NON-TRIVIAL COPIES OF $\mathbb{N}^*$

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Abstract. We show that it is consistent to have regular closed non-clopen copies of $\mathbb{N}^*$ within $\mathbb{N}^*$ and a non-trivial self-map of $\mathbb{N}^*$ even if all autohomeomorphisms of $\mathbb{N}^*$ are trivial.

1. Introduction

In this paper we are interested in the possible existence of regular closed subsets of $\mathbb{N}^*$ that are non-trivial copies of $\mathbb{N}^*$, i.e. that are not clopen. A proper subspace $K \subset \mathbb{N}^*$ is said to be a trivial copy of $\mathbb{N}^*$ if there is an embedding of $\beta\mathbb{N}$ into $\beta\mathbb{N}$ which sends the remainder $\mathbb{N}^*$ onto K. More generally, a function $h \in \mathbb{N}^{\mathbb{N}}$ is said to *induce* a function $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ on $I \subset \mathbb{N}$ if $F(a) =^* h^{-1}(a) = \{n \in \mathbb{N} : h(n) \in a\}$ for all $a \subset I$. The function F is said to be *trivial* on I if there is a such a function h. We use $\mathrm{triv}(F)$ to denote the ideal of sets on which F is trivial. Such a function F is usually a *lifting* of a homomorphism $\psi : \mathcal{P}(\mathbb{N})/\operatorname{fin} \to \mathcal{P}(\mathbb{N})/\operatorname{fin}$ in the sense that $F(a)/\operatorname{fin} = \psi(a/\operatorname{fin})$ for all $a \subset \mathbb{N}$. We would similarly say that F *induces* a homomorphism on $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. The ideal $\mathrm{triv}(\psi)$ would coincide with that of $\mathrm{triv}(F)$ for any such lifting of ψ. An ideal is a P-ideal if it is countably directed modulo the finite ideal. Dually, a closed subset of $\mathbb{N}^*$ is a P-set if its neighborhood base is countably directed.

The study of trivial versus non-trivial mappings on $\mathbb{N}^*$ has a celebrated and impactful history. Shelah's breakthrough in [19] that it was consistent that all autohomeomorphisms of $\mathbb{N}^*$ are trivial lead to many results. These include W. Just's results [15, 16] that nowhere dense P-sets are consistently not homeomorphic to $\mathbb{N}^*$ and that $\mathbb{N}^*$ need not map onto its own square. Throughout this paper we reference many of the other major developments by Shelah, Steprans, Veličković and Farah. Among these is the result by Farah [12, p. 77] that PFA implies that if $K \subset \mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$, then there is a, possibly empty, clopen subset A of $\mathbb{N}^*$ such that $A \subset K$ and $K \setminus A$ is nowhere dense.

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It was later shown in [4], answering a question of longstanding, that there are non-trivial nowhere dense copies of $\mathbb{N}^*$.

Veličković [28] introduced the poset $\mathbb{P}_2$, see Definition 2.1, which was created to produce a model of Martin's Axiom in which $\mathfrak{c} = \aleph_2$ and in which there are non-trivial autohomeomorphisms on $\mathbb{N}^*$. This was achieved by forcing over a model of PFA. Several variations of $\mathbb{P}_2$ are possible and we continue the study of the properties of $\mathbb{N}^*$ that hold in the model(s) obtained when forcing with $\mathbb{P}_2$ (and its variants $\mathbb{P}_0$ and $\mathbb{P}_1$) over a model of PFA (see also [21, 25, 8]).

Theorem 1.1. *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_0$ the following all hold:*

1. *all automorphisms on $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ are trivial,*
2. *there is a copy K of $\mathbb{N}^*$ in $\mathbb{N}^*$ such that K is regular closed with a single boundary point,*
3. *there is continuous function f from $\mathbb{N}^*$ onto $\mathbb{N}^*$ whose restriction to K is a homeomorphism,*
4. *the function $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ induced by f^{-1}, in the sense that $(F(a))^* = f^{-1}(a^*)$, is not trivial.*

Theorem 1.2. *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_1$, there are non-trivial automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.*

These results are re-stated and proven as theorems 2.4, 6.2 and 7.1. We need more definitions before we can state the other major result below. Farah [12, p. 73] defines the important notion of an ideal of $\mathcal{P}(\mathbb{N})$ being ccc over fin to mean that there is no uncountable almost disjoint family of subsets of $\mathbb{N}$ none of which are in the ideal. By Stone duality, we define a closed subset K of $\mathbb{N}^*$ to be ccc over fin if there is no uncountable family of pairwise disjoint clopen subsets of $\mathbb{N}^*$ each meeting K in a non-empty set. It is shown in [12, p. 74] that PFA implies that for each homomorphism ψ from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, which is the Stone dual of a copy of $\mathbb{N}^*$ in $\mathbb{N}^*$, $\operatorname{triv}(\psi)$ is ccc over fin. It was already shown in [21] that for automorphisms ψ on $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, $\operatorname{triv}(\psi)$ is a dense P-ideal in the models under study. A crucial step for our results above, and a result of independent interest, is the following strengthening (restated and proven as 5.9).

Theorem 1.3. *In the extension obtained by forcing over a model of PFA by any of $\mathbb{P}_0$, $\mathbb{P}_1$, or $\mathbb{P}_2$, if Φ is an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, then $\operatorname{triv}(\Phi)$ is a ccc over fin ideal.*

2. Preliminaries

Now we recall the partial order $\mathbb{P}_2$ from [28].

Definition 2.1. The partial order $\mathbb{P}_2$ is defined to consist of all 1-to-1 functions f where

(1) $\mathrm{dom}(f) = \mathrm{range}(f) \subset \mathbb{N}$,
(2) for all $i \in \mathrm{dom}(f)$ and $n \in \omega$, $f(i) \in [2^n, 2^{n+1})$ if and only if $i \in [2^n, 2^{n+1})$
(3) $\limsup_{n\to\infty} |[2^n, 2^{n+1}) \setminus \mathrm{dom}(f)| = \infty$
(4) for all $i \in \mathrm{dom}(f)$, $i = f^2(i) \neq f(i)$.

The ordering on $\mathbb{P}_2$ is $\subseteq^*$.

Definition 2.2. The poset $\mathbb{P}_1$ is defined to consist of all $\{0,1\}$-valued partial functions f such that $\mathrm{dom}(f) \subset \mathbb{N}$ and $\limsup_{n\to\infty} |[2^n, 2^{n+1}) \setminus \mathrm{dom}(f)| = \infty$. The ordering on $\mathbb{P}_1$ is $\subseteq^*$.

The poset $\mathbb{P}_0$ is the subposet of $\mathbb{P}_1$ consisting of those $f \in \mathbb{P}_1$ satisfying that for all $n \in \omega$, $f^{-1}(1) \cap [2^n, 2^{n+1})$ has size at most 1, and it is non-empty if and only if $[2^n, 2^{n+1}) \subset \mathrm{dom}(f)$.

Proposition 2.3 ([28])**.** *Let G denote a $\mathbb{P}_2$-generic filter. The collection $\mathcal{U} = \{\mathbb{N} \setminus \mathrm{dom}(f) : f \in G\}$ is an ultrafilter. This is also true for the posets $\mathbb{P}_0$ and $\mathbb{P}_1$.*

The generic ultrafilter $\mathcal{U}$ added by each of these posets is a tie-point of $\mathbb{N}^*$ (as introduced in [9] $A \underset{\mathcal{U}}{\bowtie} B$, see also [7, 8]): namely there is a cover by regular closed subsets A, B satisfying that $A \cap B = \{\mathcal{U}\}$. For a regular closed set A of $\mathbb{N}^*$, we let $\mathcal{I}_A$ denote the ideal $\{a \subset \mathbb{N} : a^* \subset A\}$. For the set-theoretically oriented reader we point out that if A, B are the regular closed sets that witness that $\mathcal{U}$ is a tie-point, then this corresponds to $\mathcal{I}_A$ and $\mathcal{I}_B$ being orthogonal ideals with the property that their union generates a maximal (proper) ideal equalling the complement of $\mathcal{U}$. It was shown in [14] that under the continuum hypothesis there is, for every ultrafilter $\mathcal{U}$, a copy A of $\mathbb{N}^*$ in which $\mathcal{U}$ is a P-point, so that with B equalling its regular closed complement ($B = \mathrm{cl}(\mathbb{N}^* \setminus A)$) then $A \underset{\mathcal{U}}{\bowtie} B$ holds. B is also a copy of $\mathbb{N}^*$ (by Parovicenko's theorem) while $\mathcal{U}$ is also a P-point in B if and only if $\mathcal{U}$ is a P-point of $\mathbb{N}^*$. The ideals $\mathcal{I}_A$ and $\mathcal{I}_B$ are closely connected to the concepts of gaps, in fact *tight gaps* [24, 1.1], in $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ and for roughly that reason, there are no tie-points in models of PFA. Using that adding Cohen reals destroys ultrafilters and preserves gaps, it is easily shown that there are no tie-points in the standard Cohen model. In fact for each of these results, stronger results were shown in [3] and [18] respectively. It is evident that $\mathcal{U}$ being a tie-point of $\mathbb{N}^*$ is equivalent to there being a continuous 2-valued function defined on $\mathbb{N}^* \setminus \{\mathcal{U}\}$ that does not extend continuously to all of $\mathbb{N}^*$ (i.e. to $\mathcal{U}$). In models of PFA [3] and the usual Cohen mode [18], every real-valued continuous function on $\mathbb{N}^* \setminus \{\mathcal{U}\}$ extends continuously to $\mathbb{N}^*$.

In the cases of G being generic for either of $\mathbb{P}_0$ and $\mathbb{P}_1$, the witnesses to $\mathcal{U}$ being a tie-point would be where $\mathcal{I}_A$ is the family $\{f^{-1}(1) : f \in G\}$, while, for $\mathbb{P}_2$, $\mathcal{I}_A = \{\{i \in \mathrm{dom}(f) : i < f(i)\} : f \in G\}$. This is discussed in [8]. One of our main motivations is to discover if A or B can be, or needs to be, homeomorphic to $\mathbb{N}^*$ as this information can be quite useful in applications (again, see [8]). An important potential application of tie-points, together with the question of whether either of the associated pair A, B is a copy of $\mathbb{N}^*$, is the possible consistency of the Banach space ℓ_∞/c_0 (i.e. $C(\mathbb{N}^*)$) not being primary (see [10] and [11, p. 577]). There are many lines of investigation that may be pursued concerning the properties of these models. An excellent example suggested by the referee would be rigidity properties of other Stone-Cech remainders such as the PFA results found in [23, 13, 6].

If PFA holds, then each of $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$ is $\aleph_1$-closed and $\aleph_2$-distributive (see [25, p.4226]). In this paper we will restrict our study to forcing with these posets individually, but the reader is referred to [25] for the method to generalize to countable support infinite products.

Theorem 2.4. *If G is $\mathbb{P}_0$-generic and $A \underset{\mathcal{U}}{\bowtie} B$ are as defined above, then there is a homeomorphism from A to $\mathbb{N}^*$ which extends to a continuous mapping with domain all of $\mathbb{N}^*$.*

Proof. Let $\psi \in \mathbb{N}^{\mathbb{N}}$ be defined so that $\psi([2^n, 2^{n+1})) = \{n\}$ for all n, and let ψ^* denote the canonical extension with domain and range $\mathbb{N}^*$. In fact, for each free ultrafilter $\mathcal{W}$, the preimage of $\mathcal{W}$ under ψ^* is the set of ultrafilters extending $\{\psi^{-1}[W] : W \in \mathcal{W}\}$. Recall that A is the closure of the set $\bigcup\{(f^{-1}(1))^* : f \in G\}$. We will simply show that $\psi^* \restriction A$ is one-to-one. Let $\mathcal{V} = \{b \subset \mathbb{N} : \psi^{-1}(b) \in \mathcal{U}\}$. By the definition of $\mathbb{P}_0$, it follows that, for each $f \in G$, $\psi^* \restriction (f^{-1}(1))^*$ is one-to-one and that $\psi(f^{-1}(1)) \notin \mathcal{V}$. It follows easily that the preimage of any point of $\mathbb{N}^* \setminus \{\mathcal{V}\}$ contains a single point in A. Now suppose that $\mathcal{W} \neq \mathcal{U}$ is in the preimage of $\mathcal{V}$. Since $\mathcal{U}$ is generated by $\{\mathbb{N} \setminus \mathrm{dom}(f) : f \in G\}$, we may choose an $f \in G$ with $\mathrm{dom}(f) \in \mathcal{W}$. Since $\psi(f^{-1}(1)) \notin \mathcal{V}$, we have that $f^{-1}(0) \in \mathcal{W}$. But now, $f^{-1}(0)$ is mod finite disjoint from each member of $\mathcal{I}_A$, which shows that $\mathcal{W} \notin A$. ☐

We recall some basic forcing notions that will be used.

Proposition 2.5 ([17, VII.8.3]). *If G is a $2^{<\omega_1}$-generic filter over a model M, then $\mathcal{P}(\mathbb{N}) \cap M = \mathcal{P}(\mathbb{N}) \cap M[G]$, ω_1 is preserved, and $\diamondsuit$ holds in $M[G]$.*

Proposition 2.6 ([17, pp. 55-57])**.** *Let $\mathcal{A} \subset \mathcal{P}(\omega)$ be a family of co-infinite sets and closed under finite unions. The almost disjoint sets partial order $\mathbb{P}_{\mathcal{A}}$ is*

$$\{\langle s, F\rangle : s \subset \omega \wedge |s| < \omega \wedge F \subset \mathcal{A} \wedge |F| < \omega\} \ ,$$

where $\langle s', F'\rangle \leq \langle s, F\rangle$ if and only if

$$s \subset s' \wedge F \subset F' \wedge (\forall a \in F)(a \cap s' \subset s) \ .$$

$\mathbb{P}_{\mathcal{A}}$ is σ-centered and there is a $\mathbb{P}_{\mathcal{A}}$-name $\dot{d}$ that is forced to be an infinite subset of ω that is almost disjoint from every element of $\mathcal{A}$.

The rest of the paper is devoted to proving the theorems listed in the first section. Let us remark that Theorem 1.3 holds for all onto homomorphisms but this requires lengthy verifications that the results for automorphisms from [21, 25, 8] also hold for onto homomorphisms. As mentioned above a homomorphism ψ from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ onto $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ is said to be trivial, if there is function $h \in \mathbb{N}^{\mathbb{N}}$ which induces ψ in the sense that $\psi(a/\operatorname{fin}) = (h^{-1}(a))/\operatorname{fin}$ for all $a \subset \mathbb{N}$. We intend to deal with automorphisms only so it will be more convenient to work with the inverse map; hence $\psi(a/fin) = h(a)/\operatorname{fin}$. We will say that ψ is not trivial at a point $x \in \mathbb{N}^*$ if no member of $\operatorname{triv}(\psi)$ is in the ultrafilter corresponding to x.

3. The auxillary poset $\mathbb{P}(\mathfrak{F})$

For this section let $\mathbb{P}$ denote any one of the posets $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$. It is known that $\mathbb{P}$ is σ-directed closed. The following partial order was introduced in [21] as a tool to uncover the forcing preservation properties of $\mathbb{P}$, such as Velickovic's result that PFA implies that $\mathbb{P}$ is $\aleph_2$-distributive (and so introduces no new ω_1-sequences of subsets of $\mathbb{N}$).

We will be using the methods from [21] to identify properties of $\mathbb{P}$ that follow from PFA. These methods consist of meeting ω_1-many dense sets in finite iterations of proper posets, the first of which is the simple poset $2^{<\omega_1}$.

This next definition is also from [21, Definition 2.2].

Definition 3.1. Let $\mathfrak{F}$ denote any filter on $\mathbb{P}$. Define $\mathbb{P}(\mathfrak{F})$ to be the partial order consisting of all $g \in \mathbb{P}$ such that there is some $f \in \mathfrak{F}$ which is almost equal to it. The ordering on $\mathbb{P}(\mathfrak{F})$ is $f \leq g$ if $f \supseteq g$.

The forcing poset $\mathbb{P}(\mathfrak{F})$ (which is just the set $\mathfrak{F}$) introduces a new total function f which extends mod finite every member of $\mathfrak{F}$. Although f will not be a member of $\mathbb{P}$ it is only because its domain does not satisfy the growth condition (3) in the definition of $\mathbb{P}$. There is a simple σ-centered poset (see Proposition 2.6) that will force an appropriate set

$I \subset \mathbb{N}$ which mod finite contains all the domains of members of $\mathfrak{F}$ and satisfies that $f \restriction I$ is a member of $\mathbb{P}$ which is below each member of $\mathfrak{F}$ (see [21, 2.1]).

A strategic choice of the filter $\mathfrak{F}$ will ensure that $\mathbb{P}(\mathfrak{F})$ is ccc and much more. Again we are lifting results from [21, Lemma 2.6] and [25, Corollary 3.2]. A poset is said to be ω^ω-bounding if every new function in ω^ω is bounded by some ground model function. Since the poset $\mathbb{P}$ is countably closed there are filters that can meet any given family of ω_1-many dense sets. In a model of PFA, there are no (ω_2, ω_2)-gaps [1, 4.3] and so, in such a model, there are ω_2-many dense sets that cannot be met by a single filter. However, if we first collapse the continuum to obtain a model of $\diamondsuit$ then, as shown in [21], we can meet all dense sets that are in the ground model.

Lemma 3.2. *In the forcing extension by $2^{<\omega_1}$, there is a maximal filter $\mathfrak{F}$ on $\mathbb{P}$ which is $\mathbb{P}$-generic over V and for which $\mathbb{P}(\mathfrak{F})$ is ccc, ω^ω-bounding, and preserves that the set of ground model reals is not meager.*

Almost all of the work we have to do is to establish additional preservation results for the poset(s) $\mathbb{P}(\mathfrak{F})$. Once these are established, we are able to apply the standard PFA type methodology as demonstrated in [21, 25] to determine properties of the forcing extension by $\mathbb{P}$. As mentioned above, we have this result from [25, Corollary 3.3].

Lemma 3.3. *In a model obtained by forcing with $\mathbb{P}$ over a model of PFA, the trivial ideal for every automorphism of $\mathcal{P}(\mathbb{N})/fin$ is a dense P-ideal.*

Corollary 3.4 (PFA)**.** *Let $\dot\Phi$ be a $\mathbb{P}$-name that is forced to be an automorphism of $\mathcal{P}(\mathbb{N})/fin$. In the forcing extension by $2^{<\omega_1}$, the trivial ideal of the valuation of $\dot\Phi$ by the filter $\mathfrak{F}$ on $\mathbb{P}$ as in Lemma 3.2 is a dense P-ideal.*

Proof. Let V be a model in which PFA holds. By Lemma 3.3, for every $b \in [\mathbb{N}]^{\aleph_0}$, there is a dense set D_b of conditions in $\mathbb{P}$ that force there is some infinite subset of b in the trivial ideal of $\dot\Phi$. Similarly, for every sequence $\vec{a} = \langle a_n : n \in \omega\rangle \subset [\mathbb{N}]^{\aleph_0}$, there is a dense set $D_{\vec{a}}$ of conditions that force that either there is a member of the trivial ideal of $\dot\Phi$ that mod finite contains each member of $\vec{a}$, or that force that $\vec{a}$ is not contained in the trivial ideal. Now let G be $2^{<\omega_1}$-generic over V and let $\mathfrak{F}$ be as in Lemma 3.2. Being $\mathbb{P}$-generic over V, the filter $\mathfrak{F}$ will meet D_{a_0} and $D_{\vec{a}}$ for every ω-sequence $\vec{a} \subset [\mathbb{N}]^{\aleph_0}$. By Proposition 2.5, $\mathcal{P}(\mathbb{N}) \cap V = \mathcal{P}(\omega) \cap V[G]$, hence the trivial ideal of the $\mathfrak{F}$-valuation of $\dot\Phi$ will be dense. Similarly, since there are no new ω-sequences of members of $[\mathbb{N}]^{\aleph_0}$, this trivial ideal will be a P-ideal. $\square$

4. σ-Borel liftings and more notation

A lifting of a map Φ from $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ to itself is any function F from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ which satisfies that $F(a)/\operatorname{fin} = \Phi(a/\operatorname{fin})$ for all $a \in \mathcal{P}(\mathbb{N})$. For each $\ell \in \mathbb{N}$ and $s \subset \ell$, let $[s;\ell] = \{x \subset \mathbb{N} : x \cap \ell = s\}$. This defines the standard Polish topology on $\mathcal{P}(\mathbb{N})$. For a set $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ and a function F on $\mathcal{P}(\mathbb{N})$, let us say that $F \restriction \mathcal{C}$ is σ-Borel if there is sequence $\{\psi_n : n \in \omega\}$ of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for each $b \in \mathcal{C}$, there is an n such that $F(b) =^* \psi_n(b)$.

We continue the analysis of $\mathbb{P}$-names from V (the ground model of PFA) where $\mathbb{P}$ is any one of $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2$. For the remainder of the paper we let H denote a $2^{<\omega_1}$-generic filter. Recall from Lemma 2.5 that forcing with $2^{<\omega_1}$ does not change the set $\mathcal{P}(\mathbb{N})$. In the forcing extension $V[H]$, following [21], we use a V-generic filter $\mathfrak{F} \subset \mathbb{P}$. In particular, fix a $\mathbb{P}$-name $\dot\Phi$ which is forced by $\mathbf{1}_{\mathbb{P}}$ to be a lifting of an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. Let F denote $\operatorname{val}_{\mathfrak{F}}(\dot\Phi)$. Of course it follows that, in $V[H]$, F is a lifting of an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. The following key result of ([8, 2.3]) was extracted from [21] and [25, Theorem 3.3].

Lemma 4.1 (PFA)**.** *For any dense P-ideal $\mathcal{I}$ on $\mathbb{N}$ and for each $\mathbb{P}(\mathfrak{F})$-generic filter G, there is an $I \in \mathcal{I}$ such that $F \restriction (V \cap [\mathbb{N} \setminus I]^\omega)$ is σ-Borel in the extension $V[H][G]$.*

One of the main results which we can extract from [25] and simply deduce from Lemma 3.3 and Lemma 4.1 is the following.

Lemma 4.2. *$F \restriction (V \cap \mathcal{P}(\mathbb{N}))$ is σ-Borel in the extension obtained by forcing with $\mathbb{P}(\mathfrak{F})$.*

We will also need several results from [8]. The following are [8, 3.1] and [8, 2.5] respectively. The basic ideas originate in [19, p. 131] and [27, p 131].

Lemma 4.3. *Assume that $b \in V \cap [\mathbb{N}]^\omega$ is such that $F \restriction [V \cap [b]^\omega]$ is σ-Borel in $V[G]$. Then, in V, there is an increasing sequence $\{n_k : k \in \omega\} \subset \omega$ such that F is trivial on each $a \in [b]^\omega$ for which there is an $r \in \mathfrak{F}$, such that $a \subset \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \operatorname{dom}(r)\}$.*

Lemma 4.4. *Let H and $\mathfrak{F}$ be as above. Then for each $\mathbb{P}(\mathfrak{F})$-name $\dot h \in \mathbb{N}^{\mathbb{N}}$ there are an increasing sequence $n_0 < n_1 < \cdots$ of integers and a condition $f \in \mathfrak{F}$ such that either*

(1) $f \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h} \restriction \bigcup\{[n_k, n_{k+1}) : k \in K\} \notin V$" *for each infinite* $K \subset \omega$ *or*
(2) *for each* $i \in [n_k, n_{k+1})$ *and each* $g < f$ *such that* g *forces a value on* $\dot{h}(i)$, $f \cup (g \restriction [n_k, n_{k+1}))$ *also forces a value on* $\dot{h}(i)$.

Furthermore, if f *forces* $\dot{h}$ *to be finite-to-one, we can arrange that for each* k *and each* $i \in [n_k, n_{k+1})$, f *forces that* $\dot{h}(i) \in [n_{k-1}, n_{k+2})$.

Next we need to use a key Lemma from [8, Lemma 3.1].

Lemma 4.5. *There is a condition* $\bar{p} \in \mathfrak{F}$ *and an increasing sequence* $\{n_k : k \in \omega\} \subset \mathbb{N}$ *such that, for each* $k \in \omega$ *there is an* $m_k \in \omega$ *such that*

(1) $[n_k, n_{k+1}) \setminus \mathrm{dom}(\bar{p})$ *is a subset of* $2^{m_k+1} \setminus 2^{m_k}$ *and has cardinality at least* k,
(2) $\bar{p}$ *forces (over* $\mathbb{P}$*) that* $\mathrm{triv}(F)$ *contains all* $a \subset \mathbb{N}$ *for which there is an* $r \in \mathfrak{F}$, *such that* $a \subset \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(r)\}$.

We will say that a sequence $\langle r_j : j \in \omega\rangle \subset \mathbb{P}$ is a fusion sequence (for $\mathbb{P}$) if, for each $j \in \omega$,

(1) $r_j \subset r_{j+1}$,
(2) there is an $m_j \in \omega$ such that $2^{m_j+1} \setminus (2^{m_j} \cup \mathrm{dom}(r_j))$ has cardinality at least j,
(3) $r_{j+1} \restriction 2^{m_j+1}$ is a subset of r_j.

If $\langle r_j : j \in \omega\rangle$ is a fusion sequence, then the function $r = \bigcup_{j\in\omega} r_j$ is a condition in $\mathbb{P}$ that is an extension of each r_j.

Fix any $p \in \mathfrak{F}$ which forces, over $\mathbb{P}$, that $\dot{\Phi}$ and, therefore F have all of the above properties. Assume also that p satisfies the requirement in Lemma 4.5. Notice that for each $q \in \mathfrak{F}$, we have a one-to-one function h_q with domain $a_q = \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(q)\}$ which witnesses that $a_q \in \mathrm{triv}(F)$. Therefore the family $\{h_q : q \in \mathfrak{F}\}$ is a σ-directed (mod finite) family of functions which, because $\mathbb{N} \notin \mathrm{triv}(F)$ in $V[H]$, has no extension in $V[H]$.

In this next proof we will use the concept of a splitting family. A family $\mathcal{S} \subset \mathcal{P}(\mathbb{N})$ is splitting if for all infinite $Y \subset \mathbb{N}$, there is an $S \in \mathcal{S}$ satisfying that each of $S \cap Y$ and $Y \setminus S$ are infinite.

Lemma 4.6. *The family* $\{\mathrm{dom}(h_q) : q \in \mathfrak{F}\}$ *generates a dense ideal in* $V[H]$ *which remains dense after forcing with* $\mathbb{P}(\mathfrak{F})$.

Proof. Let us note that from the fact that $\mathfrak{F}$ is V-generic, it follows from Proposition 2.3 the family $\{\mathrm{dom}(q) : q \in \mathfrak{F}\}$ is a maximal ideal. Then, given the definition of the family $\{a_q : q \in \mathfrak{F}\}$, and the partition of $\mathbb{N}$ by the intervals $\{[n_k, n_{k+1}) : k \in \omega\}$, it follows that the finite-to-one map sending each $[n_k, n_{k+1})$ to $\{k\}$, will send $\{a_q : q \in \mathfrak{F}\}$ to a maximal ideal $\mathcal{J}$. To finish the proof of the Lemma it suffices to prove that $\mathcal{J}$ remains dense after forcing with $\mathbb{P}(\mathfrak{F})$. Following [2, p.121], it is well known, and due to Solomon, that if $\mathcal{D} \subset \mathbb{N}^{\mathbb{N}}$ is a dominating family of strictly increasing functions, then the family $\mathcal{S} = \{S_f = \bigcup_{n\in\mathbb{N}}[f^{2n}(1), f^{2n+1}(1)) : f \in \mathcal{D}\}$ is a splitting family. The forcing $\mathbb{P}(\mathfrak{F})$ is ω^ω-bounding and so preserves that the family $\mathcal{S}$ is splitting. We finish the proof by proving that $\mathbb{P}(\mathfrak{F})$ preserves that $\mathcal{J}$ is dense. Indeed, let $Y \subset \mathbb{N}$ be any infinite set in the forcing extension by $\mathbb{P}(\mathfrak{F})$. Choose any $f \in \mathcal{D}$ so that $S_f \cap Y$ and $Y \setminus S_f$ are infinite. By symmetry, assume that $S_f \in \mathcal{J}$. It follows that Y is not almost disjoint from S_f. □

We prove, using this next well-known result, that there is a $\mathbb{P}(\mathfrak{F})$-name, $\dot{h}$, of a function in $\mathbb{N}^{\mathbb{N}}$ that is forced to mod finite extend every member of $\{h_q : q \in \mathfrak{F}\}$. Uncountable pairwise incompatible families of partial functions on $\mathbb{N}$ are similar to Luzin gaps. Such a family will not have a common mod finite extension. The following result is an easy consequence of a result of Todorcevic (see [26, Theorem 8.7] and [12, 2.2.1]). Proper posets were introduced in [19] and it was shown that countably closed posets, ccc posets, and finite iterations of proper posets are all proper.

Proposition 4.7. *If $\{h_\alpha : \alpha \in \omega_1\}$ is a family of partial functions on $\mathbb{N}$ with mod finite increasing domains, and if there is no common mod finite extension, then there is a proper poset which introduces an uncountable pairwise incompatible subfamily.*

Lemma 4.8. *There is a $\mathbb{P}(\mathfrak{F})$-name $\dot{h}$ on $\mathbb{N}$ that is forced to mod finite extend every member of $\{h_q : q \in \mathfrak{F}\}$ and which is forced to be 1-to-1 on a cofinite subset of $\mathbb{N}$.*

Proof. By Lemma 4.6, the domains of the members of the family $\{h_q : q \in \mathfrak{F}\}$ is a dense ideal after forcing with $\mathbb{P}(\mathfrak{F})$. If h is any function that mod finite extends the family $\{h_q : q \in \mathfrak{F}\}$ of 1-to-1 functions, then there is a cofinite subset of h that is 1-to-1. So it suffices to prove that there is such a function h in the extension by $\mathbb{P}(\mathfrak{F})$.

Let us assume, towards a contradiction, that forcing with $\mathbb{P}(\mathfrak{F})$ preserves that the family $\{h_q : q \in \mathfrak{F}\}$ has no common mod finite extension. Let $\dot{q}_{\omega_1}$ denote the $\mathbb{P}(\mathfrak{F})$-name of the function that equals the union of the generic filter G. We note that the sequence $\{n_k : k \in \omega\}$ is a sequence from V. Let $\dot{Q}$ be the $\mathbb{P}(\mathfrak{F})$-name of the proper poset as described in Lemma 4.7. That is, there is a $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q}$-name,

$\{(\dot{q}_\alpha, \dot{a}_\alpha, \dot{h}_\alpha, \dot{L}_\alpha) : \alpha \in \omega_1\}$, satisfying that, for each $\alpha < \beta \in \omega_1$, it is forced that

(1) $\{\dot{q}_\alpha, \dot{a}_\alpha, \dot{h}_\alpha : \alpha \in \omega_1\}$ are $\mathbb{P}(\mathfrak{F})$-names,
(2) $\dot{q}_\alpha \subset \dot{q}_{\omega_1}$,
(3) $\dot{a}_\alpha = \bigcup\{[n_k, n_{k+1}) : k \in \dot{L}_\alpha\} \subset \mathrm{dom}(\dot{q}_\alpha)$,
(4) $\dot{q}_\alpha$ forces that $\dot{h}_\alpha \in V$ induces $\dot{\Phi}$ on $\dot{a}_\alpha$,
(5) $\dot{L}_\alpha \subset^* \dot{L}_\beta$,
(6) there is an $n \in \dot{a}_\alpha \cap \dot{a}_\beta$ with $\dot{h}_\alpha(n) \neq \dot{h}_\beta(n)$.

Next we let, $\dot{\mathcal{A}}$ denote the $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q}$-name of the family $\{\mathrm{dom}(\dot{q}_\alpha) : \alpha \in \omega_1\}$. We consider a suborder, namely $\dot{R}$, of the $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q}$-name of the poset $\mathbb{P}_{\dot{\mathcal{A}}}$ of Proposition 2.6. The conditions $\langle s, F\rangle$ in $\dot{R}$ are simply required to satisfy that $\dot{q}_{\omega_1} \restriction (\mathbb{N} \setminus s) \in \mathbb{P}$. We let $\dot{d}$ denote the $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q} * \dot{R}$-name of the generic infinite subset of ω added by $\dot{R}$ (it is not important as to whether $0 \in \dot{d}$). It is easiest to describe $\dot{d}$ by its interpretation after forcing with $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q}$. The valuation of the name $\dot{d}$ in this forcing extension will be the set $\{(n, \langle s, F\rangle) : n \in s \wedge \langle s, F\rangle \in R\}$. Now let $\dot{L}$ denote the $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q} * \dot{R}$-name of the set $\{k \in \omega : [n_k, n_{k+1}) \cap \dot{d} = \emptyset\}$. It is easily checked that it is forced that, for every $\alpha \in \omega_1$, $\dot{L}_\alpha \subset^* \dot{L}$ and that $\dot{d}$ is infinite (as in Proposition 2.6). The restriction on the elements of $\dot{R}$ ensures that $\dot{q}_{\omega_1} \restriction (\mathbb{N} \setminus \dot{d})$ is forced to be an element of $\mathbb{P}$ and that, in $\mathbb{P}$, $\dot{q}_{\omega_1} \restriction (\mathbb{N} \setminus \dot{d}) \leq \dot{q}_\alpha$ for all $\alpha < \omega_1$.

Now return to the PFA model and choose a filter Γ on $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q} * \dot{R}$ that meets ω_1-many dense open sets sufficient to ensure that properties (1)-(5) of the valuations of the names $\{(\dot{q}_\alpha, \dot{a}_\alpha, \dot{h}_\alpha, \dot{L}_\alpha) : \alpha < \omega_1\}$ will all hold and so that the valuations of $\dot{L}$, $\dot{q}_{\omega_1}$, and $\dot{d}$ all have the properties mentioned above. We may also assume that the condition p of Lemma 4.5 is an element of Γ in the sense that the condition $(1, p, 1, 1)$ (as an element of $2^{<\omega_1} * \mathbb{P}(\mathfrak{F}) * \dot{Q} * \dot{R}$) is an element of Γ. Now let $\{h_\alpha : \alpha \in \omega_1\}$ be the valuations of the elements of $\{\dot{h}_\alpha : \alpha \in \omega_1\}$ by the filter Γ. Also let L be the valuation of $\dot{L}$ by Γ and let r be the valuation of the condition $\dot{q}_{\omega_1} \restriction (\mathbb{N} \setminus \dot{d})$ by Γ. We observe that $a_r = \bigcup\{[n_k, n_{k+1}) : k \in L\}$ is a subset of $\mathrm{dom}(r)$ and so, by Lemma 4.5 there is a function h_r (with domain a_r) that induces F on $\mathcal{P}(a_r)$. Choose an uncountable $\Lambda \subset \omega_1$ so that there is a $\bar{k} \in \omega$ such that $L_\alpha \setminus L \subset \bar{k}$ and $h_\alpha \restriction (a_\alpha \setminus n_{\bar{k}}) \subset h_r$ for all $\alpha \in \Lambda$. We may further suppose that $h_\alpha \restriction (a_\alpha \cap n_{\bar{k}}) = h_\beta \restriction (a_\beta \cap n_{\bar{k}})$ for all $\alpha, \beta \in \Lambda$. It should now be clear that for all $\alpha, \beta \in \Lambda$, condition (6) does not hold. $\square$

For partial functions p and s with domains contained in $\mathbb{N}$, we let $s \sqcup p$ denote the function $s \cup (p \restriction (\mathrm{dom}(p) \setminus \mathrm{dom}(s)))$. Since $\dot{h}$ is forced to mod finite extend each h_q (for $q \in \mathfrak{F}$), we have, by Lemma 4.6, that condition (1) of Lemma 4.4 fails to hold. Using this we now prove there is an extension $\bar{q}$ of the condition from Lemma 4.5 with additional properties.

Lemma 4.9. *There is a condition $\bar{q} < \bar{p}$ and a re-indexed subsequence $\langle n_k : k \in \omega \rangle$ of the sequence from Lemma 4.4, that satisfies that for each $k \in \mathbb{N}$ and for each $i \in [n_k, n_{k+1})$ such that $\bar{q}$ does not force a value on $\dot{h}(i)$ and for each $q < \bar{q}$ that does force a value on $\dot{h}(i)$, $(q \restriction [n_k, n_{k+1})) \sqcup \bar{q}$ also forces that value on $\dot{h}(i)$ and that value is in $[n_k, n_{k+1})$.*

Proof. Since $\bar{p}$ forces that $\dot{h}$ mod finite extends every h_q ($q \in \mathfrak{F}$) and that the domains are a dense ideal, it follows that condition (2) of Lemma 4.4 will hold. Let I be the set of $i \in \mathbb{N}$ such that $\bar{p}$ does not decide the value of $\dot{h}(i)$. For each $k \in \mathbb{N}$ and each $i \in [n_k, n_{k+1}) \cap I$, there is therefore only finitely many possible values for $\dot{h}(i)$. Furthermore, we may assume that $\bar{p}$ forces that $\dot{h}$ is 1-to-1 on I. By Lemma 4.4, if $k < k' \in \mathbb{N}$ and $i \in [n_k, n_{k+1}) \cap I$ and $i' \in [n_{k'}, n_{k'+1}) \cap I$, then the possible values for $\dot{h}(i)$ and those for $\dot{h}(i')$ are disjoint. Therefore, for each $k \in \mathbb{N}$, there is some $k' \in \mathbb{N}$ such that $\bar{p}$ forces that $\dot{h}(i) > n_{k+1}$ for all $i > n_{k'}$. Let $k_0 = 0$ and assume that m_0 is a bound to the possible values of $\dot{h}(i)$ for $i \in I \cap [n_{k_0}, n_{k_0+1})$. Then choose any strictly increasing sequence $\langle k_j : j \in \omega \rangle$ (with $m_0 < n_{k_1}$) satisfying that for all $0 < j \in \omega$ and $i \in [n_{k_j}, n_{k_j+1}) \cap I$, $\bar{p}$ forces that $\dot{h}(i)$ is larger than $n_{k_{j-1}+1}$ and smaller than $n_{k_{j+1}}$. Let $\bar{q}$ be any extension of $\bar{p}$ satisfying that for all $j \in \omega$

(1) $\bar{q} \restriction [n_{k_{3j+1}}, n_{k_{3j+2}}) = \bar{p} \restriction [n_{k_{3j+1}}, n_{k_{3j+2}})$ and
(2) the domain of $\bar{q}$ contains $[n_{k_{3j}}, n_{k_{3j+1}}) \cup [n_{k_{3j+2}}, n_{k_{3j+3}})$.

The desired subsequence called for in the lemma is $\langle n_{k_{3j}} : j \in \omega \rangle$. □

We end this section by summarizing and establishing the notation that has been developed thus far.

Theorem 4.10 (PFA)**.** *Let $\dot{\Phi}$ be a $\mathbb{P}$-name that is forced by $\mathbf{1}_{\mathbb{P}}$ to be a lifting of an automorphism of $\mathcal{P}(\mathbb{N})/fin$. Let H be a $2^{<\omega_1}$-generic filter. In the forcing extension $V[H]$, there is a filter $\mathfrak{F} \subset \mathbb{P}$, a condition $\bar{q} \in \mathfrak{F}$, and sequences $\{n_k, m_k : k \in \omega\}$ of integers, and a $\mathbb{P}(\mathfrak{F})$-name $\dot{h}$ such that, for each $k \in \omega$,*

(1) *$\mathfrak{F}$ is a V-generic filter for $\mathbb{P}$,*

(2) *the valuation, F, by $\mathfrak{F}$ of $\dot\Phi$ induces an automorphism of $\mathcal{P}(\mathbb{N})/\mathit{fin}$,*
(3) *if $\mathbf{1}_{\mathbb{P}}$ forces that $\dot\Phi$ is non-trivial, then F is non-trivial,*
(4) $n_k < 2^{m_k} < 2^{m_k+1} < n_{k+1}$,
(5) $\mathrm{dom}(\bar q) \supset [n_k, n_{k+1}) \setminus (2^{m_k+1} \setminus 2^{m_k})$,
(6) $2^{m_k+1} \setminus (\mathrm{dom}(\bar q) \cup 2^{m_k})$ *has cardinality at least k,*
(7) *$\bar q$ forces that $\dot h \in \mathbb{N}^{\mathbb{N}}$, and for each $i \in [n_k, n_{k+1})$, if $\bar q$ does not force a value on $\dot h(i)$, then $\bar q \Vdash n_k \leq \dot h(i) < n_{k+1}$ and for any $q < \bar q$ that does force a value on $\dot h(i)$, so does $(q \restriction (2^{m_k+1} \setminus 2^{m_k}) \sqcup \bar q$,*
(8) *for each $r \in \mathfrak{F}$, there is an $\mathbb{N}$-valued function h_r with domain $a_r = \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \mathrm{dom}(r)\}$, such that r forces that $h_r \subset^* \dot h$ and is a lifting of $F \restriction \mathcal{P}(a_r)$,*
(9) *the poset $\mathbb{P}(\mathfrak{F})$ forces that $\{a_r : r \in \mathfrak{F}\}$ generates a dense ideal.*

5. CCC OVER FIN

In this section we complete the proof of Theorem 1.3. Throughout this section we assume that $\dot\Phi$ is a $\mathbb{P}$-name as in Theorem 4.10 and we continue with the items established in Theorem 4.10.

Lemma 5.1. *If $Y = \{y_k : k \in \omega\}$ with $y_k \in [n_k, n_{k+1})$ for each k, then for each $q < \bar q$ in $\mathbb{P}(\mathfrak{F})$, there is a $p \supset q$ such that p decides $\dot h \restriction Y$.*

Proof. Since we assume that $q \supset \bar q$, the properties of Theorem 4.10 will hold. Let K be the set of k such that q does not already force a value on $\dot h(y_k)$ and let $Y' = \{y_k : k \in K\}$. Recall that $\bar q$ forces that $\dot h(y_k) \in [n_k, n_{k+1})$ for all $k \in K$. Now, using the V-genericity of $\mathfrak{F}$, choose $q' < q$ in $\mathfrak{F}$ so that q' forces a value on $F(Y')$. For each $k \in K$, choose $j_k \in F(Y') \cap [n_k, n_{k+1})$ if it is non-empty, otherwise set $j_k = 1$. This defines a function $g : Y' \to \mathbb{N}$ (i.e. $g(y_k) = j_k$) that is an element of V. Consider any $r \in \mathbb{P}(\mathfrak{F})$ with $r \leq q$. Let $a_r(K) = a_r \cap \bigcup\{[n_k, n_{k+1}) : k \in K\}$. Since r forces that $h_r \restriction (a_r(K) \cap Y') =^* \dot h \restriction (a_r(K) \cap Y')$ and that h_r is a lifting of F on $\mathcal{P}(a_r)$, it follows that r also forces that $h_r(a_r(K) \cap Y') =^* F(Y') \cap a_r(K)$. Since the ideal $\{a_r : r \in \mathfrak{F}\}$ is dense, it follows that $F(Y') \cap [n_k, n_{k+1}) = \{j_k\}$ for almost all $k \in K$. It now similarly follows that q forces that $\dot h \restriction Y' =^* g$. Clearly, by density, there is a condition $p < q$ that decides a value $m \in \mathbb{N}$ such that $\dot h(y_k) = g(y_k)$ for all $m < k \in K$. It follows that p decides $\dot h \restriction Y$. □

We proceed by contradiction using the following Lemma.

Lemma 5.2. *If $\bar q$ forces that* $\mathrm{triv}(\dot\Phi)$ *is not ccc over fin, then we may assume that there are an almost disjoint family $\{a_\alpha : \alpha < \omega_1\} \subset [\mathbb{N}]^\omega$ and ultrafilters $\{\mathcal{W}_\alpha : \alpha \in \omega_1\} \subset \mathbb{N}^*$ such that, for all $\alpha \in \omega_1$, $\bar q$ forces that*

(1) $F(a_\alpha) = b_\alpha$ *and* $\{a_\beta \cup b_\beta : \beta < \omega_1\}$ *is an almost disjoint family,*
(2) F *is not trivial at* $\mathcal{W}_\alpha$*, and*
(3) $a_\alpha \in \mathcal{W}_\alpha$.

Proof. Using that $\mathbb{P}$ is $\aleph_2$-distributive we then have that $\mathrm{triv}(F)$ is not ccc over fin. Also, we can assume that $\bar{q}$ is below some condition $p \in \mathfrak{F}$ that forces the following: the almost disjoint family $\{a_\alpha : \alpha < \omega_1\} \subset [\mathbb{N}]^\omega$ satisfies that $a_\alpha \notin \mathrm{triv}(F)$ and that $F(a_\alpha) = b_\alpha$ for all $\alpha \in \omega_1$. Notice that the family $\{b_\alpha : \alpha \in \omega_1\}$ is also an almost disjoint family. By compactness of a^*_α, we may choose an ultrafilter $\mathcal{W}_\alpha$ on $\mathbb{N}$ so that $a_\alpha \in \mathcal{W}_\alpha$ and so that F is not trivial at $\mathcal{W}_\alpha$. If there is an uncountable set of α such that $F(\mathcal{W}_\alpha) = \mathcal{W}_\alpha$, then we pass to such an uncountable subcollection as well as shrink each a_α so that the new b_α is a subset of the original a_α.

So we now assume that $F(\mathcal{W}_\alpha) \neq \mathcal{W}_\alpha$ for all $\alpha \in \omega_1$. We may now assume that a_α and b_α are almost disjoint for each α. Next, for each $\gamma \in \omega_1$, let S_γ be the set of all $\alpha \in \omega_1$ such that either $a_\gamma \in F(\mathcal{W}_\alpha)$ or $b_\gamma \in \mathcal{W}_\alpha$. If, there is some $\gamma \in \omega_1$ such that S_γ is uncountable, then we can, by passing to an uncountable subset S of S_γ and, for all $\alpha \in S$, shrinking a_α so as to ensure that either $a_\alpha \subset b_\gamma$ for all $\alpha \in S$ or that $b_\alpha \subset a_\gamma$ for all $\alpha \in S$. In either case, we obtain a new family as required in the Lemma.

The final case is that we have that S_γ is countable for every $\gamma \in \omega_1$. Recursively choose an uncountable subcollection $\{\alpha_\xi : \xi \in \omega_1\} \subset \omega_1$ so that $\alpha_\xi \notin S_{\alpha_\eta}$ for all $\eta < \xi$. First suppose there is a $\delta < \omega_1$ such that there is an uncountable set $S \subset \omega_1$ with each $\mathcal{W}_{\alpha_\xi}$ (for $\xi \in S$) being in the $\beta\mathbb{N}$ closure of the union of the family of clopen sets $\{b^*_{\alpha_\eta} : \eta < \delta\}$. It then follows that, for all $\xi \in S$, $\mathcal{W}_{\alpha_\xi}$ is not in the closure of the family $\{b^*_{\alpha_\eta} : \delta \leq \eta < \xi\}$. For each $\xi \in S\setminus\delta$, replace a_{α_ξ} by a smaller $a_{\alpha_\xi} \in \mathcal{W}_{\alpha_\xi}$ satisfying that $a_{\alpha_\xi} \cap b_{\alpha_\eta}$ is finite for all $\delta \leq \eta < \xi$. If there is no such $\delta < \omega_1$, then we can assume, by passing to an uncountable subfamily, that $\mathcal{W}_{\alpha_\xi}$ is not in the closure of the union of the family $\{b^*_{\alpha_\eta} : \eta < \xi\}$. In this case we may also assume that each $a_{\alpha_\xi} \in \mathcal{W}_{\alpha_\xi}$ is mod finite disjoint from b_{α_η} for all $\eta < \xi$. By symmetry, we may perform the same reduction so that for all $\xi < \omega_1$, b_{α_ξ} is almost disjoint from a_{α_η} for all $\eta < \xi$.

This completes the proof. It is not needed but it is interesting that since this argument can take place in the ground model of PFA, we have, by a result of Shelah reported in [5, 3.11], that we may assume that either $F(\mathcal{W}_\alpha) = \mathcal{W}_\alpha$ for all $\alpha \in \omega_1$ or the sets $\{\mathcal{W}_\alpha : \alpha \in \omega_1\}$ and $\{F(\mathcal{W}_\alpha) : \alpha \in \omega_1\}$ have disjoint closures in $\beta\mathbb{N}$. □

For each k, let $H_k = [n_k, n_{k+1}) \setminus \mathrm{dom}(\bar{q}) = [2^{m_k}, 2^{m_k+1}) \setminus \mathrm{dom}(\bar{q})$. Let $\mathcal{H}_k$ denote the set of functions s with domain contained in H_k for which there is a $q \le \bar{q}$ with $s = q \restriction H_k$. Recall that for the posets $\mathbb{P}_0$ and $\mathbb{P}_1$, the conditions are functions into 2, while for the poset $\mathbb{P}_2$, the conditions q extending $\bar{q}$ are permutations which send each H_k into itself. Therefore, with $\mathbb{P}$ being any of the three posets considered in this paper, $\mathcal{H}_k$ is a finite set of functions with domain and range contained in $2 \cup H_k$.

Definition 5.3. For each condition $q \in \mathbb{P}(\mathfrak{F})$, and each $i \in \mathbb{N}$, let $\mathrm{Orb}_q(i) = \{j : (\exists p < q)\ p \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h}(i) = j\text{”}\}$. Also let $S(k, q) = \{s \in \mathcal{H}_k : q \restriction H_k \subset s\}$.

Corollary 5.4. *Our condition $\bar{q}$ also satisfies that for each $i \in \mathbb{N}$ and $q < \bar{q}$, if $\mathrm{Orb}_q(i)$ has more than one element, there is a k such that $\{i\} \cup \mathrm{Orb}_q(i) \subset [n_k, n_{k+1})$.*

Lemma 5.5. *For each $\alpha \in \omega_1$, there are $r_\alpha < \bar{q}$ in $\mathbb{P}(\mathfrak{F})$ and $W_\alpha \subset a_\alpha$ such that $W_\alpha \in \mathcal{W}_\alpha$ and $r_\alpha \Vdash_{\mathbb{P}(\mathfrak{F})}$ “$\dot{h}[W_\alpha] \subset b_\alpha$”*

Proof. We prove that if there is no such pair r_α, W_α, then we reach a contradiction to the fact that $\bar{q} \Vdash F(a_\alpha) = b_\alpha$. We must use the genericity of $\mathfrak{F}$ and a density argument. Let $\bar{r} \supset \bar{q}$ be any extension in $\mathbb{P}$. It suffices to prove that there is a suitable $r_\alpha \supset \bar{r}$ in $\mathbb{P}$ and using genericity to conclude that there is such an r_α in $\mathfrak{F}$. Given any such $\bar{r}$, let Y be the set of $y \in a_\alpha$ such that $\bar{r}$ fails to force that $\dot{h}(y) \in b_\alpha$. Of course if Y is finite, then we have that $\bar{r} \Vdash \dot{h}[a_\alpha \setminus Y] \subset b_\alpha$. Otherwise, choose a strictly increasing sequence $\langle \ell_k : k \in \omega\rangle$ so that, for each $k \in \omega$, there is a $y_k \in [n_{\ell_k}, n_{\ell_{k+1}})$ and an $s_k \in S(\ell_k, \bar{r})$ with $s_k \sqcup \bar{r}$ forcing a value on $\dot{h}(y_k)$ that is not in b_α. We may assume that $[n_{\ell_k}, n_{\ell_k+1})$ is contained in $\mathrm{dom}(s_k \cup \bar{r})$. Choose any infinite $L \subset \omega$ so that $r = \bigcup\{s_k \sqcup \bar{r} : k \in L\}$ is a condition in $\mathbb{P}$. Notice that r forces that $\dot{h}[\{y_k : k \in L\}]$ is disjoint from b_α, and that if r were in $\mathfrak{F}$, $a_r = \mathrm{dom}(h_r)$ would contain $\{y_k : k \in L\}$. The genericity of $\mathfrak{F}$ ensures that there will be such an $r \in \mathfrak{F}$ and this is our contradiction: $r \le \bar{q}$ forces that $\dot{h}[\{y_k : k \in L\}]$ should mod finite equal $h_r[\{y_k : k \in L\}]$ and that $h_r[\{y_k : k \in L\}] \subset h_r(a_\alpha) =^* b_\alpha$. □

By strengthening the condition $\bar{q}$, we can assume that we have the following property.

Lemma 5.6. *For each integer ℓ and each condition $q < \bar{q}$, there is a condition $p < q$ and a set $I \in [\omega_1]^\ell$ such that $p < r_\alpha$ for each $\alpha \in I$.*

Proof. Since $\mathbb{P}(\mathfrak{F})$ is ccc, there is a condition $q \le \bar{q}$ forcing that there is an uncountable set of $\alpha \in \omega_1$ such that r_α is in the generic filter. □

Lemma 5.7. *For each $\alpha \in \omega_1$, there is an integer ℓ_α such that for each k and each $s_k \in S(k, r_\alpha)$, if $|H_k \setminus \operatorname{dom}(s_k)| > \ell_\alpha$, then $s_k \sqcup r_\alpha$ does not decide $\dot{h} \restriction W_\alpha \cap [n_k, n_{k+1})$.*

Proof. We first recall that r_α forces with respect to $\mathbb{P}$ that F is not trivial at $\mathcal{W}_\alpha$. Similarly, the assertion that $F \restriction a_q$ is trivial is forced by every $q < p$ with respect to the poset $\mathbb{P}$ rather than $\mathbb{P}(\mathfrak{F})$. This is simply to note that in this proof we do not have to be concerned with finding a condition in $\mathfrak{F}$.

Now suppose that such an integer ℓ_α did not exist, then we could find an infinite $K \subset \omega$ and a sequence $\langle s_k : k \in \omega\rangle \in \Pi_{k\in\omega} S(k, r)$ with $\{|H_k \setminus \operatorname{dom}(s_k)| : k \in K\}$ diverging to infinity, and such that $s_k \sqcup r$ decides $\dot{h} \restriction W_\alpha \cap [n_k, n_{k+1})$ for each k. But then of course, $q = \bigcup_{k\in K} s_k \sqcup r_\alpha$ would force that $\dot{h} \restriction W_\alpha = h_\alpha$ for some $h_\alpha \in V$. It follows easily from the assumption that F is not trivial at $\mathcal{W}_\alpha$, that there is some $q' < q$ and some infinite $W \subset W_\alpha$ such that $q' \Vdash_{\mathbb{P}}$ "$\dot{\Phi}(W) \cap h_\alpha[W] = \emptyset$". By further extending q' and possibly shrinking W, we can assume that $W \subset \operatorname{dom}(h_{q'})$. This contradicts that $h_{q'}[W]$ is supposed to be forced by q' to be (mod finite) equal to both $\dot{h}[W]$ and $\dot{\Phi}(W)$. □

By passing to an uncountable subcollection we may suppose that there is some $\bar{\ell}$ such that $\ell_\alpha = \bar{\ell}$ for all α. Now define $S'(k, q) = \{s \in S(k, q) : |H_k \setminus \operatorname{dom}(s)| > \bar{\ell}\}$.

Lemma 5.8. *There is a condition r and an infinite set K such that $\{|H_k \setminus \operatorname{dom}(r)| : k \in K\}$ diverges to infinity, and, for each $k \in K$, we can select $\{i_s : s \in S'(k, r)\} \subset [n_k, n_{k+1})$ such that $\operatorname{Orb}_r(i_s) \cap \operatorname{Orb}_r(i_{s'})$ is empty for each $s \neq s' \in S'(k, r)$, and $s \sqcup r$ does not decide $\dot{h}(i_s)$.*

Proof. We define a fusion sequence $\langle \bar{r}_j : j \in \omega\rangle$ of extensions of $\bar{q}$ as described after Lemma 4.5. The inductive requirement is that, for each $j \in \omega$, there is a k_j such that

1. $\bar{r}_{j+1} \restriction n_{k_j} = \bar{r}_j \restriction n_{k_j}$,
2. $|H_{k_j} \setminus \operatorname{dom}(\bar{r}_{j+1})| > j$,
3. there is a selection $\{i_s : s \in S'(k_j, \bar{r}_{j+1})\} \subset [n_{k_j}, n_{k_j+1})$ such that $\operatorname{Orb}_{\bar{r}_{j+1}}(i_s) \cap \operatorname{Orb}_{\bar{r}_{j+1}}(i_{s'})$ is empty for each $s \neq s' \in S'(k_j, \bar{r}_{j+1})$, and $s \sqcup \bar{r}_{j+1}$ does not decide $\dot{h}(i_s)$,
4. $k_j < k_{j+1}$.

Suppose that we have such sequences $\langle \bar{r}_j : j \in \omega\rangle$ and $\langle k_j : j \in \omega\rangle$. We let $r = \bigcup_{j\in\omega} \bar{r}_j$ be the meet of the sequence and $K = \{k_j : j \in \omega\}$. Since $r \restriction n_{k_j+1} = r_{j+1} \restriction n_{k_j+1}$ and $r \leq \bar{q}$, it follows that $S'(k_j, \bar{r}_{j+1}) = S'(k_j, r)$ and that, by item (7) of Theorem 4.10, item (3) of the fusion condition also holds with r in place of $\bar r_{j+1}$. Therefore to prove the Lemma, we just have to assume that, given $\ell\in\omega$, if we have chosen $\langle \bar r_j : j\le \ell\rangle$ and $\langle k_j : j<\ell\rangle$ we can find a suitable k_ℓ and $\bar r_{\ell+1}$.

Let L be bigger than $(\ell+2)^{\ell+2}$. Apply Lemma 5.6 to find an $r\le \bar r_\ell$ that is below r_α for each $\alpha\in I$ for some $I\subset\omega_1$ of cardinality at least L. For each $\alpha\in I$, we can assume that r decides $\dot\Phi(W_\alpha)=F(W_\alpha)$ and, by Lemma 4.9, that $\mathrm{dom}(h_r)$ contains $[n_k,n_{k+1})$ for each k such that r decides $\dot h\restriction W_\alpha\cap[n_k,n_{k+1})$. Recall that a_α and b_α were defined in Lemma 5.2. We may choose an m such that $[a_\alpha\cup F(W_\alpha)\cup b_\alpha]\cap[a_\beta\cup F(W_\beta)\cup b_\beta]\subset m$ for each $\alpha\ne\beta\in I$.

Let $K=\{k>k_{\ell-1} : |H_k\setminus \mathrm{dom}(r)|>\ell\}$. It follows from Lemma 5.7, that for each $\alpha\in I$, $k\in K$, and $s\in S'(k,r)$, there is an $i\in W_\alpha\cap[n_k,n_{k+1})$ for which $s\sqcup r$ does not decide $\dot h(i)$. Choose any $k_\ell\in K$ and redefine r to be $\left(\bar r_\ell\restriction n_{k_j}\right)\sqcup r$. Since this change to r is finite it is equivalent to the original r in the poset $\mathbb P$ and is still an extension of $\bar q$ in $\mathbb P(\mathfrak F)$. For the remainder of the paragraph we use k in place of k_ℓ. Next, choose $s_k\in S(k,r)$ with $\ell\le|H_k\setminus\mathrm{dom}(s_k)|<\ell+2$ and fix any injection from $S'(k,s_k\sqcup r)$ into I (i.e. $\{\alpha_s : s\in S'(k,s_k\cup r)\}$). For each $s\in S'(k,s_k\sqcup r)$, there is an $i_s\in W_\alpha\cap[n_k,n_{k+1})$ such that $s\sqcup r$ does not decide $\dot h(i_s)$. Since r forces that $\{i_s,\dot h(i_s)\}\subset a_{\alpha_s}\cup b_{\alpha_s}$ and for $s'\ne s$, r forces that $i_{s'},\dot h(i_{s'})\notin a_{\alpha_s}\cup b_{\alpha_s}$, we have satisfied the requirement that $i_s\notin\mathrm{Orb}_r(i_{s'})$ (the hard part was making them distinct). Now we define $\bar r_{\ell+1}$ to equal $s_k\sqcup r$. □

Theorem 5.9. *The trivial ideal,* triv(F)*, is ccc over fin.*

Proof. Let r and the sequence $\{\{i_s : s\in S'(k,r)\} : k\in K\}$ be as constructed in Lemma 5.8. Since $\{|H_k\setminus\mathrm{dom}(r)| : k\in K\}$ diverges to infinity, we may assume that $\mathrm{dom}(r)\supset[n_k,n_{k+1})$ for each $k\notin K$. We define a $\mathbb P$-name of an ultrafilter. Each $q<r$ forces that the set $X(q)=\bigcup_{k\in K}\{i_s : s\in S'(k,q)\}$ is a member. Let x be any ultrafilter extending this filter. We claim that $\dot\Phi(x)$ has no value. Assume first that there is some $q<r$ which forces that $X(r)\in\dot\Phi(x)$ (i.e. q forces that $F^{-1}(X(r))\in x$). We may then further assume that some $q'<q$ forces that $X(q)$ has some infinite subset $Y(q)\in x$ such that for some m, $(F(Y(q))\setminus m)\subset X(r)$. Of course, $Y(q)$ is just a set in V and $Y(q)\cap X(q')$ is large. Choose an infinite sequence $\{k_j : j\in\omega\}$ so that for each j, we can choose $i_j\in Y(q)\cap\{i_s : s\in S'(k_j,q')\}$ and so that $\{|H_k\setminus\mathrm{dom}(q')| : k\notin\{k_j : j\in\omega\}$ diverges to infinity. For each j, let s_{k_j} denote the member s of $S'(k_j,q')$ such that $i_j=i_s$. For each j, choose any $s'_j\in S'(k_j,q')$ which extends s_{k_j} that satisfies that $s'_j\sqcup q'\Vdash_{\mathbb P(\mathfrak F)}$ "$\dot h(i_j)\ne i_j$". Let h^* denote the function with domain $\{i_j : j\in\omega\}$

satisfying that $s'_j \sqcup q' \Vdash_{\mathbb{P}(\mathfrak{F})}$ "$\dot{h}(i_j) = h^*(i_j)$". Note that $h^*(i_j) \notin X(r)$ for all j since i_j is the only member of $\mathrm{Orb}_r(i_j)$ in $X(r)$. We can extend the condition $q' \cup \bigcup_j s'_j$ further to some q^* so that $\mathrm{dom}(q^*) \supset [n_{k_j}, n_{k_j+1})$ for each j. We observe that $J = \{i_j : j \in \omega\} \subset Y(q) \cap \mathrm{dom}(h_{q^*})$. By removing finitely many elements, we may assume that q^* forces that h^* agrees with h_{q^*} on J. However, we now have a contradiction since $h_{q^*}[J] =^* F(J) \subset^* F(Y(q)) \subset^* X(r)$.

Now assume that there is a $q < r$ which forces that $X(r)$ is not in $\dot{\Phi}(x)$. There is a $q' < q$ and a $Y(q) \subset X(q)$ such that for some $Z \subset \mathbb{N} \setminus X(r)$, q' forces that $Y(q) \in x$ and $Z = F(Y(q))$. Again select a sequence $\{k_j : j \in \omega\}$ so that for each j, $[n_{k_j}, n_{k_j+1}) \cap Y(q) \cap X(q')$ is not empty, and choose i_j from this set. We may choose this sequence so that $\{|H_k \setminus \mathrm{dom}(q')| : k \notin \{k_j : j \in \omega\}$ diverges to infinity. For each j, let $s_j \in S'(k_j, q')$ be chosen so that $i_j = i_{s_j}$. If for infinitely many j, it is possible to select $s'_j \in S'(k_j, s_j \cup q')$ so that $s'_j \cup q'$ forces that $\dot{h}(i_j)$ is not in Z, then we select such an s'_j. The proof then proceeds much as in the first case because it will allow us to obtain that for an infinite J, $\dot{h}[\{i_j : j \in J\}]$ is disjoint from Z in contradiction to $F(\{i_j : j \in J\})$ being contained in Z. In the other case, we select $z_j \in Z$ such that s_j has an extension forcing that $\dot{h}(i_j)$ is equal to z_j, but we also know that we can (and do) select s'_j extending s_j to force that $\dot{h}(i_j)$ is not equal to z_j. Applying the same arguments to the automorphism $\dot{F}^{-1}$ we may certainly select an infinite $J \subset \omega$ and a $q^* < q' \cup \bigcup_j s'_j$ so that $\{z_j : j \in J\}$ is in the range of h_{q^*}. It follows that there is a sequence $Y = \{y_j : j \in J\} \subset Y(q)$ such that $h_{q^*}(y_j) = z_j$ for each $j \in J$. Clearly then this puts $z_j \in \mathrm{Orb}_r(y_j)$ for all but finitely many j. By Lemma 5.4, we actually have that y_j is also from $[n_{k_j}, n_{k_j+1})$ and so the contradiction is that we have arranged that $\mathrm{Orb}_r(y_j)$ and $\mathrm{Orb}_r(i_j)$ are disjoint. □

6. ALL HOMEOMORPHISMS ARE TRIVIAL WHEN FORCING WITH $\mathbb{P}_0$

In this section we restrict to the case where $\mathbb{P}$ is $\mathbb{P}_0$ and we continue with the properties and notation from Theorem 4.10. For each k, we again let H_k be the set $[2^{m_k}, 2^{m_k+1}) \setminus \mathrm{dom}(\bar{q})$ and we recall that $\{|H_k| : k \in \omega\}$ diverges to infinity. Say that a condition q is standard, if for each $\ell > 0$, there are at most finitely many k such that $H_k \setminus \mathrm{dom}(q)$ has cardinality ℓ. The standard conditions are dense below $\bar{q}$ in $\mathbb{P}_0$. For a standard condition q, let $K(q)$ denote those k such that $H_k \setminus \mathrm{dom}(q)$ is not empty. It follows then that $\{|H_k \setminus \mathrm{dom}(q)| : k \in K(q)\}$ diverges to infinity. Recall that q is identically 0 on $\mathrm{dom}(q) \cap H_k$ for all $k \in K(q)$.

For a condition p and $i \in H_k \setminus \mathrm{dom}(p)$, we abuse notation and suppose that $p \cup \{(i,1)\}$ denotes the smallest condition in $\mathbb{P}_0$ that contains $p \cup \{(i,1)\}$.

Lemma 6.1. *If $p_0 < \bar{q}$ in $\mathbb{P}_0$ is a standard condition such that no extension of p_0 decides $\dot{h}(t)$ for all values of t, then there is an extension $p < p_0$ such that for all $i \in \mathbb{N} \setminus \mathrm{dom}(p)$, there is a value t_i so that for some distinct pair u_i, v_i, $p \cup \{(i,0)\} \Vdash \dot{h}(t_i) = u_i$ and $p \cup \{(i,1)\} \Vdash \dot{h}(t_i) = v_i$.*

Proof. We proceed by a simple recursion. By induction on ℓ, suppose we have chosen p_ℓ together with a family $\{i(k,j) : j < \ell\} \subset H_k \setminus \mathrm{dom}(p_\ell)$ for all $k \in K(p_\ell)$. We assume that for each $j < \ell$ and $k \in K(p_\ell)$, there is a value $t_{k,j}$ so that $p_{j+1} \cup \{(i(k,j'),0) : j' \leq j\}$ and $p_{j+1} \cup \{(i(k,j'),0) : j' < j\} \cup \{(i(k,j),1)\}$ force distinct values, $u_{k,j}, v_{k,j}$, on $\dot{h}(t_{k,j})$. As usual in such a fusion, we assume that $p_{j+1} \restriction n_{m_j} \subset p_j$ so that we will have that $\bigcup_\ell p_\ell$ is a condition. Now we may choose a sequence $\langle t_{k,\ell} : k \in K \rangle$ (for some infinite $K \subset K(p_\ell)$) such that, for each $k \in K$, $t_{k,\ell} \in [n_k, n_{k+1})$ and $p_\ell \cup \{(i(k,j),0) : j < \ell\}$ does not force a value on $\dot{h}(t_{k,\ell})$. For each $k \in K$, there are two values $\bar{\imath}^k_0, \bar{\imath}^k_1$ from $H_k \setminus (\mathrm{dom}(p_\ell) \cup \{i(k,j) : j < \ell\})$, such that $p_\ell \cup \{(\bar{\imath}^k_0, 1)\}$ and $p_\ell \cup \{(\bar{\imath}^k_1, 1)\}$ force distinct values, v^k_0, v^k_1, on $\dot{h}(t_{k,\ell})$. Using Lemma 5.1, choose $\bar{p}_{\ell+1} < p_\ell$ such that for all $k \in K(\bar{p}_{\ell+1}) \subset K$, $\{i(k,j) : j < \ell\}$ is disjoint from $\mathrm{dom}(\bar{p}_{\ell+1})$ and $\bar{p}_{\ell+1} \cup \{(i(k,j),0) : j < \ell\}$ forces a value, $u_{k,\ell}$, on $\dot{h}(t_{k,\ell})$. Suppose, without loss of generality, that $v^k_1 \neq u_{k,\ell}$ and let $i_{k,\ell} = \bar{\imath}^k_1$. It follows that $i_{k,\ell} \in \mathrm{dom}(\bar{p}_{\ell+1})$ and so define $p_{\ell+1}$ to be the condition we get by removing $i_{k,\ell}$ from the domain of $\bar{p}_{\ell+1}$ for all $k \in K = K(p_{\ell+1})$.

When the recursion is finished, we choose any increasing sequence $\{k_\ell : \ell \in \omega\}$ so that $k_\ell \in K(p_{\ell+1})$, and $p < p_0$ any condition so that $K(p) = \{k_\ell : \ell \in \omega\}$, and $H_{k_\ell} \setminus \mathrm{dom}(p) = \{i(k_\ell, j) : j < \ell\}$. Of course this implies that p is constantly 0 on each $H_{k_\ell} \cap \mathrm{dom}(p)$. For each $k = k_\ell$ and $j < \ell$, we have that $p \cup \{(i(k,j),1)\}$ forces the value $v_{k,j}$ on $\dot{h}(t_{k,j})$ because of Lemma 4.9. And similarly, since $p_{\ell+1} \restriction H_k \subset p$, we have that $p \cup \{(i(k,j'),0) : j' \leq j\}$ forces that $\dot{h}(t_{k,j}) = u_{k,j}$. Because of this, we have that if $q < p$ is such that $k \in K(q)$ and q forces a value on $\dot{h}(t_{k,j})$, then this value has to be $u_{k,j}$. We finish the construction by another more routine recursion. There should be no risk of confusion if we re-use the notation p_1, p_2 etc. for the values in this new recursion. For each $k \in K(p)$, let $j_{k,0}$ denote the largest value so that $i(k, j_{k,0}) \notin \mathrm{dom}(p)$. By Lemma 5.1, there is a condition $p_1 < p$ so that p_1 forces a value on $\dot{h} \restriction \{t_{k,j_{k,0}} : k \in K(p)\}$. Again, as discussed above, we have that p_1 forces that $\dot{h}(t_{k,j_{k,0}}) = u_{k,j_{k,0}}$ for each $k \in K(p_1)$. There is an infinite set $K_1 \subset K(p_1)$ such that there is a largest $j_{k,1} < j_{k,0}$ such that $i(k, j_{k,1}) \notin \mathrm{dom}(p_1)$. Find a condition $p_2 < p_1$ which forces a value on $\dot{h} \restriction \{t_{k,j_{k,1}} : k \in K_1\}$. Continue this induction. Again there is a sequence $\{k_\ell : \ell \in \omega\}$ such that $j_{k,\ell}$ was successfully chosen for $k = k_{\ell+1}$. We extend p to a condition p' so that $K(p') = \{k_\ell : \ell \in \omega\}$ and $H_{k_\ell} \setminus \mathrm{dom}(p')$ is equal to $\{i(k_\ell, j_{k_\ell,m}) : m < \ell\}$. We still have that $p' \cup \{(i, 1)\} \Vdash \dot{h}(t_{k,i}) = v_{k,i}$ for each $i \in H_k \setminus \mathrm{dom}(p')$, but we can now show that $p' \cup \{(i, 0)\}$ forces that $\dot{h}(t_{k,i}) = u_{k,i}$. The simplest way to do this is to consider any $i' \in H_k \setminus \mathrm{dom}(p')$ with $i' \neq i$. If $i' < i$, then the condition $p' \cup \{(i', 1)\}$ is compatible with $p_{m+1} \restriction H_k$ where m is chosen so that $i = i(k, j_{k,m})$. On the other hand if $i' > i$, then $p' \cup \{(i', 1)\}$ is compatible with $p \cup \{(i(k, j), 0) : j \leq j_{k,m}\}$. Since each of these force that $\dot{h}(t_{k,i}) = u_{k,i}$, we have that $p' \cup \{(i, 0)\}$ forces that $\dot{h}(t_{k,i}) = u_{k,i}$. □

Theorem 6.2. *In the extension obtained by forcing over a model of PFA by $\mathbb{P}_0$ all automorphisms on $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ are trivial.*

Proof. Fix a condition p as in Lemma 6.1 satisfying that for each $i \notin \mathrm{dom}(p)$ there is a t_i such that there are distinct values u_i, v_i that $p \cup \{(i, 0)\}$ and $p \cup \{(i, 1)\}$, respectively, force on $\dot{h}(t_i)$. We prove that this leads to a contradiction. Choose a condition $q < p$ and a set Y so that q forces that $F(Y) = \{u_i : i \notin \mathrm{dom}(p)\}$. Let $L_0 = \{i \notin \mathrm{dom}(q) : t_i \notin Y\}$. If L_0 is infinite, then we have a contradiction by choosing any infinite subset L' of L_0 so that $L' \cap H_k$ has at most one element for each k, and considering the condition $q' = q \cup \{(i, 0) : i \in L'\}$. We now have that q' forces that $F(\{t_i : i \in L'\}) =^* h_{q'}(\{t_i : i \in L'\})$ will be almost contained in $F(Y)$ while $\{t_i : i \in L'\}$ is disjoint from Y.

Now suppose that L_0 is finite. If $\{v_i : i \notin \mathrm{dom}(q)\} \setminus F(Y)$ is infinite, then we choose an infinite L' so that $L' \cap H_k$ is empty for infinitely many $k \in K(q)$ and so that $\{v_i : i \in L'\}$ is disjoint from $F(Y)$. Again the extension $q' = q \cup \{(i, 1) : i \in L'\}$ will force that $F(\{t_i : i \in L'\}) =^* h_{q'}(\{t_i : i \in L'\})$, but this contradicts that it is supposed to be mod finite contained in $F(Y)$.

The final case then is that there is an infinite sequence $L' \subset K(q)$ such that $K(q) \setminus L'$ is still infinite and there is a sequence of pairs $\{i_k, i'_k : k \in L'\}$ such that i_k, i'_k are distinct members of $H_k \setminus \mathrm{dom}(q)$ and $v_{i_k} = u_{i'_k}$ for each $k \in L'$. Now we have that the extension $q' = q \cup \{(i_k, 1) : k \in L'\} \supset q \cup \{(i_k, 1), (i'_k, 0) : k \in L'\}$ will force that $\dot{h}$ is not 1-to-1. □

7. More properties of the poset $\mathbb{P}_1$

It was proven in [8, Theorem 5.1] that for the generic tie-point, $A \underset{\mathcal{U}}{\bowtie} B$, added by $\mathbb{P}_1$, neither of the sets A nor B are homeomorphic to $\mathbb{N}^*$. However we now prove that there are tie-points involving copies of $\mathbb{N}^*$.

Theorem 7.1. *In a model obtained by forcing with the poset $\mathbb{P}_1$ over a model of PFA, there is a non-trivial autohomeomorphism φ of $\mathbb{N}^*$ and two regular closed copies A, B of $\mathbb{N}^*$ and a tie-point $\mathcal{W}$ such that*

1. *$\varphi[A] = B$ and $\varphi[B] = A$, and $A \cap B = \{\mathcal{W}\}$,*
2. *$\mathcal{W}$ is the only point on the boundary of each of A and B,*
3. *φ is the identity on $\mathbb{N}^* \setminus (A \cup B)$.*

Just for completeness we note that the point $\mathcal{W}$ can be expressed as a tie-point using two different covers. Let C be the closure of $\mathbb{N}^* \setminus A$ and let D be the closure of $\mathbb{N}^* \setminus B$. Then it follows from the theorem that each of $A \underset{\mathcal{W}}{\bowtie} C$ and $B \underset{\mathcal{W}}{\bowtie} D$ hold.

Since the proof of the theorem is simply a construction that constitutes the remainder of the section, we decline to work within a traditional proof environment and proceed with a series of definitions and proofs of claims.

We will define a strange sequence, $\{\dot{t}_m : m \in \omega\}$, of $\mathbb{P}_1$-names of pairs. These will code liftings of the maps between A and B (each will "pick" a point from the pair) and the mappings of each onto $\mathbb{N}^*$ (each member from the m-th pair being sent to m). The difficult part of the construction is to ensure that A and B meet in a single ultrafilter.

For each $m \in \omega$ and each function $\sigma \in 2^{[2^m, 2^{m+1})}$, we will choose a pair $a_\sigma \subset [2^m, 2^{m+1})$. To motivate the proof we make some definitions under the assumption that we have made these choices.

Definition 7.2. $\dot{t}_m$ will simply be that a condition $p \in \mathbb{P}_1$ such that $[2^m, 2^{m+1}) \subset \mathrm{dom}(p)$, will force that $\dot{t}_m$ is equal to $a_{p \restriction [2^m, 2^{m+1})}$.

Analogous to the definition of $K(p)$ in Section 6, for $p \in \mathbb{P}_1$, let $M(p)$ denote the set $\{m \in \omega : [2^m, 2^{m+1}) \not\subset \mathrm{dom}(p)\}$. Without mention, we will assume that we work with the dense set of conditions that satisfy $\{|[2^m, 2^{m+1}) \setminus \mathrm{dom}(p)| : m \in M(p)\}$ diverges to infinity.

Definition 7.3. For $p \in \mathbb{P}_1$,

1. $T(p) = \langle t_m : m \in \mathbb{N} \setminus M(p) \text{ and } p \Vdash t_m = \dot{t}_m \rangle$ is a function,
2. $A(p) = \{\min(t_m) : t_m \in T(p)\}$,
3. $h_{p,A} \in \mathbb{N}^{\mathbb{N} \setminus M(p)}$ is 1-to-1 where $h_{p,A}(m) = \min(T_p(m))$,
4. $B(p) = \{\max(t_m) : t_m \in T(p)\}$,

(5) $h_{p,B} \in \mathbb{N}^{\mathbb{N}\setminus M(p)}$ is 1-to-1 where $h_{p,B}(m) = \max(T_p(m))$, and
(6) $W(p) = \bigcup_{m\in M(p)}\{a_\sigma : \sigma \cup p \in \mathbb{P}_1 \wedge \sigma \in 2^{[2^m,2^{m+1})}\}$.

Definition 7.4. If G is a generic filter on $\mathbb{P}_1$, then
(1) A is the closure of the open set $\bigcup\{A(p)^* : p \in G\}$,
(2) B is the closure of the open set $\bigcup\{B(p)^* : p \in G\}$, and
(3) $\mathcal{W}$ is the filter generated by the family $\{W(p) : p \in G\}$.

Also let $\mathcal{M}$ be the filter generated by $\{M(p) : p \in G\}$.

Let $\psi \in \mathbb{N}^\mathbb{N}$ be the finite-to-1 mapping sending each interval $[2^m, 2^{m+1})$ to $\{m\}$.

Claim 7.4.1. If G is $\mathbb{P}_1$-generic then $\mathcal{M}$ is the ultrafilter that is the image of the ultrafilter $\mathcal{U} = \{\mathbb{N} \setminus \mathrm{dom}(f) : f \in G\}$ by ψ.

The intention is that we will define the family of a_σ's so that $\mathcal{W}$ is forced to be an ultrafilter and is the only boundary point of each of A and B. We also intend that the family of mappings $\{\left(h_{p,A}^{-1}\right)^* : p \in G\}$ will induce the homeomorphism on $A \setminus \{\mathcal{W}\}$ into $\mathbb{N}^*$ that extends to a homeomorphism from A onto $\mathbb{N}^*$ by sending $\mathcal{W}$ to $\mathcal{M}$. The analogous statements will hold for B.

Now we set about showing that there is such a sequence of names. We will define, for $m \in \omega$ and $\sigma \in 2^{[2^m,2^{m+1})}$, the value of a_σ based only on the cardinality of $\sigma^{-1}(1)$. To make these choices we now introduce the idea of an ℓ-structure for $\ell \in \omega$.

Definition 7.5. We define a 0-structure. We let $L_0 = 2$ and $n_0 = 6 = L_0 + L_0^{L_0}$ and choose any set of distinct integers $\{y_0, y_1, y_2, y_3\}$. For readability define the 0-structure with $y_i = i$ $(i < 4)$. We define the family of pairs $\{a_{\langle i\rangle} : i < n_0\}$:

$$a_{\langle 0\rangle} = \{0,1\},\quad a_{\langle 1\rangle} = \{2,3\},\quad a_{\langle 2\rangle} = \{0,2\},$$
$$a_{\langle 3\rangle} = \{0,3\},\quad a_{\langle 4\rangle} = \{1,2\},\quad a_{\langle 5\rangle} = \{1,3\}$$

Technically, to fit with the coming inductive definition, 0-structure will be the pair $\langle\{a_{\langle i\rangle} : i < n_0\}, \{\{y_0, y_1, y_2, y_3\}\}\rangle$.

Claim 7.5.1. The 0-structure satisfies that for each set Y such that $Y \cap a_{\langle i\rangle}$ is not empty for each $i < L_0$, there is a $j < n_0$ such that $Y \supset a_{\langle j\rangle}$.

This is the (very) finitary version of a positive set Y containing at least one element of the structure and is the process by which we will ensure that the above defined $\mathcal{W}$ is an ultrafilter.

Definition 7.6. By recursion on ℓ, we define $L_\ell = 2\, n_0\, n_1 \cdots n_{\ell-1}$, set $n_\ell = L_\ell + L_\ell^{L_\ell}$, and we define our ℓ-structure based on the cartesian product

$$\mathcal{N}_\ell = n_\ell \times n_{\ell-1} \times \cdots \times n_1 \times n_0 \ .$$

It is awkward, but ultimately more convenient, to have this product in descending order. For each $0 < j \leq \ell$, also let $\mathcal{N}_{\ell,j} = n_\ell \times \cdots \times n_j$.

Definition 7.7. An ℓ-structure, for $0 < \ell \in \omega$, is defined recursively as a family

$$\langle \{a_x : x \in \mathcal{N}_\ell\}, \{Y_\rho : \rho \in \bigcup_{0<j\leq\ell} \mathcal{N}_{\ell,j} \cup \{\emptyset\}\}\rangle$$

satisfying

(1) $\{a_x : x \in \mathcal{N}_\ell\} \subset [\mathbb{N}]^2$ and base set $Y_\emptyset = \bigcup\{a_x : x \in \mathcal{N}_\ell\}$,
(2) for each $\rho \in \bigcup_{0<j\leq\ell} \mathcal{N}_{\ell,j}$, Y_ρ is the union of all a_x with $x \in \mathcal{N}_\ell$ and $\rho \subset x$,
(3) for each $0 < j \leq \ell$ and $\rho \in \mathcal{N}_{\ell,j}$, the family
$$\langle \{a_x : \rho \subset x \in \mathcal{N}_\ell\}, \{Y_\psi : \rho \subset \psi \in \textstyle\bigcup_{i<\ell} \mathcal{N}_{\ell,i}\}\rangle$$
is a $(j-1)$-structure (with a prefix of ρ on each index),
(4) the elements of $\{Y_{\langle m\rangle} : m < L_\ell\}$ is a partition of $Y_\emptyset$ and $|Y_{\langle 0\rangle}| = |Y_{\langle k\rangle}|$ for all $k < n_\ell$,
(5) for each Y such that $Y \cap Y_{\langle m\rangle} \neq \emptyset$ for each $m < L_\ell$, there is a $k < n_\ell$ such that $Y \supset Y_{\langle k\rangle}$.

Claim 7.7.1. Let $\langle \{a_x : x \in \mathcal{N}_\ell\}, \{Y_\rho : \rho \in \bigcup_{0<j\leq\ell} \mathcal{N}_{\ell,j} \cup \{\emptyset\}\}\rangle$ be an ℓ-structure and $0 < \ell$. Let $Y \subset \mathbb{N}$ and let $\rho \in \mathcal{N}_{\ell,j}$ for some $1 \leq j$. Then either Y or its complement contains an element from $\{Y_{\rho^\frown\langle k\rangle} : k < n_{j-1}\}$.

Proof of Claim: Suppose that $\mathbb{N} \setminus Y$ fails to contain any of the sets from $\{Y_{\rho^\frown\langle m\rangle} : m < L_{j-1}\}$. It then follows that $Y \cap Y_{\rho^\frown\langle m\rangle} \neq \emptyset$ for all $m < L_{j-1}$. By property (5) of a $j-1$-structure, it follows that there is a $k < n_{j-1}$ such that Y contains $Y_{\rho^\frown\langle k\rangle}$. □

Claim 7.7.2. For each ℓ, the cardinality of the base set for any two ℓ-structures is the same and is at most $L_{\ell+1}$.

Proof of Claim: The proof is by induction on ℓ. By condition (4) in Definition 7.7, the cardinality of the base set for an ℓ-structure is n_ℓ times the cardinality for an $\ell-1$-structure. The base set is the union of pairs indexed by $\mathcal{N}_\ell$ and so is has cardinality at most $2|\mathcal{N}_\ell| = L_{\ell+1}$. □

Claim 7.7.3. For each $\ell > 0$, there is an ℓ-structure.

Proof of Claim: The proof is by induction and is a quite straightforward construction. Let $\{\bar{a}_x : x \in \mathcal{N}_{\ell-1}\}$ be the pairs from an $\ell-1$-structure. The definition of L_ℓ ensures that it exceeds the cardinality of $\bar{Y}_\emptyset$ the base set for this $\ell-1$ structure. Let $\{Y_{\langle m\rangle} : m < L_\ell\}$ be a pairwise disjoint family of sets of integers each of the same cardinality as $\bar{Y}_\emptyset$. Similarly, let $\{Y_{\langle k\rangle} : m \leq k < n_\ell\}$ be a family of sets, each of cardinality $|\bar{Y}_\emptyset|$, so that the last inductive assumption is satisfied. I.e. each $Y_{\langle k\rangle}$ $(m \leq k < n_\ell)$ is simply a selection of a single point from each element of $\{Y_{\langle m\rangle} : m < L_\ell\}$. For each $k < n_\ell$, fix a bijection, f_k, between $\bar{Y}_\emptyset$ and $Y_{\langle k\rangle}$ and define $a_{k\frown\bar{x}} = f_k[\bar{a}_{\bar{x}}]$ for each $\bar{x} \in \mathcal{N}_{\ell-1}$. □

For each ℓ, let c_ℓ denote the order-preserving mapping from $\mathcal{N}_\ell$ with the natural lexicographic ordering into an initial segment of $[0, L_{\ell+1})$. It is important to observe that for each $\rho \in \bigcup_{j<\ell} \mathcal{N}_{\ell,j}$, the set $[\rho] = \{x \in \mathcal{N}_\ell : \rho \subset x\}$ is an interval in the lexicographic ordering, and so, $c_\ell([\rho])$ is an interval in $[0, L_{\ell+1})$.

Definition 7.8. Here we give the definition of the family of all a_σ for Definition 7.2.

(1) For each $\ell \in \omega$, choose a fixed ℓ-structure

$$\langle \{a^\ell_x : x \in \mathcal{N}_\ell\}, \{Y^\ell_\rho : \rho \in \bigcup_{0<j\leq\ell} \mathcal{N}_{\ell,j} \cup \{\emptyset\}\}\rangle$$

so that the base set $Y^\ell_\emptyset$ is an initial segment of integers.

(2) For each integer m, choose $\ell = \ell_m$ maximal so that $L_{\ell_m+1} \leq 2^m$ (hence the interval $[2^m, 2^{m+1})$ will support an ℓ-structure). For each $\sigma \in 2^{[2^m,2^{m+1})}$ such that there is an $x \in \mathcal{N}_\ell$ with $c_\ell(x) = |\sigma^{-1}(1)|$, define a_σ to be the pair obtained by adding 2^m to each member of a^ℓ_x. If there is no such $x \in \mathcal{N}_\ell$, let $a_\sigma = \{2^m, 2^m{+}1\}$.

(3) For each m and $\sigma \in 2^{[2^m,2^{m+1})}$, the pair $a_\sigma \subset [2^m, 2^{m+1})$.

Define the condition $p_0 \in \mathbb{P}_1$ by the prescription that for all m and $\ell = \ell_m$, $p_0 \restriction [2^m, 2^{m+1})$ is the partial function which is 0 on the segment $[2^m + |Y^\ell_\emptyset|, 2^{m+1})$. This ensures that for all $\sigma \in 2^{[2^m,2^{m+1})}$ which extend $p_0 \restriction [2^m, 2^{m+1})$, there will be an $x \in \mathcal{N}_\ell$ such that $c_\ell(x) = |\sigma^{-1}(1)|$.

Claim 7.8.1. p_0 forces that $\mathcal{W}$ is an ultrafilter.

Proof of Claim: Let $q < p_0$ and $Y \subset \mathbb{N}$. We must prove that there is a $p < q$ such that Y either contains, or is disjoint from, $W(p)$. We may assume that the sequence $\{k_m = |[2^m, 2^{m+1}) \setminus \mathrm{dom}(q)| : m \in M(q)\}$ diverges to infinity. For each $m \in M(q)$, let $q_m = q \restriction [2^m, 2^{m+1})$. Also, for $m \in M(q)$, let i_m be the largest integer so that $n_{i_m} < k_m/3$ (where n_i is defined in Definition 7.6). We note that $\{i_m : m \in M(q)\}$

also diverges to infinity. It then follows that for each $m \in M(q)$ there is a $\rho_m \in \mathcal{N}_{\ell_m, i_m}$ such that the interval $c_{\ell_m}[\rho_m]$ is contained in $[|q_m^{-1}(1)|, |q_m^{-1}(1)| + k_m)$. By Claim 7.7.1, there is an extension ψ_m of ρ_m with $\psi_m \in \mathcal{N}_{\ell_m, i_m+1}$ such that Y either contains, or is disjoint from, $W_m = 2^m + Y^{\ell_m}_{\psi_m}$ (i.e. $W_m = \bigcup\{a_\sigma : \sigma \in 2^{[2^m, 2^{m+1})}$ and $|\sigma^{-1}(1)| \in c_{\ell_m}[\psi_m]\}$). By symmetry, we may assume that there is an infinite $K \subset M(q)$ such that Y contains W_m for all $m \in K$. Let $p < q$ be any condition such that $p^{-1}(0) = q^{-1}(0)$ (no more 0's are added) and for each $m \in K$, the minimum element of $c_{\ell_m}[\psi_m]$ is the number of values in $[2^m, 2^{m+1})$ which are sent to 1 by p. Notice that for $m \in K$, $|[2^m, 2^{m+1}) \setminus \operatorname{dom}(p)| > n_{i_m+2}$ and so we may assume that $M(p) = K$. In other words, p will satisfy that $W(p) = \bigcup_{m \in M(p)} W_m$. This shows that p forces that Y contains a member of $\mathcal{W}$. □

Claim 7.8.2. Let p_0 be an element of the $\mathbb{P}_1$-generic filter G. In the forcing extension $V[G]$, $\mathcal{W}$ is the unique boundary point in $\mathbb{N}^*$ of each of the open sets $\bigcup\{A(p)^* : p \in G\}$ and $\bigcup\{B(p)^* : p \in G\}$.

Proof of Claim: The family $\{W(p) : p \in G\}$ is a base for the ultrafilter $\mathcal{W}$. Let $p \in G$ be arbitrary. By Claim 7.4.1, there is a $p' \in G$ such that $M(p) \setminus M(p')$ is infinite. By Claim 7.2, for each $m \in M(p) \setminus M(p')$, $\min(T_{p'}(m))$ is an element of $W(p) \cap A(p') \cap [2^m, 2^{m+1})$. It follows that $(W(p))^* \cap (A(p'))^*$ is non-empty. This proves that $\mathcal{W}$ is in the closure of $\bigcup\{A(p)^* : p \in G\}$. It follows analogously that $\mathcal{W}$ is in the closure of $\bigcup\{B(p)^* : p \in G\}$. To prove that $\mathcal{W}$ is the unique such point, we check that $A(p') \setminus A(p)$ is contained mod finite in $W(p)$ for all $p' \in G$. Let $r \in G$ be any common extension of p and p'. Choose an integer m_0 large enough so that $p \restriction (\mathbb{N} \setminus m_0) \cup p' \restriction (\mathbb{N} \setminus m_0)$ is contained in r. Note that $\bar{r} = r \restriction (\mathbb{N} \setminus 2^{m_0})$ is also a common extension of p and p' in G. By Definition 7.3, it follows that $A(p') \setminus A(r)$ is finite and that $M(r)$ is a subset of $M(p)$. Therefore $A(r) \setminus A(p)$ is a subset of $W(p)$. □

Claim 7.8.3. The Stone-Cech extension ψ^* of the finite-to-1 function ψ from Claim 7.7.1, satisfies that each of $\psi^* \restriction A$ and $\psi^* \restriction B$ are homeomorphisms onto $\mathbb{N}^*$.

Proof of Claim: It suffices to note that for each $p \in G$, $h_{p,A}$ is one-to-one and $h_{p,A} \subset \psi$. This implies that $\psi^* \restriction (A(p))^*$ is 1-to-1. In addition $\psi^*(\mathcal{W}) = \mathcal{M}$ is not an element of $\psi^*[(A(p))^*]$. □

Definition 7.9. Let $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ be defined as follows:

(1) $\varphi \restriction A$ is equal to $(\psi^* \restriction B)^{-1} \circ \psi^* \restriction A$.
(2) $\varphi \restriction B$ is equal to $(\psi^* \restriction A)^{-1} \circ \psi^* \restriction B$.
(3) $\varphi \restriction (\{\mathcal{W}\} \cup \mathbb{N}^* \setminus (A \cup B))$ is the identity function.

Claim 7.9.1. The mapping φ from Definition 7.9 is a homeomorphism.

Proof. The sets A, B, and $\{\mathcal{W}\} \cup (\mathbb{N}^* \setminus (A \cup B))$ is a closed cover of $\mathbb{N}^*$. Clearly φ is 1-to-1 and continuous on each of these closed sets. By Claim 7.7.1, $\varphi(\mathcal{W}) = \mathcal{W}$ holds in each of the three cases. □

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