Ultrafilters in the random real model

Alan Dow * Osvaldo Guzmán †

Abstract

We prove that P-points (even strong P-points) and Gruff ultrafilters exist in any forcing extension obtained by adding fewer than $\aleph_\omega$-many random reals to a model of CH. These results improve and correct previous theorems that can be found in the literature.

1 Introduction

Ultrafilters[1] play a fundamental role in infinite combinatorics, set-theoretic topology and model theory. From constructing compactifications of topological spaces and analyzing convergence, to proving Ramsey theorems, finding nontrivial elementary embeddings of the universe, or building nonstandard models of a theory, applications of ultrafilters are ubiquitous across these areas of mathematics. Several important classes of ultrafilters on countable sets have been introduced and studied over the years. A particularly notable example are the P-points, which were introduced by Walter Rudin in [45], to prove that under the Continuum Hypothesis (CH), the space $\omega^* = \beta\omega \setminus \omega$ is not homogeneous (the same conclusion was later established without assuming CH by Frolík in [24] and further refined by Kunen in [34] who explicitly constructed ultrafilters with distinct topological types). Since then, special classes of ultrafilters on countable sets became a central topic of study and research.

Although it is a straightforward theorem of ZFC that there are (non-principal) ultrafilters on the natural numbers, the existence of ultrafilters with interesting topological or combinatorial properties is far more subtle. Moreover, their existence is often independent. The first major result of this kind was obtained by Kunen in [35] where he proved that Ramsey ultrafilters consistently do not exist. Some time later, Miller proved in [41] (see also [42]) that Q-points may not exist and Shelah constructed a model without P-points (see [55] and [47]). More recently, it was proved by Cancino and Zapletal (see [11]) that it is consistent that every (non-principal) ultrafilter on ω is Tukey top. For more results

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[1] Undefined concepts will be reviewed in later sections.

regarding the existence or non existence of special ultrafilters, the reader may consult [54], [48], [6], [20], [23], [12], [9] or [27] among many others. As these results illustrate, the existence of special classes of ultrafilters is a major concern for set theorists and topologists.

Random forcing was introduced by Solovay in [49] and models obtained by adding more than ω_1 many random reals to a model of CH are often called *random models* (or *random real models*). Since random forcing is one of the most well-known and studied forcing notions, one might expect that the structure of ultrafilters in the random models is very well understood. However, this is far from the case. The previously mentioned Theorem of Kunen in [35] actually shows that there are no Ramsey ultrafilters in the random models. On the other hand, there are Q-points in those models since the dominating number is equal to ω_1 (see [3]). Now, the important question is: *What about P-points?* This is where the story gets complicated. In an unpublished note, Kunen proved that if you add ω_1 Cohen reals to a model of CH and then any number of random reals, you will get a P-point. In particular, P-points exists in some random models. The general case (without the preliminary Cohen reals) was later addressed by Cohen[2] [15]. He defined a combinatorial object called *pathway,* (which is very similar to Roitman´s Model Hypothesis (MH), see [44] and [1]) proved that the existence of a pathway entails that there is a P-point and that pathways exist after adding any number of random reals to a model of CH. Unfortunately, it was later discovered that the proof of the existence of pathways is flawed[3] (see [21] and [12]). No further progress was made on this problem until the publication of [19], where the first author proved that there is a P-point if ω_2-many random reals are added to a model of $\mathsf{CH} + \square_{\omega_1}$ (see [44] for further results). In the present work, we improve this result by showing that there are P-points (and more) if less than $\aleph_\omega$ random reals are added to a model of the Continuum Hypothesis (the principle $\square_{\omega_1}$ is no longer needed). The proof follows closely the argument from [19], but through a careful analysis of the interaction of countable subsets within countable elementary submodels, we are able to avoid the use of $\square_{\omega_1}$ and extend the result beyond ω_2. We introduce a new type of combinatorial object, which we call a *multiple $\mathfrak{d}$-pathway.* This notion has some resemblance to Cohen's pathways, the hypothesis MH and to the *generalized pathways* introduced by Fernández-Bretón in [22]. We will prove that the existence of a multiple $\mathfrak{d}$-pathway entails the existence of P-points (even strong P-points) and Gruff ultrafilters. Finally, it will be proved that multiple $\mathfrak{d}$-pathways exist if less than $\aleph_\omega$ random or Cohen reals are added to a model of CH.

The structure of the paper is as follows: after reviewing some notation and preliminaries, in Section 4, multiple $\mathfrak{d}$-pathways will be introduced and we will

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[2] It is worth pointing out that this is not the same Cohen who intoduced forcing and after whom the Cohen reals are named.

[3] Despite of the mistake, the paper [15] is very valuable. The introduction of pathways is very important and the construction of a P-point from a pathway is correct.

prove some of their most fundamental properties. In Section 5 a P-point is constructed from a multiple $\mathfrak{d}$-pathway. This construction will be further refined in Section 6 to get a strong P-point. Although in theory, the reader can skip Section 5 and jump to Section 6, we do not recommend it, since the construction in Section 5 is the best example to understand how to perform transfinite recursions using a multiple $\mathfrak{d}$-pathway. Section 7 contains our last application of multiple $\mathfrak{d}$-pathways, the construction of a Gruff ultrafilter. In Section 8 we develop some combinatorial results regarding countable elementary submodels that will be needed later. In Section 9 we prove that there are multiple $\mathfrak{d}$-pathways in the models obtained by adding less than $\aleph_\omega$ many random or Cohen reals. Although we are mainly interested in the random reals, the proof for Cohen reals is exactly the same.

2 Notation

For a set X, we denote by $\mathcal{P}(X)$ its power set. We say that $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *filter on* X if $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$, for every $A, B \subseteq X$, if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$ and if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$. A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal on* X if $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$, for every $A, B \subseteq X$, if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$ and if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$. If $\mathcal{B}$ is a family of subsets of X, denote $\mathcal{B}^* = \{X \setminus B \mid B \in \mathcal{B}\}$. It is easy to see that if $\mathcal{F}$ is a filter then $\mathcal{F}^*$ is an ideal (called the *dual ideal of* $\mathcal{F}$) and if $\mathcal{I}$ an ideal then $\mathcal{I}^*$ is a filter (called the dual filter of $\mathcal{I}$). If $\mathcal{I}$ is an ideal on X, define $\mathcal{I}^+ = \mathcal{P}(X) \smallsetminus \mathcal{I}$, which is called the family of *$\mathcal{I}$-positive sets.* The *Fréchet filter* is the filter of cofinite subsets. An *ultrafilter* is a maximal filter that extends the Fréchet filter (so in this work, all ultrafilters are non-principal). If $\mathcal{U}$ is an ultrafilter, we say that $\mathcal{B} \subseteq \mathcal{U}$ *is a base of* $\mathcal{U}$ if every element of $\mathcal{U}$ contains one of $\mathcal{B}$. We say that a family $\mathcal{P} \subseteq \mathcal{P}(X)$ is *centered* if the intersection of any finite collection of its elements is infinite.

By $\mathfrak{c}$ we denote the cardinality of the set of real numbers. For any two sets A and B, we say $A \subseteq^* B$ (A *is an almost subset of* B or A *is almost contained in* B) if $A \setminus B$ is finite. For $\mathcal{P} \subseteq [\omega]^\omega$ and $A \in \mathcal{P}(\omega)$, we say that A *is a pseudointersection of* $\mathcal{P}$ if it is almost contained in all elements of $\mathcal{P}$. For $f, g \in \omega^\omega$, define $f \leq g$ if and only if $f(n) \leq g(n)$ for every $n \in \omega$ and $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. A family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is not bounded with respect to $\leq^*$. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family and the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family. We say $\mathcal{S} = \{f_\alpha \mid \alpha \in \mathfrak{b}\} \subseteq \omega^\omega$ is a *scale* if $\mathcal{S}$ is dominating and $f_\alpha \leq^* f_\beta$ whenever $\alpha < \beta$. It is easy to see that $\mathfrak{b} = \mathfrak{d}$ is equivalent to the existence of a scale. A function $f \in \omega^\omega$ is *unbounded over a model* M if $f \nleq^* g$ for every $g \in M$ and *is dominating over* M if it dominates every element of $M \cap \omega^\omega$.

$\mathcal{P}(\omega)$ will have its natural topology, which is homeomorphic to 2^ω. In this way, the topology of $\mathcal{P}(\omega)$ has for a subbase the sets of the form $\langle n\rangle_0 = \{A \subseteq \omega \mid n \notin A\}$ and $\langle n\rangle_1 = \{A \subseteq \omega \mid n \in A\}$, for $n \in \omega$.

A *Polish space* is a separable and completely metrizable space. The Baire space (ω^ω) and the Cantor space (2^ω) are examples of Polish spaces. We will need the concepts of F_σ, G_δ, Borel, analytic, coanalytic and projective subsets of a Polish space, which can be found in [33] or [50]. We say that $T \subseteq 2^{<\omega}$ is a *tree* if it is closed under taking initial segments and $f \in 2^\omega$ is a *branch of* T if $f \restriction n \in T$ for every $n \in \omega$. The set of all branches of T is denoted by $[T]$. It is well known that the compact subsets of 2^ω correspond to branches of subtrees of $2^{<\omega}$ (see [33]). For $n \in \omega$, the *n-level of the tree* T is denoted by T_n.

Let X be a topological space. We say that $C \subseteq X$ *is crowded* if it does not have an isolated point. A *perfect subset of* X is a closed, non-empty crowded subset of X. On the other hand, $S \subseteq X$ *is scattered* if it does not contain a non-empty crowded subset.

We will work extensively with elementary submodels. The reader is invited to consult [17] for their most important properties and to learn how to apply them in topology and set theory. For κ a cardinal, by $\mathsf{H}(\kappa)$ we denote the collection of all subsets whose transitive closure has size less than κ. For M a countable elementary submodel of $\mathsf{H}(\kappa)$ (and $\kappa > \omega_1$), *the height of* M is $\delta_M = M \cap \omega_1$. It is easy to see that it is always a countable ordinal. We will fix $\trianglelefteq$ a well order of $\mathsf{H}(\kappa)$ and by $\mathsf{Sub}(\kappa)$ we denote the set of all countable $M \subseteq \mathsf{H}(\kappa)$ such that $(M, \in, \trianglelefteq)$ is an elementary submodel of $(\mathsf{H}(\kappa), \in, \trianglelefteq)$. The well ordering will play a key role in some of our arguments.

For A a set of the ordinal numbers, we will denote by $\mathsf{OT}(A)$ its order type. If f is a function, by $\mathsf{dom}(f)$ we denote its domain and $\mathsf{im}(f)$ is its image.

3 Forcing preliminaries

We review some preliminaries on forcing that will be needed in Section 9. Naturally, we assume the reader is already familiar with the method of forcing as presented in [36].

Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. By $V^{\mathbb{P}}$ we denote the class of all $\mathbb{P}$-names as defined in [36]. An isomorphism $F : \mathbb{P} \longrightarrow \mathbb{Q}$ can be extended recursively to a bijection between $V^{\mathbb{P}}$ and $V^{\mathbb{Q}}$ (which we will also denote as F) by letting $F(\dot{a}) = \{(F(\dot{b}), F(p)) \mid (\dot{b}, p) \in \dot{a}\}$. The proof of the following result can essentially be found in [30] or [31] and is easy to prove by induction on the rank of names.

Proposition 1 *Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders and $F : \mathbb{P} \longrightarrow \mathbb{Q}$ an isomorphism. For every $p \in \mathbb{P}$, φ a formula, $\dot{a}_0, ..., \dot{a}_n \in V^{\mathbb{P}}$ and sets $b_0, ..., b_m$, the following are equivalent:*

1. *$p \Vdash$ "$\varphi(\dot{a}_0, ..., \dot{a}_n, b_0, ..., b_m)$".*
2. *$F(p) \Vdash$ "$\varphi(F(\dot{a}_0), ..., F(\dot{a}_n), b_0, ..., b_m)$".*

We now review the standard method for adding many random or Cohen reals using finite support. Proofs of the results mentioned below can be found in [37]. For a set I, we will always equip 2^I with its usual Tychonoff topology (where $2 = \{0, 1\}$ is a discrete space).

Definition 2 *Let I, J be two infinite sets and $\triangle : I \longrightarrow J$ an injective function.*

1. *Define $\triangle' : 2^J \longrightarrow 2^I$ given by $\triangle'(f) = f \circ \triangle$ for $f \in 2^J$.*
2. *Define $\triangle_* : \mathcal{P}(2^I) \longrightarrow \mathcal{P}(2^J)$ where $\triangle_*(B) = \{f \in 2^J \mid f \circ \triangle \in B\}$ (in other words, $\triangle_*(B) = (\triangle')^{-1}(B)$).*

It is easy to see that if $\triangle : I \longrightarrow J$ is a bijection, then $\triangle'$ is a homeomorphism. The following proposition follows from standard computations and diagram chasing arguments:

Proposition 3 *Let I, J, K be three infinite sets and $\triangle : I \longrightarrow J$, $\sigma : J \longrightarrow K$ bijective functions.*

1. *$(Id_I)_* = Id_{2^I}$ (where Id_X denotes the identity mapping of a set X).*
2. *$(\sigma \circ \triangle)_* = \sigma_* \circ \triangle_*$.*
3. *$(\triangle^{-1})_* = (\triangle_*)^{-1}$.*
4. *$\triangle_*$ is a bijection.*

We say $B \subseteq 2^I$ is *Baire* if it belongs to the smallest σ-algebra that contains all clopen sets. Denote by $\mathsf{Baire}(2^I)$ the collection of all the Baire subsets of 2^I. If I is countable, then the notion of Borel and Baire coincide, but they are not the same if I is uncountable. If $\triangle : I \longrightarrow J$ is bijective, then $\triangle_* \restriction \mathsf{Baire}(2^I)$ is a bijection between $\mathsf{Baire}(2^I)$ and $\mathsf{Baire}(2^J)$.

Definition 4 *Let I be an infinite set.*

1. *$\mathcal{M}_I$ denotes the σ-ideal of meager sets in 2^I.*
2. *$\mathcal{N}_I$ denotes the σ-ideal of null sets in 2^I (where 2^I has the standard product measure).*
3. *Cohen forcing on I (denoted by $\mathbb{C}(I)$) is the set of all Baire subsets of 2^I that are not in $\mathcal{M}_I$. We order $\mathbb{C}(I)$ by inclusion: $A \leq B$ if and only if $A \subseteq B$.*

4. *Random forcing on* I *(denoted by* $\mathbb{B}(I)$*) is the set of all Baire subsets of* 2^I *that are not in* $\mathcal{N}_I$*. We order* $\mathbb{B}(I)$ *by inclusion:* $A \leq B$ *if and only if* $A \subseteq B$*.*

Note that as we defined them, nor $\mathbb{C}(I)$ or $\mathbb{B}(I)$ are separative. Alternatively, we could take their quotients. This choice does not affect the content of the paper. If $\triangle : I \longrightarrow J$ is bijective, then $\triangle_* : \mathbb{C}(I) \longrightarrow \mathbb{C}(J)$ and $\triangle_* : \mathbb{B}(I) \longrightarrow \mathbb{B}(J)$ are isomorphism of partial orders. If $K \subseteq I$, then $\mathbb{B}(K)$ $(\mathbb{C}(K))$ is isomorphic to a regular suborder of $\mathbb{B}(I)$ $(\mathbb{C}(I))$, and for convenience we will regard them as actual suborders. Furthermore, if $\dot{a}$ is a $\mathbb{B}(I)$-name for a subset of ω, we can find a countable $K \subseteq I$ such that $\dot{a}$ is a $\mathbb{B}(K)$-name (and the same for Cohen forcing). We will be using all these facts implicitly.

4 Multiple $\mathfrak{d}$-pathways

Multiple $\mathfrak{d}$-pathways will be introduced in this section and we will prove some of their most fundamental properties. Before proceeding, we introduce some definitions.

Definition 5 *Let* X *be a set and* $n \in \omega$*. A relation* $R \subseteq X^n \times \omega^\omega$ *is* $\leq^*$-adequate *if for every* $x_1, ..., x_n \in X$*, there is* $f \in \omega^\omega$ *such that for every increasing* $g \in \omega^\omega$*, if* $g \nleq^* f$*, then the relation* $R(x_1, ..., x_n, g)$ *holds.*

A function f as above will be called an *R-control for* $(x_1, ..., x_n)$. It is not hard to find $\leq^*$-adequate relations and several examples will be provided in the text.

Definition 6 *Let* $M_0, ..., M_n$ *be countable elementary submodels of some* $\mathsf{H}(\kappa)$*. We will say that the sequence* $\langle M_0, ..., M_n \rangle$ *is* δ-increasing *if* $\delta_{M_i} \leq \delta_{M_{i+1}}$ *for each* $i < n$*.*

It is worth pointing out that in our work, the sequence $\langle M_0, ..., M_n \rangle$ is typically not an $\in$-chain (which is often the case when working with models as side conditions, see [51] and [52]) and it will often be the case that $\delta_{M_i} = \delta_{M_{i+1}}$ for some $i < n$. When discussing a specific δ-increasing sequence $\langle M_0, ..., M_n \rangle$, it will be understood that $n \in \omega$ and we will write δ_i instead of δ_{M_i} in case there is no risk of confusion.

We can finally introduce the main definition of this section:

Definition 7 *Let* $\kappa > \mathfrak{c}$ *be a regular cardinal,* $\mathcal{B} = \{f_\alpha \mid \alpha < \omega_1\} \subseteq \omega^\omega$ *a family of increasing functions and* $\mathcal{S} \subseteq \mathsf{Sub}(\kappa)$ *a stationary subset of* $[\mathrm{H}(\kappa)]^\omega$ *such that every model in* $\mathcal{S}$ *has* $\mathcal{B}$ *as an element. We say that* $(\mathcal{B}, \mathcal{S})$ *is a* multiple

$\mathfrak{d}$-pathway *if for every δ-increasing sequence $\langle M_0, ..., M_n \rangle$ of models from $\mathcal{S}$, for every projective and $\leq^*$-adequate relation $R \subseteq (\omega^\omega)^{n+2}$ with $R \in M_n$ and $x_i \in M_i$ for $i \leq n$, we have that $R(x_0, ..., x_n, f_{\delta_n})$ holds.*

Several comments regarding the definition are in order:

1. The term "multiple" indicates that there are typically several models that share the same height, which is not the case in the Model Hypothesis of Roitman.
2. For our applications, the stationarity of $\mathcal{S}$ is only used to ensure that every real (and hence every countable ordinal) appears in some model of $\mathcal{S}$.
3. The relation R is required to be projective. The key feature of this requirement is that it can be coded by a real, so several strengthenings or weakenings are possible. For the applications of P-points and strong P-points, Borel relations are enough, but it appears that more is needed for Gruff ultrafilters.
4. Following the approach of Fernández-Bretón in [22], it is possible to define a notion of a multiple pathway for more cardinal invariants of the continuum. We do not pursue this approach here, since we do not have applications for other cardinal invariants. However, the study of multiple pathways parametrized by cardinal invariants of the continuum might prove fruitful in the future.

To avoid constant repetition, when working with a multiple $\mathfrak{d}$-pathway $(\mathcal{B}, \mathcal{S})$, we will always use $\{f_\alpha \mid \alpha < \omega_1\}$ to refer to the sequence $\mathcal{B}$. We have the following simple result regarding multiple pathways:

Proposition 8 *Let $(\mathcal{B}, \mathcal{S})$ be a multiple $\mathfrak{d}$-pathway. For every $M \in \mathcal{S}$, the function f_{δ_M} is unbounded over M. In particular, the existence of a multiple $\mathfrak{d}$-pathway implies that $\mathfrak{b} = \omega_1$.*

Proof. Let $M \in \mathcal{S}$. Define the relation $R \subseteq (\omega^\omega)^2$ where $R(g, f)$ if $f \nleq^* g$. Clearly $R \in M$, is a G_δ relation and is $\leq^*$-adequate. Since obviously $\langle M \rangle$ is δ-increasing, it follows that every real in M is R related to f_{δ_M}. Finally, since the models in a multiple $\mathfrak{d}$-pathway cover ω^ω, we conclude that $\mathcal{B}$ is unbounded, hence $\mathfrak{b} = \omega_1$. ■

The definition of multiple $\mathfrak{d}$-pathway only mentions relations on the Baire space, however, we can extend it to any Polish space, as the next lemma shows:

Lemma 9 *Let $(\mathcal{B}, \mathcal{S})$ be a multiple $\mathfrak{d}$-pathway and $\langle M_0, ..., M_n \rangle$ a δ-increasing sequence of models of $\mathcal{S}$. For every Polish space $X \in M_0 \cap ... \cap M_n$, $R \subseteq X^{n+1} \times \omega^\omega$ projective and $\leq^*$-adequate relation in M_n and $x_i \in X \cap M_i$ for $i \leq n$, the relation $R(x_0, ..., x_n, f_{\delta_n})$ holds.*

Proof. We first find a continuous surjection $H : \omega^{\omega} \longrightarrow X$ with $H \in M_0 \cap ... \cap M_n$ (since X is a Polish space, there is a continuous surjection from ω^{ω} to X, we now use the well order of $\mathsf{H}(\kappa)$ to find one, denoted H, that is in all our models). Define the relation $P \subseteq (\omega^{\omega})^{n+2}$ where $P(g_0, ..., g_n, f)$ holds just in case $R(H(g_0), ..., H(g_n), f)$ is true. Note that $P \in M_n$ and is $\leq^*$-adequate. Moreover, it is easy to see that P is a continuous preimage of R, so P is projective as well. H is surjective and it is in every M_i, so we can find $g_i \in M_i \cap \omega^{\omega}$ such that $H(g_i) = x_i$. Since $(\mathcal{B}, \mathcal{S})$ is a multiple $\mathfrak{d}$-pathway, we know that $P(g_0, ..., g_n, f_{\delta_n})$ is true, which means that $R(x_0, ..., x_n, f_{\delta_n})$ is true as well. ■

This covers the basics of multiple $\mathfrak{d}$-pathways and we are ready to move on to applications.

5 A P-point from a multiple $\mathfrak{d}$-pathway

An ultrafilter $\mathcal{U}$ on ω is a P-*point* if every countable subfamily of $\mathcal{U}$ has a pseudointersection in $\mathcal{U}$. Without a doubt, the class of P-points is among the most important and studied families of ultrafilters on countable sets. Note that ultrafilters that are not P-points are very easy to construct (for example, every ultrafilter extending the dual filter of the density zero ideal). In this way, the challenge is in constructing P-points. Shelah was the first to show that it is consistent that there are no P-points (see [55] and [47]). On the other hand, several set theoretic axioms imply the existence of a P-point, for example the equality $\mathfrak{d} = \mathfrak{c}$, the inequality $\mathfrak{u} < \mathfrak{d}$ (see [3]) or the parametrized diamond $\diamondsuit(\mathfrak{r})$ from [43]. None of these principles hold in the random real models, which makes the construction of P-points in such models very interesting. We will now build a P-point from a multiple $\mathfrak{d}$-pathway. Our approach is inspired by Theorem 5.7 of [19].

Definition 10 *Let* $F : \omega \longrightarrow \mathcal{P}(\omega)$ *and* $g \in \omega^{\omega}$. *Define* $F^g = \bigcup_{n\in\omega} F(n) \cap g(n)$.

It is clear that if F is $\subseteq$-decreasing, then F^g is a (possibly finite) pseudointersection of $\mathsf{im}(F)$. Note that if $f \leq g$, then $F^f \subseteq F^g$. It is trivial to see that if F is the constant function with value $A \subseteq \omega$ and $g \in \omega^{\omega}$ is increasing, then $F^g = A$. The following results are easy and we leave them to the reader.

Lemma 11 *Let* $F : \omega \longrightarrow [\omega]^{\omega}$ *be* $\subseteq$*-decreasing. There is* $f \in \omega^{\omega}$ *such that for every increasing* $g \in \omega^{\omega}$, *if* $g \nleq^* f$, *then* F^g *is infinite.*

Lemma 12

1. *Let* $n \in \omega$. *The set* $R_n \subseteq \mathcal{P}(\omega)^{n+1}$ *consisting of all* $(A_0, ..., A_n)$ *such that* $A_0 \cap ... \cap A_n$ *is infinite, is* G_{δ}.

2. *The set* $R \subseteq \mathcal{P}(\omega)^{\omega}$ *consisting of all functions* F *such that* $\mathsf{im}(F)$ *is centered, is* G_{δ}.

We will need the following:

Lemma 13 *Let* $(\mathcal{B}, \mathcal{S})$ *be a multiple* $\mathfrak{d}$*-pathway and* $\langle M_0, ..., M_n \rangle$ *a* δ*-increasing sequence of models of* $\mathcal{S}$ *and* $m \leq n$ *be the least such that* $\delta_m = \delta_n$. *For every* $i \leq n$, *assume* $F_i : \omega \longrightarrow [\omega]^{\omega} \in M_i$ *is* $\subseteq$*-decreasing.*

If $\left\{ F_0^{f_{\delta_0}}, ..., F_{m-1}^{f_{\delta_{m-1}}} \right\} \cup \mathsf{im}(F_m) \cup ... \cup \mathsf{im}(F_n)$ *is centered, then* $\bigcap_{i \leq n} F_i^{f_{\delta_i}}$ *is infinite.*

Proof. Define the relation $R \subseteq (\mathcal{P}(\omega)^{\omega})^{n+1} \times \omega^{\omega}$ where $R(G_0, ..., G_n, f)$ holds in case one of the following conditions is true:

1. $\left\{ G_0^{f_{\delta_0}}, ..., G_{m-1}^{f_{\delta_{m-1}}} \right\} \cup \mathsf{im}(G_m) \cup ... \cup \mathsf{im}(G_n)$ is not centered.
2. $G_0^{f_{\delta_0}} \cap ... \cap G_{m-1}^{f_{\delta_{m-1}}} \cap G_m^f \cap ... \cap G_n^f$ is infinite.

Since $f_{\delta_0}, ..., f_{\delta_{m-1}} \in M_n$, we get that $R \in M_n$. By Lemma 12 (or rather by its proof), the first clause is an F_{σ} condition and the second one is G_{δ}, so R is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$, although we only care that it is Borel. Moreover, it is $\leq^*$-adequate by Lemma 11 and Proposition 8. The conclusion follows since $(\mathcal{B}, \mathcal{S})$ is a multiple $\mathfrak{d}$-pathway. ■

We can now prove:

Theorem 14 *If there is a multiple* $\mathfrak{d}$*-pathway, then there is a* P*-point.*

Proof. Fix $(\mathcal{B}, \mathcal{S})$ a multiple $\mathfrak{d}$-pathway. Define $D = \{\delta_M \mid M \in \mathcal{S}\}$ and for every $\delta \in D$, let $W_{\delta} = \cup \{\mathcal{P}(\omega) \cap M \mid M \in \mathcal{S} \wedge \delta_M \leq \delta\}$. By recursion over $\delta \in D$, we will define families $\mathcal{U}_{\delta}$, $\mathcal{A}_{\delta}$ and $\mathcal{P}_{\delta}$ with the following properties:

1. $\mathcal{U}_{\delta}$, $\mathcal{P}_{\delta}$ and $\mathcal{A}_{\delta}$ are subsets of $[\omega]^{\omega}$.
2. $\mathcal{U}_{\gamma} \subseteq \mathcal{U}_{\delta}$ and $\mathcal{A}_{\gamma} \subseteq \mathcal{A}_{\delta}$ for $\gamma \in D \cap \delta$.
3. $\mathcal{A}_{\delta} \subseteq W_{\delta}$.
4. $\mathcal{P}_{\delta}$ is the collection of all $F^{f_{\delta}}$ for which there is $M \in \mathcal{S}$ with $\delta_M = \delta$, $F : \omega \longrightarrow \mathcal{A}_{<\delta}$ is $\subseteq$-decreasing and belongs to M (where $\mathcal{A}_{<\delta} = \bigcup_{\xi \in D \cap \delta} \mathcal{A}_{\xi}$).
5. $\mathcal{U}_{\delta} = \bigcup \{\mathcal{P}_{\gamma} \mid \gamma \in D \cap (\delta + 1)\}$.
6. $\mathcal{U}_{\delta} \cup \mathcal{A}_{\delta}$ is centered.

7. $\mathcal{A}_\delta$ is $\subseteq$-maximal with respect to points 3 and 6.

Assume we are at step $\delta \in D$ and $\mathcal{U}_\gamma$, $\mathcal{A}_\gamma$ and $\mathcal{P}_\gamma$ have been defined for all $\gamma \in D \cap \delta$. In case δ is the minimum of D, we have $\mathcal{U}_\delta = \mathcal{P}_\delta = \emptyset$. Choose $\mathcal{A}_\delta \subseteq W_\delta$ any maximal centered set extending the Fréchet filter. Now consider the case where δ is not the least member of D. Note that $\mathcal{U}_\delta$ and $\mathcal{P}_\delta$ are defined from $\mathcal{A}_{<\delta}$, so we only need to find $\mathcal{A}_\delta$. Define $\mathcal{U}_{<\delta} = \bigcup\limits_{\xi \in D \cap \delta} \mathcal{U}_\xi$ and note that $\mathcal{U}_{<\delta} \cup \mathcal{A}_{<\delta}$ is centered, since (by the recursion hypothesis) it is an increasing union of centered sets. We now prove the following:

Claim 15 *$\mathcal{U}_\delta \cup \mathcal{A}_{<\delta}$ is centered.*

Let $B_0, ..., B_n \in \mathcal{U}_\delta \cup \mathcal{A}_{<\delta}$, for every $i \leq n$, we find $M_i \in \mathcal{S}$ and $F_i \in M_i$ in the following way:

1. In case $B_i \in \mathcal{A}_{<\delta}$, choose M_i for which $\delta_i = \delta_{M_i} < \delta$ and $B_i \in M_i$. Let $F_i : \omega \longrightarrow [\omega]^\omega$ be the constant function with value B_i.
2. If $B_i \in \mathcal{U}_\delta$, choose M_i with $\delta_i = \delta_{M_i} \leq \delta$ and $F_i : \omega \longrightarrow \mathcal{A}_{<\delta_i} \in M_i$ that is $\subseteq$-decreasing and $B_i = F_i^{f_{\delta_i}}$.

It might be possible that for some $i \leq n$ both clauses apply (in other words, $B_i \in \mathcal{U}_\delta \cap \mathcal{A}_{<\delta}$). If that is the case, we can choose to follow either one of them. For each $i \leq n$, we have the following:

1. $B_i = F_i^{f_{\delta_i}}$.
2. $F_i \in M_i$, is $\subseteq$-decreasing and $\mathsf{im}(F_i) \subseteq \mathcal{A}_{<\delta}$.

By taking a reenumeration and possibly picking more elements of $\mathcal{U}_\delta \cup \mathcal{A}_{<\delta}$, we may assume that $\langle M_0, ..., M_n \rangle$ is δ-increasing and $\delta_n = \delta$. Let $m \leq n$ be the least such that $\delta_m = \delta$. We claim that $H = \left\{ F_0^{f_{\delta_0}}, ..., F_{m-1}^{f_{\delta_{m-1}}} \right\} \cup \mathsf{im}(F_m) \cup ... \cup \mathsf{im}(F_n)$ is contained in $\mathcal{U}_{<\delta} \cup \mathcal{A}_{<\delta}$. Pick $i \leq n$. We have the following cases:

1. If $B_i \in \mathcal{A}_{<\delta}$, then $F_i^{\delta_i} = B_i$ so there is nothing to do.
2. If $B_i \in \mathcal{U}_\delta$ and $i < m$, then $F_i : \omega \longrightarrow \mathcal{A}_{<\delta_i}$ which implies that $F_i^{f_{\delta_i}} \in \mathcal{U}_{\delta_i} \subseteq \mathcal{U}_\delta$.
3. If $B_i \in \mathcal{U}_\delta$ and $m \leq i$, there is nothing to do since we already noted that $\mathsf{im}(F_i) \subseteq \mathcal{A}_{<\delta}$.

We already pointed out that $\mathcal{U}_{<\delta} \cup \mathcal{A}_{<\delta}$ is centered, so H is centered as well. We are now in position to invoke Lemma 13 and conclude that $\bigcap\limits_{i \leq n} F_i^{f_{\delta_i}}$ is infinite. Since $B_i = F_i^{f_{\delta_i}}$, this finishes the proof of the claim. We can now apply Zorn's Lemma and find $\mathcal{A}_\delta \subseteq W_\delta$ extending $\mathcal{A}_{<\delta}$, such that its union with $\mathcal{U}_\delta$ is centered and it is maximal with these properties.

After completing the recursion, define $\mathcal{U} = \bigcup\limits_{\delta \in D} \mathcal{U}_\delta$ and $\mathcal{A} = \bigcup\limits_{\delta \in D} \mathcal{A}_\delta$. We will now prove the following:

Claim 16

1. *$\mathcal{U} \cup \mathcal{A}$ is centered.*
2. *$\mathcal{U} = \mathcal{A}$.*
3. *$\mathcal{U}$ is an ultrafilter.*
4. *$\mathcal{U}$ is a* P*-point.*

To see the first point, simply note that $\mathcal{U} \cup \mathcal{A}$ is the increasing union of the sets $\mathcal{U}_\delta \cup \mathcal{A}_\delta$ and since we already knew those are centered, it follows that $\mathcal{U} \cup \mathcal{A}$ is centered as well. We will now prove $\mathcal{U} \subseteq \mathcal{A}$. Let $B \in \mathcal{U}$, find $\delta \in D$ such that $B \in W_\delta$ (recall that $\mathcal{S}$ is stationary). Since $B \in \mathcal{U}$, clearly $\mathcal{U}_\delta \cup \{B\} \cup \mathcal{A}_\delta$ is centered. By the maximality of $\mathcal{A}_\delta$ that $B \in \mathcal{A}_\delta$. We will now show that $\mathcal{A} \subseteq \mathcal{U}$. Let $A \in \mathcal{A}$ and $\delta \in D$ such that $A \in \mathcal{A}_\delta$. Since $\mathcal{S}$ is stationary, we can find $\gamma \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \gamma$ and $\delta, A \in M$ (so $\delta < \gamma$). Let F be the constant sequence with value A. We get that $A = F^{f_\gamma} \in \mathcal{P}_\gamma$, so $B \in \mathcal{U}$.

It is time to prove that $\mathcal{U}$ is an ultrafilter. Let $A, B, C, E \subseteq \omega$ such that $A, B \in \mathcal{U}$ and $A \subseteq C$. Find $\delta \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \delta$ and $A, B, C, E \in M$. In this way, $A \cap B, C, E$ and $\omega \setminus E$ are in W_δ. Since $\mathcal{U} = \mathcal{A}$ is centered, it follows that $\mathcal{U}_\delta \cup \{A \cap B, C\} \cup \mathcal{A}_\delta$ is centered. By the maximality of $\mathcal{A}_\delta$, we get that $A \cap B, C \in \mathcal{A}_\delta$. Moreover, either $\mathcal{U}_\delta \cup \mathcal{A}_\delta \cup \{E\}$ is centered or $\mathcal{U}_\delta \cup \mathcal{A}_\delta \cup \{\omega \setminus E\}$ is centered, so the maximality of $\mathcal{A}_\delta$ entails that $E \in \mathcal{A}_\delta$ or $\omega \setminus E \in \mathcal{A}_\delta$. Finally, recall that in the first step of the recursion we made sure that $\mathcal{U}$ contains the Fréchet filter.

It remains to prove that $\mathcal{U}$ is a P-point. Pick $F : \omega \longrightarrow \mathcal{U}$ that is $\subseteq$-decreasing. For every $n \in \omega$, choose $\delta_n \in D$ such that $F(n) \in \mathcal{A}_{\delta_n}$. We now find $\delta \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \delta$ and $F, \{\delta_n \mid n \in \omega\} \in M$. In this way, $F \in M$ and it is a decreasing sequence of elements of $\mathcal{A}_{<\delta}$, so $F^{f_\delta} \in \mathcal{P}_\delta \subseteq \mathcal{U}$. ■

6 A strong P-point from a multiple $\mathfrak{d}$-pathway

A P-point is an ultrafilter for which we can diagonalize against countably many of its elements, while a strong P-point is an ultrafilter for which we can diagonalize against countably many of its compact subsets. Since every singleton is compact, it follows that strong P-points are P-points. It is not difficult to show that Ramsey ultrafilters are never strong P-points. The mere existence of a P-point is not enough to imply either the existence of a strong P-point, nor of a P-point that is not strong[4]. In the Miller model every P-point is strong, while in Shelah's model of only one Ramsey ultrafilter (see [47]), every P-point is Ramsey, and therefore not strong. Naturally, both types of P-points exist under CH ($\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ is enough). In [4], Blass, Hrušák and Verner proved that strong P-points are precisely the ultrafilters whose Mathias forcing does not add dominating reals. This makes them very useful for constructing models where the bounding number is small, while other invariants like the splitting number or variants of the almost disjointness number are large (see [8], [46], [47], [18], [7], [5], [26], [25] or [28]). Strong P-points were introduced by Laflamme in [39] with the purpose of studying the collection of all F_σ filters as a forcing notion. We will obtain a strong P-point from a multiple $\mathfrak{d}$-pathway.

Definition 17 *An ultrafilter $\mathcal{U}$ on ω is a* strong P-point *if for every sequence $\langle \mathcal{C}_n \rangle_{n\in\omega}$ of compact subsets of $\mathcal{U}$, there is a partition $P = \{P_n \mid n \in \omega\}$ of ω into intervals such that the set $\{A \subseteq \omega \mid \forall n \in \omega \exists X_n \in \mathcal{C}_n \, (A \cap P_n = X_n \cap P_n)\}$ is contained in $\mathcal{U}$.*

For convenience, we will denote $\mathsf{fin} = [\omega]^{<\omega} \setminus \{\emptyset\}$. Given an ultrafilter $\mathcal{F}$ on ω, we define the filter $\mathcal{F}^{<\omega}$ on fin that is generated by $\{[A]^{<\omega} \setminus \{\emptyset\} \mid A \in \mathcal{F}\}$. It is not hard to see for $X \subseteq \mathsf{fin}$, we have that $X \in (\mathcal{F}^{<\omega})^+$ if and only if for every $A \in \mathcal{F}$, there is $s \in X$ such that $s \subseteq A$. The following theorem combines results from Blass, Chodounský, Hrušák, Minami, Repovš, Verner and Zdomskyy (see [4], [29] and [13]). We will only need the equivalence between 1) and 3).

Theorem 18 *Let $\mathcal{U}$ be an ultrafilter on ω. The following are equivalent:*

1. *$\mathcal{U}$ is a strong P-point.*
2. *The Mathias forcing of $\mathcal{U}$ does not add dominating reals.*
3. *$\mathcal{U}^{<\omega}$ is a P^+ filter, this means that every countable $\subseteq$-decreasing sequence contained in $(\mathcal{U}^{<\omega})^+$ has a pseudointersection in $(\mathcal{U}^{<\omega})^+$.*
4. *$\mathcal{U}$ is a Menger subset of $\mathcal{P}(\omega)$.*

We now need to prove some results regarding compact sets that will be used later.

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[4] Note that a weak P-point is not the same as a P-point that is not strong.

Definition 19 *Let $X \subseteq \mathsf{fin}$. Define:*

1. $\mathcal{C}(X) = \{A \subseteq \omega \mid \forall s \in X\,(A \cap s \neq \emptyset)\}$.
2. $T_X \subseteq 2^{<\omega}$ *is the set consisting of all* $s \in 2^{<\omega}$ *such that for every* $u \in X$, *if* $u \subseteq \mathsf{dom}(s)$, *then* $u \cap s^{-1}(\{1\}) \neq \emptyset$.

It is easy to see that $\mathcal{C}(X)$ is a compact subset of $\mathcal{P}(\omega)$, T_X is a tree and $\mathcal{C}(X) = [T_X]$ (we are identifying a set with its characteristic function). The relevance of these notions is the following lemma, which can be found in [28] or [25]:

Lemma 20 *Let $\mathcal{U}$ be an ultrafilter on ω and $X \subseteq \mathsf{fin}$. $X \in (\mathcal{U}^{<\omega})^+$ if and only if $\mathcal{C}(X) \subseteq \mathcal{U}$.*

We now prove the following:

Lemma 21 *Let $X, Y \subseteq \mathsf{fin}$. The following are equivalent:*

1. $\mathcal{C}(X) \cup \mathcal{C}(Y)$ *is centered.*
2. *For every $n, m \in \omega$ and every $s_1, ..., s_n \in (T_X)_m$, $z_1, ..., z_n \in (T_Y)_m$ there is $k > m$ such that for every $\overline{s}_1, ..., \overline{s}_n \in (T_X)_{k+1}$, $\overline{z}_1, ..., \overline{z}_n \in (T_Y)_{k+1}$ for which $s_i \subseteq \overline{s}_i$ and $z_i \subseteq \overline{z}_i$ for every $i \leq n$, it is the case that $\overline{s}_i(k) = \overline{z}_i(k) = 1$ for every $i \leq n$.*

Proof. It is easy to see that 2) implies 1). For the other implication, assume that 2) fails. Let $n, m \in \omega$ and $s_1, ..., s_n \in (T_X)_m$, $z_1, ..., z_n \in (T_Y)_m$ that witness the failure of 2). Let S be the set of $(\overline{s}_1, ..., \overline{s}_n, \overline{z}_1, ..., \overline{z}_n)$ such that there is $k > m$ for which $\overline{s}_1, ..., \overline{s}_n \in (T_X)_{k+1}$, $\overline{z}_1, ..., \overline{z}_n \in (T_Y)_{k+1}$, $s_i \subseteq \overline{s}_i$ and $z_i \subseteq \overline{z}_i$ for every $i \leq n$, but it is not true that $\overline{s}_i(k) = \overline{z}_i(k) = 1$ for every $i \leq n$. S has a natural tree ordering, it is finitely branching and by our assumption, it has infinite height. By invoking König's Lemma (see [38], [32]), we know that S has a cofinal branch. We can now find a finite subset of $\mathcal{C}(X) \cup \mathcal{C}(Y)$ with finite intersection. ■

With the lemma (or rather its generalization), we get:

Corollary 22 *Let $n \in \omega$ and define $R \subseteq \mathcal{P}\left([\omega]^{<\omega}\right)^n$ as the set of all $(X_1, ..., X_n)$ such that $\bigcup\limits_{i \leq n} \mathcal{C}(X_i)$ is centered. R is a G_δ relation.*

We now introduce the following notion:

Definition 23 *For a function $F : \omega \longrightarrow \mathcal{P}(\mathsf{fin})$ and $g \in \omega^\omega$, we define the set $F^g = \bigcup\limits_{n \in \omega} F(n) \cap \mathcal{P}(g(n))$.*

If F is $\subseteq$-decreasing, then F^g is a pseudointersection of $\mathsf{im}(F)$. Note that if $f \leq g$, then $F^f \subseteq F^g$. It is trivial to see that if F is the constant function with value $A \subseteq \mathsf{fin}$ and f is increasing, then $F^f = A$. Lastly, if $B \subseteq \omega$, then $\mathcal{C}([B]^1)$ consists of all supersets of B. The following lemma has appeared in [28] (Lemma 57) and in [25] (Lemma 3):

Lemma 24 *Let $\mathcal{F}$ be a filter, $\mathcal{D} \subseteq \mathcal{F}$ compact and $X_1, ..., X_n \subseteq \mathcal{P}(\mathsf{fin})$ with $\mathcal{C}(X_1), ..., \mathcal{C}(X_n) \subseteq \mathcal{F}$. For every $i \leq n$, there is $Y_i \in [X_i]^{<\omega}$ such that for every $F \in \mathcal{D}$ and $A_i^1, ..., A_i^n \in \mathcal{C}(Y_i)$, we have that $F \cap \bigcap\limits_{i,j \leq n} A_i^j \neq \emptyset$.*

It will be convenient to introduce the following notion:

Definition 25 *Let $\mathcal{A}$ be a family of compact subsets of $\mathcal{P}(\omega)$. We say that $\mathcal{A}$ is* compatible *if $\bigcup \mathcal{A}$ is centered.*

We can now prove the following:

Lemma 26 *Let $\mathcal{D} \subseteq \mathcal{P}(\omega)$ be a compact set, $n \in \omega$ and $F_i : \omega \longrightarrow \mathcal{P}(\mathsf{fin})$ for each $i \leq n$ such that $\{\mathcal{D}\} \cup \{\mathcal{C}(F_i(k)) \mid i \leq n \wedge k \in \omega\}$ is compatible. There is $h \in \omega^\omega$ such that for every increasing $g \in \omega^\omega$, if $g \nleq^* h$, then $\mathcal{D} \cup \mathcal{C}(F_0^g) \cup ... \cup \mathcal{C}(F_n^g)$ is compatible.*

Proof. We may assume that $F_i(k) \subseteq [\omega \setminus k]^{<\omega}$ for every $k \in \omega$ and $i \leq n$. With the aid of Lemma 24, we can find an increasing $h \in \omega^\omega$ such that for every $m \in \omega$ the following holds: For every $E_0, ..., E_m \in \mathcal{D}$ and $A_i^1, ..., A_i^m \in F_m(i) \cap \mathcal{P}(h(m))$ for every $i \leq n$, we have that $E_0 \cap ... \cap E_m \cap \bigcap\limits_{i \leq n,\ j \leq m} A_i^j$ is non-empty. It is easy to see that h has the desired property. ■

We only need one more preliminary result:

Lemma 27 *Let $(\mathcal{B}, \mathcal{S})$ be a multiple $\mathfrak{d}$-pathway, $\langle M_0, ..., M_n \rangle$ a δ-increasing sequence of models in $\mathcal{S}$ and $m \leq n$ the least such that $\delta_m = \delta_n$. For every $i \leq n$, let $F_i : \omega \longrightarrow \mathcal{P}(\mathsf{fin}) \in M_i$ that is $\subseteq$-decreasing.*

If $\{\mathcal{C}(F_0^{f_{\delta_0}}), ..., \mathcal{C}(F_{m-1}^{f_{\delta_{m-1}}})\} \cup \{\mathcal{C}(F_i(k)) \mid m \leq i \leq n \wedge k \in \omega\}$ is compatible, then $\bigcup\limits_{i \leq n} \mathcal{C}(F_i^{f_{\delta_i}})$ is centered.

Proof. Define the relation $R \subseteq (\mathcal{P}(\mathsf{fin})^\omega)^{n+1} \times \omega^\omega$ where $R(G_0, ..., G_n, f)$ holds just in case one of the following conditions is met:

1. $\{\mathcal{C}(G_0^{f_{\delta_0}}), ..., \mathcal{C}(G_{m-1}^{f_{\delta_{m-1}}})\} \cup \{\mathcal{C}(G_i(k)) \mid m \leq i \leq n \wedge k \in \omega\}$ is not compatible.
2. $\mathcal{C}(G_0^{f_{\delta_0}}) \cup ... \cup \mathcal{C}(G_{m-1}^{f_{\delta_{m-1}}}) \cup \mathcal{C}(G_m^f) \cup ... \cup \mathcal{C}(G_n^f)$ is centered.

Since $f_{\delta_0}, ..., f_{\delta_{m-1}} \in M_n$, we conclude that $R \in M_n$. By Corollary 22, the first clause is an F_σ condition and the second one is G_δ, so R is both $F_{\sigma\delta}$ and $G_{\delta\sigma}$, hence it is Borel. Moreover, it is $\leq^*$-adequate by Lemma 26. The conclusion of the lemma follows since $(\mathcal{B}, \mathcal{S})$ is a multiple $\mathfrak{d}$-pathway. ■

We now prove the main result of the section:

Theorem 28 *If there is a multiple* $\mathfrak{d}$*-pathway, then there is a strong* P*-point.*

Proof. Let $(\mathcal{B}, \mathcal{S})$ be a multiple $\mathfrak{d}$-pathway. Define $D = \{\delta_M \mid M \in \mathcal{S}\}$ and for every $\delta \in D$, define W_δ as the set of all $\mathcal{C}(X)$ such that $X \subseteq \mathsf{fin}$ and there is $M \in \mathcal{S}$ such that $\delta_M \leq \delta$ and $X \in M$. By recursion over $\delta \in D$, we will define $\mathcal{U}_\delta$, $\mathcal{A}_\delta$ and $\mathcal{P}_\delta$ such that:

1. $\mathcal{U}_\delta \subseteq [\omega]^\omega$ while $\mathcal{P}_\delta$ and $\mathcal{A}_\delta$ are families of compact subsets of $\mathcal{P}(\omega)$.
2. $\mathcal{U}_\gamma \subseteq \mathcal{U}_\delta$ and $\mathcal{A}_\gamma \subseteq \mathcal{A}_\delta$ for every $\gamma \in D \cap \delta$.
3. $\mathcal{A}_\delta \subseteq W_\delta$.
4. $\mathcal{P}_\delta$ is the collection of all $\mathcal{C}(F^{f_\delta})$ such that $F : \omega \longrightarrow \mathcal{P}(\mathsf{fin})$ is $\subseteq$-decreasing and there is $M \in \mathcal{S}$ with the property that $\delta_M = \delta$, $F \in M$ and $\mathcal{C}(F(n)) \in \mathcal{A}_{<\delta}$ for every $n \in \omega$ (where $\mathcal{A}_{<\delta} = \bigcup\limits_{\gamma \in D \cap \delta} \mathcal{A}_\xi$).
5. $\mathcal{U}_\delta = \bigcup \{\mathcal{P}_\gamma \mid \gamma \in D \cap (\delta + 1)\}$.
6. $\mathcal{U}_\delta \cup \bigcup \mathcal{A}_\delta$ is centered.
7. $\mathcal{A}_\delta$ is maximal with respect to points 3 and 6.

Assume we are at step $\delta \in D$ and $\mathcal{U}_\gamma$, $\mathcal{A}_\gamma$ and $\mathcal{P}_\gamma$ have been defined for all $\gamma \in D \cap \delta$. In case δ is the minimum of D, we have $\mathcal{U}_\delta = \mathcal{P}_\delta = \emptyset$. Choose $\mathcal{A}_\delta \subseteq W_\delta$ any maximal compatible set of compact sets such that $\bigcup \mathcal{A}_\delta$ extends the Fréchet filter. Now consider the case where δ is not the least member of D. Note that $\mathcal{U}_\delta$ and $\mathcal{P}_\delta$ are defined from $\mathcal{A}_{<\delta}$, so we only need to find $\mathcal{A}_\delta$. Define $\mathcal{U}_{<\delta} = \bigcup\limits_{\xi \in D \cap \delta} \mathcal{U}_\xi$ and note that $\mathcal{U}_{<\delta} \cup \bigcup \mathcal{A}_{<\delta}$ is centered, since (by the recursion hypothesis) it is an increasing union of centered sets. We now prove the following:

Claim 29 $\mathcal{U}_\delta \cup \bigcup \mathcal{A}_{<\delta}$ *is centered.*

Let $B_0, ..., B_n \in \mathcal{U}_\delta \cup \bigcup \mathcal{A}_{<\delta}$, for every $i \leq n$, we find $M_i \in \mathcal{S}$ and $F_i \in M_i$ in the following way:

1. If $B_i \in \bigcup \mathcal{A}_{<\delta}$, let M_i such that $\delta_i = \delta_{M_i} < \delta$ and there is $X_i \in M_i$ such that $B_i \in \mathcal{C}\left(X_i\right)$ and $\mathcal{C}\left(X_i\right) \in \mathcal{A}_{\delta_i}$. Let $F_i : \omega \longrightarrow \mathcal{P}(\mathsf{fin})$ be the constant sequence with value X_i.
2. If $B_i \in \mathcal{U}_\delta$, let M_i such that $\delta_i = \delta_{M_i} \leq \delta$ and $F_i : \omega \longrightarrow \mathcal{P}(\mathsf{fin}) \in M_i$ such that $B_i \in \mathcal{C}(F_i^{f_{\delta_i}})$ and each $\mathcal{C}\left(F_i\left(k\right)\right)$ is in $\mathcal{A}_{<\delta_i}$.

It might be possible that for some $i \leq n$ both clauses apply. If that is the case, we can choose to follow either one of them. For each $i \leq n$, we have the following:

1. $B_i \in \mathcal{C}(F_i^{f_{\delta_i}})$.
2. $F_i : \omega \longrightarrow \mathcal{P}(\mathsf{fin}) \in M_i$ and is $\subseteq$-decreasing.
3. $\mathcal{C}\left(F_i\left(k\right)\right) \in \mathcal{A}_{<\delta}$ for all $k \in \omega$.

By taking a reenumeration and possibly picking more elements of $\mathcal{U}_\delta \cup \bigcup \mathcal{A}_{<\delta}$, we may assume that $\langle M_0, ..., M_n \rangle$ is δ-increasing and $\delta_n = \delta$. Let $m \leq n$ be the least such that $\delta_m = \delta$. We claim that $H = \mathcal{C}(F_0^{f_{\delta_0}}) \cup ... \cup \mathcal{C}(F_{m-1}^{f_{\delta_{m-1}}}) \cup \bigcup \left\{\mathcal{C}\left(F_i\left(k\right)\right) \mid m \leq i \leq n \wedge k \in \omega\right\}$ is contained in $\mathcal{U}_{<\delta} \cup \bigcup \mathcal{A}_{<\delta}$. Pick $i \leq n$. We have the following cases:

1. If $B_i \in \bigcup \mathcal{A}_{<\delta}$, we have that $\mathcal{C}\left(X_i\right) \in \mathcal{A}_{<\delta}$ and $F_i^{\delta_i} = X_i$.
2. If $B_i \in \mathcal{U}_\delta$ and $i < m$, then $F_i : \omega \longrightarrow \mathcal{A}_{<\delta_i}$ which implies that $F_i^{f_{\delta_i}} \in \mathcal{U}_{\delta_i} \subseteq \mathcal{U}_\delta$.
3. If $B_i \in \mathcal{U}_\delta$ and $m \leq i$, there is nothing to do since we already noted that $\mathcal{C}\left(F_i\left(k\right)\right) \in \mathcal{A}_{<\delta}$ for all $k \in \omega$.

Recall that $\mathcal{U}_{<\delta} \cup \bigcup \mathcal{A}_{<\delta}$ is centered, so H is centered as well. We are now in position to invoke Lemma 27 and conclude that $\bigcup\limits_{i \leq n} \mathcal{C}(F_i^{f_{\delta_i}})$ is centered. Since $B_i \in \mathcal{C}(F_i^{f_{\delta_i}})$, this finishes the proof of the claim. We use Zorn's Lemma and find $\mathcal{A}_\delta \subseteq W_\delta$ extending $\mathcal{A}_{<\delta}$ such that $\mathcal{U}_\delta \cup \bigcup \mathcal{A}_\delta$ is centered and it is maximal with these properties.

After completing the recursion, define $\mathcal{U} = \bigcup\limits_{\delta \in D} \mathcal{U}_\delta$ and $\mathcal{A} = \bigcup\limits_{\delta \in D} \mathcal{A}_\delta$. We will now prove the following:

Claim 30

1. *$\mathcal{U} \cup \bigcup \mathcal{A}$ is centered.*

2. $\mathcal{U} = \bigcup \mathcal{A}$.

3. $\mathcal{U}$ *is an ultrafilter.*

4. If $X \in (\mathcal{U}^{<\omega})^{+}$, *then* $\mathcal{C}(X) \in \mathcal{A}$.

5. $\mathcal{U}$ *is a strong* P*-point.*

Since $\mathcal{U} \cup \bigcup \mathcal{A}$ is equal to the union of $\mathcal{U}_\delta \cup \bigcup \mathcal{A}_\delta$ and these are increasing and centered, it follows that $\mathcal{U} \cup \bigcup \mathcal{A}$ is centered. We prove that $\mathcal{U} \subseteq \bigcup \mathcal{A}$. Let $B \in \mathcal{U}$, $\delta \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \delta$ and $B \in M$. Let $X = [B]^1$ and recall that $\mathcal{C}(X) = \{A \subseteq \omega \mid B \subseteq A\}$ and $\mathcal{C}(X) \in W_\delta$. Clearly $\mathcal{U}_\delta \cup \mathcal{A}_\delta \cup \mathcal{C}(X)$ is centered, so by the maximality of $\mathcal{A}_\delta$, we get that $\mathcal{C}(X) \in \mathcal{A}_\delta$, hence $B \in \bigcup \mathcal{A}$. Now, take $A \in \bigcup \mathcal{A}$. Find $\delta \in D$ and $Y \in W_\delta$ such that $A \in \mathcal{C}(Y)$. Let F be the constant sequence with value Y. We now find $\gamma \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \gamma$ and $\delta, F \in M$. We have that $\mathcal{C}(Y) = \mathcal{C}\left(F^{f_\gamma}\right) \subseteq \mathcal{U}_\gamma$.

We now prove that $\mathcal{U}$ is an ultrafilter. Let $A, B, C, E \subseteq \omega$ such that $A, B \in \mathcal{U}$ and $A \subseteq C$. Find $\delta \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \delta$ and $A, B, C, E \in M$. Let $X = [A \cap B]^1$, which is in W_δ. Since $\mathcal{U} = \bigcup \mathcal{A}$ is centered, it follows that $\mathcal{U}_\delta \cup \mathcal{C}(X) \cup \bigcup \mathcal{A}_\delta$ is centered. By the maximality of $\mathcal{A}_\delta$, we get that $\mathcal{C}(X) \in \mathcal{A}_\delta$. Since $A \cap B$, $C \in \mathcal{C}(X)$, it follows that $A \cap B, C \in \mathcal{U}$. Moreover, we know that either $\mathcal{U}_\delta \cup \bigcup \mathcal{A}_\delta \cup \mathcal{C}([E]^1)$ is centered or $\mathcal{U}_\delta \cup \bigcup \mathcal{A}_\delta \cup \mathcal{C}([\omega \setminus E]^1)$ is centered, so by the maximality of $\mathcal{A}_\delta$, we obtain that $\mathcal{C}\left([E]^1\right) \in \mathcal{A}_\delta$ or $\mathcal{C}\left([\omega \setminus E]^1\right) \in \mathcal{A}_\delta$. Finally, recall that in the first step of the recursion we made sure that $\mathcal{U}$ contains the Fréchet filter.

Let $X \in (\mathcal{U}^{<\omega})^{+}$. Since $\mathcal{U}$ is an ultrafilter, we know that $\mathcal{C}(X) \subseteq \mathcal{U}$ by Lemma 20. Find $\delta \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \delta$ and $X \in M$. Clearly $\mathcal{U}_\delta \cup \mathcal{C}(X) \cup \bigcup \mathcal{A}_\delta$ is centered, so by the maximality of $\mathcal{A}_\delta$ we conclude that $\mathcal{C}(X) \in \mathcal{A}_\delta$.

It only remains to prove that $\mathcal{U}$ is a strong P-point. We use Theorem 18. Let $F : \omega \longrightarrow (\mathcal{U}^{<\omega})^{+}$ be $\subseteq$-decreasing. By the previous point of the claim, for every $n \in \omega$, we can find $\delta_n \in D$ such that $\mathcal{C}(F(n)) \in \mathcal{A}_{\delta_n}$. We now choose $\delta \in D$ and $M \in \mathcal{S}$ with $\delta_M = \delta$ such that $F \in M$ and $\delta_n < \delta$ for every $n \in \omega$. In this way, $\mathcal{C}(F^{f_\delta}) \in \mathcal{P}_\delta$ and then it is contained in $\mathcal{U}$. We conclude that $F^{f_\delta} \in (\mathcal{U}^{<\omega})^{+}$ by applying Lemma 20 once again. ■

7 A Gruff ultrafilter from a multiple $\mathfrak{d}$-pathway

We now turn our attention to ultrafilters on the rational numbers[5], which we denote by $\mathbb{Q}$. A particularly nice combinatorial feature of the rational numbers

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[5] When discussing the rational numbers, we always assume it is equipped with its usual topology.

(which is not longer true for the the real numbers), is that it has no *Bernstein subsets*. In other words, if we split the rational numbers into two pieces, then at least one of them contains a perfect subset[6]. This property motivates the following definition:

Definition 31 *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{Q}$. We say that $\mathcal{U}$ is a* Gruff ultrafilter *if it has a base of perfect sets.*

Gruff ultrafilters were introduced by van Douwen in [53] while studying $\beta\mathbb{Q}$ (the Čech-Stone compactification of $\mathbb{Q}$) who asked if there are such ultrafilters. He was able to to prove they exist in case that $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ and later Copláková and Hart in [16] obtained the same conclusion under $\mathfrak{b} = \mathfrak{c}$. Both of these results were improved by Fernández-Bretón and Hrušák in [23] where a Gruff ultrafilter is obtained from $\mathfrak{d} = \mathfrak{c}$. Fernández-Bretón and Hrušák were also interested in the existence of Gruff ultrafilters in the random model and constructed one using a pathway (see also [22]). Unfortunately, as discussed before, it is not known if pathways exist in the random model. We will now build a Gruff ultrafilter from a multiple $\mathfrak{d}$-pathway. Our approach takes inspiration from [23] ([22]) and [19]. Apart from the papers already mentioned, the reader may consult [**?**], [14] and [40] to learn more about Gruff ultrafilters and [10] for more on combinatorics of scattered subsets of the rationals.

Denote by scatt the ideal of scattered subsets of $\mathbb{Q}$ and by bscatt the ideal generated by both the scattered sets and the bounded (from above) subsets of $\mathbb{Q}$. When constructing a Gruff ultrafilter, it is sometimes more convenient to build one which has a base of perfect unbounded subsets (see [23]). Given $A \subseteq \mathbb{Q}$, the *crowded kernel of* A (denoted by $K(A)$) is the union of all the crowded subsets of A. The following are simple remarks regarding this notion:

Lemma 32 *Let $A \subseteq \mathbb{Q}$.*

1. *If $A \in \mathsf{scatt}^+$, then $K(A)$ is crowded.*
2. *$K(A)$ is the largest crowded subset contained in A.*
3. *If $A \in \mathsf{bscatt}^+$, then $K(A)$ is crowded and unbounded.*
4. *The symmetric difference between A and $K(A)$ is in scatt.*
5. *If $\mathcal{F}$ is a filter on $\mathbb{Q}$ such that $\mathsf{scatt}^* \subseteq \mathcal{F}$ and $A \in \mathcal{F}$, then $K(A) \in \mathcal{F}$.*

We will now recall a very useful notion from [23]. From now on, fix an enumeration $\mathbb{Q} = \{q_n \mid n \in \omega\}$. For $q \in \mathbb{Q}$ and $r > 0$, we denote by $B(q,r)$ the open ball of q with radius r. Given a function $f \in \omega^\omega$ and $n \in \omega$, denote $J_f(n) =$

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[6] Recall that perfect sets are non-empty.

$B(q_n, \frac{\sqrt{2}}{k})$, where k is the least natural number such that $q_m \notin B(q_n, \frac{\sqrt{2}}{k})$ for every $m \leq f(n)$ such that $m \neq n$ (the purpose of $\sqrt{2}$ is only to ensure that $J_f(n)$ is a clopen subset of $\mathbb{Q}$, evidently we can use any other positive irrational number). Intuitively, we are making $J_f(n)$ as large as possible with the restriction that it can not include any q_m for which $m \leq f(n)$ and $m \neq n$.

Definition 33 *Let $X \subseteq \mathbb{Q}$ and $f \in \omega^\omega$. Define $X(f) = \mathbb{Q} \setminus \bigcup\limits_{n \notin X} J_f(n)$.*

The following two results can be found in [23]:

Lemma 34 *Let $X, Y \subseteq \mathbb{Q}$ and $f, g \in \omega^\omega$.*

1. *$X(f)$ is a closed subset of X.*
2. *If $X \subseteq Y$, then $X(f) \subseteq Y(f)$.*
3. *$X(f) \cap Y(f) = (X \cap Y)(f)$.*
4. *If $f \leq g$, then $X(f) \subseteq X(g)$.*

Proposition 35 *Let $X \subseteq \mathbb{Q}$ be crowded and unbounded. There is $h \in \omega^\omega$ such that for every increasing $g \in \omega^\omega$, if $g \nleq^* h$, then $X(g)$ is perfect and unbounded.*

Using Proposition 35, for every $X \subseteq \mathbb{Q}$ that is crowded and unbounded, we fix a function $h_X \in \omega^\omega$ such that for every increasing $g \in \omega^\omega$, if $g \nleq^* h_X$, then $X(g)$ is perfect and unbounded.

For a rational number $q \in \mathbb{Q}$, define $\langle q \rangle_0 = \{A \subseteq \mathbb{Q} \mid q \notin A\}$ and $\langle q \rangle_1 = \{A \subseteq \mathbb{Q} \mid q \in A\}$. We endow $\mathcal{P}(\mathbb{Q})$ with the topology that has as a subbase the family $\{\langle q \rangle_i \mid q \in \mathbb{Q} \wedge i < 2\}$. Evidently, $\mathcal{P}(\mathbb{Q})$ with this topology is homeomorphic to $\mathcal{P}(\omega)$. The following lemma refers to this topology.

Lemma 36

1. *The collection of all crowded unbounded subsets of $\mathbb{Q}$ is G_δ.*
2. *The ideal bscatt is coanalytic.*

We now prove the following:

Lemma 37 *Let $(\mathcal{B}, \mathcal{S})$ be a multiple $\mathfrak{d}$-pathway and $\langle M_0, ..., M_n \rangle$ a δ-increasing sequence of models from $\mathcal{S}$. Let $m \leq n$ be the first one such that $\delta_m = \delta_n$. For every $i \leq n$, pick $X_i \in M_i \cap$ bscatt^+.*

If $X_0(f_{\delta_0}) \cap ... \cap X_{m-1}(f_{\delta_{m-1}}) \cap X_m \cap ... \cap X_n \in \mathsf{bscatt}^+$, then $\bigcap\limits_{i \leq n} X_i(f_{\delta_i}) \in$ bscatt^+.

Proof. Define the relation $R \subseteq \mathcal{P}(\mathbb{Q})^{n+1} \times \omega^\omega$ where $R(Y_0, ..., Y_n, f)$ holds in case one of the following conditions is met:

1. $Y_0(f_{\delta_0}) \cap ... \cap Y_{m-1}\left(f_{\delta_{m-1}}\right) \cap Y_m \cap ... \cap Y_n \in \mathsf{bscatt}$.
2. $Y_0(f_{\delta_0}) \cap ... \cap Y_{m-1}\left(f_{\delta_{m-1}}\right) \cap Y_m(f) \cap ... \cap Y_n(f) \in \mathsf{bscatt}^+$.

Since $f_{\delta_0}, ..., f_{\delta_{m-1}} \in M_n$, we conclude that $R \in M_n$. By Lemma 36 (or rather by its proof), the first clause is coanalytic and the second one is analytic, so R is projective. We now prove that it is $\leq^*$-adequate. Let $Y_0, ..., Y_n \subseteq \mathbb{Q}$, if $Z = Y_0(f_{\delta_0}) \cap ... \cap Y_{m-1}\left(f_{\delta_{m-1}}\right) \cap Y_m \cap ... \cap Y_n \in \mathsf{bscatt}$, there is nothing to do, so assume otherwise. We claim that $h_{K(Z)}$ is an R-control for $(Y_0, ..., Y_n)$. To see this, pick $g \in \omega^\omega$ increasing such that $g \nleq^* h_{K(Z)}$. We now have the following:

$$\begin{array}{rcl} K(Z)(g) & \subseteq & Z(g) \\ & = & \bigcap\limits_{i<m} (Y_i(f_{\delta_i}))(g) \cap \bigcap\limits_{m \leq i \leq n} Y_i(g) \\ & \subseteq & \bigcap\limits_{i<m} (Y_i(f_{\delta_i})) \cap \bigcap\limits_{m \leq i \leq n} Y_i(g) \end{array}$$

By Proposition 35, we know that $K(Z)(g)$ is perfect and unbounded, so $\bigcap\limits_{i<m} (Y_i(f_{\delta_i})) \cap \bigcap\limits_{m \leq i \leq n} Y_i(g)$ is not in bscatt. The conclusion of the lemma follows since $(\mathcal{B}, \mathcal{S})$ is a multiple $\mathfrak{d}$-pathway. ■

We now proceed to prove the main result of the section.

Theorem 38 *If there is a multiple $\mathfrak{d}$-pathway, then there is a Gruff ultrafilter.*

Proof. Let $(\mathcal{B}, \mathcal{S})$ be a multiple $\mathfrak{d}$-pathway. Define $D = \{\delta_M \mid M \in \mathcal{S}\}$ and for $\delta \in D$, denote $W_\delta = \bigcup\limits_{\delta_M \leq \delta} M \cap \mathcal{P}(\mathbb{Q})$. By recursion over $\delta \in D$, we shall find $\mathcal{U}_\delta, \mathcal{A}_\delta$ and $\mathcal{P}_\delta$ with the following properties:

1. $\mathcal{U}_\delta$ and $\mathcal{P}_\delta$ are families of perfect and unbounded sets.
2. $\mathcal{A}_\delta \subseteq W_\delta \cap$ bscatt^+.
3. If $\gamma \in D \cap \delta$, then $\mathcal{A}_\gamma \subseteq \mathcal{A}_\delta$ and $\mathcal{U}_\gamma \subseteq \mathcal{U}_\delta$.
4. $\mathcal{P}_\delta$ is the family of all $X(f_\delta)$ for which there is $M \in \mathcal{S}$ for which $\delta_M = \delta$ and $X \in M \cap \mathcal{A}_{<\delta}$ is crowded (where $\mathcal{A}_{<\delta} = \bigcup\limits_{\gamma \in D \cap \delta} \mathcal{A}_\gamma$).
5. $\mathcal{U}_\delta = \bigcup \{\mathcal{P}_\xi \mid \xi \in D \cap (\delta + 1)\}$.
6. $\mathcal{U}_\delta \cup \mathcal{A}_\delta$ generates a filter contained in bscatt^+.
7. $\mathcal{A}_\delta$ is maximal with respect to points 2 and 6.

Before starting the construction, note that $\mathcal{A}_\delta$ will have the following property: If $B \in \mathcal{A}_\delta$, then $K(B) \in \mathcal{A}_\delta$. To see this, let $M \in \mathcal{S}$ such that $B \in M$ and $\delta_M \leq \delta$. Since $B \in M$, we get that $K(B) \in M$, hence $K(B) \in W_\delta$. Call $\mathcal{F}$ the filter generated by $\mathcal{U}_\delta \cup \mathcal{A}_\delta \cup$ bscatt^*. Since $B \in \mathcal{F}$, then $K(B) \in \mathcal{F}$ (see Lemma 32) which implies that $\mathcal{U}_\delta \cup \mathcal{A}_\delta \cup \{K(B)\}$ generates a filter contained in bscatt^+. By the maximality of $\mathcal{A}_\delta$, we conclude that $K(B) \in \mathcal{A}_\delta$.

Assume we are at step $\delta \in D$ and $\mathcal{U}_\gamma$, $\mathcal{A}_\gamma$ and $\mathcal{P}_\gamma$ have been defined for all $\gamma \in D \cap \delta$. In case δ is the minimum of D, we have $\mathcal{U}_\delta = \mathcal{P}_\delta = \emptyset$. Choose $\mathcal{A}_\delta \subseteq W_\delta \cap$ scatt^+ any maximal centered set extending the filter of cobounded subsets of $\mathbb{Q}$. Now consider the case where δ is not the least member of D. Note that $\mathcal{U}_\delta$ and $\mathcal{P}_\delta$ are defined from $\mathcal{A}_{<\delta}$, so we only need to find $\mathcal{A}_\delta$, but first we need to prove that both $\mathcal{U}_\delta$ and $\mathcal{P}_\delta$ consist of perfect and unbounded sets. It is enough to prove it for $\mathcal{P}_\delta$. Let $M \in \mathcal{S}$ with $\delta_M = \delta$ and $X \in M \cap \mathcal{A}_{<\delta}$ is crowded. Moreover, X is also unbounded since $\mathcal{A}_{<\delta}$ extends the filter of cobounded sets. We need to prove that $X(f_\delta)$ is perfect and unbounded. Since $X \in M$, it follows that h_K is also in M. Since f_δ is unbounded over M, we get that $X(f_\delta)$ is perfect and unbounded by Proposition 35.

Define $\mathcal{U}_{<\delta} = \bigcup\limits_{\xi \in D \cap \delta} \mathcal{U}_\xi$ and note that $\mathcal{U}_{<\delta} \cup \mathcal{A}_{<\delta}$ generates a filter contained in bscatt^+ by the recursion hypothesis. We now prove the following:

Claim 39 *$\mathcal{U}_\delta \cup \mathcal{A}_{<\delta}$ generates a filter contained in bscatt^+.*

Let $B_0, ..., B_n \in \mathcal{U}_\delta \cup \mathcal{A}_{<\delta}$, for each $i \leq n$, we find $M_i \in \mathcal{S}$ and $X_i \in M_i$ in the following way:

1. In case $B_i \in \mathcal{A}_{<\delta}$, choose M_i for which $\delta_i = \delta_{M_i} < \delta$ and $B_i \in M_i$. Let $X_i = K(B_i)$.
2. If $B_i \in \mathcal{U}_\delta$, choose M_i with $\delta_i = \delta_{M_i} \leq \delta$ and $X_i \in M_i \cap \mathcal{A}_{<\delta_i}$ crowded such that $B_i = X_i(f_{\delta_i})$.

It might be possible that for some $i \leq n$ both clauses apply. If that is the case, we use either of them. For each $i \leq n$, we have the following:

1. $X_i \in M_i \cap \mathcal{A}_{<\delta}$ and is both perfect and unbounded.
2. $X_i(f_{\delta_i}) \subseteq B_i$.

By taking a reenumeration and possibly picking more elements of $\mathcal{U}_\delta \cup \mathcal{A}_{<\delta}$, we may assume that $\langle M_0, ..., M_n \rangle$ is δ-increasing and $\delta_n = \delta$. Let $m \leq n$ be the least such that $\delta_m = \delta$. We claim that $X_0(f_{\delta_0}), ..., X_{m-1}(f_{\delta_{m-1}}), X_m, ..., X_n \in \mathcal{U}_{<\delta} \cup \mathcal{A}_{<\delta}$. Pick $i \leq n$. We have the following cases:

1. If $B_i \in \mathcal{A}_{<\delta}$, then $X_i = K(B_i) \in \mathcal{A}_{\delta_i}$, so $X_i(f_{\delta_i}) \in \mathcal{U}_{\delta_i}$.
2. If $B_i \in \mathcal{U}_\delta$ and $i < m$, then $X_i \in \mathcal{A}_{\delta_i}$, so $X_i(f_{\delta_i}) \in \mathcal{U}_{\delta_i}$.
3. If $B_i \in \mathcal{U}_\delta$ and $m \leq i$, we already knew that $X_i \in \mathcal{A}_{<\delta}$.

Recall that $\mathcal{U}_{<\delta} \cup \mathcal{A}_{<\delta}$ generates a filter contained in bscatt^+, so $X_0(f_{\delta_0}) \cap ... \cap X_{m-1}(f_{\delta_{m-1}}) \cap X_m \cap ... \cap X_n \in \mathsf{bscatt}^+$. We are now in position to call Lemma 37 and conclude that $\bigcap\limits_{i\leq n} X_i(f_{\delta_i}) \in \mathsf{bscatt}^+$. Since $X_i(f_{\delta_i}) \subseteq B_i$, this finishes the proof of the claim. We now invoke Zorn's Lemma and find $\mathcal{A}_\delta \subseteq W_\delta$ extending $\mathcal{A}_{<\delta}$ as desired.

After completing the recursion, define $\mathcal{A} = \bigcup\limits_{\delta\in D} \mathcal{A}_\delta$ and $\mathcal{U}$ as the set of all $B \subseteq \mathbb{Q}$ for which there is $U \in \bigcup\limits_{\delta\in D} \mathcal{U}_\delta$ for which $B \subseteq U$. We will now prove the following:

Claim 40

1. *$\mathcal{U} \cup \mathcal{A}$ generates a filter contained in bscatt^+.*
2. *$\mathcal{U} = \mathcal{A}$.*
3. *$\mathcal{U}$ is an ultrafilter.*
4. *$\mathcal{U}$ is a Gruff ultrafilter.*

The first point is easy, we now prove that $\mathcal{U} = \mathcal{A}$. We will first see that $\mathcal{U} \subseteq \mathcal{A}$. Let $U \in \mathcal{U}$, find $\delta \in D$ and $M \in \mathcal{S}$ such that there is $B \in \mathcal{U}_\delta$ for which $B \subseteq U$ and $B, U \in M$. It is clear that $\mathcal{U}_\delta \cup \mathcal{A}_\delta \cup \{U\}$ generates a filter contained in bscatt^+. Since $U \in W_\delta$, it follows by the maximality of $\mathcal{A}_\delta$ that $U \in \mathcal{A}_\delta$. We now prove that $\mathcal{A} \subseteq \mathcal{U}$. Let $A \in \mathcal{A}_\delta$ for some $\delta \in D$. We now choose $\gamma \in D$ and $M \in \mathcal{S}$ such that $\delta_M = \gamma$ and $A, \delta \in M$. Since $A \in \mathcal{A}_{<\gamma}$, it follows that $K = K(A)$ is also in $\mathcal{A}_{<\gamma}$. In this way, $K(f_\gamma) \in \mathcal{U}_\gamma$ and then $A \in \mathcal{U}$.

The proof that $\mathcal{U}$ is an ultrafilter is similar to arguments used in the proof of Theorems 14 and 28. Finally, it is Gruff since $\bigcup\limits_{\delta\in D} \mathcal{U}_\delta$ is a base of $\mathcal{U}$ consisting of perfect sets. ■

8 Combinatorics of elementary submodels

Our current goal now is to prove that multiple $\mathfrak{d}$-pathways may consistently exist. We will derive several combinatorial results concerning countable elementary submodels which are the new insight for the main theorems of the paper.

The results in this section do not directly refer to pathways and may be of independent interest.

Fix a regular cardinal $\kappa > \mathfrak{c}$ and $\trianglelefteq$ a well order of $\mathsf{H}(\kappa)$. The following result is well-known, we prove it for completeness.

Lemma 41 (CH) *Assume that* $M, N \in \mathsf{Sub}(\kappa)$ *and* $\delta_M \leq \delta_N$. $\mathsf{H}(\omega_1) \cap M \subseteq \mathsf{H}(\omega_1) \cap N$.

Proof. Let $g : \omega_1 \longrightarrow \mathsf{H}(\omega_1)$ be the $\trianglelefteq$-minimal bijection, so it is in both M and N. Since $\mathsf{H}(\omega_1) \cap M = g[\delta_M]$ and $\mathsf{H}(\omega_1) \cap N = g[\delta_N]$, the result follows. ■

We now extend the previous lemma:

Lemma 42 (CH) *Let* $M, N \in \mathsf{Sub}(\kappa)$ *with* $\delta_M \leq \delta_N$. *If* $A \in M \cap N$ *and is a countable subset of the ordinals, then* $\mathcal{P}(A) \cap M \subseteq N$.

Proof. Let $B \in \mathcal{P}(A) \cap M$ and $\gamma = \mathsf{OT}(A) < \omega_1$. Denote by $e : A \longrightarrow \gamma$ the (unique) order isomorphism. Since $A \in M \cap N$, it follows that $e \in M \cap N$. Clearly $e[B] \in \mathsf{H}(\omega_1) \cap M$, so $e[B] \in N$ by Lemma 41. Since e^{-1} is also in N, it follows that $B \in N$. ■

If A is a set of ordinals, we denote by $\overline{A}$ its closure in the usual order topology. It is easy to see that the closure of a countable set is also countable. In particular, if $M \in \mathsf{Sub}(\kappa)$ and $A \in M$ is a countable set of ordinals, then $\overline{A} \subseteq M$. For us, *a partition* P is simply a collection of pairwise disjoint sets ($\emptyset \in P$ is allowed) and a partition for a set A is a partition whose union is A.

Lemma 43 (CH) *Let* $M, N \in \mathsf{Sub}(\kappa)$ *with* $\delta_M \leq \delta_N$, $n \in \omega$ *and* $A, B \in [\omega_n]^{\leq\omega}$ *such that* $A \in M$ *and* $B \in N$.

1. *There is a partition* $P = \{A_0, A_1\} \in M$ *of* A *such that* $A_0 \in N$ *and* $A_1 \cap B = \emptyset$.
2. $A \cap B \in N$.

Proof. Note that the second point is a trivial consequence of the first. We prove the first point by induction over n. For $n \leq 1$, we have that $A \in N$ by Lemma 41, so we simply let $A_0 = A \cap B$ and $A_1 = A \setminus B$. Assume the lemma is true for n, we prove that it is true for $n+1$ as well. Denote $\beta = \bigcup \overline{A \cap B} + 1$ and note that $\beta \in M \cap N$. Fix $h : \beta \longrightarrow \omega_n$ be the $\trianglelefteq$-least injective function, clearly $h \in M \cap N$. Define $C = h[A \cap \beta]$ and $D = h[B \cap \beta]$, we have that $C \in M$ and $D \in N$. We can now apply the inductive hypothesis and find a partition $\{C_0, C_1\} \in M$ of C such that $C_0 \in N$ and $C_1 \cap D = \emptyset$. Letting $A_0 = h^{-1}(C_0)$ and $A_1 = A \setminus h^{-1}(C_0)$, we have that $\{A_0, A_1\} \in M$ and $A_0 \in N$. We only need to prove that $A_1 \cap B = \emptyset$. Assume that there is $\alpha \in A_1 \cap B$, it follows that $\alpha < \beta$ and $h(\alpha) \in C \setminus C_0 = C_1$. In this way, $h(\alpha) \in C_1 \cap D$, but this is a contradiction since $C_1 \cap D = \emptyset$. ■

The following definition plays a similar role to the finite partitions used in [19].

Definition 44 *Let $\langle M_0, ..., M_n\rangle$ be a δ-increasing sequence of models from $\mathsf{Sub}(\kappa)$. We say that $\mathcal{P} = \langle P_i \mid i \leq n\rangle$ is* a coherent sequence of partitions *for $\langle M_0, ..., M_n\rangle$ if for every $i \leq n$, the following conditions hold:*

1. *$P_i \in M_i$ and is a finite partition consisting of countable subsets of the ordinals.*
2. *If $i < j$, then $P_i \cap M_j \subseteq P_j$.*
3. *$\bigcup_{j \leq n} P_j$ is a partition.*

The next lemma illustrates how to construct non-trivial coherent sequences of partitions.

Lemma 45 (CH) *Let $\langle M_0, ..., M_n\rangle$ be a δ-increasing sequence of models from $\mathsf{Sub}(\kappa)$, $l \in \omega$ and $A_i \in M_i \cap [\omega_l]^{\leq\omega}$ for every $i \leq n$. There is $\mathcal{P} = \langle P_i \mid i \leq n\rangle$ a coherent sequence of partitions for $\langle M_0, ..., M_n\rangle$ such that $A_i \subseteq \cup P_i$ for every $i \leq n$.*

Proof. We proceed by induction over n. For $n = 0$ we simple take $P_0 = \{A_0\}$ and we are done. Assume the lemma is true for n, we will see it is true for $n+1$ as well. Find $\langle P_i \mid i \leq n\rangle$ a coherent sequence of partitions for $\langle M_0, ..., M_n\rangle$ such that $A_i \subseteq \cup P_i$ for every $i \leq n$.

Claim 46 *There is $\langle R_i \mid i \leq n\rangle$ a coherent sequence of partitions for $\langle M_0, ..., M_n\rangle$ with the following properties:*

1. *$\cup P_j \subseteq \cup R_j$ for every $j \leq n$.*
2. *For every $B \in \bigcup_{i \leq n} R_i$, we have that $B \cap A_{n+1} \in M_{n+1}$.*

Denote $P = \bigcup_{i \leq n} P_i$. Pick $C \in P$ and $i \leq n$ the minimal one for which $C \in P_i$. We can apply Lemma 43 and find a partition $\{C_0, C_1\} \in M_i$ of C such that $C_0 \in M_n$ and $A_{n+1} \cap C_1 = \emptyset$. Define $R = \{C_u \mid C \in P \wedge u \in 2\}$ and for every $i \leq n$, denote $R_i = R \cap M_i$. We claim that $\langle R_i \mid i \leq n\rangle$ is as desired. Clearly R is a partition, if $B \in R$ then $B \cap A_{n+1} \in M_{n+1}$ and if $i < j$, then $R_i \cap M_j = R \cap M_i \cap M_j = R_j \cap M_i \subseteq R_j$. It remains to prove that $\cup P_j \subseteq \cup R_j$ for every $j \leq n$. Let $C \in P_j$ and find $i \leq j$ the first one for which $C \in P_i$. Note that $C \in M_i \cap M_j$ and $\delta_i \leq \delta_j$, so by Lemma 42, we know that $\mathcal{P}(C) \cap M_i \subseteq M_j$, which entails that $C_0, C_1 \in M_j$, hence $C_0, C_1 \in R_j$. This finishes the proof of the claim.

Fix $\langle R_i \mid i \leq n \rangle$ as above. Define $R_{n+1} = (M_n \cap \bigcup\limits_{i \leq n} R_i) \cup \{A_{n+1} \setminus \bigcup\limits_{i \leq n} R_i\}$. It is easy to see that $\langle R_i \mid i \leq n+1 \rangle$ is as desired. ■

We now prove the final result of this section, which will enable us to transfer certain names across elementary submodels.

Proposition 47 (CH) *Let $\langle M_0, ..., M_n \rangle$ be a δ-increasing sequence of models from $\mathsf{Sub}(\kappa)$, $l \in \omega$ and $\mathcal{P} = \langle P_i \mid i \leq n \rangle$ a coherent sequence of partitions for $\langle M_0, ..., M_n \rangle$ where $P_i \subseteq [\omega_l]^{\leq \omega}$. There is a bijection $\triangle : \omega_l \longrightarrow \omega_l$ with the following properties:*

1. *If $A \in \bigcup\limits_{i \leq n} P_i$, then $\triangle[A] \in M_n$ and $\triangle \restriction A$ is order preserving.*
2. *If $A \in P_n$, then $\triangle \restriction A$ is the identity mapping.*

Proof. Take an enumeration $\{A_0, ..., A_m\}$ of all elements of $\bigcup\limits_{i<n} P_i$ that are not in P_n and denote $Y = \bigcup \bigcup\limits_{i \leq n} P_i$. Choose $\beta < \delta_n$ such that $\mathsf{OT}(Y) < \beta$ and $Y \cap \omega_1 \subseteq \beta$. For each $k \leq m$, denote $\gamma_k = \mathsf{OT}(A_k)$ and $e_k : A_k \longrightarrow \gamma_k$ the (unique) order isomorphism. Define $\triangle_k : A_k \longrightarrow \omega_1$ where $\triangle_k(\alpha) = \beta(k+1) + e_k(\alpha)$ and note that $\mathsf{im}(\triangle_k) = [\beta(k+1), \beta(k+1) + \gamma_k)$, which belongs to M_n since $\beta, \gamma_k \in M_n$. Moreover, note that if $k \neq r$, then $\mathsf{im}(\triangle_k) \cap \mathsf{im}(\triangle_r) = \emptyset$ and if $A \in P_n$, then $A \cap \omega_1 \subseteq \beta$, while $\mathsf{im}(\triangle_k) \cap \beta = \emptyset$ for every $k \leq m$, so $\mathsf{im}(\triangle_k)$ and A are disjoint. In this way, we can extend $\bigcup\limits_{k \leq m} \triangle_k$ to a permutation of ω_l that fixes every element of P_n. ■

9 Forcing multiple $\mathfrak{d}$-pathways

We now apply the results from the previous section to establish the existence of multiple $\mathfrak{d}$-pathways in certain random and Cohen models. Having that goal in mind, we need to introduce a couple of definitions.

Definition 48 *Let $\mathbb{P}$ be a partial order and $\dot{a}, \dot{b}$ two $\mathbb{P}$-names. We say that $\dot{a}$ and $\dot{b}$* are equivalent *if $\mathbb{P} \Vdash$ "$\dot{a} = \dot{b}$".*

We now introduce the main definition of this Section.

Definition 49 *Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$* has the transformation property *if for every large enough regular κ, $\langle M_0, ..., M_n \rangle$ a δ-increasing sequence of models in $\mathsf{Sub}(\kappa)$ where $\mathbb{P} \in M_i$ for every $i \leq n$, $\dot{a}_0 \in M_0, ..., \dot{a}_n \in M_n$ that are $\mathbb{P}$-names for subsets of ω, there is an automorphism $H : \mathbb{P} \longrightarrow \mathbb{P}$ such that:*

1. *$H(\dot{a}_n) = \dot{a}_n$.*

2. *For every $i < n$, we have that $H(\dot{a}_i)$ is equivalent to a $\mathbb{P}$-name that is in M_n.*

Note that the automorphism H is not required to be in M_n. We recommend that the reader to consult Section 3 as we will be using the notation and results from there.

Theorem 50 (CH) *Let $\mathbb{P}$ be a ccc forcing that does not add dominating reals and has the transformation property. $\mathbb{P}$ forces that there is a multiple $\mathfrak{d}$-pathway.*

Proof. Choose $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_1\} \subseteq \omega^\omega$ a scale of increasing functions, κ a large enough regular cardinal such that $\mathbb{P} \in \mathsf{H}(\kappa)$. Define $\mathcal{S}_0 = \{M \mid M \in \mathsf{Sub}(\kappa) \wedge \mathcal{B}, \mathbb{P} \in M\}$, which is stationary. We claim that if $G \subseteq \mathbb{P}$ is a generic filter, then in $V[G]$ we will have that $(\mathcal{B}, \mathcal{S})$ is a multiple $\mathfrak{d}$-pathway, where $\mathcal{S} = \{M[G] \mid M \in \mathcal{S}\}$. Since $\mathbb{P}$ is ccc, $\mathcal{S}$ is forced to be stationary, $M[G] \cap V = M$ and $\delta_{M[G]} = \delta_M$ for every $M \in \mathcal{S}_0$ (see [47] and [52]).

Let $p \in \mathbb{P}$, $\langle M_0, ..., M_n \rangle$ a δ-increasing sequence of models in $\mathcal{S}_0$, $\dot{x}_0 \in M_0, ..., \dot{x}_n \in M_n$ that are $\mathbb{P}$-names for elements of ω^ω and $\dot{R} \in M_n$ a name for a projective and $\leq^*$-adequate relation. Since a projective relation can be coded by a subset of ω (and also any element of ω^ω), by the transformation property, we can find an automorphism $H : \mathbb{P} \longrightarrow \mathbb{P}$ such that $H(\dot{x}_i)$ is equivalent to a name in M_n for every $i \leq n$, $H(\dot{x}_n) = \dot{x}_n$ and $H(\dot{R}) = \dot{R}$. Since $\dot{R}$ is forced to be $\leq^*$-adequate, there is $\dot{g} \in M_n$ a $\mathbb{P}$-name that is forced to be an $\dot{R}$-control for $(H(\dot{x}_0), ..., H(\dot{x}_n))$. Since $\mathbb{P}$ does not add dominating reals and is ccc, we know that $H(p) \Vdash$"$f_{\delta_n} \not\leq^* \dot{g}$". Therefore, we know that $H(p) \Vdash$"$H(\dot{R})(H(\dot{x}_0), ..., H(\dot{x}_n), f_{\delta_n})$" and since H is an isomorphism, with the aid of Proposition 1, we conclude that
$p \Vdash$"$\dot{R}(\dot{x}_0, ..., \dot{x}_n, f_{\delta_n})$". ■

Both Cohen and random forcings are ccc and do not add dominating reals (random forcing does not even add unbounded reals, see [2]). Our next goal is to show that they have the transformation property. For the remainder of the section, l will denote a natural number, $\mathbb{P}(I)$ will be either Cohen or random forcing (where $I \subseteq \omega_l$) and κ a large enough regular cardinal.

Proposition 51 (CH) *Let $M, N \in \mathsf{Sub}(\kappa)$ with $\delta_M \leq \delta_N$, $I \in M \cap [\omega_l]^\omega$ and $\dot{a} \in M$ a $\mathbb{P}(I)$-name for a subset of ω. If $\triangle : \omega_l \longrightarrow \omega_l$ a permutation for which there is $P \in M$ a finite partition of I such that for every $A \in P$ we have that $\triangle \restriction A$ is order preserving and $\triangle[A] \in N$, then $\triangle_*(\dot{a})$ is equivalent to a name in N.*

Proof. Take an enumeration $P = \{P_i \mid i \leq n\}$ and choose $\beta < \delta_M$ a limit ordinal larger than $\mathsf{OT}(I)$. For each $i \leq n$, denote $\gamma_i = \mathsf{OT}(P_i)$ and $e_i : P_i \longrightarrow \gamma_i$ the only isomorphism. Since $\triangle$ is order preserving in each P_i, we know that γ_i is isomorphic to $\triangle[P_i]$ as well. Let $\widehat{e}_i : \triangle[P_i] \longrightarrow \gamma_i$ be the only isomorphism. Note that $e_i \in M$ and $\widehat{e}_i \in N$. We now define the function:

$$h : I \longrightarrow \omega_1 \qquad\qquad g : \triangle\left[I\right] \longrightarrow \omega_1$$

Such that for every $\alpha \in P_i$:

$$h\left(\alpha\right) = \beta i + e_i\left(\alpha\right) \qquad \begin{aligned} g\left(\triangle\left(\alpha\right)\right) &= \beta i + \widehat{e_i}\left(\triangle\left(\alpha\right)\right) \\ &= \beta i + e_i\left(\alpha\right) \\ &= h\left(\alpha\right) \end{aligned}$$

Clearly $h = g \circ \triangle$ and $\mathsf{im}(h) = \mathsf{im}(g)$. We have the isomorphisms $h_* : \mathbb{P}\left(I\right) \longrightarrow \mathbb{P}\left(h\left[I\right]\right)$ and $g_* : \mathbb{P}\left(\triangle\left[I\right]\right) \longrightarrow \mathbb{P}\left(h\left[I\right]\right)$. Note that $h, h_* \in M$ and $g, g_* \in N$. Denote $\dot{b} = h_*\left(\dot{a}\right)$, which is a $\mathbb{P}\left(h\left[I\right]\right)$-name. Since $h\left[I\right]$ is a countable subset of ω_1, since $\mathbb{P}\left(h\left[I\right]\right)$ is ccc, we can code $\dot{b}$ as an element of $M \cap \mathsf{H}(\omega_1)$ and by Lemma 41, we conclude that $\dot{b} \in N$. In this way, we know that $\dot{c} = g_*^{-1}(\dot{b})$ is a $\mathbb{P}\left(\triangle\left[I\right]\right)$-name that is in N. In this way, in order to prove that $\triangle_*\left(\dot{a}\right)$ is in N, it is enough to show that it is equal to $\dot{c}$. Let $p \in \mathbb{P}\left(\triangle\left[I\right]\right)$ and $n \in \omega$. Using Propositions 1 and 3, we obtain the following:

$$\begin{aligned} p \Vdash \text{“}n \in \dot{c}\text{”} \quad &\longleftrightarrow \quad p \Vdash \text{“}n \in g_*^{-1}(\dot{b})\text{”} \\ &\longleftrightarrow \quad g_*\left(p\right) \Vdash \text{“}n \in \dot{b}\text{”} \\ &\longleftrightarrow \quad g_*\left(p\right) \Vdash \text{“}n \in h_*\left(\dot{a}\right)\text{”} \\ &\longleftrightarrow \quad h_*^{-1} \circ g_*\left(p\right) \Vdash \text{“}n \in \dot{a}\text{”} \\ &\longleftrightarrow \quad \left(h^{-1} \circ g\right)_*\left(p\right) \Vdash \text{“}n \in \dot{a}\text{”} \\ &\longleftrightarrow \quad \triangle_*^{-1}\left(p\right) \Vdash \text{“}n \in \dot{a}\text{”} \\ &\longleftrightarrow \quad p \Vdash \text{“}n \in \triangle_*\left(\dot{a}\right)\text{”} \end{aligned}$$

We conclude that $\triangle_*\left(\dot{a}\right)$ and $\dot{c}$ are equivalent. Since $c \in N$, the proof is finished. ■

We can finally prove:

Proposition 52 (CH) *$\mathbb{P}\left(\omega_l\right)$ has the transformation property.*

Proof. Let κ be a regular large enough cardinal, $\langle M_0, ..., M_n\rangle$ a δ-increasing sequence of models from $\mathsf{Sub}(\kappa)$ and $\dot{a}_0 \in M_0, ..., \dot{a}_n \in M_n$ be $\mathbb{P}\left(\omega_l\right)$ names for subsets of ω. For every $i \leq n$, find $A_i \in M_i \cap \left[\omega_l\right]^{\omega}$ such that $\dot{a}_i$ is a $\mathbb{P}\left(A_i\right)$-name. We now use Lemma 45 to summon $\mathcal{P} = \langle P_i \mid i \leq n\rangle$ a coherent sequence of partitions for $\langle M_0, ..., M_n\rangle$ such that $A_i \subseteq \cup P_i$ for every $i \leq n$. Denote $I_i = \cup P_i$ for every $i \leq n$. Clearly $I_i \in M_i$ and $\dot{a}_i$ is a $\mathbb{P}\left(I_i\right)$-name. We now invoke Proposition 47 to find a permutation $\triangle : \omega_l \longrightarrow \omega_1$ such that is the identity in every element of P_n and for every $B \in \bigcup P_i$ it is the case that $\triangle\left[B\right] \in M_n$ and $\triangle \restriction B$ is order preserving. Finally, by Proposition 51, we know that $\triangle_*\left(\dot{a}_i\right)$ is equivalent to a name in M_n for every $i \leq n$. Moreover, since $\triangle \restriction I_n$ is the identity, we get that $\triangle_*\left(\dot{a}_n\right) = \dot{a}_n$. ■

We can now conclude:

Corollary 53 (CH) *Let $l < \omega$. Both $\mathbb{C}\left(\omega_l\right)$ and $\mathbb{B}\left(\omega_l\right)$ force that there is a multiple $\mathfrak{d}$-pathway.*

In particular:

Theorem 54 (CH) *Let $l < \omega$. $\mathbb{B}(\omega_l)$ force that there is a strong P-point and a Gruff ultrafilter.*

Of course this is also true for Cohen forcing, but it is not new since the existence of a Gruff ultrafilter and a strong P-point follow from $\mathfrak{d} = \mathfrak{c}$ (see [23] and [26]).

10 Open Questions

We now list some questions we do not know how to solve. The most important one is the following:

Problem 55 *Are there P-points (Gruff ultrafilters, strong P-points) in every model obtained by adding any number of random reals to a model of CH?*

It would be enough to provide a positive answer to the following:

Problem 56 *Does CH imply that $\mathbb{C}(\kappa)$ and $\mathbb{B}(\kappa)$ have the transformation property for any cardinal κ?*

In [19] the first author proved that there will be an ultrafilter that does not contain a nowhere dense P-subfilter (equivalently, ω^* can not be covered by nowhere dense P-sets) after adding ω_2 Cohen reals to a model of $\mathsf{CH}+\ \square_{\omega_1}$. Building on the ideas developed in this paper and in [19], we were able to construct such an ultrafilter after adding fewer than $\aleph_\omega$ Cohen reals over a model of CH, this result will appear in a forthcomming paper.

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Alan Dow
Department of Mathematics and Statistics, UNC Charlotte.
adow@charlotte.edu

Osvaldo Guzmán
Centro de Ciencias Matemáticas, UNAM.
oguzman@matmor.unam.mx