SPACES OF CONTINUOUS FUNCTIONS OVER A Ψ-SPACE

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Dedicated to our friend Aleksander Vladimirovich Arhangel'skiĭ

ABSTRACT. The Lindelöf property of the space of continuous real-valued continuous functions is studied. A consistent example of an uncountable Ψ-like space is constructed for which the space of continuous real-valued functions with the pointwise convergence topology is Lindelöf.

All spaces considered in this paper will be Tychonoff. For a space X, $C_p(X)$ denotes, as usual, the space of all continuous real-valued functions with the topology of pointwise convergence, i.e., the topology of $C_p(X)$ is inherited from Tychonoff product $\mathbb{R}^X$.

It is well known that the Lindelöf property is met in the space $C_p(X)$ very rarely. If X is separable metrizable, then $C_p(X)$ is Lindelöf. Except for this classical one, there was no other theorem about Lindelöf function spaces for quite some time. Theorems in the literature proceed in the converse implication, they deduce properties of X from the fact that $C_p(X)$ is Lindelöf, cf. [A]. Quite recently, Raushan D. Buzyakova discovered another class of spaces, having Lindelöf space of continuous functions: For any ordinal α with the usual ordinal topology, if $X = \alpha \setminus \{\beta \in \alpha : \mathrm{cf}\,(\beta) > \omega\}$, then $C_p(X)$ is Lindelöf ([B]).

Our aim is to find other spaces, which are far from being metrizable, and still have the space of continuous functions Lindelöf. We are certainly motivated by the questions raised in ([B] and [A]). We were eventually led to study Ψ- and Ψ-like spaces from this point of view. Our main goal is to present two examples under the set-theoretical principle $\diamondsuit$ with this property. As a result we are able to answer some questions from [B].

Let $\mathcal{A}$ be an infinite maximal almost disjoint family on ω. A Ψ-space is a space $\Psi(\mathcal{A})$, whose underlying set is $\omega \cup \mathcal{A}$ and the topology is given by: All points from ω are isolated, the neighborhood basis at $A \in \mathcal{A}$ consists of all sets $\{A\} \cup A \setminus K$, where K is a finite subset of ω [GJ, Exercise 5I]. If we relax maximality and consider only an uncountable almost disjoint family $\mathcal{A}$, then we shall call the resulting space $\Psi(\mathcal{A})$ a Ψ-like space.

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The Lindelöf property of $C_p(X)$ always fails for a Ψ-space X. We believe this to be a new result.

Proposition 1. *If $\mathcal{A}$ is a MAD family on ω, then $C_p(\Psi(\mathcal{A}))$ is not Lindelöf.*

Proof. Fix an arbitrary maximal almost disjoint family $\mathcal{A}$ on ω. For $A \in \mathcal{A}$, let $V_A = \{f \in C_p(\Psi(\mathcal{A})) : f(A) \neq 0\}$. For $k < m < \omega$, let $V_{k,m} = \{f \in C_p(\Psi(\mathcal{A})) :$ If $k \leq n < m$, then $f(n) < \frac{1}{2}$ and $f(m) < \frac{1}{1+k}\}$. Let $\mathcal{V} = \{V_A : A \in \mathcal{A}\} \cup \{V_{k,m} : k < m < \omega\}$.

We shall show that $\mathcal{V}$ is an open cover of $C_p(\Psi(\mathcal{A}))$ without a countable subcover.

If f is a continuous function on $\Psi(\mathcal{A})$, then either there is some $A \in \mathcal{A}$ with $f(A) \neq 0$; in this case, f is then in V_A. Or for every $A \in \mathcal{A}$, $f(A) = 0$, and, by the maximality of $\mathcal{A}$, $\lim_{n\to\infty} f(n)$ exists and equals to 0. So there is some k such that $f(n) < \frac{1}{2}$ for all $n \geq k$ and there is some $m > k$ such that $f(n) < \frac{1}{1+k}$ for all $n \geq m$. For this pair k, m, $f \in V_{k,m}$.

Consider a countable subfamily $\mathcal{W} \subseteq \mathcal{V}$. Since $\mathcal{A}$ is uncountable, there is some $A \in \mathcal{A}$ with $V_A \notin \mathcal{W}$. Consider the following function g: $g(n) = 1$ for all $n \in A$, $g(A) = 1$, $g(n) = \frac{1}{1+|A\cap n|}$ for $n \notin A$, $g(B) = 0$ for $B \in \mathcal{A} \setminus \{A\}$. Clearly, g is a continuous function on $\Psi(\mathcal{A})$. Since for every $B \in \mathcal{A}$, if $B \neq A$, then $g(B) = 0$, we have that g cannot belong to V_B for $V_B \in \mathcal{W}$.

If $k < m < \omega$, then $g \notin V_{k,m}$: If there is some $n \in A$, $k \leq n < m$, then $g(n) = 1$ and so $g \notin V_{k,m}$. But if for all n, $k \leq n < m$, we have $n \notin A$, then $A \cap m \subseteq k$ and we have either $g(m) = 1$ or $g(m) \geq \frac{1}{1+k}$, depending on whether m belongs to A or not. In both cases, $g \notin V_{k,m}$.

So $C_p(\Psi(\mathcal{A}))$ is not Lindelöf. □

It is perhaps of some interest that if one restricts to the subspace of two-valued continuous functions, namely the space $C_p(\Psi(\mathcal{A}), \{0,1\})$, then it may or may not be Lindelöf, depending on the set-theory, as illustrated in the following two statements.

Theorem 2. *Assume $\diamondsuit$. Then there is a maximal almost disjoint family $\mathcal{A}$ on ω such that $C_p(\Psi(\mathcal{A}), \{0,1\})$ is Lindelöf.*

Proof. Our aim is to construct the MAD family $\mathcal{A}$ in such a way that every continuous two-valued function on $\Psi(\mathcal{A})$ is almost constant on the subspace $\mathcal{A}$, i.e., either the set $\{A \in \mathcal{A} : f(A) = 0\}$ or the set $\{A \in \mathcal{A} : f(A) = 1\}$ is finite. Throughout the proof, we shall restrict our attention only to functions with $\{A \in \mathcal{A} : f(A) = 1\}$ being a finite set.

We shall construct an almost disjoint family $\mathcal{A}$ by induction and use an enumeration $\mathcal{A} = \{A_\alpha : \omega \leq \alpha < \omega_1\}$. If φ is a finite function on a subset of ω_1 with values in $\{0,1\}$, we shall interpret it as a code for an open set $V(\varphi) \subseteq C_p(\Psi(\mathcal{A}), \{0,1\})$ by the rule $V(\varphi) = \{f \in C_p(\Psi(\mathcal{A}), \{0,1\}) : f(k) = \varphi(k)$ for all $k \in \mathrm{dom}\,(\varphi) \cap \omega, f(A_\alpha) = \varphi(\alpha)$ for all $\alpha \in \mathrm{dom}\,(\varphi) \setminus \omega\}$.

Fix an enumeration $\{\varphi_\alpha : \alpha < \omega_1\}$ of the set of all finite functions $\bigcup\{2^K : K \in [\omega_1]^{<\omega}\}$. This, of course, gives also an enumeration $V_\alpha = V(\varphi_\alpha)$ of the open basis of the space of continuous two-valued functions on the future $\Psi(\mathcal{A})$.

Next, fix some enumeration $\{M_\alpha : \alpha \in \omega_1\}$ of $\mathcal{P}(\omega)$, which is via characteristic functions also an enumeration of $\{0,1\}^\omega$. Let every subset of ω appear in this enumeration cofinally many times.

Also, for every β, $\omega + \omega \leq \beta < \omega_1$, choose and fix some bijection $b_\beta : \omega \longrightarrow \beta \setminus \omega$.

Finally, let $\langle S_\beta : \omega + \omega \leq \beta < \omega_1 \rangle$ be a diamond sequence.

We shall proceed by transfinite induction. Start with an arbitrary infinite partition of ω, say $A_{\omega+n} = \{2^n \cdot (2k+1) - 1 : k \in \omega\}$ for $n \in \omega$ and let $\mathcal{A}_{\omega+\omega} = \{A_{\omega+n} : n \in \omega\}$.

In each step β, $\omega + \omega \leq \beta < \omega_1$, we shall construct first two strictly increasing sequences $q_\beta(n)$, $k_\beta(n)$ of integers. We shall consider two cases, depending on the behaviour of the diamond instance S_β. Given a finite set $\mathcal{B} \subseteq \mathcal{A}_\beta$(=$\{A_\alpha : \omega \leq \alpha < \beta\}$), natural number ℓ and a finite set $K \subseteq \ell$, define a function $f_{(\mathcal{B},\ell,K)}$ on $\Psi(\mathcal{A}_\beta)$ by the rule $f(A) = 1$ for every $A \in \mathcal{B}$, $f(n) = 1$ for every $n \in K$ and also for every $n \in A$, $n \geq \ell$ with $A \in \mathcal{B}$. For the remaining $n \in \omega$ and $A \in \mathcal{A}_\beta$, the value of f at n (at A, resp.) will be 0.

Case 1: For every $\alpha \in S_\beta$, $\mathrm{dom}\,(\varphi_\alpha) \subseteq \beta$ and the family $\{V_\alpha : \alpha \in S_\beta\}$ covers all functions $f_{(\mathcal{B},\ell,K)}$, where $\mathcal{B} \in [\mathcal{A}_\beta]^{<\omega}$, $\ell \in \omega$ and $K \subseteq \ell$.

Put $q_\beta(0) = 0$, next, if $q_\beta(n)$ is known, we have a finite set of functions

$$F_n^\beta = \{f_{(\mathcal{B},\ell,K)} : \mathcal{B} \subseteq \{A_\alpha : \alpha \in b_\beta[q_\beta(n)]\}, \ell \leq q_\beta(n), K \subseteq \ell\}.$$

For each $f \in F_n^\beta$, choose $\alpha(f) \in S_\beta$ with $f \in V_{\alpha(f)}$ and let $q_\beta(n+1)$ be the smallest integer bigger than $q_\beta(n)$ such that for every $f \in F_n^\beta$, $\mathrm{dom}\,(\varphi_{\alpha(f)}) \cap \omega \subseteq q_\beta(n+1)$.

Case 2: Not Case 1. Put $q_\beta(n) = n$ in this case.

We already know sequences k_α for $\omega+\omega \leq \alpha < \beta$. Choose the sequence k_β in such a way that for every function g from the countable list $\{q_\beta\} \cup \{k_\alpha : \omega+\omega \leq \alpha < \beta\}$ there is some j such that for every $n \geq j$, the set of values $\{g(i) : k_\beta(n) < g(i) < k_\beta(n+1)\}$ is of size at least n.

Finally, it remains to define the set A_β. Since the set $\{A_\alpha : \omega \leq \alpha < \beta\}$ is countable, one may reenumerate it as $\{B_n : n \in \omega\}$. A standard induction allows one to pick the n'th point of A_β outside of the union $\bigcup_{i<n} B_i$, to ensure that for every $n \in \omega$, $A_\beta \cap k_\beta(n+1) \setminus k_\beta(n)$ contains at most one point and also, whenever possible, to get $|A_\beta \cap M_\beta| = \omega$ and $|A_\beta \setminus M_\beta| = \omega$.

This completes the inductive definitions.

Clearly, we arrived in a maximal almost disjoint family. Almost disjointness follows from the inductive definitions; if $X \in [\omega]^\omega$, then X appeared in our enumeration as M_β. If $M_\beta \cap A_\alpha$ was finite for all $\alpha < \beta$, then it was possible to get $A_\beta \cap M_\beta$ infinite. So $\mathcal{A}$ is maximal.

Let us show that no continuous two-valued function f on $\Psi(\mathcal{A})$ can satisfy $|\{A \in \mathcal{A} : f(A) = 0\}| = \omega = |\{A \in \mathcal{A} : f(A) = 1\}|$. Consider a set $X \subseteq \omega$ such that both sets $\{A \in \mathcal{A} : |X \cap A| = \omega\}$, $\{A \in \mathcal{A} : |A \setminus X| = \omega\}$ are infinite. Then there is some $\beta < \omega_1$ such that $M_\beta = X$ and, since the set X was listed cofinally many times, we have also that $\{A \in \mathcal{A}_\beta : |A \cap X| = \omega\}$ and $\{A \in \mathcal{A}_\beta : |A \setminus X| = \omega\}$ are infinite. But this means that the set A_β was chosen so that $A_\beta \cap X$ is infinite as well as $A_\beta \setminus X$. Consequently, if $f^{-1}(0) \supseteq X$ and $f^{-1}(1) \supseteq \omega \setminus X$, then the mapping f is discontinous at A_β.

It remains to show that $C_p(\Psi(\mathcal{A}), \{0,1\})$ is Lindelöf. Let $\mathcal{V}$ be an open cover of $C_p(\Psi(\mathcal{A}), \{0,1\})$. We may and shall assume that there is a subset $I \subseteq \omega_1$ such that $\mathcal{V} = \{V_\alpha : \alpha \in I\}$.

Consider the following set $C \subseteq \omega_1$: $\beta \in C$ iff

(i) for each $\alpha \in \beta \cap I$, $\mathrm{dom}\,(\varphi_\alpha) \subseteq \beta$;

(ii) whenever $\mathcal{B}$ is a finite subset of $\mathcal{A}_\beta$, $\ell < \omega$ and $K \subseteq \ell$, then there is an $\alpha \in \beta \cap I$ with $f_{(\mathcal{B},\ell,K)} \in V_\alpha$.

The set C is obviously closed unbounded in ω_1.

The set $S = \{\beta \in \omega_1 : S_\beta = I \cap \beta\}$ is a stationary subset of ω_1, so select an ordinal $\beta \in C \cap S$. For this β we have that $\mathcal{W} = \{V_\alpha : \alpha \in S_\beta\}$ is a countable subset of $\mathcal{V}$. Let us prove that it covers all functions from $C_p(\Psi(\mathcal{A}), \{0,1\})$ which attain the value 1 at finite number of members of $\mathcal{A}$ only.

To this end, pick such a continuous f on $\Psi(\mathcal{A})$ arbitrarily. Denote by $\mathcal{B}$ the set of all $A \in \mathcal{A}_\beta$ with $f(A) = 1$ and by $\mathcal{D}$ the set of all $A \in \mathcal{A} \setminus \mathcal{A}_\beta$ with $f(A) = 1$.

The set $\mathcal{B} \cup \mathcal{D}$ is finite. If $\mathcal{D}$ is empty, it is enough to select $\ell < \omega$ so big that that for all $n \geq \ell$, $f(n) = 1$ if and only if $n \in A$ for some $A \in \mathcal{B}$, and to put $K = \ell \cap f^{-1}(1)$. We have now that $f = f_{(\mathcal{B},\ell,K)}$ and by (ii) and by the fact that $\beta \in C$, f belongs to some member of $\mathcal{W}$.

If the set $\mathcal{D}$ is nonempty, $|\mathcal{D}| = m > 0$, then $\mathcal{D} = \{A_{\gamma(1)}, \ldots, A_{\gamma(m)}\}$ with each $\gamma(i)$ bigger or equal to β.

Notice that for every $\gamma \geq \beta$, our construction of the set A_γ guaranteed that for each $n < \omega$, $|A_\gamma \cap k_\gamma(n+1) \setminus k_\gamma(n)| \leq 1$. Also, we made sure that there was some $j = j(\gamma)$ with $\{k_\beta(i) : k_\gamma(n) < k_\beta(i) < k_\gamma(n+1)\}| \geq n$ whenever $n \geq j$. So, if j is bigger that $\max\{j(\gamma(1)), \ldots, j(\gamma(m))\}$ and i is so big that $k_\beta(i) > \max\{k_{\gamma(1)}(j), \ldots, k_{\gamma(m)}(j)\}$, then for every $A \in \mathcal{D}$, $|A \cap k_\beta(i+1) \setminus k_\beta(i)| \leq 2$ (typically, $|A \cap k_\beta(i+1) \setminus k_\beta(i)| \leq 1$, but one must make allowance for the case when $k_\beta(i) < k_\gamma(n) < k_\beta(i+1)$ and the two consecutive points of A_γ were chosen in intervals $k_\gamma(n) \setminus k_\beta(i)$, $k_\beta(i+1) \setminus k_\gamma(n)$).

Let $p \in \omega$ be such that $(\bigcup \mathcal{B} \cap \bigcup \mathcal{D}) \setminus p = \emptyset$ and for every $n \geq p$, $f(n) = 1$ if and only if $n \in \bigcup \mathcal{B} \cup \bigcup \mathcal{D}$.

Let $r \in \omega$ be big enough for $b_\beta[r]$ to contain all $\alpha < \beta$ with $A_\alpha \in \mathcal{B}$.

Choose now $n \in \omega$ such that $n > 2m+1$, $k_\beta(n) > \max\{k_{\gamma(1)}(j), \ldots, k_{\gamma(m)}(j)\}$, $k_\beta(n) \geq r$, $k_\beta(n) \geq p$ and $|\{q_\beta(i) : k_\beta(n) < q_\beta(i) < k_\beta(n+1)\}| \geq n$. The number of intervals $q_\beta(i+1) \setminus q_\beta(i)$, which are contained in the interval $k_\beta(n+1) \setminus k_\beta(n)$ is bigger than $2m$ and there are only m many sets in $\mathcal{D}$, each meeting an inteval $k_\beta(n+1) \setminus k_\beta(n)$ in at most two points. Consequently, there must be some $\tilde{\imath}$ with $k_\beta(n) < q_\beta(\tilde{\imath}) < q_\beta(\tilde{\imath}+1) < k_\beta(n+1)$ and with $A \cap q_\beta(\tilde{\imath}+1) \setminus q_\beta(\tilde{\imath}) = \emptyset$ for all $A \in \mathcal{D}$.

Put $\ell = q_\beta(\tilde{\imath})$ and $K = \ell \cap f^{-1}(1)$. The mapping $f_{(\mathcal{B},\ell,K)}$ belongs to some V_α with $\alpha \in S_\beta$. Since $\beta \in C$, we have that $V_\alpha \in \mathcal{W}$. We, however, have also that $f_{(\mathcal{B},\ell,K)}$ belongs to the set $F^\beta_{\tilde{\imath}}$, which implies that α could be chosen to satisfy that $\operatorname{dom}(\varphi_\alpha) \cap \omega \subseteq q_\beta(\tilde{\imath}+1)$. By the choice of $\tilde{\imath}$, no $A \in \mathcal{D}$ meets the interval $q_\beta(\tilde{\imath}+1) \setminus q_\beta(\tilde{\imath})$, so f and $f_{(\mathcal{B},\ell,K)}$ agree on $\operatorname{dom}(\varphi_\alpha)$. This however means that $f \in V_\alpha$ and concludes the proof. □

Remark. In the previous example, all two-valued continuous functions on $\Psi(\mathcal{A})$ attained one of the values on a compact (or empty) subset of $\Psi(\mathcal{A})$. Hence for this $\mathcal{A}$, $\beta(\Psi(\mathcal{A})) \setminus \Psi(\mathcal{A})$ consists of precisely one point. It should be remarked that a slightly more complicated construction can provide the family $\mathcal{A}$ such that the Čech-Stone remainder of the resulting space is homeomorphic to any compact 0-dimensional metric space given in advance.

The opposite situation occurs if $\mathfrak{b} > \omega_1$. An explicit statement follows.

Proposition 3. *Assume $\mathfrak{b} > \omega_1$. If $\mathcal{A}$ is a MAD family on ω, then $C_p(\Psi(\mathcal{A}), \{0,1\})$ is not Lindelöf.*

Proof. Enumerate an uncountable subset of $\mathcal{A}$ as $\{A_\alpha : \alpha < \omega_1\}$. Let e_α be an increasing bijection from ω onto A_α. Since $\mathfrak{b} > \omega_1$, there is a function $f \in \omega^\omega$ dominating all e_α, $\alpha < \omega_1$. We are allowed to assume that f is strictly increasing.

If we define a mapping $g \in \omega^\omega$ by $g(0) = f(0)$, $g(n+1) = f(g(n)+1)$, then for each $\alpha < \omega_1$ there is some $j = j(\alpha)$ such that for every $n \geq j$, $A_\alpha \cap g(n+1) \setminus g(n) \neq \emptyset$.

Let j_0 be the minimal $j \in \omega$ such that $j = j(\alpha)$ for uncountably many $\alpha \in \omega_1$ and let $\mathcal{B} = \{A_\alpha : \alpha < \omega_1, j(\alpha) = j_0\}$.

The cover $\mathcal{V}$ which does not have a countable subcover consists of all sets

$V_A = \{f \in C_p(\Psi(\mathcal{A}), \{0,1\}) : f(A) = 1\}$ for A in $\mathcal{A}$,

$V(K) = \{f \in C_p(\Psi(\mathcal{A}), \{0,1\}) : f(n) = 1$ for all $n \in K$ and $f(n) = 0$ for all $n \in g(m(K)) \setminus K\}$, where K is a finite subset of ω and $m(K) = j_0 + 1$, if $\max K < j_0$, otherwise $m(K) = j+1$ for a minimal j satisfying $j_0 \leq \max K < j$.

Notice that $\mathcal{V}$ is a cover of $C_p(\Psi(\mathcal{A}), \{0,1\})$: If $f(A) = 1$ for some $A \in \mathcal{A}$, then $f \in V(A)$. If $f(A) = 0$ for all $A \in \mathcal{A}$, then by maximality of $\mathcal{A}$, then $K = f^{-1}(1)$ must be finite and $f \in V(K)$ then.

If $\mathcal{W}$ is a countable subfamily of $\mathcal{V}$, then there is some $B \in \mathcal{B}$ with $V(B) \notin \mathcal{W}$. However, a characteristic function h_B ($h_B(x) = 1$ iff $x \in B$ or $x = B$) is a continuous two-valued function and belongs to no $V(A) \in \mathcal{W}$, but it also belongs to no $V(K)$ for a finite $K \subseteq \omega$, because $h_B(n) = 1$ for some $n \in g(m(K)) \setminus g(m(K)-1)$. □

According to Proposition 1, we have to relax maximality, if we wish to get an almost disjoint family $\mathcal{A}$ with $C_p(X)$ Lindelöf. In what follows, we shall deal with a standard example of a (nonmaximal) uncountable almost disjoint family, namely, with the family of all branches in a full binary tree. Some notation is needed.

Let $\Sigma = \bigcup_{n\in\omega} 2^n$ and let $X = \Sigma \cup 2^\omega$. We shall equip X with two topologies, $\mathcal{T}$ (=tree topology) and $\mathcal{C}$ (=cone topology). In both topologies, the set Σ is the set of isolated points of X. If $x \in 2^\omega$, then the set $\{\{x \restriction n : k \leq n \leq \omega\} : k \in \omega\}$ is a neighborhood basis at x it the topology $\mathcal{T}$, and $\{y \in X : y \supseteq x \restriction k\} : k \in \omega\}$ is a neighborhood basis at x in the topology $\mathcal{C}$.

Notice that the subspace 2^ω of the space $(X, \mathcal{C})$ is homeomorphic to the Cantor set.

Lemma 4. *The space $C_p(X, \mathcal{T})$ is not Lindelöf.*

Proof. For $n \in \omega$, let $V_n = \{f \in C_p(X, \mathcal{T}) : (\forall p \in 2^n)\, f(p) \in (-1,1)\}$.

For $x \in 2^\omega$, let $V_x = \{f \in C_p(X, \mathcal{T}) : f(x) \in \mathbb{R} \setminus \{0\}\}$.

If $A \subseteq \Sigma$, $A = \{p, q\}$ and neither $p \subseteq q$ nor $q \subseteq p$ (i.e., A is an antichain in a tree order of Σ), let $V_A = \{f \in C_p(X, \mathcal{T}) : f(p) \in \mathbb{R} \setminus \{0\}, f(q) \in \mathbb{R} \setminus \{0\}\}$.

Let $\mathcal{V} = \{V_n : n \in \omega\} \cup \{V_x : x \in 2^\omega\} \cup \{V_A : A \subseteq \Sigma, |A| = 2, A$ is an antichain$\}$.

Clearly, $\mathcal{V}$ is a collection of open subsets of $C_p(X, \mathcal{T})$. Let us check that $\mathcal{V}$ is a cover of $C_p(X, \mathcal{T})$. If $f \in C_p(X, \mathcal{T})$, then either for some $n \in \omega$ and all $p \in 2^n$, $f(p) \in (-1,1)$. Then $f \in V_n$. Or for every $n \in \omega$, there is some $p \in 2^n$ with $|f(p)| \geq 1$. If there are two incomparable p, q like that, then $f \in V_{\{p,q\}}$. The remaining possibility is that for each $n \in \omega$ there is a unique $p_n \in 2^n$ with $|f(p_n)| \geq 1$ and for any $n < m < \omega$, $p_n \subseteq p_m$. Let $x = \bigcup_{n\in\omega} p_n$ then. By $\mathcal{T}$-continuity, $|f(x)| \geq 1$. So for this x, $f \in V_x$.

If $\mathcal{W}$ is a countable subset of $\mathcal{V}$, then there is some $x \in 2^\omega$ with $V_x \notin \mathcal{W}$. Define $f(p) = 1$ for all $p \subseteq x$, $f(p) = 0$ otherwise. Then $f \in C_p(X, \mathcal{T})$ and $f \notin \bigcup \mathcal{W}$. □

The following result is the main result of the paper and is the main step in answering question 3.5 of [B] (see Example 8).

Theorem 5. *Assume $\diamondsuit$. Then there is an uncountable subset $Z \subseteq 2^\omega$ such that $C_p(\Sigma \cup Z, \mathcal{T})$ is Lindelöf.*

Proof. We shall construct the set Z by a transfinite induction to ω_1. We have two topologies on X, $\mathcal{C}$ and $\mathcal{T}$, and we know that $C_p(X, \mathcal{T})$ is not Lindelöf. So both inclusions $C_p(X, \mathcal{C}) \subset C_p(X, \mathcal{T}) \subset C_p(\Sigma \cup Z, \mathcal{T})$ are proper. To keep the necessary control, we shall consider all real-valued functions defined on Σ. Of course, not all of them continuously extend to points from 2^ω. Given an $\varepsilon > 0$ and $f : \Sigma \longrightarrow \mathbb{R}$, let us denote $\mathrm{Osc}\,(f, \varepsilon, \mathcal{C})$ ($\mathrm{Osc}\,(f, \varepsilon, \mathcal{T})$, resp.) the set of all $x \in 2^\omega$ such that for every $\mathcal{C}$-open neighborhood ($\mathcal{T}$-open neighborhood, resp.) U of x there are $p, q \in U \cap \Sigma$ with $|f(p) - f(q)| \geq \varepsilon$. Next, put $\mathrm{Osc}\,(f, \mathcal{C}) = \bigcup_{\varepsilon > 0} \mathrm{Osc}\,(f, \varepsilon, \mathcal{C})$ and similarly, $\mathrm{Osc}\,(f, \mathcal{T}) = \bigcup_{\varepsilon > 0} \mathrm{Osc}\,(f, \varepsilon, \mathcal{T})$. Clearly, each set $\mathrm{Osc}\,(f, \varepsilon, \mathcal{C})$ is a closed subset of 2^ω with the usual topology of a Cantor set, and $\mathrm{Osc}\,(f, \mathcal{C})$ is an F_σ-set in 2^ω. Also, since the topology $\mathcal{T}$ is finer than the topology $\mathcal{C}$, we have $\mathrm{Osc}\,(f, \mathcal{T}) \subseteq \mathrm{Osc}\,(f, \mathcal{C})$.

Enumerate $\mathbb{R}^\Sigma$ as $\{f_\alpha : \alpha < \omega_1\}$.

Fix a countable basis $\mathcal{B}$ for the reals. The basis of Tychonoff product $\mathbb{R}^X$ consists of all sets of the form

$$V(\langle x_0, x_1, \ldots, x_k \rangle, \langle B_0, B_1, \ldots, B_k \rangle) = \{f \in \mathbb{R}^X : (\forall i \leq k)\, f(x_i) \in B_i\},$$

where $k \in \omega$, $x_0, x_1, \ldots, x_k \in X$ and $B_0, B_1, \ldots, B_k \in \mathcal{B}$.

Enumerate this basis for $\mathbb{R}^X$ as $\{V_\alpha : \alpha < \omega_1\}$.

Finally, let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$-sequence on ω_1 and let $\mathcal{V}_\alpha = \{V_\beta : \beta \in S_\alpha\}$.

During the induction, we shall define points x_α, meager sets M_α and mappings g_α as follows:

Assume x_β, M_β and g_β are known for all $\beta < \alpha < \omega_1$. Consider the family $\mathcal{V}_\alpha$ first. If there is a mapping $g : \Sigma \longrightarrow \mathbb{R}$ such that

(i) $\mathrm{Osc}\,(g, \mathcal{T}) \cap \{x_\beta : \beta < \alpha\} = \emptyset$, and
(ii) $\mathrm{Osc}\,(g, \mathcal{C})$ is a meager subset of 2^ω, and
(iii) a $\mathcal{T}$-continuous extension of g to the set $2^\omega \setminus \mathrm{Osc}\,(g, \mathcal{T})$ does not belong to $\bigcup \mathcal{V}_\alpha$,

denote this g as g_α and put $M_\alpha = \mathrm{Osc}\,(g_\alpha, \mathcal{C})$. If there is no mapping with the required properties, denote by g_α an arbitrary constant mapping and put $M_\alpha = \emptyset$.

Next, consider a mapping f_α. If there is a point $x \in 2^\omega \setminus \bigcup_{\beta \leq \alpha} M_\beta$, $x \notin \{x_\beta : \beta < \alpha\}$, such that f_α cannot be $\mathcal{T}$-continuously extended to x, denote this x as x_α. If there is no point like this,

let $x_\alpha \in 2^\omega$ be an arbitrary point not belonging to $\{x_\beta : \beta < \alpha\} \cup \bigcup_{\beta \leq \alpha} M_\beta$.

This completes the inductive definitions. It remains to denote $Z = \{x_\alpha : \alpha < \omega_1\}$.

We need to show that $C_p(\Sigma \cup Z, \mathcal{T})$ is Lindelöf. However, we need some information about continuous functions on $(\Sigma \cup Z, \mathcal{T})$ before.

Claim. *Let $f \in \mathbb{R}^\Sigma$. Then either* $\mathrm{Osc}\,(f \restriction \Sigma, \mathcal{C})$ *is a meager subset or 2^ω or for every countable family $\{D_n : n \in \omega\}$ of nowhere dense subsets of 2^ω, the set* $\mathrm{Osc}\,(f, \mathcal{T}) \setminus \bigcup_{n \in \omega} D_n$ *contains a perfect set.*

If the set $\mathrm{Osc}\,(f, \mathcal{C})$ is not meager, then there is some $\varepsilon > 0$ such that $\mathrm{Int}\,(\mathrm{Osc}\,(f, \varepsilon, \mathcal{C})) \neq \emptyset$. So there is some $p_\emptyset \in \Sigma$ such that $U_\emptyset = \{y \in 2^\omega : p_\emptyset \subseteq y\} \subseteq \mathrm{Int}\,(\mathrm{Osc}\,(f, \varepsilon, \mathcal{C})) \setminus D_0$.

Induction step: Suppose that for $n \in \omega$ and all $\sigma \in 2^n$ we have found points $p_\sigma \in \Sigma$ and pairwise disjoint open sets $U_\sigma = \{y \in 2^\omega : p_\sigma \subseteq y\}$ with the property that for $\sigma \subseteq \varrho$, $p_\sigma \subseteq p_\varrho$ and, consequently, $U_\varrho \subseteq U_\sigma$.

For $\sigma \in 2^n$, we have two disjoint $\mathcal{C}$-open sets: $G_0 = \{y \in 2^\omega : p_\sigma{}^\frown 0 \subseteq y\}$ and $G_1 = \{y \in 2^\omega : p_\sigma{}^\frown 1 \subseteq y\}$. Since the set D_n is closed and nowhere dense, there are some $t_i \in \Sigma$, $t_i \supseteq p_\sigma{}^\frown i$ for $i = 0, 1$ with $W_{t_i} = \{y \in 2^\omega : t_i \subseteq y\}$ disjoint from D_n. Since both the sets W_{t_0} and W_{t_1} are subsets of $\operatorname{Osc}(f, \varepsilon, \mathcal{C})$, there are two points $s_i, q_i \in W_{t_i}$ with $|f(s_i) - f(q_i)| \geq \varepsilon$, $i = 0, 1$. Let $p_{\sigma^\frown i}$ be the point from $\{s_i, q_i\}$ which satisfies $|f(p_\sigma) - f(p_{\sigma^\frown i})| \geq \frac{\varepsilon}{2}$. Put $U_{\sigma^\frown i} = \{y \in 2^\omega : p_{\sigma^\frown i} \subseteq y\}$. This completes the inductive definitions.

The set $P = \{\bigcup_{n\in\omega} p_{\varphi\restriction n} : \varphi \in 2^\omega\}$ is a perfect subset of 2^ω, disjoint from all D_n, $n \in \omega$, and the mapping f cannot be $\mathcal{T}$-continuously extended to points of P, since the sequence $\langle f(y \restriction n) : n \in \omega\rangle$ not Cauchy whenever $y \in P$.

The Claim is proved.

As an immediate consequence we have that every $f \in C_p(\Sigma \cup Z, \mathcal{T})$ the set $\operatorname{Osc}(f \restriction \Sigma, \mathcal{C})$ is meager. Indeed, $f \restriction \Sigma$ appears in our enumeration as f_α. The family $\{M_\beta : \beta \leq \alpha\}$ is a countable family of meager subsets of 2^ω, so if $\operatorname{Osc}(f_\alpha, \mathcal{C})$ were not meager, then by Claim, the point x_α would belong to $Z \cap \operatorname{Osc}(f_\alpha, \mathcal{T})$, and f would not be $\mathcal{T}$-continuous then.

It remains to show that $C_p(\Sigma \cup Z, \mathcal{T})$ is Lindelöf. Let $\mathcal{V}$ be an open cover of $C_p(\Sigma \cup Z, \mathcal{T})$. We assume that $\mathcal{V}$ consists of basic open sets and so for some $I \subseteq \omega_1$, $\mathcal{V} = \{V_\beta : \beta \in I\}$.

By $\diamondsuit$, the set $S = \{\alpha \in \omega_1 : \{V_\beta : \beta \in I \cap \alpha\} = \mathcal{V}_\alpha\}$ is stationary.

For $k, n \in \omega$, $y_0, y_1, \dots, y_k \in \Sigma \cup Z$, $G_0, G_1, \dots, G_n$ basic open subsets of $(X, \mathcal{C})$, and $B_0, B_1, \dots, B_k, B'_0, B'_1, \dots, B'_n \in \mathcal{B}$, let us denote

$$K(\langle y_0, \dots, y_k, G_0, G_1, \dots, G_n\rangle, \langle B_0, \dots, B_k, B'_0, B'_1, \dots, B'_n\rangle) = \\ = \{f \in C_p(\Sigma \cup Z, \mathcal{T}) : (\forall i \leq k)\, f(y_i) \in B_i \text{ and } (\forall i \leq n)\, f[G_i] \subseteq B'_i\}.$$

Define a set $C \subseteq \omega_1$ by: $\alpha \in C$ iff whenever $\{y_0, y_1, \dots, y_k\} \cap Z \subset \{x_\gamma : \gamma < \alpha\}$ and $K(\langle y_0, \dots, y_k, G_0, \dots, G_n\rangle, \langle B_0, B_1, \dots, B_k, B'_0, B'_1, \dots, B'_n\rangle)$ is nonempty and contained in V for some $V \in \mathcal{V}$, then there is such a $V \in \mathcal{V}_\alpha$.

It is again easy to see that the set C is closed unbounded in ω_1. Pick an $\alpha \in S \cap C$. Since $\alpha \in S$, we have that $\{V_\beta : \beta \in I \cap \alpha\} = \mathcal{V}_\alpha$, so $\mathcal{V}_\alpha$ is a countable subset of $\mathcal{V}$. So we need to show that the family $\mathcal{V}_\alpha$ covers $C_p(\Sigma \cup Z, \mathcal{T})$.

Suppose $\mathcal{V}_\alpha$ does not cover, i.e. there is a mapping $f \in C_p(\Sigma \cup Z, \mathcal{T}) \setminus \bigcup \mathcal{V}_\alpha$; denote by g the restriction $f \restriction \Sigma$. Since f is $\mathcal{T}$-continuous mapping on $\Sigma \cup Z$, we have $\operatorname{Osc}(g, \mathcal{T}) \cap Z = \emptyset$, which in particular means that $\operatorname{Osc}(g, \mathcal{T}) \cap \{x_\beta : \beta < \alpha\} = \emptyset$.

As a consequence of the Claim we know that $\operatorname{Osc}(g, \mathcal{C})$ is a meager subset of 2^ω. Also, the mapping f is a subset of a $\mathcal{T}$-continuous extension $\tilde{g}$ of a mapping g and f does not belong to $\bigcup \mathcal{V}_\alpha$, so $\tilde{g}$ does not belong to $\bigcup \mathcal{V}_\alpha$ as well.

We have verified that in the α-th step of the induction, all demands (i), (ii), (iii) were satisfied. Consider now the mapping g_α defined in this step. The mapping g_α can be $\mathcal{T}$-continuously extended to all $\{x_\beta : \beta < \alpha\}$ by (i) and can be even $\mathcal{C}$-continuously extended to all $\{x_\beta : \alpha \leq \beta < \omega_1\}$, because $\operatorname{Osc}(g_\alpha, \mathcal{C}) \cap \{x_\beta : \alpha \leq \beta < \omega_1\} = \emptyset$ by (ii) and by the choice of points x_β for $\alpha \leq \beta < \omega_1$. Let us call this extension h.

Since h is in $C_p(\Sigma \cup Z, \mathcal{T})$, there is some $V \in \mathcal{V}$ containing the mapping h. Let us write $V = V(\langle x_{\alpha(0)}, \dots, x_{\alpha(k)}, x_{\beta(0)}, \dots, x_{\beta(n)} \rangle, \langle B_0, \dots, B_k, B'_0, \dots, B'_n \rangle)$, where all $\alpha(i) < \alpha$ and all $\beta(i) \geq \alpha$.

For each $i \leq n$, let a real $\varepsilon(i) > 0$ be so small that $\{t \in \mathbb{R} : |f(x_{\beta(i)}) - t| \leq \varepsilon(i)\} \subseteq B'_i$, put $\varepsilon = \frac{1}{2}\min\{\varepsilon(0), \dots, \varepsilon(n)\}$. The set $\mathrm{Osc}\,(h, \varepsilon, \mathcal{C})$ is closed and meager and disjoint from the set $\{x_{\beta(0)}, \dots, x_{\beta(n)}\}$. So for each $i \leq n$ there is some $\mathcal{C}$-neighborhood G_i of a point $x_{\beta(i)}$, $G_i \cap \mathrm{cl}_{(X,\mathcal{C})}(\mathrm{Osc}\,(h, \varepsilon, \mathcal{C})) = \emptyset$.

If $x \in G_i$, then $|h(x) - h(x_{\beta(i)})| \leq \varepsilon$ and consequently $h(x) \in B'_i$. Therefore

$$h \in K(\langle x_{\alpha(0)}, \dots, x_{\alpha(k)}, G_0 \dots, G_n \rangle, \langle B_0, \dots, B_k, B'_0, \dots, B'_n \rangle).$$

Next, the inclusion

$$K(\langle x_{\alpha(0)}, \dots, x_{\alpha(k)}, G_0 \dots, G_n \rangle, \langle B_0, \dots, B_k, B'_0, \dots, B'_n \rangle) \subseteq V$$

trivially holds, because any function, which maps the set G_i onto a subset of B'_i, must map point $x_{\beta(i)} \in G_i$ into B'_i, too.

Since $\alpha \in C$ there must be some $V_\beta \in \mathcal{V}_\alpha$ with

$$h \in K(\langle x_{\alpha(0)}, \dots, x_{\alpha(k)}, G_0 \dots, G_n \rangle, \langle B_0, \dots, B_k, B'_0, \dots, B'_n \rangle) \subseteq V_\beta\ .$$

However, h is a $\mathcal{T}$-continuous extension of g_α, $h \in V_\beta$ and $V_\beta \in \mathcal{V}_\alpha$, which contradicts (iii) of our choice of g_α.

So $\mathcal{V}_\alpha$ is a cover of $C_p(\Sigma \cup Z, \mathcal{T})$ and consequently $C_p(\Sigma \cup Z, \mathcal{T})$ is Lindelöf. □

To provide a counterpart to Proposition 1, we have a strong belief that it may be consistent that no uncountable almost disjoint family $\mathcal{A}$ has Lindelöf $C_p(\Psi(\mathcal{A}))$. We can, however, prove the following weaker statement only.

Proposition 6. *Assume $\mathfrak{b} > \omega_1$. If $\mathcal{A}$ is an almost disjoint family on ω of size ω_1, then $C_p(\Psi(\mathcal{A}))$ is not Lindelöf. If, in addition $2^\omega < 2^{\omega_1}$, then $C_p(\Psi(\mathcal{A}), \{0,1\})$ is not Lindelöf.*

Proof. The space $\Psi(\mathcal{A})$ is separable and its closed discrete subspace $\mathcal{A}$ is of size ω_1. We shall consider two cases.

Case 1. The space $\Psi(\mathcal{A})$ is not normal.

We shall show that a closed subspace $C_p(\Psi(\mathcal{A}), \{0,1\})$ of $C_p(\Psi(\mathcal{A}))$ is not Lindelöf in this case.

Denote $\mathcal{B}$ and $\mathcal{C}$ two disjoint closed subsets of $\mathcal{A}$, which cannot be separated.

Observe that the assumption $\mathfrak{b} > \omega_1$ implies that whenever $\mathcal{A}_0$ is a countable subset of $\mathcal{A}$, then there is a continuous two-valued function on $\Psi(\mathcal{A})$, separating $\mathcal{A}_0$ from $\mathcal{A} \setminus \mathcal{A}_0$. — This is clear if $\mathcal{A}_0$ is finite. If $\mathcal{A}_0 = \{A_n : n \in \omega\}$ and each $A_n = \{a_{n,k} : k \in \omega\}$ with $a_{n,k} < a_{n,k+1}$ for all $k \in \omega$, define for $A \in \mathcal{A} \setminus \mathcal{A}_0$ a mapping $f_A \in \omega^\omega$ by the rule $f_A(n) = \min\{k : A \cap A_n \subseteq \{a_{n,i} : i < k\}\}$. Since $\mathfrak{b} > \omega_1$, there is a mapping $g \in \omega^\omega$ dominating all f_A, $A \in \mathcal{A} \setminus \mathcal{A}_0$. Let $h : \Psi(\mathcal{A}) \longrightarrow \{0,1\}$ be defined by $h(A) = 1$ for all $A \in \mathcal{A}_0$ and $h(a_{n,k}) = 1$ for all $n \in \omega$ and all $k \geq g(n)$ with $a_{n,k} \notin \bigcup_{i<n} A_i$, $g(x) = 0$ for the remaining $x \in \Psi(\mathcal{A})$. Clearly h is continuous and separates $\mathcal{A}_0$ from $\mathcal{A} \setminus \mathcal{A}_0$.

Applying the observation, we conclude that both $\mathcal{B}$ and $\mathcal{C}$ must be uncountable. So enumerate $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ and $\mathcal{C} = \{C_\alpha : \alpha < \omega_1\}$.

For $\alpha < \omega_1$, let

$$U_\alpha = \{f \in C_p(\Psi(\mathcal{A}), \{0,1\}) : f(B_\alpha) = 0\}$$
$$V_\alpha = \{f \in C_p(\Psi(\mathcal{A}), \{0,1\}) : f(B_\alpha) = f(C_\alpha) = 1\}$$

The collection $\mathcal{V} = \{U_\alpha, V_\alpha : \alpha < \omega_1\}$ is an open cover of $C_p(\Psi(\mathcal{A}), \{0,1\})$: By the choice of $\mathcal{B}$ and $\mathcal{C}$, there is no function separating them. So if $f \in C_p(\Psi(\mathcal{A}), \{0,1\})$, then there is some $\alpha < \omega_1$ with $f(B_\alpha) = f(C_\alpha)$ and consequently, $f \in U_\alpha \cup V_\alpha$ for this α.

However, no countable subcollection $\mathcal{W} \subseteq \mathcal{V}$ covers $C_p(\Psi(\mathcal{A}), \{0,1\})$: If α is such that $\mathcal{W} \subseteq \{U_\beta, V_\beta : \beta < \alpha\}$, applying the observation once more, find a continuous mapping f such that $f(B_\beta) = 1$ and $f(C_\beta) = 0$ for all $\beta < \alpha$. Clearly, $f \notin \bigcup \mathcal{W}$.

Case 2. The space $\Psi(\mathcal{A})$ is normal.

Now we shall show that a closed subspace $C_p(\Psi(\mathcal{A}), \omega)$ is not Lindelöf.

Enumerate $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$. For $\alpha < \beta < \omega_1$ and $n \in \omega$, let

$$U_{\alpha,\beta,n} = \{f \in C_p(\Psi(\mathcal{A}), \omega) : f(A_\alpha) = f(A_\beta) = n\}.$$

The collection $\mathcal{V} = \{U_{\alpha,\beta,n} : \alpha < \beta < \omega_1,\ n \in \omega\}$ is an open cover of $C_p(\Psi(\mathcal{A}), \omega)$. Apparently, each set $U_{\alpha,\beta,n}$ is an open subset of $C_p(\Psi(\mathcal{A}), \omega)$. If $f : \Psi(\mathcal{A}) \longrightarrow \omega$ is an arbitrary function, then there must be some $n \in \omega$ with $\{\alpha < \omega_1 : f(A_\alpha) = n\}$ uncountable. So there are $\alpha < \beta < \omega_1$ with $f \in U_{\alpha,\beta,n}$.

If $\mathcal{W}$ is a countable subset of $\mathcal{V}$, put

$$\gamma = \sup\{\beta \in \omega_1 : \text{for some } \alpha < \beta \text{ and } n \in \omega, U_{\alpha,\beta,n} \in \mathcal{W}\} + 1.$$

Since γ is a countable ordinal, there is a bijection $b : \gamma \longrightarrow \omega$. Define a mapping $g : \mathcal{A} \longrightarrow \omega$ by the rule $g(A_\alpha) = b(\alpha)$ for $\alpha < \gamma$ and $g(A_\alpha) = 0$ for $\gamma \leq \alpha < \omega_1$. The subspace $\mathcal{A}$ is a discrete subspace, so the mapping g is continuous, and $\mathcal{A}$ is a closed subspace in a normal space $\Psi(\mathcal{A})$, so g has a continuous extension f to the whole space $\Psi(\mathcal{A})$. It should be clear now that the mapping f cannot belong to any member of $\mathcal{W}$, since every possibility when $f \in U_{\alpha,\beta,n}$ implies $n = 0$ and $\beta \geq \gamma$.

In both cases, we succeeded to find a closed subspace of $C_p(\Psi(\mathcal{A}))$, which is not Lindelöf. The first statement in the proposition follows. The second statement follows immediately from Jones' Lemma, that with the hypothesis $2^\omega < 2^{\omega_1}$, a separable space with an uncountable closed discrete subset will not be normal. Therefore, the space $\Psi(\mathcal{A})$ will not be normal, and, as is shown in Case 1, $C_p(\Psi(\mathcal{A}), \{0,1\})$ is not Lindelöf. □

A very similar proof may serve to answer Question 3.1 from [B] — *Let X be countably compact and first countable. Assume also that the closure of any countable set is countable in X. Is then $C_p(X)$ Lindelöf?* — in the negative. In fact, the example below answers negatively also Buzyakova's questions 3.2 and 3.3.

Example 7. *There is a locally countable, locally compact, first countable, countably compact, zero-dimensional space X such that each countable subset of X has a countable closure and $C_p(X)$ is not Lindelöf.*

Proof. The space in question was already constructed by Jerry Vaughan in 1979 with its main properties summarized in the title [V], but its space of continuous functions has never been examined.

We shall not repeat the construction from Vaughan's paper, since we need only a few properties, as indicated below, to show that $C_p(X)$ is not Lindelöf.

The underlying set of the space X is $\omega_1\times(\omega+1)\times\{0,1\}$ equipped with a topology $\mathcal{T}$ such that the mapping $\pi : (X,\mathcal{T}) \longrightarrow \omega_1 \times (\omega+1)$ defined by $\pi(\alpha,\beta,0) = \pi(\alpha,\beta,1) = (\alpha,\beta)$ for all $\alpha\in\omega_1$, $\beta\in\omega+1$ is open and continuous. Consequently, for each $\alpha<\omega_1$, a subspace $(\alpha+1)\times(\omega+1)\times\{0,1\}$ is compact, and hence, the closure of every countable subset is countable.

Next, the sets $F = \{(\alpha,\omega,0) : \alpha<\omega_1\}$ and $H = \{(\alpha,\omega,1) : \alpha<\omega_1\}$ are both closed and cannot be separated, which makes the space $(X,\mathcal{T})$ nonnormal.

Let us show that the space $C_p(X)$ is not Lindelöf. Let for $\alpha<\omega_1$

$$\begin{aligned} U_\alpha &= \{f\in C_p(X) : f(\alpha,\omega,0)\neq 0 \text{ and } f(\alpha,\omega,1)\neq 0\}\\ V_\alpha &= \{f\in C_p(X) : f(\alpha,\omega,0)\neq 1 \text{ and } f(\alpha,\omega,1)\neq 1\}\\ W_\alpha &= \{f\in C_p(X) : f(\alpha,\omega,0)\neq 0 \text{ and } f(\alpha,\omega,1)\neq 1\}. \end{aligned}$$

The family $\mathcal{V} = \{U_\alpha : \alpha<\omega_1\}\cup\{V_\alpha : \alpha<\omega_1\}\cup\{W_\alpha : \alpha<\omega_1\}$ consists of open subsets of $C_p(X)$. Let us show that it covers $C_p(X)$. Fix an $\alpha<\omega_1$. If $f\in C_p(X)$ is such that $f\notin U_\alpha\cup V_\alpha\cup W_\alpha$, then necessarily $f(\alpha,\omega,0)=0$ and $f(\alpha,\omega,1)=1$. This, however, cannot happen for all $\alpha<\omega_1$, since otherwise the mapping f would separate closed sets F and H.

The family $\mathcal{V}$ has no countable subcover: Indeed, if $\mathcal{V}_0\subseteq\mathcal{V}$ is countable, put $\gamma = \sup\{\alpha : U_\alpha\in\mathcal{V}_0 \text{ or } V_\alpha\in\mathcal{V}_0 \text{ or } W_\alpha\in\mathcal{V}_0\}+1$. The set $(\gamma+1)\times(\omega+1)\times\{0,1\}$ is compact, hence normal, so there is a continuous mapping $g : (\gamma+1)\times(\omega+1)\times\{0,1\}\longrightarrow\mathbb{R}$ with $g(\alpha,\omega,0)=0$ and $g(\alpha,\omega,1)=1$ for $\alpha<\gamma+1$. However, the set $(\gamma+1)\times(\omega+1)\times\{0,1\}$ as a preimage of an open set $(\gamma+1)\times(\omega+1)$ under a continuous mapping π is open. So the mapping f, which agrees with g on $(\gamma+1)\times(\omega+1)\times\{0,1\}$ and equals to 0 in all remaining points from X, is continuous. Apparently, $f\in C_p(X)\setminus\bigcup\mathcal{V}_0$. So $C_p(X)$ is not Lindelöf. □

Buzyakova attributes her Question 3.5 to Arhangel'skiĭ. *Let $C_p(X\setminus\{x\})$ be Lindelöf for a space X and let x have a countable tightness in X. Is $C_p(X)$ Lindelöf? What if X is first countable?* We do not know the answer to the first-countable case, but we have already presented the space $X\setminus\{x\}$ in Theorem 5.

Example 8. *Assume $\diamondsuit$. Then there is a space X containing a point x such that x has a countable tightness, $X\setminus\{x\}$ is first countable, $C_p(X\setminus\{x\})$ is Lindelöf while $C_p(X)$ is not.*

Proof. Let X be a one-point compactification of the space $(\Sigma\cup Z,\mathcal{T})$ constructed in the proof of Theorem 5, denote $\{x\} = X\setminus(\Sigma\cup Z)$. We have proved above that $C_p(X\setminus\{x\})$ is Lindelöf. Clearly, the space X is countably tight at x.

It remains to show that $C_p(X)$ is not Lindelöf. The proof will be an analogy of the proof of Lemma 4, we shall even use the same kind of an open cover:

For $n\in\omega$, let $V_n = \{f\in C_p(X) : (\forall p\in 2^n)\, f(p)\in(-1,1)\}$.

For $z\in Z$, let $V_z = \{f\in C_p(X) : f(z)\in\mathbb{R}\setminus\{0\}\}$.

If $A\subseteq\Sigma$, $A=\{p,q\}$ and A is an antichain in a tree order of Σ, let $V_A = \{f\in C_p(X) : f(p)\in\mathbb{R}\setminus\{0\}, f(q)\in\mathbb{R}\setminus\{0\}\}$.

Let $\mathcal{V} = \{V_n : n\in\omega\}\cup\{V_z : z\in Z\}\cup\{V_A : A\subseteq\Sigma, |A|=2, A \text{ is an antichain}\}$.

The family $\mathcal{V}$ is an open cover of $C_p(X)$. Let $f\in C_p(X)$ be arbitrary. If $f(x)\neq 0$, then by the continuity at x there must ve some $z\in Z$ with $f(z)\neq 0$, too. So $f\in V_z$ for this z.

If $f(x) = 0$, we shall repeat the reasoning from the proof of Lemma 4: Either there is some $n \in \omega$ such that for all $p \in 2^n$, $f(p) \in (-1, 1)$ and $f \in V_n$ then. Or there are two incomparable $p, q \in \Sigma$ with $|f(p)| \geq 1$, $|f(q)| \geq 1$ and we have $f \in V_{\{p,q\}}$. Finally, if $f \notin V_n$ for all $n \in \omega$ and $f \notin V_A$ for each two-element antichain A in Σ, then there is a function $y \in 2^\omega$ with $|f(y \restriction n)| \geq 1$ for all $n \in \omega$. In this last case, if $y \in Z$, then $f \in V_y$. But y must belong to Z, for otherwise x is a cluster point of the set $\{y \restriction n : n \in \omega\}$ and $f(x) = 0$, which contradicts the continuity of f.

Consider now a countable subfamily $\mathcal{W} \subseteq \mathcal{V}$. Then there is some $z \in Z$ with $V_z \notin \mathcal{W}$. The mapping f with value 1 at z and at all $z \restriction n$, $n \in \omega$, and with value 0 in all remaining points from X, is continuous and does not belong to $\bigcup \mathcal{W}$. □

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