THE INDEPENDENCE OF GCH AND A COMBINATORIAL PRINCIPLE RELATED TO BANACH-MAZUR GAMES

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Abstract. It was proved recently that Telgársky's conjecture, which concerns partial information strategies in the Banach-Mazur game, fails in models of $\mathsf{GCH} + \square$. The proof introduces a combinatorial principle that is shown to follow from $\mathsf{GCH} + \square$, namely:

$\triangledown$: Every separative poset $\mathbb{P}$ with the κ-cc contains a dense sub-poset $\mathbb{D}$ such that $|\{q \in \mathbb{D} : p \text{ extends } q\}| < \kappa$ for every $p \in \mathbb{P}$.

We prove this principle is independent of GCH and CH, in the sense that $\triangledown$ does not imply CH, and GCH does not imply $\triangledown$ assuming the consistency of a huge cardinal.

We also consider the more specific question of whether $\triangledown$ holds with $\mathbb{P}$ equal to the weight-$\aleph_\omega$ measure algebra. We prove, again assuming the consistency of a huge cardinal, that the answer to this question is independent of $\mathsf{ZFC} + \mathsf{GCH}$.

1. Introduction

Telgársky's conjecture states that for each $k \in \mathbb{N}$, there is a topological space X such that the player NONEMPTY has a winning $(k+1)$-tactic, but no winning k-tactic, in the Banach-Mazur game on X. Recently, the first two authors, along with David Milovich and Lynne Yengulalp, proved that it is consistent for this conjecture to fail [1]. The proof introduces the following combinatorial principle, which implies the failure of Telgársky's conjecture:

$\triangledown$: Every separative poset $\mathbb{P}$ with the κ-cc contains a dense sub-poset $\mathbb{D}$ such that $|\{q \in \mathbb{D} : p \text{ extends } q\}| < \kappa$ for every $p \in \mathbb{P}$.

In [1], the consistency of $\triangledown$ is proved from $\mathsf{GCH} + \square$ via the construction of what are called κ-sage Davies trees, which are defined in Section 2 below. The existence of arbitrarily long κ-sage Davies trees implies $\triangledown$ holds for κ-cc posets. It is also proved in [1] that $\triangledown$ implies $\mathfrak{b} = \aleph_1$, or more generally that $\triangledown$ implies there is no decreasing sequence of length ω_2 in $\mathcal{P}(\omega)/\text{fin}$. Therefore $\triangledown$ is independent of ZFC.

But this raises the question of the relationship between $\triangledown$ and GCH, specifically whether either of these statements implies the other. The purpose of this paper is to answer this question in the negative by showing that GCH does not imply $\triangledown$, and $\triangledown$ does not imply CH.

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In Section 2, we prove that when Cohen reals are added by forcing, the existence of arbitrarily long κ-sage Davies trees in the ground model suffices to guarantee that $\triangledown$ holds for κ-cc posets in the extension. Thus adding Cohen reals to a model of $\mathsf{GCH} + \Box$ produces a model of $\triangledown + \neg\mathsf{CH}$.

On the other hand, we show in Section 3 that the Chang conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies that $\triangledown$ fails. This is done by directly constructing a ccc poset $\mathbb{P}$ (a modified product of $\aleph_\omega$ Hechler forcings) and then using $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ to show it violates $\triangledown$. As $\mathsf{GCH}+(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is consistent relative to a huge cardinal [5], this shows that GCH does not imply $\triangledown$ unless huge cardinals are inconsistent. We note that finding a model of $\mathsf{GCH} + \neg\triangledown$ requires large cardinals. In fact, the proof of the consistency of $\triangledown$ in [1] only uses $\mathsf{GCH}+\Box$-for-singulars, and the consistency of GCH plus the failure of $\Box$ at any singular cardinal is known to have significant large cardinal strength [2].

In Section 4 we consider the more specific question of whether $\triangledown$ holds with $\mathbb{P}$ equal to the weight-$\aleph_\omega$ measure algebra. We prove that the answer to this question is also independent of $\mathsf{ZFC} + \mathsf{GCH}$. Once again Chang's conjecture for $\aleph_\omega$ comes into the proof, and so the result is established modulo the consistency of a huge cardinal.

2. $\triangledown$ does not imply CH

A Davies tree is a sequence $\langle M_\alpha \colon \alpha < \nu \rangle$ of elementary submodels of some large fragment H_θ of the set-theoretic universe such that the M_α enjoy certain coherence and covering properties. (These sequences are called "trees" because they are usually constructed by enumerating the leaves of a tree of elementary submodels of H_θ.) These structures provide a unified framework for carrying out a wide variety of constructions in infinite combinatorics. They were introduced by R. O. Davies in [3], and an excellent survey of their many uses can be found in Daniel and Lajos Soukup's paper [10].

Also in [10], the Soukups construct a countably closed version of a Davies tree called a "sage Davies tree" using $\mathsf{GCH} + \Box$. These structures were generalized in [1] by constructing $<\kappa$-closed versions of these trees for uncountable κ, called κ-sage Davies trees. Roughly, κ-sage Davies trees of length ν allow us to take an object of size ν with "critical substructures" of size $<\kappa$ (such as a ν-sized poset with the κ-cc), and to approximate the large object (size ν) with a sequence of smaller ones (size κ). It was proved in [1] that $\mathsf{GCH} + \Box$ implies the existence of arbitrarily long κ-sage Davies trees for every regular cardinal κ.

In this section, we show that if we begin with a model of set theory containing arbitrarily long κ-sage Davies trees, then, after adding any number of Cohen reals by forcing, $\triangledown$ holds in the extension for separative κ-cc posets. It follows that $\triangledown$ is consistent with any permissible value of $2^{\aleph_0}$.

Given a poset $\mathbb{P}$, recall that the *Souslin number* of $\mathbb{P}$, denoted $S(\mathbb{P})$, is the minimum value of κ such that $\mathbb{P}$ has no antichains of size κ. Erdős and Tarski proved in [4] that $S(\mathbb{P})$ is a regular cardinal for every poset $\mathbb{P}$.

For every poset $\mathbb{P}$, let $\triangledown(\mathbb{P})$ denote the statement that $\triangledown$ holds for $\mathbb{P}$, i.e., that there is a dense sub-poset $\mathbb{D}$ of $\mathbb{P}$ with $|\{d \in \mathbb{D} \colon p \text{ extends } d\}| < S(\mathbb{P})$ for every $p \in \mathbb{P}$.

In what follows, H_θ denotes the set of all sets hereditarily smaller than some very big cardinal θ. Given two sets M and N, we write $M \prec N$ to mean that $(M, \in)$ is an elementary submodel of $(N, \in)$. A set M is called $<\kappa$*-closed* if $M^{<\kappa} \subseteq M$. If M satisfies (enough of) ZFC, this is equivalent to the property $[M]^{<\kappa} \subseteq M$.

Definition 2.1. Let κ, ν be infinite cardinals and let p be some set. A *κ-sage Davies tree for ν over p* is a sequence $\langle M_\alpha \colon \alpha < \nu \rangle$ of elementary submodels of $(H_\theta, \in)$, for some "big enough" regular cardinal θ, such that

(1) $p \in M_\alpha$, M_α is $<\kappa$-closed, and $|M_\alpha| = \kappa$ for all $\alpha < \nu$.
(2) $[\nu]^{<\kappa} \subseteq \bigcup_{\alpha<\nu} M_\alpha$.
(3) For each $\alpha < \nu$, there is a set $\mathcal{N}_\alpha$ of elementary submodels of H_θ such that $|\mathcal{N}_\alpha| < \kappa$, each $N \in \mathcal{N}_\alpha$ is $<\kappa$-closed and contains p, and

$$\textstyle\bigcup_{\xi<\alpha} M_\xi = \bigcup \mathcal{N}_\alpha.$$

(4) $\langle M_\xi \colon \xi < \alpha \rangle \in M_\alpha$ for each $\alpha < \nu$.
(5) $\bigcup_{\alpha<\nu} M_\alpha$ is a $<\kappa$-closed elementary submodel of H_θ.

The following fact is proved in [1, Theorem 3.20]:

Theorem 2.2. *Assume* GCH $+ \square$. *Let κ, ν be infinite regular cardinals with $\kappa < \nu$. For any set p, there is a κ-sage Davies tree for ν over p.*

In fact, the proof in [1] uses a weak version of $\square$ related to the Very Weak Square principle articulated by Foreman and Magidor in [6]. The following fact, which we will use below, is Lemma 3.7 in [1].

Lemma 2.3. *Let κ, ν be regular cardinals with $\kappa < \nu$, let p be any set, and let $\langle M_\alpha \colon \alpha < \nu \rangle$ be a κ-sage Davies tree for ν over p. If $\alpha < \beta < \nu$, then*

$$\alpha \in M_\beta \quad \Leftrightarrow \quad M_\alpha \in M_\beta \quad \Leftrightarrow \quad M_\alpha \subseteq M_\beta.$$

In addition to the five properties listed above that define a κ-sage Davies tree, it will be convenient here to have trees with one additional property:

(6) For every $\alpha < \nu$, there is a well ordering $\sqsubset_\alpha$ of M_α with order type κ such that if $\alpha < \beta < \mu$ and $\alpha \in M_\beta$, then $\sqsubset_\alpha \in M_\beta$.

It turns out that this property of κ-sage Davies trees is already a consequence of properties (1) through (5).

Lemma 2.4. *Let κ, ν be regular cardinals with $\kappa < \nu$ and let p be some set. Every κ-sage Davies tree for ν over p satisfies property* (6).

Proof. First observe that if $\alpha < \nu$ then $M_\alpha \in M_{\alpha+1}$. This is because $\langle M_\xi : \xi < \alpha+1 \rangle \in M_{\alpha+1}$ by definition, and this implies $M_\alpha \in M_{\alpha+1}$ because M_α is definable from $\langle M_\xi : \xi < \alpha+1 \rangle$.

Because $|M_\alpha| = \kappa$, there is (in H_θ) a well ordering of M_α with order type κ. By elementarity, there is some such well ordering of M_α in $M_{\alpha+1}$. For each $\alpha < \mu$, fix a well ordering $\sqsubset_\alpha$ of M_α with order type κ such that $\sqsubset_\alpha \in M_{\alpha+1}$. If $\alpha < \beta < \nu$ and $\alpha \in M_\beta$, then $\alpha + 1 \in M_\beta$ and therefore $M_{\alpha+1} \subseteq M_\beta$ by the previous lemma. In particular, $\sqsubset_\alpha \in M_\beta$. □

It will be convenient to work with complete Boolean algebras rather than arbitrary posets when proving $\triangledown$ holds in Cohen extensions. This restriction is justified by the following lemma.

Lemma 2.5. *$\triangledown$ holds if and only if it holds for every poset of the form $\mathbb{P} = \mathbb{B} \setminus \{\mathbf{0}\}$, where $\mathbb{B}$ is a complete Boolean algebra.*

Proof. This is proved in [1, Lemma 2.10]. Roughly, the "only if" direction is obvious because posets of the form $\mathbb{B} \setminus \{\mathbf{0}\}$ are always separative, and the "if" direction is proved by showing that if $\mathbb{P}$ is separative, then $\triangledown(\mathbb{P})$ is equivalent to $\triangledown$(the Boolean completion of $\mathbb{P}$). □

Given a complete Boolean algebra $\mathbb{B}$, $S(\mathbb{B})$ denotes the Souslin number of the poset $\mathbb{B} \setminus \{\mathbf{0}\}$. Given $J \subseteq \mathbb{B}$, $\bigwedge J$ denotes the infimum of J in $\mathbb{B}$ and $\bigvee J$ denotes the supremum of J in $\mathbb{B}$.

Lemma 2.6. *Let $\mathbb{B}$ be a complete Boolean algebra and let $J \subseteq \mathbb{B}$. Then there is some $J' \subseteq J$ with $|J'| < S(\mathbb{B})$ such that $\bigwedge J' = \bigwedge J$ and $\bigvee J' = \bigvee J$.*

Proof. If we delete the "and $\bigvee J' = \bigvee J$" from the end of the lemma, then it becomes a special case of [1, Lemma 3.2]. If we delete the "$\bigwedge J' = \bigwedge J$ and" instead, then it follows from the previous sentence via de Morgan's laws. Thus given $J \subseteq \mathbb{B}$, there is some $J'_\wedge \subseteq J$ with $|J'_\wedge| < S(\mathbb{B})$ such that $\bigwedge J'_\wedge = \bigwedge J$, and there is some $J'_\vee \subseteq J$ with $|J'_\vee| < S(\mathbb{B})$ such that $\bigvee J'_\vee = \bigvee J$. Then $J' = J'_\wedge \cup J'_\vee$ satisfies the conclusion of the lemma. □

Lemma 2.7. *Let $\mathbb{B}$ be a complete Boolean algebra and let $X \subseteq \mathbb{B}$ with $|X| = S(\mathbb{B})$. Then there is some $Y \subseteq X$ with $|X \setminus Y| < S(\mathbb{B})$ such that $\bigwedge Y = \bigwedge (Y \setminus Z)$ for every $Z \subseteq Y$ with $|Z| < S(\mathbb{B})$.*

Proof. Let $\kappa = S(\mathbb{B})$. Fix $X \subseteq \mathbb{B} \setminus \{\mathbf{0}\}$ with $|X| = \kappa$, and let $\{b_\alpha : \alpha < \kappa\}$ be an enumeration of X with order type κ. Let $c_\alpha = \bigwedge \{b_\xi : \xi \geq \alpha\}$ for each $\alpha < \kappa$, and note that $\alpha \leq \alpha'$ implies $c_\alpha \leq c_{\alpha'}$. By Lemma 2.6, there is some $\beta < \kappa$ such that $\bigvee \{c_\alpha : \alpha < \kappa\} = \bigvee \{c_\alpha : \alpha < \beta\}$. (This uses the fact that κ is regular: as mentioned above, the Souslin number of a poset is always a regular cardinal.) Because the c_α form a non-decreasing sequence in $\mathbb{B}$, this means $c_\alpha = c_\beta$ for all $\alpha \geq \beta$. Let $Y = \{b_\xi : \xi \geq \beta\}$. If $Z \subseteq Y$ with $|Z| < \kappa$, then there is some α with $\beta \leq \alpha < \kappa$ such that $Z \subseteq \{b_\xi : \xi < \alpha\}$. But then

$$c_\beta = \bigwedge Y \leq \bigwedge (Y \setminus Z) \leq \bigwedge \{b_\xi : \xi \geq \alpha\} = c_\alpha = c_\beta.$$

Therefore $\bigwedge (Y \setminus Z) = c_\beta$ for any $Z \subseteq Y$ with $|Z| < \kappa$. □

If $\mathbb{F}$ is a forcing poset and A is a set, recall that a *nice name* for a subset of A is a subset $\dot{X}$ of $A \times \mathbb{F}$ such that for each $a \in A$, $\left\{p \in \mathbb{F} \colon (a,p) \in \dot{X}\right\}$ is an antichain in $\mathbb{F}$. Given $B \subseteq A$, $\dot{X} \restriction B = \dot{X} \cap (B \times \mathbb{F})$. We adopt the convention of deleting a dot to denote the evaluation of a name. For example, if $\dot{X}$ is a nice $\mathbb{F}$-name for a subset of μ, then we write $\mathbf{1}_{\mathbb{F}} \Vdash$ "$X \subseteq \mu$."

Lemma 2.8. *Let $\mathbb{F}$ be a ccc notion of forcing, let $\dot{\trianglelefteq}$ be an $\mathbb{F}$-name for a relation on some infinite cardinal μ, and suppose that $\mathbf{1}_{\mathbb{F}} \Vdash$ "$(\mu, \trianglelefteq)$ is a complete Boolean algebra with $S(\mu, \trianglelefteq) = \kappa$." Let $p \in \mathbb{F}$ and let $\dot{X}$ be a nice name for a subset of μ. If $p \Vdash$ "$|X| = \kappa$" then there is some $\dot{Y} \subseteq \dot{X}$ with $|\dot{Y} \setminus \dot{X}| < \kappa$ such that $p \Vdash$ "$\bigwedge Y = \bigwedge (Y \setminus Z)$ for any $Z \subseteq \mu$ with $|Z| < \kappa$."*

Proof. As μ is infinite, κ must be a regular uncountable cardinal. Because $\mathbb{F}$ has the ccc, we know that for every $\mathbb{F}$-name $\dot{W}$ for a subset of μ, if $q \in \mathbb{F}$ and $q \Vdash$ "$|W| < \kappa$", then there is some $A \subseteq \mu$ (in the ground model) such that $|A| < \kappa$ and $q \Vdash$ "$W \subseteq A$."

By Lemma 2.7, and the existential completeness lemma, there is a name $\dot{Y}_0$ for a subset of μ such that $p \Vdash$ "$Y_0 \subseteq X$ and $|X \setminus Y_0| < \kappa$ and $\bigwedge Y_0 = \bigwedge (Y_0 \setminus Z)$ for every $Z \subseteq Y$ with $|Z| < \kappa$." By the previous paragraph, there is some $A \subseteq \mu$ (in the ground model) such that $|A| < \kappa$ and $p \Vdash$ "$X \setminus Y_0 \subseteq A$." Furthermore, $p \Vdash$ "$\bigwedge ((X \setminus A) \setminus Z) = \bigwedge X \setminus (A \cup Z) = \bigwedge Y_0 = \bigwedge Y_0 \setminus A = \bigwedge X \setminus A$ for any $Z \subseteq \mu$ with $|Z| < \kappa$."

Let $\dot{Y} = \dot{X} \restriction (\mu \setminus A)$. Clearly $\dot{Y} \subseteq \dot{X}$ and $p \Vdash$ "$Y = X \setminus A$." Because $\dot{X}$ is a nice name and $\mathbb{F}$ has the ccc, $\left\{q \in \mathbb{F} \colon (q,a) \in \dot{X}\right\}$ is countable for every $a \in A$; therefore $|\dot{X} \setminus \dot{Y}| \leq \aleph_0 \cdot |A| < \kappa$. Finally, because $p \Vdash$ "$Y = X \setminus A$", the last assertion of the lemma follows from the last sentence of the previous paragraph. $\square$

Given a cardinal λ, let $\mathrm{Fn}(\lambda, 2)$ denote the poset of finite partial functions $\lambda \to \{0,1\}$, the standard forcing poset for adding λ Cohen reals.

Theorem 2.9. *Suppose V is a model of $\mathsf{GCH} + \square$ (or, more generally, suppose V is a model satisfying the conclusion of Theorem 2.2). If λ is any cardinal and G is $\mathrm{Fn}(\lambda, 2)$-generic over V, then $V[G] \models \triangledown$.*

Proof. Let μ, κ be infinite cardinals, and let $\dot{\trianglelefteq}$ be a $\mathrm{Fn}(\lambda, 2)$-name such that $\emptyset \Vdash$ "$(\mu, \trianglelefteq)$ is a complete Boolean algebra with $S(\mu, \trianglelefteq) = \kappa$." Note that this implies κ is regular and uncountable. Let ν be a regular uncountable cardinal with $\lambda, \mu \leq \nu$ and with $\kappa < \nu$. Without loss of generality, we may and do assume that 0 (the ordinal) is equal to $\mathbf{0}$ (the $\trianglelefteq$-least element of μ). More precisely, we assume $\emptyset \Vdash$ "$\mathbf{0}_{(\mu, \trianglelefteq)} = 0$."

We work momentarily in the ground model. Applying Theorem 2.2, let $\langle M_\alpha \colon \alpha < \nu \rangle$ be a κ-sage Davies tree for ν over $(\mu, \dot{\trianglelefteq})$. Applying Lemma 2.4, fix for each $\alpha < \nu$ some well ordering $\sqsubset_\alpha$ of M_α with order type κ such that if $\alpha < \beta < \nu$ and $\alpha \in M_\beta$, then $\sqsubset_\alpha \in M_\beta$.

For each $x \in \bigcup_{\alpha<\nu} M_\alpha$, the *level* of x, denoted $\mathrm{Lev}(x)$, is defined as the least $\alpha < \nu$ such that $x \in M_\alpha$. Let $\sqsubset$ denote the well-order of $\bigcup_{\alpha<\nu} M_\alpha$ defined as follows:

- if $\mathrm{Lev}(x) < \mathrm{Lev}(y)$, then $x \sqsubset y$.
- if $\mathrm{Lev}(x) = \mathrm{Lev}(y) = \alpha$, then $x \sqsubset y$ if and only if $x \sqsubset_\alpha y$.

We write $x \sqsubseteq y$ to mean that either $x \sqsubset y$ or $x = y$.

We now define, via recursion, a sequence $\langle d_\gamma \colon \gamma < \mu \rangle$ of members of μ. Simultaneously, we also define a sequence $\langle I_\gamma \colon \gamma < \mu \rangle$ of $<\kappa$-sized subsets of μ, and a sequence $\left\langle \dot{J}_\gamma \colon \gamma < \mu \right\rangle$ of nice names. These definitions take place in the extension $V[G]$, and we do not claim that any of these sequences is a member of the ground model V. For the base case, let $d_0 = 0$ and let $I_0 = \dot{J}_0 = \emptyset$. For the recursive step, fix $\gamma < \mu$ and suppose that d_β, I_β, and $\dot{J}_\beta$ are already defined for each $\beta \sqsubset \gamma$. If there is some $\beta \sqsubset \gamma$ such that $0 \neq d_\beta \trianglelefteq \gamma$, then set $d_\gamma = 0$ and set $I_\gamma = \dot{J}_\gamma = \emptyset$. If there is no such β, then let I_γ denote the $\sqsubseteq$-minimal set in the ground model V with the following two properties:

- I_γ is a $<\kappa$-sized subset of μ.
- In $V[G]$, there is some $J \subseteq I_\gamma$ such that $0 \neq \bigwedge J \trianglelefteq \gamma$.

Note that I_γ is well-defined because $\{\gamma\} \in V$ and $\{\gamma\}$ has both these properties. (Note that this implies $I_\gamma \sqsubseteq \{\gamma\}$.) Because of the second property of I_γ listed above, there is a nice name $\dot{J}$ in the ground model V for a subset of I_γ such that, for some $p \in G$, we have $p \Vdash$ "$(\dot{J})_G = J \subseteq I_\gamma$ and $0 \neq \bigwedge J \trianglelefteq \gamma$." Let $\dot{J}_\gamma$ denote the $\sqsubseteq$-minimal nice $\mathrm{Fn}(\lambda, 2)$-name in V with this property. Finally, let $d_\gamma = \bigwedge (\dot{J}_\gamma)_G$.

(Note: Because the I_γ's and the $\dot{J}_\gamma$'s are defined in the extension, we have in the ground model a name $\dot{I}_\gamma$ and a name $\ddot{J}_\gamma$ for a nice name for a subset of $\dot{I}_\gamma$ that is forced (by $\emptyset$) to be the γ^{th} element of the sequence constructed above. In particular, $p \Vdash$ "$(\ddot{J}_\gamma)_G = \dot{J}_\gamma \subseteq I_\gamma = (\dot{I}_\gamma)_G$" for some $p \in G$. Recall our convention of deleting a dot to denote the evaluation of a name!)

Let $\mathbb{D} = \{d_\gamma \colon \gamma < \mu \text{ and } d_\gamma \neq 0\}$. We claim that this set $\mathbb{D}$ is a witness to the fact that $\triangledown(\mu, \trianglelefteq)$ holds in $V[G]$.

To see that $\mathbb{D}$ is a dense subset of $(\mu, \trianglelefteq)$, fix some nonzero $\gamma < \mu$. If $d_\gamma \neq 0$, then $d_\gamma \in \mathbb{D}$ and $d_\gamma \trianglelefteq \gamma$. If $d_\gamma = 0$, then this means there is some $\beta \sqsubset \gamma$ such that $0 \neq d_\beta \trianglelefteq \gamma$, and so $d_\beta \in \mathbb{D}$ and $d_\beta \trianglelefteq \gamma$. Either way, some member of $\mathbb{D}$ is $\trianglelefteq \gamma$. As γ was arbitrary, $\mathbb{D}$ is dense.

For the more difficult part of the proof, we must show that every $\delta \in \mu \setminus \{0\}$ has the property that $|\{d \in \mathbb{D} \colon \delta \trianglelefteq d\}| < \kappa$. Aiming for a contradiction, let us suppose otherwise. Fix some $\delta \in \mu \setminus \{0\}$ such that $|\{d \in \mathbb{D} \colon \delta \trianglelefteq d\}| \geq \kappa$. Let $S = \{\gamma < \mu \colon d_\gamma \in \mathbb{D} \text{ and } \delta \trianglelefteq d_\gamma\}$.

Observe that $\beta \neq \gamma$ implies $d_\beta \neq d_\gamma$ whenever $d_\beta, d_\gamma \in \mathbb{D}$. (This is because if $\beta \sqsubset \gamma$, then $d_\gamma \neq 0$ implies $d_\beta \ntrianglelefteq \gamma$ while $d_\gamma \trianglelefteq \gamma$.) Therefore the map

$\gamma \mapsto d_\gamma$ is injective on S, and we may think of S simply as an indexing set for $\{d \in \mathbb{D}\colon \delta \trianglelefteq d\} = \{d_\gamma\colon \gamma \in S\}$.

Claim. *There is some $I \subseteq \mu$ such that $I_\gamma = I$ for $\geq\kappa$-many $\gamma \in S$.*

Proof of claim. Aiming for a contradiction, let us assume the claim is false. Let ζ denote the least ordinal $<\nu$ with the property that $\mathrm{Lev}(I_\gamma) < \zeta$ for $\geq\kappa$-many $\gamma \in S$. Some such ζ must exist because $|S| \geq \kappa$ and ν is a regular cardinal with $\kappa < \nu$.

By part (3) of our definition of a κ-sage Davies tree, there is a collection $\mathcal{N}$ of $<\kappa$-closed elementary submodels of H_θ such that $|\mathcal{N}| < \kappa$ and $\bigcup \mathcal{N} = \bigcup_{\xi<\zeta} M_\xi$. By our choice of ζ and the regularity of κ, some $N \in \mathcal{N}$ has the property that $I_\gamma \in N$ for $\geq\kappa$-many $\gamma \in S$. Fix some such N, let $S_N = \{\gamma \in S\colon I_\gamma \in N\}$, and let $\mathbb{D}_N = \{d_\gamma\colon \gamma \in S_N\}$. Note that $\bigwedge \mathbb{D}_N \neq 0$ because $\delta \trianglelefteq \bigwedge \mathbb{D} \trianglelefteq \bigwedge \mathbb{D}_N$.

Applying Lemma 2.6, there is some $T \subseteq S_N$ with $|T| < \kappa$ such that $\bigwedge \mathbb{D}_N = \bigwedge \{d_\gamma\colon \gamma \in T\}$. Let $I_0 = \bigcup \{I_\gamma\colon \gamma \in T\}$. Then I_0 is a subset of $N \cap \mu$ in $V[G]$, and $|I_0| < \kappa$. Because $\mathrm{Fn}(\lambda, 2)$ has the ccc, there is a subset I of $N \cap \mu$ in V with $I_0 \subseteq I$ and $|I| \leq |I_0| \cdot \aleph_0 < \kappa$. Because N is $<\kappa$-closed in V, we have $I \in N$.

For each $\gamma \in T$, there is a subset $J_\gamma = (\dot{J}_\gamma)_G$ of I_γ with $\bigwedge J_\gamma = d_\gamma$. Note that $\bigwedge \{d_\gamma\colon \gamma \in T\} = \bigwedge_{\gamma\in T} \bigwedge J_\gamma = \bigwedge\left(\bigcup_{\gamma\in T} J_\gamma\right)$, and let $J = \bigcup_{\gamma\in T} J_\gamma$. Now $J \subseteq I$, and $\bigwedge J = \bigwedge \{\bigwedge J_\gamma\colon \gamma \in T\} = \bigwedge \{d_\gamma\colon \gamma \in T\} = \bigwedge \mathbb{D}_N$. Furthermore, $0 \neq \bigwedge \mathbb{D}_N \trianglelefteq d_\gamma$ for each $\gamma \in S_N$. Thus, for each $\gamma \in S_N$, there is a subset J of I such that $0 \neq \bigwedge J \trianglelefteq d_\gamma \trianglelefteq \gamma$.

This shows that I satisfies the conditions in the definition of I_γ whenever $\gamma \in S_N$. It follows that $I_\gamma \sqsubseteq I$ for all $\gamma \in S_N$. Now, our definition of $\sqsubseteq$ entails that I has $<\kappa$-many $\sqsubseteq$-predecessors in $\mathrm{Lev}(I)$, and each predecessor $I' \sqsubseteq I$ has $<\kappa$-many $\gamma \in S_N$ with $I_\gamma = I'$ (by our assumption at the beginning of the proof of this claim). Therefore $\mathrm{Lev}(I_\gamma) = \mathrm{Lev}(I)$ for only $<\kappa$-many $\gamma \in S_N$. As $\mathrm{Lev}(I_\gamma) \leq \mathrm{Lev}(I)$ for all $\gamma \in S_N$ and $|S_N| \geq \kappa$, it follows that $\mathrm{Lev}(I_\gamma) < \mathrm{Lev}(I)$ for $\geq\kappa$-many $\gamma \in S_N$. But $\mathrm{Lev}(I) < \zeta$, because $I \in N \subseteq \bigcup_{\xi<\zeta} M_\xi$, so this contradicts our choice of ζ. $\square$

Fix some $I \subseteq \mu$ with $|I| < \kappa$ that satisfies the conclusion of the above claim. By replacing S with a size-κ subset of $\{\gamma \in S\colon I_\gamma = I\}$ if necessary, we may (and do) assume that $|S| = \kappa$, $I_\gamma = I$ for all $\gamma \in S$, and $\delta \trianglelefteq d_\gamma$ for all $\gamma \in S$.

Let ζ denote the least ordinal $<\nu$ such that there are κ-many $\gamma \in S$ with $\mathrm{Lev}(\dot{J}_\gamma) < \zeta$. (Some such ζ must exist because ν is a regular cardinal with $|S| = \kappa < \nu$.) By replacing S with $\left\{\gamma \in S\colon \mathrm{Lev}(\dot{J}_\gamma) < \zeta\right\}$ if necessary, we may (and do) assume that $\mathrm{Lev}(\dot{J}_\gamma) < \zeta$ for all $\gamma \in S$.

Recall that the sequence $\left\langle \dot{J}_\gamma\colon \gamma < \mu \right\rangle$ was defined in the extension, not in the ground model. In the ground model, we have a sequence $\left\langle \ddot{J}_\gamma\colon \gamma < \mu \right\rangle$ of names for nice names, representing the sequence $\left\langle \dot{J}_\gamma \colon \gamma < \mu \right\rangle$ constructed in the extension, meaning that $\emptyset \Vdash$ "$(\ddot{J}_\gamma)_G = \dot{J}_\gamma$ for each $\gamma < \mu$."

We now work in the ground model V. Let $\dot{S}$ be a nice $\mathrm{Fn}(\lambda, 2)$-name for S, and fix some $p \in \mathrm{Fn}(\lambda, 2)$ such that

$$\begin{aligned} p \Vdash \quad & |S| = \kappa, \\ & I_\gamma = I \text{ for all } \gamma \in S, \\ & \delta \trianglelefteq d_\gamma \text{ for all } \gamma \in S, \\ & \mathrm{Lev}((\ddot{J}_\gamma)_G) < \zeta \text{ for all } \gamma \in S, \text{ and} \\ & \text{if } \zeta' < \zeta \text{ then } \left| \left\{ \gamma \in S \colon \mathrm{Lev}(\dot{J}_\gamma) < \zeta' \right\} \right| < \kappa. \end{aligned}$$

Let q be an arbitrary extension of p in $\mathrm{Fn}(\lambda, 2)$.

Claim. *There is a nice name $\dot{S}' = \{(\gamma_\alpha, q_\alpha) \colon \alpha < \kappa\} \subseteq \dot{S}$, a condition $r \supseteq q$, and a sequence $\left\langle \dot{K}_{\gamma_\alpha} \colon \alpha < \kappa \right\rangle$ (in the ground model V) of nice names for subsets of I, such that* $\mathrm{dom}(q_\alpha) \cap \mathrm{dom}(q_\beta) = \emptyset$ *for all $\alpha \neq \beta$ in κ, and*

$$r \Vdash \quad |S'| = \kappa \quad \textit{and} \quad \dot{J}_\gamma = (\ddot{J}_\gamma)_G = \dot{K}_\gamma \textit{ for all } \gamma \in S'.$$

Furthermore, if $\dot{T}$ is any size-κ subset of $\dot{S}'$ and $t \supseteq r$, then the above statement remains true when $\dot{S}'$ is replace by $\dot{T}$ and r is replaced by t.

Proof of claim. Because $\dot{S}$ is a nice name for a subset of μ and $\mathrm{Fn}(\lambda, 2)$ has the ccc, we may write $\dot{S} = \{(\gamma_\alpha, p_\alpha) \colon \alpha < \kappa\}$, where $\gamma_\alpha < \mu$ and $p_\alpha \in \mathrm{Fn}(\lambda, 2)$ for all α, and where any particular ordinal appears only countably many times among the γ_α, i.e., $|\{\alpha < \kappa \colon \gamma_\alpha = \gamma\}| \leq \aleph_0$ for every $\gamma < \mu$.

Letting $\dot{S}_1 = \dot{S} \setminus \{(\gamma_\alpha, p_\alpha) \colon p_\alpha \perp q\}$, it is clear that $q \Vdash S_1 = S$. Note that $|\dot{S}_1| = \kappa$, because $q \Vdash$ "$S_1 = S$ and $|S| = \kappa$."

For every $(\gamma_\alpha, p_\alpha) \in \dot{S}_1$, p_α is compatible with q and $q \cup p_\alpha \Vdash$ "$(\ddot{J}_{\gamma_\alpha})_G$ is a nice name (in V) for a subset of I and $\mathrm{Lev}((\ddot{J}_{\gamma_\alpha})_G) < \zeta$." For each such α, we may therefore choose some $q^0_\alpha \supseteq q \cup p_\alpha$ that decides $\ddot{J}_{\gamma_\alpha}$; that is, we choose some $q^0_\alpha \supseteq q \cup p_\alpha$ and some nice name $\dot{K}_{\gamma_\alpha} \in V$ with $\mathrm{Lev}(\dot{K}_{\gamma_\alpha}) < \zeta$ such that $q^0_\alpha \Vdash$ "$\dot{J}_{\gamma_\alpha} = (\ddot{J}_{\gamma_\alpha})_G = \dot{K}_{\gamma_\alpha}$."

By the Δ-system lemma, there is some $D \subseteq \left\{\alpha \colon (\gamma_a, p_\alpha) \in \dot{S}_1\right\}$ with $|D| = \kappa$ such that $\left\{\mathrm{dom}(q^0_\alpha) \colon \alpha \in D\right\}$ is a Δ-system with root R. (We allow for the possibility that this is a "degenerate" Δ-system with $\mathrm{dom}(q^0_\alpha) = R$ for all $\alpha < \kappa$.) By the pigeonhole principle, there is some $r : R \to 2$ and some $E \subseteq D$ with $|E| = \kappa$ such that $q^0_\alpha \restriction R = r$ for all $\alpha \in E$. Note that $r \supseteq q \supseteq p$, because $q^0_\alpha \supseteq q$ for each α. Let $\dot{S}_2 = \left\{(\gamma_\alpha, q^0_\alpha) \colon \alpha \in E\right\}$. By relabelling and re-indexing the members of $\dot{S}_2$, we may write $\dot{S}_2 = \left\{(\gamma_\alpha, q^0_\alpha) \colon \alpha < \kappa\right\}$. Finally, let $q_\alpha = q^0_\alpha \setminus r$ for all α and let $\dot{S}' = \{(\gamma_\alpha, q_\alpha) \colon \alpha < \kappa\}$. It is clear that $r \Vdash$ "$\dot{S}' = \dot{S}_2$", and this implies $r \Vdash$ "$\dot{J}_\gamma = (\ddot{J}_\gamma)_G = \dot{K}_\gamma$ for all $\gamma \in S'$." Clearly $\mathrm{dom}(q_\alpha) \cap \mathrm{dom}(q_\beta) = \emptyset$ for all $\alpha \neq \beta$ in κ.

Finally, suppose $\dot{T} \subseteq \dot{S}'$ with $|\dot{T}| = \kappa$, and fix $t \in \mathrm{Fn}(\lambda, 2)$ with $t \supseteq r$. That $t \Vdash |T| = \kappa$ follows from the fact that the domains of the q_α's are pairwise disjoint (so that any generic filter must include κ of the q_α's), together with the fact that any particular ordinal appears only countably many times among the γ_α. We have $t \Vdash$ "$\dot{J}_\gamma = (\ddot{J}_\gamma)_G = \dot{K}_\gamma$ for all $\gamma \in T$" because $r \Vdash$ "$\dot{J}_\gamma = (\ddot{J}_\gamma)_G = \dot{K}_\gamma$ for all $\gamma \in S'$" and $t \Vdash$ "$T \subseteq S'$." $\square$

Fix some nice name $\dot{S}'$ as in the claim above.

By part (3) of our definition of a sage Davies tree, there is a collection $\mathcal{N}$ of $<\kappa$-closed closed elementary submodels of H_θ with $|\mathcal{N}| < \kappa$ such that $\bigcup \mathcal{N} = \bigcup_{\xi<\zeta} M_\xi$. By the pigeonhole principle, some $N \in \mathcal{N}$ has the property that $\dot{K}_{\gamma_\alpha} \in N$ for κ-many $\alpha < \kappa$. Fix some such N.

Let $\dot{S}'_N = \left\{(\gamma_\alpha, q_\alpha) \in \dot{S}' \colon \dot{K}_{\gamma_\alpha} \in N\right\}$. Applying Lemma 2.8, there is some $\dot{T} \subseteq \dot{S}'_N$ with $\left|\dot{S}'_N \setminus \dot{T}\right| < \kappa$ such that

$$\begin{aligned} r \Vdash \quad & \bigwedge \{\bigwedge K_\gamma \colon \gamma \in T\} = \bigwedge \{\bigwedge K_\gamma \colon \gamma \in T \setminus Z\} \\ & \text{for any } Z \subseteq \mu \text{ with } |Z| < \kappa. \end{aligned}$$

By re-labelling and re-indexing the q_α and γ_α one final time, let us write $\dot{T} = \{(q_\alpha, \gamma_\alpha) \colon \alpha < \kappa\}$.

Claim. *For any $\alpha < \kappa$ and any s compatible with q_α, if $s \Vdash$ "$i \in K_{\gamma_\alpha}$" then $q_\alpha \cup s \Vdash$ "for any $j \in I$ with $j \ntrianglelefteq i$, there is some $i' \in I$ such that $j \ntrianglelefteq i'$ and $i' \in K_\gamma$ for κ-many $\gamma \in T$."*

Proof of claim. For the proof of this claim, it is more convenient to work in a generic extension. Suppose s is compatible with q_α and $s \Vdash$ "$i \in K_{\gamma_\alpha}$", and let $V[H]$ be an arbitrary $\mathrm{Fn}(\lambda, 2)$-generic extension of V with $q_\alpha \cup s \in H$.

Fix $j \in I$ with $j \ntrianglelefteq i$. Because $q_\alpha \in H$, we have $\gamma_\alpha \in T$. Therefore $\bigwedge \{\bigwedge K_\gamma \colon \gamma \in T\} \trianglelefteq \bigwedge K_{\gamma_\alpha} \trianglelefteq i$. As $j \ntrianglelefteq i$, we have $j \ntrianglelefteq \bigwedge \{\bigwedge K_\gamma \colon \gamma \in T\}$. By our choice of T, we also have $j \ntrianglelefteq \bigwedge \{\bigwedge K_\gamma \colon \gamma \in T \setminus Z\}$ for any $<\kappa$-sized $Z \subseteq T$. This implies there are κ-many $\gamma \in T$ such that $j \ntrianglelefteq \bigwedge K_\gamma$. For each such γ, there is some $i' \in I$ such that $i' \in K_\gamma$ and $j \ntrianglelefteq i'$. By the pigeonhole principle, using the fact that $|I| < \kappa$, there is some particular $i' \in I$ with $j \ntrianglelefteq i'$ such that $i' \in K_\gamma$ for κ-many $\gamma \in T$.

Thus any generic extension $V[H]$ with $q_\alpha \cup s \in H$ satisfies "for any $j \in I$ with $j \ntrianglelefteq i$, there is some $i' \in I$ such that $j \ntrianglelefteq i'$ and $i' \in K_\gamma$ for κ-many $\gamma \in T$." The claim follows. $\square$

Given $i \in I$ and $\alpha < \kappa$, we write "$i \in \mathrm{supp}(\dot{K}_{\gamma_\alpha})$" to mean $(i, s) \in \dot{K}_{\gamma_\alpha}$ for some $s \in \mathrm{Fn}(\lambda, 2)$. Let

$$I_\kappa = \left\{i \in I \colon i \in \mathrm{supp}(\dot{K}_{\gamma_\alpha}) \text{ for } \kappa\text{-many values of } \alpha\right\}.$$

Note that $I_\kappa \subseteq N$ (because each $\dot{K}_{\gamma_\alpha}$ is in N). Let

$$\dot{K} = \{(i, s) \in N \colon i \in I_\kappa \text{ and } s \Vdash \text{“}i \in K_\gamma \text{ for infinitely many } \gamma \in T\text{”}\}.$$

Notice that $\dot{K} \subseteq N$, although we cannot claim $\dot{K} \in N$. The following claim gives us the next best thing to having $\dot{K} \in N$.

Claim. *There is a nice name $\dot{J}$ for a subset of I, with $\dot{J} \in N$, such that $\emptyset \Vdash$ "$J = K$."*

Proof of Claim. For each $i \in \mathrm{supp}(\dot{K})$, fix an antichain $\mathcal{A}_i$ in $\mathrm{Fn}(\lambda, 2) \cap N$ such that $s \Vdash$ "$i \in K$" for every $s \in \mathcal{A}_i$, and $\mathcal{A}_i$ is maximal with respect to this property (i.e., if $t \in \mathrm{Fn}(\lambda, 2) \cap N$ and $t \Vdash$ "$i \in K$", then t is compatible with some member of $\mathcal{A}_i$). Let $\dot{J} = \left\{(i, s) \colon i \in \mathrm{supp}(\dot{K}) \text{ and } s \in \mathcal{A}_i\right\}$.

Clearly $\dot{J}$ is a nice name for a subset of I. Note that $i \in \mathrm{supp}(\dot{K})$ implies $i \in N$. So if $(i, s) \in \dot{J}$, then $i, s \in N$, which implies $(i, s) \in N$. Thus $\dot{J} \subseteq N$. Also $|I| < \kappa$ and $|\mathcal{A}_i| = \aleph_0 < \kappa$ for each i, which implies $|\dot{J}| < \kappa$. Because N is $<\kappa$-closed, $\dot{J} \in N$.

It is clear from our construction that $\emptyset \Vdash$ "$J \subseteq K$." For the other direction, suppose $t \in \mathrm{Fn}(\lambda, 2)$ and $t \Vdash$ "$i \in K$." Let t' be any extension of t. Because $\dot{K} \subseteq N$, it is clear that $t' \Vdash$ "$i \in K$" implies $t' \cap N \Vdash$ "$i \in K$." By our choice of $\mathcal{A}_i$, this means $t' \cap N$ is compatible with some $s \in \mathcal{A}_i$; but $s \in N$, so t' is also compatible with s. Hence $t' \not\Vdash$ "$i \notin J$." Because this is true for every $t' \supseteq t$, this shows $t \Vdash$ "$i \in J$." Hence any condition forcing $i \in K$ also forces $i \in J$. It follows that $\emptyset \Vdash$ "$K \subseteq J$" as claimed. $\square$

If $t \supseteq r$ and, for some $i \in I \cap N$, $t \Vdash$ "$i \in K_\gamma$ for infinitely many $\gamma \in T$", then $t \cap N \Vdash$ "$i \in K_\gamma$ for infinitely many $\gamma \in T$." To see this, note first that $t \Vdash$ "$i \in K_\gamma$ for infinitely many $\gamma \in T$" just means that for any $t' \supseteq t$, there are infinitely many values of α such that there is some t_α compatible with t' and $(t_\alpha, i) \in \dot{K}_{\gamma_\alpha}$. But because $\dot{K}_{\gamma_\alpha} \subseteq N$ for every α (which means that the t_α's in the previous sentence are always in N), this fact evidently does not change when we replace t with $t \cap N$.

Claim. *For each $\alpha < \kappa$, $q_\alpha \cup r \Vdash$ "$\bigwedge K \trianglelefteq \bigwedge K_{\gamma_\alpha}$."*

Proof of Claim. Fix $\alpha < \kappa$. Let $i, j \in I$, and let s be any extension of $q_\alpha \cup r$ such that $s \Vdash$ "$i \in K_{\gamma_\alpha}$ and $j \ntrianglelefteq i$." By a previous claim, $q_\alpha \cup s = s \Vdash$ "for any $j' \in I$ with $j' \ntrianglelefteq i$, there is some $i' \in I$ such that $j' \ntrianglelefteq i'$ and $i' \in K_\gamma$ for κ-many $\gamma \in T$." In particular, $s \Vdash$ "there is some $i' \in I$ such that $j \ntrianglelefteq i'$ and $i' \in K_\gamma$ for κ-many $\gamma \in T$."

Let s' be any extension of s. There is some $t \supseteq s'$ that decides the value of i' in the previous paragraph: i.e., there is some particular $i' \in I$ such that $t \Vdash$ "$i' \in K_\gamma$ for κ-many $\gamma \in T$." Thus $i' \in I_\kappa$, and $t \Vdash$ "$i' \in K_\gamma$ for infinitely many $\gamma \in T$." By the paragraph preceding this claim, $t \cap N \Vdash$ "$i' \in K_\gamma$ for infinitely many $\gamma \in T$." Hence $(t \cap N, i') \in \dot{K}$. In particular, $t \Vdash$ "$i' \in K$." But also $t \Vdash$ "$j \ntrianglelefteq i'$", and so $t \Vdash$ "$j \ntrianglelefteq \bigwedge K$." Thus for any $s' \supseteq s$, some extension of s' forces "$j \ntrianglelefteq \bigwedge K$." It follows that $s \Vdash$ "$j \ntrianglelefteq \bigwedge K$."

But s was an arbitrary extension of $q_\alpha \cup r$ having the property that, for some $i, j \in I$, $s \Vdash$ "$i \in K_{\gamma_\alpha}$ and $j \ntrianglelefteq i$." Therefore $q_\alpha \cup r \Vdash$ "if $i, j \in I$ and $i \in K_{\gamma_\alpha}$ and $j \ntrianglelefteq i$, then $j \ntrianglelefteq \bigwedge K$." This implies $q_\alpha \cup r \Vdash$ "$\bigwedge K \trianglelefteq \bigwedge K_{\gamma_\alpha}$." $\square$

In a generic extension $V[H]$ with $r \in H$, we have $\gamma \in T$ if and only if $q_\alpha \in H$ for some $\alpha < \kappa$ with $\gamma_\alpha = \gamma$, in which case $\dot{J}_\gamma = \dot{K}_{\gamma_\alpha}$ and (by the previous claim) $\bigwedge K \trianglelefteq \bigwedge K_{\gamma_\alpha}$. Therefore

$$(*) \qquad r \Vdash \quad \bigwedge K \trianglelefteq \bigwedge J_\gamma \text{ for all } \gamma \in T.$$

Claim. *$r \Vdash$ "$\delta \trianglelefteq \bigwedge K$."*

Proof of Claim. We will prove separately that $r \Vdash$ "$\delta \trianglelefteq \bigwedge \{\bigwedge K_\gamma \colon \gamma \in T\}$" and that $r \Vdash$ "$\bigwedge \{\bigwedge K_\gamma \colon \gamma \in T\} \trianglelefteq \bigwedge K$."

For the first of these assertions, note that $p \Vdash$ "$\delta \trianglelefteq \bigwedge (\dot{J}_\gamma)_G$ for all $\gamma \in S$", that $r \supseteq p$, and that $r \Vdash$ "$\dot{J}_\gamma = \dot{K}_\gamma$ for all $\gamma \in T$ and $T \subseteq S$." It follows that $r \Vdash$ "$\delta \trianglelefteq \bigwedge K_\gamma$ for all $\gamma \in T$", and therefore $r \Vdash$ "$\delta \trianglelefteq \bigwedge \{\bigwedge K_\gamma \colon \gamma \in T\}$."

For the second assertion, first note that, by the definition of $\dot{K}$, if $i \in I$ then $r \Vdash$ "if $i \in K$ then $i \in K_\gamma$ for infinitely many $\gamma \in T$." Hence for every $i \in I$, $r \Vdash$ "if $i \in K$ then $\bigwedge \{K_\gamma \colon \gamma \in T\} \trianglelefteq i$"; so $r \Vdash$ "for all $i \in I$, if $i \in K$ then $\bigwedge \{K_\gamma \colon \gamma \in T\} \trianglelefteq i$." Hence $r \Vdash$ "$\bigwedge \{\bigwedge K_\gamma \colon \gamma \in T\} \trianglelefteq \bigwedge K$." □

From the last few claims, we see that there is a nice name $\dot{J} \in N$ for a subset of I such that

$$r \Vdash \quad J = K \text{ and } 0 \neq \delta \trianglelefteq \bigwedge K \trianglelefteq \bigwedge J_\gamma \trianglelefteq \gamma \text{ for all } \gamma \in T.$$

So $r \Vdash$ "if $\gamma \in T$, then $\dot{J}$ satisfies all the criteria in the definition of $\dot{J}_\gamma$." Consequently, $r \Vdash$ "$\dot{J}_\gamma \sqsubseteq \dot{J}$ for all $\gamma \in T$." However, $\left|\left\{x \in \mathrm{Lev}(\dot{J}) \colon x \sqsubseteq \dot{J}\right\}\right| < \kappa$, and $\dot{J}_\gamma \sqsubseteq \dot{J}$ implies $\mathrm{Lev}(\dot{J}_\gamma) \leq \mathrm{Lev}(\dot{J})$. Therefore

$$r \Vdash \quad \mathrm{Lev}(\dot{J}_\gamma) < \mathrm{Lev}(\dot{J}) \text{ for all but } {<}\kappa\text{-many } \gamma \in T.$$

Also $r \Vdash$ "$T \subseteq S$ and $|T| = \kappa$" and therefore

$$r \Vdash \quad \mathrm{Lev}(\dot{J}_\gamma) < \mathrm{Lev}(\dot{J}) \text{ for } \kappa\text{-many } \gamma \in S.$$

But $\dot{J} \in N \subseteq \bigcup_{\xi<\zeta} M_\xi$, which implies that $\mathrm{Lev}(\dot{J}) < \zeta$. This contradicts our choice of ζ and p, because p forces the minimality of ζ, and $r \supseteq p$. □

Corollary 2.10. *$\triangledown + \neg$CH is consistent relative to* ZFC.

The proof of Theorem 2.9 uses a hypothesis stronger than $\triangledown$ in V in order to show that $\triangledown$ holds in $V[G]$. This leaves open the question of whether such a strong hypothesis in the ground model is really necessary.

Question 2.11. *Is $\triangledown$ preserved by Cohen forcing?*

3. GCH does not imply $\triangledown$

In this section we show that GCH does not imply $\triangledown$. As mentioned in the introduction, large cardinals are necessary for constructing a model of GCH $+ \neg\triangledown$. Another feature of our proof is that the poset $\mathbb{P}$ for which we show $\triangledown(\mathbb{P})$ fails has size $\aleph_{\omega+1}$. This feature is also necessary, in the sense that no smaller poset can work in the presence of GCH. While in certain models there are smaller posets where $\triangledown$ fails ($\triangledown$ can fail for a size-$\aleph_2$ poset

[1, Theorem 4.1], although $\triangledown$ always holds for posets of size $\leq\aleph_1$ [1, remark 2.9]), GCH implies that $\triangledown$ holds for all posets of size $\leq\aleph_\omega$.

Consider the following statement:

> For every model M for a countable language $\mathcal{L}$ that contains a unary predicate A, if $|M| = \kappa^+$ and $|A| = \kappa$ then there is an elementary submodel $M' \prec M$ such that $|M'| = \mu^+$ and $|M' \cap A| = \mu$.

This statement, abbreviated by writing $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$, is an instance of *Chang's conjecture*. In this section we will consider the case $\kappa = \aleph_\omega$, $\mu = \aleph_0$. This particular instance of Chang's conjecture is known as *Chang's conjecture for* $\aleph_\omega$ and is abbreviated $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

The usual Chang conjecture, which is the assertion $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$, is equiconsistent with the existence of an ω_1-Erdős cardinal. Chang's conjecture for $\aleph_\omega$ requires even larger cardinals. $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ was first proved consistent relative to a hypothesis a little weaker than the existence of a 2-huge cardinal in [8]. Recently this was improved to a huge cardinal in [5]. The precise consistency strength of $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is an open problem, but significant large cardinal strength is known to be needed. This is because $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies the failure of $\square_{\aleph_\omega}$ (see [9], in particular Fact 4.2 and the remarks after it), and the failure of $\square_{\aleph_\omega}$ carries significant consistency strength (see [2]).

Theorem 3.1. *If* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ *holds, then* $\triangledown$ *fails.*

Proof. We will describe a separative ccc poset $\mathbb{P}$, and then use the Chang conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ to prove that this poset violates $\triangledown$. The members of $\mathbb{P}$ have the form (p, f, A), where

- $p \in \mathbb{H}^{\aleph_\omega}$, where $\mathbb{H}^{\aleph_\omega}$ denotes the finite-support product of $\aleph_\omega$ Hechler forcings. The product is indexed by the ordinal ω_ω.
- f is a function $\omega \to \omega$, but not the constant function $n \mapsto 0$.
- A is a countably infinite subset of ω_ω and $A \supseteq \mathrm{supp}(p)$.

Given $(q, g, B), (p, f, A) \in \mathbb{P}$, we say that (q, g, B) extends (p, f, A) whenever

- q extends p in $\mathbb{H}^{\aleph_\omega}$,
- $g(n) \geq f(n)$ for all $n \in \omega$,
- $B \supseteq A$,
- if $\alpha \in A \cap (\mathrm{supp}(q) \setminus \mathrm{supp}(p))$, then $q(\alpha)$ extends $\langle \emptyset, f\rangle$ in $\mathbb{H}$.

Alternatively, one may think of $\mathbb{P}$ as a sub-poset of the countable support product of $\aleph_\omega$ Hechler forcings, consisting of those conditions r with infinite support such that for all but finitely many coordinates of $\mathrm{supp}(r)$, the $r(\alpha)$'s are all required to have an empty working part and the same side condition. Under this interpretation, a condition $(p, f, A) \in \mathbb{P}$ corresponds to the condition r in $\mathbb{H}^{\aleph_\omega}_{\mathrm{ctbl}}$ having countable support A, and with $r(\alpha) = \langle \emptyset, f\rangle$ for all $\alpha \in A \setminus \mathrm{supp}(p)$.

We begin by verifying that $\mathbb{P}$ is a separative ccc poset.

Claim. $\mathbb{P}$ *is separative.*

Proof of claim. Let $(q,g,B),(p,f,A) \in \mathbb{P}$ and suppose that (q,g,B) is not an extension of (p,f,A). As there are four parts to the definition of "extension" in $\mathbb{P}$, this can mean one of four things.

If q does not extend p in $\mathbb{H}^{\aleph_\omega}$, then because $\mathbb{H}^{\aleph_\omega}$ is separative, there is some $r \in \mathbb{H}^{\aleph_\omega}$ that extends q but is incompatible with p. By extending r further if necessary, we may assume $r(\alpha)$ extends $\langle \emptyset, g\rangle$ for all $\alpha \in \operatorname{supp}(r)$, and thereby ensure that (r,g,B) is an extension of (q,g,B). Clearly (r,g,B) is incompatible with (p,f,A), because r is incompatible with p.

If $B \not\supseteq A$, then let $\alpha \in A \setminus B$. Let h be any condition in $\mathbb{H}$ incompatible with $\langle \emptyset, f\rangle$. (Note that some such h exists because f is not the constant function $n \mapsto 0$.) Let $q' = q \cup \{(\alpha,h)\}$ and $B' = B \cup \{\alpha\}$. Then (q',g,B') is a condition extending (q,g,B); but our choice of α and h guarantees that (q',g,B') is incompatible with (p,f,A).

If $B \supseteq A$ but $g(n) < f(n)$ for some $n \in \omega$, then let $\alpha \in A \setminus (\operatorname{supp}(p) \cup \operatorname{supp}(q))$ and let h be any condition in $\mathbb{H}$ extending $\langle \emptyset, g\rangle$ but incompatible with $\langle \emptyset, f\rangle$ (e.g., $h = \langle g \restriction (n+1), g\rangle$). Let $q' = q \cup \{(\alpha,h)\}$. Then (q',g,B) is a condition extending (q,g,B), but it is incompatible with (p,f,A).

Finally, suppose there is some $\alpha \in A \cap (\operatorname{supp}(q) \setminus \operatorname{supp}(p))$ such that $q(\alpha)$ does not extend $\langle \emptyset, f\rangle$ in $\mathbb{H}$. Then, because $\mathbb{H}$ is separative, there is some $r \in \mathbb{H}$ that extends $q(\alpha)$ but is incompatible with $\langle \emptyset, f\rangle$. Define $q' \in \mathbb{H}^{\aleph_\omega}$ to be identical to q, except that $q'(\alpha) = r$. Then (q',g,B) extends (q,g,B) and is incompatible with (p,f,A). □

Claim. $\mathbb{P}$ *has the ccc.*

Proof of claim. Suppose $\mathcal{A}$ is an uncountable collection of conditions in $\mathbb{P}$. Let $\mathcal{B} = \{p \colon (p,f,A) \in \mathcal{A} \text{ for some } f \text{ and } A\}$ denote the corresponding collection of conditions in $\mathbb{H}^{\aleph_\omega}$. Because $\mathbb{H}^{\aleph_\omega}$ has the ccc, some two conditions in $\mathcal{B}$ are compatible in $\mathbb{H}^{\aleph_\omega}$. But then the two corresponding conditions in $\mathcal{A}$ are also compatible: for if $(q,g,B),(p,f,A) \in \mathbb{P}$ and r is a common extension of p and q in $\mathbb{H}^{\aleph_\omega}$, then we may further extend r, if necessary, so that for each $\alpha \in \operatorname{supp}(r)$, $r(\alpha)$ is an extension of both $\langle \emptyset, f\rangle$ and $\langle \emptyset, g\rangle$. Then $(r, \max\{f,g\}, A \cup B)$ is a common extension of (q,g,B) and (p,f,A) in $\mathbb{P}$. □

It remains to show that $(\aleph_{\omega+1},\aleph_\omega) \twoheadrightarrow (\aleph_1,\aleph_0)$ implies that for any dense $\mathbb{D} \subseteq \mathbb{P}$, there is some condition in $\mathbb{P}$ that extends uncountably many members of $\mathbb{D}$. Let $\mathbb{D}$ be a dense sub-poset of $\mathbb{P}$.

To begin, note that for each countable $A \subseteq \omega_\omega$, some member of $\mathbb{D}$ extends a condition of the form (p,f,A). This implies that

$$\left\{B \subseteq \omega_\omega \colon (q,g,B) \in \mathbb{D} \text{ for some } q \in \mathbb{H}^{\aleph_\omega} \text{ and } g \in \omega^\omega\right\}$$

is cofinal in the poset $([\omega_\omega]^\omega, \subseteq)$. The cofinality of this poset is well-known to be $>\aleph_\omega$. Hence $|\mathbb{D}| \geq \aleph_{\omega+1}$.

Let $H = \omega_\omega \cup (\mathbb{H}^{\aleph_\omega} \times \omega^\omega)$, and note that $|H| = \aleph_\omega$.

Let $(M, \in)$ be a model of (a sufficiently large fragment of) ZFC such that $H \subseteq M$, $\mathbb{D} \in M$, and $|M| = |M \cap \mathbb{D}| = \aleph_{\omega+1}$. (Such a model can be obtained in the usual way, via the downward Löwenheim-Skolem Theorem.) Let $\phi : M \to M \cap \mathbb{D}$ be a bijection, and consider the model $(M, \in, \phi, H)$ for the 3-symbol language consisting of a binary relation, a unary function, and a unary predicate. Applying the Chang conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, there exists some $M' \subseteq M$ such that $|M'| = \aleph_1$, $H' = M' \cap H$ is countable, and $(M', \in, \phi, H') \prec (M, \in, \phi, H)$.

Let $\mathbb{D}' = \mathbb{D} \cap M'$. By elementarity, the restriction of ϕ to M' is a bijection $M' \to \mathbb{D}'$, and so $|\mathbb{D}'| = \aleph_1$.

Let $B = \omega_\omega \cap M'$. As $B \subseteq H'$, we have $|B| = \aleph_0$. Note that $(p, f, A) \in \mathbb{D}'$ implies $A \in M'$, and therefore (because A is countable, and M' models (enough of) ZFC) $A \subseteq M'$. Therefore $(p, f, A) \in \mathbb{D}'$ implies $A \subseteq B$.

Furthermore, $(p, f, A) \in \mathbb{D}'$ implies $(p, f) \in M'$, which implies $(p, f) \in H'$. Therefore

$$\big\{(p, f) \colon (p, f, A) \in \mathbb{D}' \text{ for some } A \subseteq B\big\}$$

is countable. But $\mathbb{D}'$ is uncountable, so by the pigeonhole principle, there is some pair $(p, f) \in \mathbb{H}^{\aleph_\omega} \times \omega^\omega$ such that $\{A \subseteq B \colon (p, f, A) \in \mathbb{D}'\}$ is uncountable.

Finally, note that (p, f, B) is a condition in $\mathbb{P}$, and that (p, f, B) extends (p, f, A) whenever $A \subseteq B$. Therefore (p, f, B) extends uncountably many conditions in $\mathbb{D}$. $\square$

Corollary 3.2. $\mathsf{GCH} + \neg\triangledown$ *is consistent relative to a huge cardinal.*

4. The measure algebra of weight $\aleph_\omega$

In [1, Section 4], it is observed that MA implies $\triangledown$ fails for the weight-$\aleph_0$ measure algebra. In fact, this was the first known example of a poset for which $\triangledown$ consistently fails. The results contained in this section and the previous one grew from trying to discover whether $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies $\triangledown$ fails for the weight-$\aleph_\omega$ measure algebra. (As mentioned in the previous section, $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ does not imply the failure of $\triangledown$ for any poset of size $\leq \aleph_\omega$, so this makes the weight-$\aleph_\omega$ measure algebra a natural place to look.) We still do not know whether $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies the failure of $\triangledown$ for the weight-$\aleph_\omega$ measure algebra. But we show below that $\mathsf{GCH} + (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is consistent with the failure of $\triangledown$ for the weight-$\aleph_\omega$ measure algebra.

Given some set A, 2^A denotes the set of all functions $A \to 2$. The product measure μ on 2^A is defined by setting

$$\mu\big(\big\{f \in 2^A \colon f(\alpha) = 0\big\}\big) = \mu\big(\big\{f \in 2^A \colon f(\alpha) = 1\big\}\big) = \tfrac{1}{2}$$

for all $\alpha \in A$. More precisely, this coordinate-wise assignment extends naturally to a pre-measure on the clopen subsets of 2^A, and this extends, via Carathéodory's Theorem, to a countably additive measure on the smallest

σ-algebra containing all the clopen subsets of 2^A. We denote this σ-algebra by $\mathcal{B}_A$.

Now suppose $A = \kappa$ is an infinite cardinal number, and let M_κ denote the quotient of $\mathcal{B}_\kappa$ by the ideal of sets having μ-measure 0. Then M_κ is a σ-complete Boolean algebra, called *the measure algebra of weight* κ.

Given $X \subseteq 2^\kappa$ and $A \subseteq \kappa$, we say that X is *supported* on A if there is some $Y \subseteq 2^A$ such that $X = Y \times 2^{\kappa \setminus A}$. It is easy to check that if $X \neq \emptyset$ and X is supported on every A in some collection $\mathcal{A} \subseteq \mathcal{P}(\kappa)$, then X is supported on $\bigcap \mathcal{A}$. Therefore there is a smallest $A \subseteq \kappa$ on which X is supported, and we denote this set by $\mathrm{supp}(X)$.

Lemma 4.1. *Every member of $\mathcal{B}_\kappa$ is supported on a countable subset of κ. In fact, $X \in \mathcal{B}_\kappa$ if and only if $X = Y \times 2^{\kappa \setminus A}$ for some countable $A \subseteq \kappa$ and some Borel $Y \subseteq 2^A$.*

Proof. Let $\mathcal{B}$ denote the set of all X such that $X = Y \times 2^{\kappa \setminus A}$ for some countable $A \subseteq \kappa$ and some Borel $Y \subseteq 2^A$. It is clear that $\mathcal{B}$ is a σ-algebra containing all the basic clopen subsets of 2^κ; hence $\mathcal{B}_\kappa \subseteq \mathcal{B}$. Conversely, if $A \subseteq \kappa$ is countable, then $\mathcal{B}_\kappa$ contains $C \times 2^{\kappa \setminus A}$ for every clopen $C \subseteq 2^A$, because $C \times 2^{\kappa \setminus A}$ is clopen in 2^κ. It follows that $\mathcal{B}_\kappa$ must contain $Y \times 2^{\kappa \setminus A}$ for every Borel $Y \subseteq 2^A$. Hence $\mathcal{B} \subseteq \mathcal{B}_\kappa$. □

Let $\mathbb{A}$ denote the amoeba forcing. Conditions in $\mathbb{A}$ are open subsets of 2^ω with measure $< \frac{1}{2}$, and the extension relation on $\mathbb{A}$ is $\subseteq$. Let $\mathbb{A}^\omega$ denote the finite support product of ω copies of $\mathbb{A}$.

Lemma 4.2. *Let V be a model of* ZFC *and let G be an $\mathbb{A}^\omega$-generic filter over V. In $V[G]$, there is a countable collection $\mathcal{C}$ of non-null closed subsets of 2^ω such that if B is any non-null Borel subset of 2^ω whose Borel code is in V, then there is some $C \in \mathcal{C}$ such that $C \subseteq B$.*

Proof. Each $p \in G$ is a sequence of open subsets of 2^ω in V, all but finitely many of which are $\emptyset$. For each $p \in G$ and $n \in \omega$, let $\widetilde{p}(n)$ denote the reinterpretation of $p(n)$ in $V[G]$; i.e., $\widetilde{p}(n)$ is the $V[G]$-interpretation of the Borel code of $p(n)$ in V. For each $n \in \omega$, let $U_n = \bigcup_{p \in G} \widetilde{p}(n)$, and define $\mathcal{C} = \{2^\omega \setminus (U_n \cup U_m) \colon m, n \in \omega\}$.

It is straightforward to show that each U_n is an open set with measure $\frac{1}{2}$. Fix $m, n \in \omega$. The set of all $p \in \mathbb{A}^\omega$ with $p(m) \cap p(n) \neq \emptyset$ is dense. Therefore $U_m \cap U_n \neq \emptyset$, and because both these sets are open, $\mu(U_m \cap U_n) > 0$. Hence

$$\begin{aligned}\mu(2^\omega \setminus (U_m \cup U_n)) &= 1 - \mu(U_m \cup U_n) \\ &= 1 - (\mu(U_m) + \mu(U_n) - \mu(U_m \cap U_n)) \\ &= \mu(U_m \cap U_n) > 0.\end{aligned}$$

Thus $\mathcal{C}$ is a countable collection of non-null closed subsets of 2^ω.

Let B be a non-null Borel set in V. Then $\mu(2^\omega \setminus B) < 1$, and this implies there is an open $W \subseteq 2^\omega$ such that $\mu(W) < 1$ and $2^\omega \setminus B \subseteq W$. Any open set of measure < 1 can be split into two open sets of measure $< \frac{1}{2}$, so in particular there are open $V_1, V_2 \subseteq 2^\omega$ such that $\mu(V_1) < \frac{1}{2}$, $\mu(V_2) < \frac{1}{2}$, and $V_1 \cup V_2 = W$. Now the set of all $p \in \mathbb{A}^\omega$ with $p(m) = V_1$ and $p_n = V_2$ for some $m, n \in \omega$ is dense. Therefore there exist some $m, n \in \omega$ and some $p \in G$ such that $p(m) = V_1$ and $p(n) = V_2$. Letting $\widetilde{B}$, $\widetilde{V}_1$, and $\widetilde{V}_2$ denote the $V[G]$-interpretations of the Borel codes for B, V_1, and V_2, respectively, we have $2^\omega \setminus \widetilde{B} \subseteq \widetilde{V}_1 \cup \widetilde{V}_2 \subseteq \widetilde{p}(m) \cup \widetilde{p}(n)$. Hence $\widetilde{B} \supseteq 2^\omega \setminus (U_n \cup U_m) \in \mathcal{C}$. □

Theorem 4.3. *It is consistent, relative to a huge cardinal, that* GCH *holds and that* $\triangledown$ *fails for* $M_{\aleph_\omega}$.

Proof. Let V be a model of GCH plus $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. Recall that the existence of such a model is consistent relative to a huge cardinal.

Let $\mathbb{A}$ denote the amoeba forcing, and let $\mathbb{P}$ denote the length-ω_1, finite support iteration of $\mathbb{A}^\omega$. Let G be a V-generic filter on $\mathbb{P}$. We claim that $V[G]$ is the desired model of GCH where $\triangledown(M_{\aleph_\omega})$ fails.

A standard argument shows $V[G] \models$ GCH. Therefore, to prove the theorem we must show that $\triangledown(M_{\aleph_\omega})$ fails in $V[G]$. Because $M_{\aleph_\omega}$ has the ccc, this amounts to showing that for any dense sub-poset $\mathbb{D}$ of $M_{\aleph_\omega} \setminus \{\mathbf{0}\}$, some member of $M_{\aleph_\omega} \setminus \{\mathbf{0}\}$ has uncountably many members of $\mathbb{D}$ above it.

We obserbve that $\mathbb{P}$ has the ccc, and it is know that $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is preserved by ccc forcing. (This fact is considered folklore, but a proof can be found in [5, Lemma 13].) Hence $V[G] \models (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

It follows from Lemma 4.1 that every $X \in \mathcal{B}_{\omega_\omega}$ can be represented in a canonical fashion by a pair (A, a), where $A = \mathrm{supp}(X)$ is countable, and a is some canonical code for the Borel subset Y of 2^A such that $X = Y \times 2^{\omega_\omega \setminus A}$. Let us call the pair (A, a) the *code* for X.

For each $\alpha < \omega_1$, let G_α denote (as usual) the restriction of G to the first α coordinates of $\mathbb{P}$. For each $\alpha < \omega_1$, let $B^\alpha_{\omega_\omega}$ denote the set of all those members of B_{ω_ω} whose code is in $V[G_\alpha]$. For every $X \in \mathcal{B}_{\omega_\omega}$, the code for X consists of a countable set of ordinals and a countable sequence of integers. This implies there is some $\alpha < \omega_1$ such that the code for X is a member of $V[G_\alpha]$. Hence $B_{\omega_\omega} = \bigcup_{\alpha<\omega_1} B^\alpha_{\omega_\omega}$.

Working in $V[G]$, let $\mathbb{D}$ be a dense sub-poset of $M_{\aleph_\omega}$. In what follows, it is easier to work with members of $\mathcal{B}_{\omega_\omega}$ rather than with their equivalence classes in $M_{\aleph_\omega}$. For each $Z \in \mathbb{D}$, fix some $X_Z \in \mathcal{B}_{\omega_\omega}$ representing Z. Let $\mathbb{E} = \{X_Z \colon Z \in \mathbb{D}\}$, and observe that $|\mathbb{E}| = |\mathbb{D}|$. Because every dense sub-poset of $M_{\aleph_\omega}$ has cardinality $> \aleph_\omega$ (see e.g. [7, Theorem 6.13]), $|\mathbb{E}| > \aleph_\omega$. Also $|\mathbb{E}| \leq 2^{\aleph_\omega} = \aleph_{\omega+1}$, and therefore $|\mathbb{E}| = \aleph_{\omega+1}$.

Because $B_{\omega_\omega} = \bigcup_{\alpha<\omega_1} B^\alpha_{\omega_\omega}$ and $|\mathbb{E}| = \aleph_{\omega+1}$, there is some $\alpha < \omega_1$ such that $\left|\mathbb{E} \cap B^\alpha_{\omega_\omega}\right| = \aleph_{\omega+1}$. Fix some such α, and let $\mathbb{E}_\alpha = \mathbb{E} \cap B^\alpha_{\omega_\omega}$.

Let $(M, \in)$ be a model of (a sufficiently large fragment of) ZFC such that $\omega_\omega \subseteq M$, $\mathbb{E}_\alpha \in M$, and $|M| = |M \cap \mathbb{E}_\alpha| = \aleph_{\omega+1}$. (Such a model can be obtained in the usual way, via the downward Löwenheim-Skolem Theorem.) Let $\phi : M \to M \cap \mathbb{E}_\alpha$ be a bijection, and consider the model $(M, \in, \phi, \omega_\omega)$ for the 3-symbol language consisting of a binary relation, a unary function, and a unary predicate. Applying the Chang conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, there exists some $M' \subseteq M$ such that $|M'| = \aleph_1$, $M' \cap \omega_\omega$ is countable, and $(M', \in, \phi, \omega_\omega) \prec (M, \in, \phi, \omega_\omega)$.

Let $\mathbb{E}'_\alpha = \mathbb{E}_\alpha \cap M'$. By elementarity, the restriction of ϕ to M' is a bijection $M' \to \mathbb{E}'_\alpha$, and so $|\mathbb{E}'_\alpha| = \aleph_1$.

Let $A = \omega_\omega \cap M'$. If $X \in \mathbb{E}'_\alpha$, then $\mathrm{supp}(X) \in M'$, and therefore (because $\mathrm{supp}(X)$ is countable, and M' models (enough of) ZFC) $\mathrm{supp}(X) \subseteq M'$. Hence $X \in \mathbb{E}'_\alpha$ implies $\mathrm{supp}(X) \subseteq A$.

By Lemma 4.2, in $V[G]$ there is a countable collection $\mathcal{C}$ of non-null closed subsets of 2^A such that if B is any non-null Borel subset of 2^A whose Borel code is in $V[G_\alpha]$, then there is some $C \in \mathcal{C}$ such that $C \subseteq B$. (Strictly speaking, our lemma gives us such a family in $V[G_{\alpha+1}]$. But by reinterpreting the Borel codes of the members of that family in $V[G]$, we obtained the desired collection $\mathcal{C}$.) In particular, every $X \in \mathbb{E}'_\alpha$ contains $C \times 2^{\omega_\omega \setminus A}$ for some $C \in \mathcal{C}$. By the pigeonhole principle, there is some particular $C \in \mathcal{C}$ such that $X \supseteq C \times 2^{\omega_\omega \setminus A}$ for uncountably many $X \in \mathbb{E}'_\alpha$.

Moving from representatives back to equivalence classes, $[C \times 2^{\omega_\omega \setminus A}] \neq [\emptyset]$ because C is non-null in 2^A, and $[C \times 2^{\omega_\omega \setminus A}] \leq [X]$ for uncountably many $X \in \mathbb{E}'_\alpha$. Hence $[C \times 2^{\omega_\omega \setminus A}] \in M_{\aleph_\omega} \setminus \{\mathbf{0}\}$ and $[C \times 2^{\omega_\omega \setminus A}]$ extends uncountably many members of $\mathbb{D}$. Because $\mathbb{D}$ was an arbitrary dense sub-poset of $M_{\aleph_\omega}$, and because $M_{\aleph_\omega}$ has the ccc, we conclude that $\triangledown(M_{\aleph_\omega})$ fails. $\square$

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