the countably closed poset ${}^{<\omega_1}\omega_2$. In the extension $V[G_1]$, CH holds and we have added no new countable sets. We therefore have that Y is still countably compact. It is shown in the previous section that in $V[G_1]$ there is a maximal filter $\mathcal{F}$ of closed subsets of Y such that $z \in \overline{F}^X$ for all $F \in \mathcal{F}$, and such that $\mathcal{F}$ has a base consisting of separable sets. In particular, then, $\mathcal{F} \cap V$ is a base for $\mathcal{F}$.

Again using that CH holds in $V[G_1]$, we may choose a continuous increasing chain $\{M_\alpha : \alpha \in \omega_1\}$ of countable elementary submodels of $H(\omega_2)$ such that $Y, \mathcal{F}, \{W_y : y \in Y\}$ are elements of M_0 as well as some fixed well-ordering $\prec$ of $H(\omega_1)$. Note that $H(\omega_1) \subset \bigcup_{\alpha \in \omega_1} M_\alpha$. For each $\alpha \in \omega_1$, let $\mathsf{Tr}(M_\alpha) = \bigcap(\mathcal{F} \cap M_\alpha)$. Since $\mathcal{F}$ has a base of separable sets, we have that $\mathsf{Tr}(M_\alpha) \subset \overline{F \cap M_\alpha}$ for each $F \in \mathcal{F} \cap M_\alpha$. For each $\alpha \in \omega_1$, let x_α be the $\prec$-least element of $\mathsf{Tr}(M_\alpha)$ and let W_α be used to denote W_{x_α}. For a set $J \subset \omega$, say that J is $\mathcal{F}$-large providing $\overline{\omega \setminus J} \notin \mathcal{F}$. We let $\mathcal{I}$ denote the dual ideal of subsets of ω whose complements are $\mathcal{F}$-large. Needless to say, $\mathcal{I}_\alpha = \mathcal{I} \cap M_\alpha$ is countable.

Fact 1. For each $\alpha \in \omega_1$, there is a sequence $\{J(\alpha, \ell) : \ell \in \omega\} \in M_{\alpha+1}$ consisting of infinite elements of $\mathcal{I}_\alpha^\perp$ such that every neighborhood of z contains at least one member of the sequence $\{J(\alpha, \ell) \setminus n : \ell, n \in \omega\}$.

Proof of Fact. Since $\mathcal{I}_\alpha$ is countable, we can actually work in V rather than $V[G_1]$. Let Y_α equal the set $\bigcup\{\overline{J} \setminus J : J \in \mathcal{I}_\alpha^\perp\}$. We first show that z is in the closure of Y_α. Suppose that $z \in U$ for some open $U \subset X$. It is immediate that $(U \cap \omega) \setminus I$ is infinite for all $I \in \mathcal{I}_\alpha$. Therefore there is a $J \in \mathcal{I}_\alpha^\perp$ such that $J \subset U$. Of course this means that $\overline{U} \cap \overline{J} \setminus J$ is non-empty and, since X is Tychonoff, we have that $z \in \overline{Y_\alpha}^X$. Since $Y_\alpha \cup \{z\}$ has countable π-character, there is a family $\{U(\alpha, \ell) : \ell \in \omega\}$ of open subsets of X satisfying that, for each $\ell \in \omega$, $U(\alpha, \ell) \cap Y_\alpha$ is not empty, and that, for each open U with $z \in U$, there is an $\ell \in \omega$ such that $U(\alpha, \ell) \cap Y_\alpha \subset U$. Since $U(\alpha, \ell) \cap Y_\alpha$ is not empty and $U(\alpha, \ell)$ is open, we can choose some infinite $J(\alpha, \ell) \subset U(\alpha, \ell)$ that is in $\mathcal{I}_\alpha^\perp$. Clearly this shows that there is a sequence $\{J(\alpha, \ell) : \ell \in \omega\} \subset \mathcal{I}^\perp$ satisfying the statement of the Fact. □

Here is the outline of the rest of the proof. We will force a cub $C \subset \omega_1$ so that $\{x_\alpha : \alpha \in C\}$ is a free sequence as witnessed by the family $\{W_\alpha : \alpha \in C\}$ in the sense that for each $\alpha \in C$ there is a $\beta \in C \cap \alpha$ such that $\{x_\delta : \delta \in C \cap (\beta, \alpha)\} \subset W_\alpha$. The countable tightness of X will guarantee that z is not a limit of the sequence $\{x_\alpha : \alpha \in C\}$. But there is more. We will also, simultaneously, force a sequence $\{L_\alpha : \alpha \in C\}$ of infinite subsets of ω satisfying that L_α is mod finite contained in W_δ for all $\delta \in C$ such that $x_\alpha \in W_\delta$. The family $\{L_\delta : \delta \in C\}$ will be chosen to play the same role as in the proof of Theorem 2.1. We will also have that z is not a limit of the set $\bigcup\{\overline{L_\alpha}\setminus\omega : \alpha\in C\}$. Finally, we also ensure that for each $\ell\in\omega$, the sequence of pairs $\{L_\alpha, J(\alpha,\ell) : \alpha\in C\}$ form a Hausdorff-Luzin type gap. So, again as in Theorem 2.1, if U is a neighborhood of z, there will be an $\ell\in\omega$ such that $J(\alpha,\ell)$ is mod finite contained in U for uncountably many $\alpha\in C$. This means that U will also meet uncountably many L_α ($\alpha\in C$) in an infinite set, contradicting that z is not a limit point of the set $\bigcup\{\overline{L_\alpha}\setminus\omega : \alpha\in C\}$.

Let κ be a sufficiently large regular cardinal and let $\mathcal{E}$ denote the family of all countable elementary submodels of $H(\kappa)$ that include the set $\{\mathcal{F}, Y, \{M_\alpha, x_\alpha, W_\alpha : \alpha\in\omega_1\}, \{J(\alpha,\ell) : \alpha\in\omega_1, 0<\ell\in\omega\}\}$. Let $\mathbb{M}$ denote the family of all finite $\in$-chains of members of $\mathcal{E}$.

We define our poset $\mathbb{P}$. A condition $p\in\mathbb{P}$ consists of a set $\mathcal{M}_p\cup C_p\cup L^p$ where

(1) $\mathcal{M}_p\in\mathbb{M}$,
(2) $C_p=\{M\cap\omega_1 : M\in\mathcal{M}_p\}$
(3) L^p is a finite subset of $C_p\times\omega$.

For each $p\in\mathbb{P}$ and each $\delta\in C_p$ we let $W(p,\delta)$ denote the open set $\bigcap\{W_\alpha : \alpha\in C_p \text{ and } x_\delta\in W_\alpha\}$. Then, we define $p<q$ for $p,q\in\mathbb{P}$ providing

(1) $\mathcal{M}_p\supset\mathcal{M}_q$ and $L^p\supset L^q$ (i.e. $p\supset q$),
(2) for $\beta\in C_p\setminus C_q$ with $\beta<\max(C_q)$, $x_\beta\in W(q,\min(C_q\setminus\beta))$,
(3) for each $\delta\in C_q$, $L^p\cap(\{\delta\}\times\omega)\setminus L^q$ is contained in $W(q,\delta)$,
(4) for each $\delta\in C_q$, $\ell<|C_q|$ and $\beta\in C_p\cap\delta\setminus C_q$, there is a $(\beta,j)\in L^p$ such that $|C_q|<j\in J(\delta,\ell)$.

If $q\in\mathbb{P}$ is an element of M for any $M\in\mathcal{E}$, then $p=q\cup\{M\}$ is an extension of q in $\mathbb{P}$. It follows that if $G\subset\mathbb{P}$ is generic, then $C=\bigcup\{C_p : p\in G\}$ is uncountable. We will not actually need that C is forced to be closed so we will omit that discussion.

Now we prove that $\mathbb{P}$ is proper. Suppose that $\theta>\kappa$ is a sufficiently large regular cardinal. Let M be a countable elementary submodel of $H(\theta)$ such that $\mathbb{P}\in M$. Choose and $q\in\mathbb{P}\cap M$. Following [12], to show that $\mathbb{P}$ is proper we will show that $q\cup\{M\cap H(\kappa)\}$ is an M-generic condition. To do so, suppose that $D\in M$ is a dense open subset of $\mathbb{P}$ and that $p\in D$ is an extension of $q\cup\{M\cap H(\kappa)\}$.

Let $\{\delta_0,\delta_1,\ldots,\delta_{m-1}\}$ enumerate $C_p\setminus M$ in increasing order. Note that p is an extension of $p\cap M$. Let $p\sim r$ indicate that

(1) $r\in D$ and $r<p\cap M$,
(2) $C_r\setminus C_{p\cap M}=\{\beta_0^r,\ldots,\beta_{m-1}^r\}$ listed in increasing order satisfies that $\max(C_{p\cap M})<\beta_0^r$,
(3) for each $i<m-1$, $\{k : x_{\beta_i^r}\in W_{\beta_k^r}\}$ equals $\{k : x_{\delta_i}\in W_{\delta_k}\}$,

(4) $L^r \cap (\beta_0^r \times \omega)$ equals $L^p \cap M$.

Our plan (relatively standard) is to find $p \sim r \in M$ satisfying that $x_{\beta_i^r} \in W(p, \delta_0)$ for each $i < m$. However, we will also need to find a sequence $\langle t_i : i < m\rangle \subset [\omega \setminus |C_p|]^{<\aleph_0}$ so that, for each i, $t_i \subset W(r, \beta_i^r)$ and $t_i \cap J(\delta_i, \ell)$ is not empty for each $\ell < |C_p|$. We then take the extension r_1 of r defined to be $r \cup \bigcup\{\{\beta_i^r\} \times t_i : i < m\}$. This, together with the fact that r extends $p \cap M$ ensures that $r_1 \cup p$ extends each of p and r. Thus finding r as above will suffice to prove that $\mathbb{P}$ is proper.

For each (possibly empty) m-tuple $\vec{t}$ of finite subsets of ω, let $D_{\vec{t}} = \{r : p \sim r \text{ and } (\forall i < m)\ t_i \subset W(r, \beta_i^r)\}$. Also let $\mathbb{M}_{\vec{t}} = \{M^r_{\beta_0^r} : r \in D_{\vec{t}}\}$, and define T to be the set of $\vec{t}$ satisfying that $\mathbb{M}_{\vec{t}}$ is a stationary subset of $[H(\kappa)]^\omega$. For any $i < m$ and $j \in \omega$ we use the notation $\vec{t}^{\,i}_j$ as introduced in Theorem 2.1.

Claim 3. $T \in M$ and $\emptyset \in T$ because $\mathbb{M}_\emptyset$ is stationary.

Proof of Claim 3. First we note that $D_\emptyset \in M$ since it is definable from any $r \in D_\emptyset \cap M$. For each $\vec{t}$ the set $D_{\vec{t}}$ is also in M. Furthermore, since $H(\kappa) \in M$, we also have that $\mathbb{M}_{\vec{t}} \in M$ for all such $\vec{t}$. It therefore follows that $T \in M$. If $\mathbb{M}_\emptyset$ were not stationary in $[H(\kappa)]^\omega$, then M would contain a cub avoiding $\mathbb{M}_\emptyset$. However, there is no such cub in M since $M \cap H(\kappa)$ is in $\mathbb{M}_\emptyset$. □

Claim 4. If $\vec{t} \in T$ and $i < m$, then the set $J_{\vec{t},i} = \{j : \vec{t}^{\,i}_j \in T\}$ is $\mathcal{F}$-large.

Proof of Claim 4. Similar to Claim 1, we may choose a (sufficiently large) countable $M' \prec H(\theta)$ such that $M' \cap H(\kappa) \in \mathbb{M}_{\vec{t}}$. In particular we assume that $J_{\vec{t},i} \in M'$ as well as D (and thus $D_{\vec{t}}$). Choose any $r \in D_{\vec{t}}$ such that $M^r_{\beta_0^r} = M' \cap H(\kappa)$. Since $\mathbb{M}_{\vec{t}^{\,i}_j} \in M'$ for each $j \in \omega$, it is immediate that $M^r_{\beta_0^r}$ is a witness to the fact that $\mathbb{M}_{\vec{t}^{\,i}_j}$ is stationary for each $j \in W(r, \beta_i^r) \cap \omega$. This shows that $W(r, \beta_i^r) \cap \omega \subset J_{\vec{t},i}$ and that $x_{\beta_i^r}$ is not in the closure of $\omega \setminus J_{\vec{t},i}$. Since $J_{\vec{t},i} \in M^r_{\beta_i^r}$, it follows that it is $\mathcal{F}$-large. □

Claim 5. There is a $\langle t_i : i < m\rangle \in T$ such that for all $i < m$, $\delta \in C_p \setminus M$ and all $\ell < |C_p|$, the set $t_i \cap J(\delta, \ell) \setminus |C_p|$ is not empty.

Proof of Claim 5. Choose any $\vec{t} \in T$, $i < m$, and any $\ell < |C_p|$ and $\delta \in C_p \setminus M$. The set $J_{\vec{t},i}$ is an $\mathcal{F}$-large element of M. There is a $\beta \in M$ such that $J_{\vec{t},i} \in M_\beta$. Therefore $\omega \setminus J_{\vec{t},i}$ is in $\mathcal{I}_\delta$. Since $J(\delta, \ell)$ was chosen from $\mathcal{I}_\delta^\perp$, we have that $J(\delta, \ell) \cap J_{\vec{t},i}$ is infinite. The claim now follows from a straightforward finite induction. □

Now we fix a $\vec{t} = \langle t_i : i < m\rangle$ as claimed in Claim 5 and we work with the collection $D_{\vec{t}}$. The rest of the proof follows the standard steps associated with the application to topology of the elementary submodels as side conditions method. Let $E = \{\vec{\beta}_r = \{\beta_0^r, \ldots, \beta_{m-1}^r\} : r \in D_{\vec{t}}\}$. Note that $E \in H(\kappa) \cap M$.

When we write $\{\beta_0, \ldots, \beta_i\}_<$ (for some integer i), we intend that each β_k $(k < i)$ is an ordinal (in ω_1) and that the set is enumerated in increasing order. For a subset E_0 of E and an integer $i \le m$, we make an obvious, but notationally clumsy, definition:

$$\text{let } E_0[i] = \{\{\beta_0, \beta_1, \ldots, \beta_{i-1}\} : \{\beta_0, \beta_1, \ldots, \beta_{m-1}\}_< \in E_0\}\ .$$

A subset E_0 of E will be said to be ω_1-branching if for each $i < m$ and $\{\beta_0, \beta_1, \ldots, \beta_{i-1}\} \in E_0[i]$ (where $\{\beta_0, \beta_1, \ldots, \beta_{i-1}\}$ denotes the empty set when $i = 0$)

$$E_0[i, \{\beta_0, \ldots, \beta_{i-1}\}] = \{\beta : \{\beta_0, \ldots, \beta_{i-1}\} \cup \{\beta\} \in E_0[i+1]\}$$

is uncountable.

Claim 6. There is an ω_1-branching $E_0 \subset E$.

Proof. Choose any $r \in D_{\vec{t}} \cap M$ such that $E \in M^r_{\beta_0^r}$ and let $\vec{\beta}^r = \{\beta_0^r, \ldots, \beta_{m-1}^r\}_<$. For each $i < m$, let $\vec{\beta}_i^r$ denote $\{\beta_0^r, \ldots, \beta_{i-1}^r\}$. Define $E_{m-1} \subset E$ to be all those elements $\{\beta_0, \ldots, \beta_{m-2}, \beta_{m-1}\}_< \in E$ satisfying that $E[m-1, \{\beta_0, \ldots, \beta_{m-2}\}]$ is uncountable. We note that $\vec{\beta}^r$ is in E_{m-1} because $E[m-1, \vec{\beta}_{m-1}^r]$ is an element of $M^r_{\beta^r_{m-1}}$ with $\beta_{m-1}^r \in E[m-1, \vec{\beta}_i^r]$ witnessing that it is not contained in $M^r_{\beta^r_{m-1}}$.

We proceed by induction to define a descending sequence E_i for $i = m-1, m-2, \ldots, 0$ satisfying that $\vec{\beta}^r \in E_i$ and that for all $\{\beta_0, \beta_1, \ldots, \beta_{m-1}\}_< \in E_i$ and $i \le j < m$, the set $E_i[j, \{\beta_0, \beta_1, \ldots, \beta_{j-1}\}]$ is uncountable. Given E_i with $0 < i$, define $\{\beta_0, \ldots, \beta_{m-1}\}_< \in E_i$ providing $E_i[i-1, \{\beta_0, \ldots, \beta_{i-2}\}_<]$ is uncountable. The fact that $\vec{\beta}^r$ is in E_{i-1} is proven just as in the proof that $\vec{\beta}^r \in E_{m-1}$.

Clearly E_0 has the desired property. □

We recall that $M \cap \omega_1 = \delta_0 = \delta_0^p$, and we prove the next claim.

Claim 7. If $S \in M$ is an uncountable subset of ω_1, then there is an $\alpha \in S \cap M$ such that $x_\alpha \in W(p, \delta_0)$.

Proof of Claim 7. For any $S \subset \omega_1$, let $X[S]$ denote the set $\{x_\alpha : \alpha \in S\}$. We note that the closure of $X[S]$ is an element of $\mathcal{F}$ for all uncountable $S \subset \omega_1$. In addition, since $\mathcal{F}$ has a base of separable sets and the space Y has countable tightness, there is a $\beta \in \omega_1$ such that the closure of $X[S\cap\beta]$ is in $\mathcal{F}$. Now if $S\in M$ is an uncountable subset of ω_1, it follows by elementarity that there is a $\beta\in M$ such that the closure of $X[S\cap\beta]$ is in $\mathcal{F}$. Furthermore, there is a $\gamma\in M$ such that $S\cap\beta\in M_\gamma$. Finally we note that since $\gamma<\delta$, x_{δ_0} is a limit point of $X[S\cap\beta]$, and so there is an $\alpha\in S\cap\beta$ such $x_\alpha\in W(p,\delta_0)$. □

Choose any ω_1-branching $E_0\subset E$ with $E_0\in M$. It follows that $E_0[0,\emptyset]$ is an uncountable subset of ω_1 that is an element of M. Therefore, by Claim 7, there is a $\beta_0\in M\cap E_0[0,\emptyset]$ such that $x_{\beta_0}\in W(p,\delta_0)$. Similarly, $E_0[1,\{\beta_0\}]$ is uncountable and an element of M so there is a $\beta_1\in E_0[1,\{\beta_0\}]\cap M$ such that x_{β_1} is also in $W(p,\delta_0)$. Proceeding in this way, there is a $\{\beta_0,\beta_1,\ldots,\beta_{m-1}\}_<\in E_0\cap M$ such that $x_{\beta_i}\in W(p,\delta_0)$ for each $i<m$.

This completes the proof that $\mathbb{P}$ is proper. The proof of the theorem is completed by applying PFA to the proper poset $\mathbb{Q}={}^{<\omega_1}\omega_2*\mathbb{P}$ just as was done in Theorem 2.1. □

We note that Question 5 from [5] remains unsolved.

Question 3.1. Is it consistent (or even a consequence of PFA) that every space of countable tightness is C-closed?

Example 3.1. There is an example (well-known) of a countably compact space of countable tightness that does not have countable π-character. The space X is the standard Σ-product consisting of all elements of the product space 2^{ω_1} that do not take on value 1 in an uncountable set of coordinates. It is easily seen that X is countably compact (in fact, countable subsets have compact closure). We prove that X has countable tightness. Let A be a subset of X with limit $z\in X$. Choose any countable elementary submodel M with z,A and ω_1 in M. We show that z is a limit of $A\cap M$. For a finite set H of ω_1, let $[z\restriction H]$ denote the usual basic open set in 2^{ω_1}. Consider any finite $H\subset\omega_1$. It follows that $[z\restriction(H\cap M)]$ is an element of M and so there is an $a\in A\cap M\cap[z\restriction(H\cap M)]$. But now each of z and a have value 0 on every coordinate outside of M and so $a\in[z\restriction H]$.

References

[1] A. V. Arhangel′skiĭ, *Bicompacta that satisfy the Suslin condition hereditarily. Tightness and free sequences*, Dokl. Akad. Nauk SSSR **199** (1971), 1227–1230 (Russian). MR0288718

[2] Zoltán T. Balogh, *On compact Hausdorff spaces of countable tightness*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 755–764, DOI 10.2307/2046929. MR930252

[3] Z. Balogh, A. Dow, D. H. Fremlin, and P. J. Nyikos, *Countable tightness and proper forcing*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 1, 295–298, DOI 10.1090/S0273-0979-1988-15649-2. MR940491

[4] Alan Dow, *A countably compact, countably tight, nonsequential space*, General topology and applications (Middletown, CT, 1988), Lecture Notes in Pure and Appl. Math., vol. 123, Dekker, New York, 1990, pp. 71–80. MR1057625

[5] Alan Dow and Todd Eisworth, *CH and the Moore-Mrowka problem*, Topology Appl. **195** (2015), 226–238, DOI 10.1016/j.topol.2015.09.025. MR3414886

[6] Alan Dow, *Compact C-closed spaces need not be sequential*, submitted (2015).

[7] Todd Eisworth, *Totally proper forcing and the Moore-Mrówka problem*, Fund. Math. **177** (2003), no. 2, 121–137, DOI 10.4064/fm177-2-2. MR1992528

[8] ———, *Countable compactness, hereditary π-character, and the continuum hypothesis*, Topology Appl. **153** (2006), no. 18, 3572–3597, DOI 10.1016/j.topol.2006.03.021. MR2270606

[9] A. Hajnal and I. Juhász, *On hereditarily α-Lindelöf and α-separable spaces. II*, Fund. Math. **81** (1973/74), no. 2, 147–158. Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, II. MR0336705

[10] Mohammad Ismail and Peter Nyikos, *On spaces in which countably compact sets are closed, and hereditary properties*, Topology Appl. **11** (1980), no. 3, 281–292, DOI 10.1016/0166-8641(80)90027-9. MR585273

[11] D. V. Rančin, *Tightness, sequentiality and closed coverings*, Dokl. Akad. Nauk SSSR **232** (1977), no. 5, 1015–1018 (Russian). MR0436074

[12] Stevo Todorčević, *A note on the proper forcing axiom*, Axiomatic set theory (Boulder, Colo., 1983), Contemp. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1984, pp. 209–218, DOI 10.1090/conm/031/763902, (to appear in print). MR763902

[13] ———, *Free sequences*, Topology Appl. **35** (1990), no. 2-3, 235–238, DOI 10.1016/0166-8641(90)90108-E. MR1058803

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