The small Dowker space problem

Carolina Seminar
9/6/23

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Plan for today

I'll be reporting on the following papers:

- Assaf Rinot and R.S., *A guessing principle from a Souslin tree, with applications to topology*, accepted to Topology Appl, 2023.
- Assaf Rinot, R.S and Stevo Todorčević, *A new small Dowker space*, accepted to Periodica Mathematica Hungarica, 2023.

Motivation

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In contrast, the Sorgenfrey line $\mathbb{R}_l$ is a regular Lindelöf (hence normal) space whose square is not normal (hence, non-Lindelöf).

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Question (C. H. Dowker, 1951)

Is there a normal topological space whose product with the unit interval is not normal?

Such a space is called Dowker.

The Dowker space problem

Theorem (C. H. Dowker, 1951)

A normal space $\mathbb{X}$ is Dowker iff there exists a $\subseteq$-decreasing sequence $\langle D_n \mid n < \omega \rangle$ of closed sets s.t.:

1. $\bigcap_{n<\omega} D_n = \emptyset$;
2. *if, for every $n < \omega$, U_n is some open set covering D_n, then* $\bigcap_{n<\omega} U_n \neq \emptyset$.

The Dowker space problem (cont.)

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If there is a Souslin tree, then there is a Dowker space of size $\aleph_1$.

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Does the existence of a Dowker space follow from ZFC?

The Dowker space problem (cont.)

Theorem (M. E. Rudin, 1972)

There exists a Dowker space of size $(\aleph_{\omega+1})^{\aleph_0}$.

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The Dowker space problem (cont.)

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Whose space is actually smaller?

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Conjecture (M. E. Rudin, 1990)

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In [54], we present a new sufficient condition, namely, the following weakening of the continuum hypothesis:

Definition (Broverman-Ginsburg-Kunen-Tall, 1978)

⫯ asserts there is a list $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ of infinite subsets of ω_1 such that for every uncountable $B \subseteq \omega_1$, there is $\alpha < \omega_1$ with $A_\alpha \subseteq B$.

Diagram of implications

Along the way, we unify the above-mentioned results, factoring the Dowker space constructions through a new ‘guessing’ principle that we call $\clubsuit_{\mathsf{AD}}$.

$$\begin{tikzcd}
 & & \diamondsuit \arrow[dl] \arrow[d] \arrow[dr] & \\
 & \clubsuit \arrow[ddr] & \mathsf{CH} \arrow[d] \arrow[dll] & \text{Souslin tree} \arrow[ddl] \\
\stackrel{\bullet}{\vert} \arrow[drr] & & \text{Luzin set} \arrow[d] & \\
 & & \clubsuit_{\mathsf{AD}} \arrow[d] \arrow[dr] & \\
 & & \text{Dowker space of size } \aleph_1 & \neg\mathsf{PFA}
\end{tikzcd}$$

We denote by L the set of all nonzero limit countable ordinals. For every infinite ordinal $\beta < \omega_1$, we denote by α_β the unique $\alpha \in \mathsf{L}$ such that $\alpha \leq \beta < \alpha + \omega$.

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$\clubsuit_{\mathrm{AD}}$ asserts there is a matrix $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega\rangle$ such that:

0. for every $\alpha \in \mathsf{L}$, $\langle A_{\alpha,n} \mid n < \omega\rangle$ consists of pairwise disjoint cofinal subsets of α;

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2. for all $(\alpha, n) \neq (\beta, m)$, $\sup(A_{\alpha,n} \cap A_{\beta,m}) < \alpha$.

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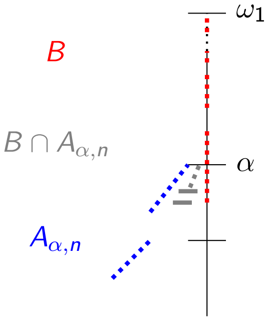
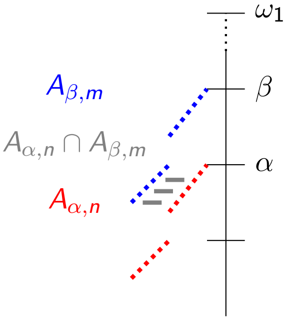


2. for all $(\alpha, n) \neq (\beta, m)$, $\sup(A_{\alpha,n} \cap A_{\beta,m}) < \alpha$.



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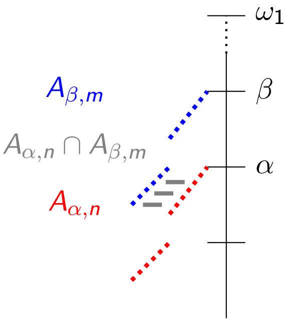


Disjointifying initial segments

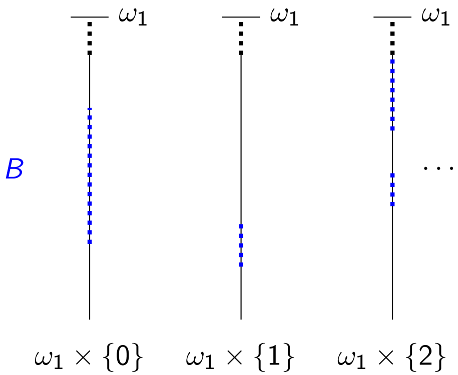
For every $\epsilon < \omega_1$, there exists a map $f : (\mathsf{L} \cap \epsilon) \times \omega \to \epsilon$ such that

1. $f(\alpha, n) < \alpha$;
2. $\{A_{\alpha,n} \setminus f(\alpha, n) \mid (\alpha, n) \in \mathrm{dom}(f)\}$ is a pairwise disjoint family.

Constructing the space

A few promises

Our space $\mathbb{X} = (X, \tau)$ will have underlying set $\omega_1 \times \omega$.
For all $B \subseteq X$ and $j < \omega$, we write $\pi_j(B) := \{\xi < \omega_1 \mid (\xi, j) \in B\}$ for its j^{th}-section.



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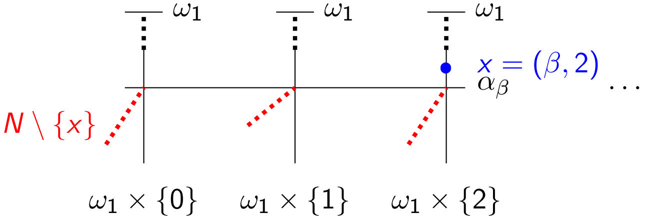

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Consequence 1

For all $\delta < \omega_1$ and $n < \omega$, $\delta \times n$ is τ-open, and $D_n := \omega_1 \times (\omega \setminus n)$ is τ-closed. $\langle D_n \mid n < \omega \rangle$ is $\subseteq$-decreasing, and $\bigcap_{n<\omega} D_n = \emptyset$.
The first part implies that $\mathbb{X}$ is not Lindelöf.

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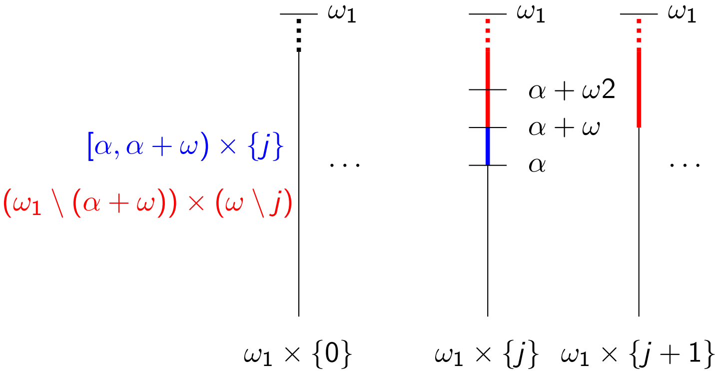
Consequence 2: The domino effect

For every $(\alpha, j) \in \mathrm{L} \times \omega$, the τ-closure of the strip $[\alpha, \alpha + \omega) \times \{j\}$ covers the following tail of ω_1 times a tail of ω:

$$(\omega_1 \setminus (\alpha + \omega)) \times (\omega \setminus j).$$


ω_1
ω_1
ω_1
$\alpha + \omega 2$
$\alpha + \omega$
α
$[\alpha, \alpha + \omega) \times \{j\}$
$\cdots$
$\cdots$
$(\omega_1 \setminus (\alpha + \omega)) \times (\omega \setminus j)$
$\omega_1 \times \{0\}$
$\omega_1 \times \{j\}$
$\omega_1 \times \{j+1\}$

The actual construction

Re-index $\langle A_{\alpha,n} \mid \alpha \in \mathsf{L}, n < \omega\rangle$ as $\langle A^j_{\beta,n} \mid \omega \leq \beta < \omega_1, j \leq n < \omega\rangle$ such that, for every $\alpha \in \mathsf{L}$,

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*The epsilons are there to ensure that the outcome space $\mathbb{X}$ is T_1.

Indeed, $\bigcap \mathcal{N}_x = \{x\}$.

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and $\pi_j(N^\epsilon_x \setminus \{x\}) = A^j_{\beta,n} \setminus \epsilon$ is a cofinal subset of α_β, as promised.

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Consequence 3

Given $(\beta, n) \in X \setminus \omega \times \omega$ and $B \subseteq X$, if there exists $j < \omega$ such that $\sup(A^j_{\beta,n} \cap \pi_j(B)) = \alpha_\beta$, then $(\beta, n) \in \mathrm{cl}(B)$.

Verifications

Lemma

Every τ-closed uncountable $B \subseteq X$ contains a 'tail', i.e., there is $(\gamma, j) \in \mathsf{L} \times \omega$ such that $(\omega_1 \setminus \gamma) \times (\omega \setminus j) \subseteq B$.

Proof. Given an uncountable $B \subseteq X$, find the least $j < \omega$ such that $\pi_j(B)$ is uncountable.

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In particular, $\sup(A^j_{\beta,j} \cap \pi_j(B)) = \alpha$ for all $\beta \in [\alpha, \alpha+\omega)$. So, the τ-closure of the countable set $Y := B \cap (\alpha \times \{j\})$ covers $[\alpha, \alpha+\omega) \times \{j\}$.

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The above proof shows that the space is hereditary separable, so altogether $\mathbb{X}$ is an S-space.

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For each $n < \omega$, $F_n := X \setminus U_n$ is a closed set disjoint from D_n.
Since D_n is uncountable, F_n must be countable. So $\bigcup_{n<\omega} F_n$ is countable, and hence $\bigcap_{n<\omega} U_n = X \setminus \bigcup_{n<\omega} F_n$ is nonempty. ■

The consistency of the negation of our guessing principle

Theorem

Suppose that $\mathrm{PID}_{\aleph_1}$ *holds and* $\mathfrak{b} > \omega_1$*. Then, for any stationary* $S \subseteq \omega_1$*,* $\clubsuit_{\mathsf{AD}}(\{S\}, 1, 1)$ *fails.*

Combinatorial principles

An ideal $\mathcal{I}$ consisting of countable sets is said to be a **P-ideal** iff every countable family of sets in the ideal admits a pseudo-union in the ideal. That is, for every sequence $\langle X_n \mid n < \omega \rangle$ of elements of $\mathcal{I}$, there exists $X \in \mathcal{I}$ such that $X_n \setminus X$ is finite for all $n < \omega$.

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Definition (Todorčević)

The P-ideal dichotomy (PID) asserts that for every P-ideal $\mathcal{I}$ consisting of countable subsets of some set Z, either:

1. there is an uncountable $B \subseteq Z$ such that $[B]^{\aleph_0} \subseteq \mathcal{I}$, or
2. there is a sequence $\langle B_n \mid n < \omega \rangle$ such that $\bigcup_{n<\omega} B_n = Z$ and, for each $n < \omega$, $[B_n]^{\aleph_0} \cap \mathcal{I} = \emptyset$.

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We denote by $\mathrm{PID}_{\aleph_1}$ the restriction of the above principle to $Z := \omega_1$. This special case was first introduced and studied by Abraham and Todorčević.

Combinatorial principles

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Suppose that $\mathsf{PID}_{\aleph_1}$ *holds and* $\mathfrak{b} > \omega_1$. *Then, for any stationary* $S \subseteq \omega_1$, $\clubsuit_{\mathsf{AD}}(\{S\}, 1, 1)$ *fails.*

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Proof. Towards a contradiction, suppose that $S \subseteq \omega_1$ is stationary, and that $\vec{A} = \langle A_\alpha \mid \alpha \in S\rangle$ is a $\clubsuit_{\mathrm{AD}}(\{S\}, 1, 1)$-sequence. Let

$$\mathcal{I} := \{X \in [\omega_1]^{\leq\aleph_0} \mid \forall \alpha \in \mathrm{acc}(\omega_1) \cap S[A_\alpha \cap X \text{ is finite}]\}.$$

It is clear that $\mathcal{I}$ is an ideal.

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Fix a bijection $e : \omega \leftrightarrow \biguplus_{n<\omega} X_n$. Then, for all $\alpha \in S$, define a function $f_\alpha : \omega \to \omega$ via

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By the definition of f_α, $\beta \in e[f_\alpha(n)]$. But $f_\alpha(n) < f(n)$, so that $\beta \in e[f_\alpha(n)] \subseteq e[f(n)]$, contradicting the fact that $\beta \in X$.

$$f_\alpha(n) := \min\{m < \omega \mid X_n \cap A_\alpha \subseteq e``m\}.$$

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Let $B \subseteq \omega_1$ be uncountable.

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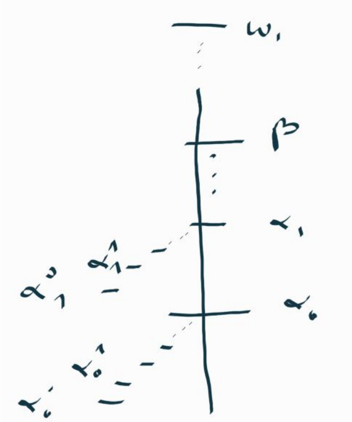

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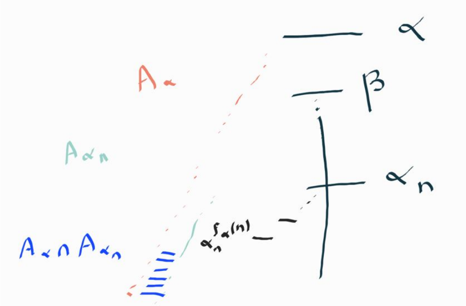

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In particular, $X \in [B]^{\aleph_0}$.

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Altogether, $\mathcal{I}$ is a P-ideal for which the two alternatives of $\mathrm{PID}_{\aleph_1}$ fail. This is a contradiction.

Thank you!