P-FILTERS AND COHEN, RANDOM, AND LAVER FORCING

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ABSTRACT. We answer questions about P-filters in the Cohen, random, and Laver forcing extensions of models of CH. In the case of the $\aleph_2$-random real poset, we prove that if $\square_{\aleph_1}$ also holds in the ground model, then there are P-points of ω^* in the extension. The majority of the paper investigates the question of whether ω^* can be covered by nowhere dense P-sets. We prove that this is not the case if $\aleph_2$-Cohen reals are added to a model of CH in which $\square_{\omega_1}$ holds, and that it is the case in the standard Laver extension. We also answer questions formulated by P. Nyikos about interactions between ultrafilter orderings of ω^ω and mod finite scales. We show they have connections to ultrafilters having non-meager P-subfilters.

1. INTRODUCTION

A P-point of a topological space X is a point with the property that is in the interior of every G_δ containing it. P-sets have been defined similarly. A subset of a topological space is nowhere dense if its closure has empty interior. Interest in nowhere dense P-sets may have begun when van Douwen and van Mill showed [7] that if a compact space K can be covered by nowhere dense P-sets, then the space $\omega \times K$ has no remote points (defined in [13]). This was the first example concerning remote points of its kind. The example of a space K cited in [7] was the space of uniform ultrafilters on ω_2. This raised the natural question of whether ω^* could be covered by nowhere dense P-sets. This was soon shown to be independent: it can not under CH ([15]) and consistently it can ([1]). In section 4, we produce a model in which CH fails and ω^* can not be covered by nowhere dense P-sets. This result was known to the author in 2005 but since that time it is shown in [10], that there is a model of MA plus $\mathfrak{c} = \aleph_2$ in which ω^* can not be covered by nowhere dense P-sets. We do not know the status under PFA.

By standard duality, a P-set in ω^* corresponds to a P-filter which we now introduce. A set a is almost included in a set b, denoted $a \subset^* b$, if $a \setminus b$ is finite. A set a is a pseudointersection for a family of sets if a is infinite and is almost included in every element of the family. A filter (base) on ω is a P-filter (base) if every countable subset of the filter has a pseudointersection that is also in the filter (base). A filter is nowhere dense if it is a free filter and has no pseudointersection. The statement that ω^* can be covered by nowhere P-sets is the Stone dual of the statement that every ultrafilter on ω has a nowhere dense P-filter. The dual notion of a P-filter is a P-ideal, and the dual notion of a nowhere dense filter is a dense ideal.

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A (descending) tower is a subset of $[\omega]^{\aleph_0}$ that is well-ordered by reverse almost inclusion and has no pseudointersection (equivalently, no proper end-extension in the well-ordering). A tower is a special type of nowhere dense P-filter base. A nowhere dense P-filter with character $\aleph_1$ will have a tower base. The NCF principle ([4]) implies that every ultrafilter has a nowhere dense P-subfilter, and, NCF plus $\mathfrak{c} = \aleph_2$ implies that every ultrafilter has a tower base. Another of the early interests in the study of nowhere dense P-filters is the connection to the (still unresolved) Scarborough-Stone problem as to whether there is a product of sequentially compact Tychonoff spaces that is not countably compact (see [21]). Let us also mention that [16] has many results concerning non-meager P-filters. One of these results is that it follows from $\diamondsuit$ that there is an ultrafilter with no non-meager P-subfilter. We note that the question raised in [16, Question 8] as to whether $\diamondsuit$ can be weakened to CH is answered in the affirmative in [15].

Concerning P-points of ω^*, a gap has been discovered in the paper [6] (see [5], see also [12]) and so it is not known if P-points exist in all models obtained by adding more than $\aleph_1$-many random reals to models of CH. It has been shown that if $\aleph_1$-many Cohen reals are added first, then P-points do exist in all random real forcing extension. In this paper, we show that having $\square_{\omega_1}$ in the ground model ensures that P-points exist if $\aleph_2$-random reals are added.

2. Iterated forcing and nowhere dense P-filters

We will be interested in proper forcings that preserve when a filter is a nowhere dense P-filter. Every proper forcing preserves the property of being a P-filter and so the property we seek is that of not adding a pseudointersection to any nowhere dense P-filter. A better known and studied property is known as being tower preserving. A poset P will be said to be tower preserving if each tower in the ground model remains a tower in the forcing extension by P. Since this is equivalent for nowhere dense P-filters of character $\aleph_1$, this notion will suffice for this paper.

The notion of a poset being tower preserving (in fact being nowhere dense P-filter preserving) fits the R-bounding preservation scheme in [23] (see also [22, VI3.10]) and so we have this preservation theorem from [22].

Proposition 2.1. *A countable support iteration of proper posets that are tower preserving is also tower preserving.*

In the case of ccc posets and finite support iterations, this next result is taken from [8].

Proposition 2.2. *A finite support iteraton of ccc tower preserving posets is tower preserving.*

Baumgartner showed that Mathias forcing [8, 3.2] and Hechler forcing [2] are tower preserving . It is folklore that Cohen forcing is tower preserving (but also follows from the fact that Hechler is tower preserving). It is implicit in [9, 3.3] that Laver forcing is tower preserving but this may not be the earliest reference (see also Corollary 7.5). Miller forcing (i.e. rational perfect set forcing) is also tower preserving (see [19]). It seems clear that we can state without references that each of these tower preserving posets is proper.

The following two iteration theorems are quite standard (for example see [11, 4.5] for a proof).

Proposition 2.3 (CH)**.** *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ be a finite support iteration of ccc posets of cardinality at most $\aleph_1$ with P_{ω_2} being the limit. If $\dot{\mathcal{A}}$ is a P_{ω_2}-name of a nowhere dense P-ideal on ω, then for any elementary submodel $M \prec H((2^{\aleph_2})^+)$, such that $\dot{\mathcal{A}}, P_{\omega_2} \in M$, $M^\omega \subset M$, and $|M| = \aleph_1$ then P_λ forces that $M \cap \dot{\mathcal{A}}$ is a tower, where $\lambda = M \cap \omega_2$.*

Proposition 2.4 (CH)**.** *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ be a countable support iteration of proper posets of cardinality at most $\aleph_1$ with P_{ω_2} being the limit. If $\dot{\mathcal{A}}$ is a P_{ω_2}-name of a nowhere dense P-ideal on ω and if $\dot{\mathcal{U}}$ is a P_{ω_2}-name of an ultrafilter on ω, then for any elementary submodel $M \prec H((2^{\aleph_2})^+)$, such that $\dot{\mathcal{A}}, \dot{\mathcal{U}}, P_{\omega_2} \in M$, $M^\omega \subset M$, and $|M| = \aleph_1$ then, for $\lambda = M \cap \omega_2$, P_λ forces that $M \cap \dot{\mathcal{A}}$ is a tower and $M \cap \dot{\mathcal{U}}$ is an ultrafilter.*

The following result can be used to show that every ultrafilter has a nowhere dense P-subfilter is consistent with either of $\mathfrak{s} < \mathfrak{b} = \mathfrak{d} = \mathfrak{c}$ and $\mathfrak{s} = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$. Of course, we already know that NCF can hold in models of $\mathfrak{s} = \mathfrak{b} < \mathfrak{d} = \mathfrak{c}$ ([4]) and $\mathfrak{b} < \mathfrak{s} = \mathfrak{d} = \mathfrak{c}$ ([3]).

Theorem 2.5 (CH)**.** *If $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ is a countable support iteration, of proper tower preserving posets of cardinality $\aleph_1$ and satisfies that $P_\lambda \Vdash \dot{Q}_\alpha$ is Miller forcing for all λ in a stationary subset of $\{\lambda \in \omega_2 : cf(\lambda) = \omega_1\}$, then P_{ω_2} forces that every ultrafilter on ω extends a tower of cofinality ω_1.*

Proof. Let S be the stationary set of λ in $\{\lambda \in \omega_2 : \mathrm{cf}(\lambda) = \omega_1\}$ such that $P_\lambda \Vdash \dot{Q}_\lambda$ is Miller forcing. It is shown in [4, Corollary 1], that for each $\lambda \in S$, P_λ forces that there is a $\dot{Q}_\lambda$-name $\dot{f}_\lambda$ of a finite-to-one function in ω^ω satisfying that $\dot{f}_\lambda[\mathcal{U}] = \dot{f}_\lambda[\mathcal{W}]$ for all ultrafilters $\mathcal{U}, \mathcal{W}$ on ω. Furthermore, Miller showed [19, 4.1,4.2] that if $\dot{\mathcal{W}}$ is a P_λ-name of a P-point ultrafilter on ω, then $P_{\lambda+1}$ forces that the filter generated by $\dot{\mathcal{W}}$ is again a P-point ultrafilter.

Now consider $\dot{\mathcal{U}}$ any P_{ω_2}-name of an ultrafilter on ω. By Proposition 2.4, there is a $\lambda \in S$ such that P_λ forces that $\dot{\mathcal{U}}_\lambda$ is an ultrafilter on ω where $\dot{\mathcal{U}}_\lambda$ is the set of pairs $(p, \dot{U})$ such that $p \in P_\lambda$, $\dot{U}$ is a P_λ-name, and $p \Vdash \dot{U} \in \dot{\mathcal{U}}$. Let G be a P_{ω_2}-generic filter and let, for $\mu < \omega_2$, $G_\mu = G \cap P_\mu$. Since $V[G_\lambda]$ is a model of CH, there is a P-point ultrafilter $\mathcal{W}$ in $V[G_\lambda]$. Let $f_\lambda = \mathrm{val}_{G_{\lambda+1}}(\dot{f}_\lambda)$ and $\mathcal{U}_\lambda = \mathrm{val}_{G_\lambda}(\dot{\mathcal{U}}_\lambda)$. It follows from the cited references in the first paragraph of the proof that $\mathcal{W}$ generates a P-point ultrafilter in $V[G_{\lambda+1}]$ and that $f_\lambda[\mathcal{W}] = f_\lambda[\mathcal{U}_\lambda]$. A finite-to-one image of an ultrafilter is an ultrafilter, so let $\mathcal{V}$ denote the ultrafilter $f_\lambda[\mathcal{W}]$. This is a P-point ultrafilter in a model of CH, and so contains a tower $\{a_\alpha : \alpha \in \omega_1\}$. This implies that the filter generated by $\mathcal{U}_\lambda$ also contains the tower $\{f_\lambda^{-1}[a_\alpha] : \alpha \in \omega_1\}$. By assumption, $P_\mu \Vdash \dot{Q}_\mu$ is tower preserving for all $\mu < \omega_2$, hence $\{f_\lambda^{-1}[a_\alpha] : \alpha \in \omega_1\} \subset \mathrm{val}_G(\dot{\mathcal{U}})$ remains a tower in $V[G]$. □

3. Square and models

In this section we establish some model theoretic consequences of the conjunction "CH + $\square_{\omega_1}$" that we can use in our forcing constructions in the following two sections. Throughout this section we assume that CH and $\square_{\aleph_1}$ hold.

For any set A of ordinals, A' denotes the set of limit ordinals α such that $A \cap \alpha$ is cofinal in α. Let $\lim(\omega_2)$ denote the set of limit ordinals in ω_2. Let $\vec{C} = \{C_\alpha : \alpha \in \lim(\omega_2) \cap \omega_2\}$ be a $\square_{\aleph_1}$-sequence (which means that each $C_\alpha \subset \alpha$ is a closed unbounded set of order-type at most ω_1 and, for $\beta \in C'_\alpha$, $C_\beta = C_\alpha \cap \beta$). Since we are assuming CH, we may fix a 1-to-1 function g_0 from ω_1 onto $H(\aleph_1)$ and a sequence $\vec{g} = \langle g_\alpha : \alpha \in \omega_2 \rangle$, where, for each $\alpha < \omega_1$, $g_\alpha = g_0$ and for $\omega_1 \leq \alpha \in \omega_2$, g_α is a bijection from α to ω_1.

For a countable elementary submodel M of $H(\kappa)$ (for any $\kappa \geq \omega_2$) let $\delta_M = M \cap \omega_1$ and $\mu_M = \sup(M \cap \omega_2)$. Also let ζ_M be the least countable ordinal satisfying that $g_0(\zeta_M)$ is a strictly increasing function in ω^ω and satisfies that $f <^* g_0(\zeta_M)$ for all $f \in M \cap \omega^\omega$.

Definition 3.1. *Let $\mathbb{M}$ denote the set of countable elementary submodels M of $(H(\aleph_2), \in, <_1, \vec{C}, \vec{g})$ satisfying that the order-type of C_{μ_M} is equal to δ_M. For $\delta \in \omega_1$, let $\mathbb{M}_\delta = \{M \in \mathbb{M} : \delta_M = \delta\}$.*

Proposition 3.2. *Let $M_1, M_2 \in \mathbb{M}$.*

1. *If $\delta_{M_1} = \delta_{M_2}$, then $\zeta_{M_1} = \zeta_{M_2}$.*
2. *If $\delta_{M_1} < \delta_{M_2}$, then $\zeta_{M_1} \in M_2$*

Proof. Let $Z = \{\zeta \in \omega_1 : g_0(\zeta) \in \omega^\omega\}$. For each $M \in \mathbb{M}$, $M \cap \omega^\omega = \{g_0(\zeta) : \zeta \in Z \cap \delta_M\}$. Therefore, if $\delta_{M_1} = \delta_{M_2}$, then $M_1 \cap \omega^\omega = M_2 \cap \omega^\omega$ and this implies that $\zeta_{M_1} = \zeta_{M_2}$. If $\delta_{M_1} < \delta_{M_2}$, then $Z \cap \delta_{M_1}$ is an element of M_2 and so, by elementarity, ζ_{M_1} is in M_2. ☐

Definition 3.3. *Let $D = \{\delta_M : M \in \mathbb{M}\}$ and, for each $\delta \in D$, let $h_\delta = g(\zeta_M)$ where $M \in \mathbb{M}$ and $\delta_M = \delta$.*

Lemma 3.4. *For each countable $S \subset H(\aleph_2)$, there is an $M \in \mathbb{M}$ such that $S \in M$ (simply $\mathbb{M}$ is a stationary subset of $[H(\aleph_2)]^{\aleph_0}$).*

Proof. Let $\{M_\gamma : \gamma \in \omega_1\}$ be a continuous $\in$-chain of countable elementary submodels of $(H(\aleph_2), \in, <_1, \vec{C}, \vec{g})$ such that $S \in M_0$. Now let $\tilde{M}$ be a countable elementary submodel of $H(\aleph_3)$ such that $\{M_\gamma : \gamma \in \omega_1\} \in \tilde{M}$. Note that $\bigcup\{M_\gamma \cap \omega_2 : \gamma \in \omega_1\}$ is an ordinal, λ, of cofinality ω_1 and that $\lambda \in \tilde{M}$. Let $\delta = \delta_{\tilde{M}}$ and $\mu = \mu_{M_\delta}$. Clearly $\mu \in C'_\lambda$ and $C_\lambda \cap \mu = C_\mu$ is a subset of $\tilde{M}$. By elementarity, the order-type of C_μ is not less than δ; so in fact the order-type of C_μ is δ. Also by elementarity, μ is equal to μ_{M_δ} and $\delta = \delta_{M_\delta}$. ☐

The main idea of the proof is this next simply property of $\mathbb{M}$.

Lemma 3.5. *If $M_1, M_2 \in \mathbb{M}$ and $\delta_1 = \delta_{M_1} \leq \delta_2 = \delta_{M_2}$, then for all $I \in M_1 \cap [\omega_2]^{\aleph_0}$, $I \cap M_2 \in M_2$ and is an initial segment of I.*

Proof. For each $\alpha \in (I \cup I') \cap M_2$, we have that $g_{\alpha+1} \in M_1 \cap M_2$ and $I \cap (\alpha+1) \in M_1$. Since there is $\xi < \delta_1$ such that $g_0(\xi) = I \cap (\alpha+1)$, it follows that $I \cap (\alpha+1)$ is also in M_2. Let $\zeta = \sup(I \cap M_2)$, by the above it suffices to prove that $\zeta \in M_2$. Since $C_\zeta \in M_1$, we have that the order-type of C_ζ is less than δ_1. Since $M_2 \in \mathbb{M}$ and $\delta_1 \leq \delta_2$, it then follows that $\zeta < \mu_{M_2}$, and so we may let λ be the minimum element of $M_2 \cap \omega_2 \setminus \zeta$. It follows that $C_\lambda \cap M_2$ is cofinal in ζ and so $C_\zeta = C_\lambda \cap \zeta$. By elementarity, $\zeta \in M_2$ since it is the unique ordinal γ in C'_λ such that the order-type of C_γ is equal to the order-type of C_μ. ☐

Lemma 3.6. *Let $M_1, \ldots, M_n$ be in $\mathcal{M}$ and enumerated so that $\delta_{M_i} \leq \delta_{M_{i+1}}$ for each $1 \leq i \leq n$. Suppose that, for each $1 \leq i \leq n$, $\mathcal{I}_i$ is a finite pairwise disjoint* *subset of* $M_i \cap [\omega_2]^{\aleph_0}$ *and satisfies that for each* $1 \leq i \leq j \leq n$ *and each* $I \in \mathcal{I}_i$*, if* $I \cap M_j$ *is not empty, then* $I = I \cap M_j \in \mathcal{I}_j$*.*

Then, for any $\lambda \in M_n$*, there is an injection* ψ *from* $\bigcup\{\bigcup \mathcal{I}_i : 1 \leq i \leq n\}$ *into* $M_n \cap \omega_2$ *satisfying that*

(1) ψ *extends the identity function on* $\bigcup \mathcal{I}_n$*,*
(2) $\psi \restriction I$ *is order-preserving into* $\omega_2 \setminus \lambda$*, and* $\psi[I] \in M_n$ *for each* $I \in \bigcup_{1 \leq i < n} \mathcal{I}_i$*.*

Proof. Let $\{\mathcal{I}_i : 1 \leq i \leq n\}$ be given as in the statement of the Lemma. We proceed by induction on n. Certainly there is nothing to prove in the case that $n = 1$.

We first prove the induction case when $n = 2$. If $M_1 \cap \omega_2 \subset M_2$ then $\mathcal{I}_1 \subset \mathcal{I}_2$ and there is nothing to prove. Otherwise, there is no loss to assume that there is some $I \in \mathcal{I}_1$ such that $I \setminus M_2$ is not empty. We let $\{J_k : k < m\}$ be any enumeration of those members of $\mathcal{I}_1$ that are disjoint from M_2. Let $\xi_k < \delta_1 \leq \delta_2$ be the order-type of J_k for each $k < m$. Fix any $\lambda \in M_2 \cap \omega_2$ large enough so that $I \subset \lambda$ for all $I \in \mathcal{I}_2$. Let ψ be the unique injection extending the identity function on $\bigcup \mathcal{I}_2$ where, ψ maps J_0 order-preserving onto $[\lambda, \lambda + \xi_0)$, and similarly J_k onto $[\lambda + \xi_0 + \cdots + \xi_{k-1}, \lambda + \xi_0 + \cdots + \xi_k)$. Clearly $\psi[J_k] \in M_2$ for each $k < m$.

Next, we apply the induction hypothesis on the pair $\mathcal{I}_{n-1}$ and $\mathcal{I}_n$, and we choose an injection ψ_n so that conditions (1)-(2) are satisfied. Again, choose any $\lambda \in M_n \cap \omega_2$ large enough so that $\psi_n[\bigcup \mathcal{I}_{n-1}] \cup \bigcup \mathcal{I}_n$ is a subset of λ. We are ready to define the mapping $\psi \supset \psi_n$. Clearly ψ extends the identity mapping on $\bigcup \mathcal{I}_n$ and property (2) will hold for all $I \in \mathcal{J}_{n-1}$.

For each $1 \leq i < n - 1$, let $\mathcal{J}_i$ be the elements of $\mathcal{I}_i$ that are disjoint from $M_{i+1} \cup \cdots \cup M_n$. For each $1 \leq i < n - 1$, let σ_i be any injection of $\bigcup \mathcal{J}_i^{n-1}$ onto a countable ordinal β_i, chosen so that, for each $J \in \mathcal{J}_i^{n-1}$, $\sigma_i^{-1}[J]$ is an interval and $\sigma_i^{-1} \restriction J$ is order-preserving. For notational convenience, let $\beta_0 = 0$. For each $1 \leq i < n - 1$ and each $\xi \in \bigcup \mathcal{J}_i^{n-1}$, define $\psi(\xi)$ to be $\lambda + \Sigma_{j<i}\beta_j + \sigma_i(\xi)$. It should be clear that ψ is well-defined and is an injection. Let I be any element of $\bigcup\{\mathcal{I}_i : i < n\}$. Choose j maximal such that $I \cap M_j$ is not empty. By assumption, $I = I \cap M_j \in \mathcal{I}_j$. If $j = n$, then $I \subset M_n$ and $\psi \restriction I$ is order-preserving. If $j = n - 1$, then $\psi \restriction I = \psi_n \restriction I$ is order-preserving by the induction hypothesis. If $j < n - 1$, the $\psi \restriction I = \sigma_i^{-1} \restriction I$ is assumed to be order-preserving. □

4. Cohen model and nowhere dense P-filters

In this section we prove the following theorem.

Theorem 4.1 ($\square_{\omega_1}$ and CH)**.** *In the forcing extension by* $Fn(\omega_2, 2)$*, there is an ultrafilter on* ω *that has no nowhere dense P-subfilter.*

For a set I, $\mathrm{Fn}(I, 2)$ denotes the usual Cohen poset consisting of finite partial functions from I into 2, as in [14]. The following lemma follows directly from Proposition 2.3.

Lemma 4.2 (CH)**.** *In the forcing extension by* $Fn(\omega_2, 2)$*, every nowhere dense P-filter has a nowhere dense P-subfilter of character* $\aleph_1$*.*

Definition 4.3. (1) *A* $Fn(\omega_2, 2)$*-name,* $\dot{a}$*, is a canonical name of a subset of* ω *if, for each* $n \in \omega$*, there is an antichain* $b_n(\dot{a}) \subset Fn(\omega_2, 2)$ *such that* $\dot{a}$ *is equal to* $\{(b_n(\dot{a}), \check{n}) : n \in \omega\}$*. (Shelah's definition of name). The support,* $\mathrm{supp}(\dot{a})$*, of a canonical name* $\dot{a}$ *is defined as* $\bigcup\{\mathrm{dom}(b_n(\dot{a})) : n \in \omega\}$*.*

(2) *If ψ is a permutation on ω_2, then we let $\widehat{\psi}$ denote the canonical lifting of ψ to a mapping on $Fn(\omega_2, 2)$ and to canonical $Fn(\omega_2, 2)$-names where, for $p \in Fn(\omega_2, 2)$, $\mathrm{dom}(\widehat{\psi}(p)) = \psi[\mathrm{dom}(p)]$ and $\widehat{\psi}(p)(\psi(\xi)) = p(\xi)$ for all $\xi \in \mathrm{dom}(p)$, and $b_n(\widehat{\psi}(\dot{a})) = \{\widehat{\psi}(p) : p \in b_n(\dot{a})\}$ for any canonical $Fn(\omega_2, 2)$-name $\dot{a}$ and $n \in \omega$.*

Proposition 4.4. *If $\{\dot{a}_i : i < \ell\}$ is a finite set of canonical $Fn(\omega_2, 2)$-names and $p \in Fn(\omega_2, 2)$, then p forces $\bigcap\{\dot{a}_i : i < \ell\}$ is infinite if and only if $\widehat{\psi}(p)$ forces that $\bigcap\{\widehat{\psi}(\dot{a}_i) : i < \ell\}$ is infinite.*

Proof. Let $\mathcal{Y} = \{\dot{y}_\xi : \xi \in \omega_2\}$ be a listing of all the canonical $\mathrm{Fn}(\omega_2, 2)$-names of infinite subsets of ω such that each such name is listed cofinally often. Let $\mathcal{T}$ be the collection of all functions $\tau \in \omega_2^{\omega_1}$ satisfying

(1) τ is a strictly increasing function,
(2) for $\alpha < \beta \in \omega_1$, 1 forces that $\dot{y}_{\tau(\beta)} \subset^* \dot{y}_{\tau(\alpha)}$,
(3) for all $\xi \in \omega_2$, there is an $\alpha \in \omega_1$ 1 forces that $\dot{y}_\xi \setminus \dot{y}_{\tau(\alpha)}$ is infinite.

Claim 1. For each $\tau \in \mathcal{T}$, 1 forces that $\mathcal{F}_\tau = \{\dot{y}_{g(\alpha)} : \alpha \in \omega_1\}$ is a base for a nowhere dense P-filter.

Claim 2. If $\dot{\mathcal{F}}$ is a $\mathrm{Fn}(\omega_2, 2)$-name and $p \in \mathrm{Fn}(\omega_2, 2)$ forces that $\dot{\mathcal{F}}$ is a nowhere dense P-filter, then there is a $\tau \in \mathcal{T}$ such that p forces that $\mathcal{F}_\tau$ is a subfilter of $\dot{\mathcal{F}}$.

For each $\tau \in \mathcal{T}$, choose any $M(\tau) \in \mathbb{M}$ satisfying that $\{\mathcal{Y}, \tau\} \in M(\tau)$ and let $\delta_\tau = \delta_{M(\tau)}$ and $\zeta_\tau = \tau(\delta_\tau)$. For each $\tau \in \mathcal{T}$, let $\dot{u}_\tau$ denote the canonical $\mathrm{Fn}(\omega_2, 2)$-name of $\omega \setminus \dot{y}_{\zeta_\tau}$. To be more precise, let $\dot{u}_\tau$ be equal to $\dot{y}_\xi$ where ξ is the minimum element of ω_2 satisfying that 1 forces that $\dot{y}_\xi = \omega \setminus \dot{y}_{\zeta_\tau}$.

Claim 3. For each $\xi \in M(\tau) \cap \omega_2$, there is an $\alpha \in \omega_1$ such that 1 forces that $\dot{y}_\xi \setminus \dot{y}_{\tau(\alpha)}$ is infinite.

Claim 4. Assume that $\tau_1, \ldots, \tau_\ell$ are elements of $\mathcal{T}$ enumerated so that $\delta_{\tau_1} \leq \delta_{\tau_2} \leq \cdots \leq \delta_{\tau_\ell}$. Then 1 forces that there is a sequence $\alpha_i < \delta_i$ $(1 \leq i \leq \ell)$ such that 1 forces that $\omega \setminus (\dot{a}_1 \cup \cdots \cup \dot{a}_\ell)$ is infinite, where, $\dot{a}_i = \dot{y}_{\tau_i(\alpha_i)}$ for $1 \leq i \leq \ell$.

Proof of Claim: For each $1 \leq i \leq \ell$, let M_i be used to denote $M(\tau_i)$. We proceed by induction on ℓ. So assume that $\alpha_1 \cdots \alpha_{\ell-1}$ have been found so that $\alpha_i < \delta_i$ and that 1 forces that $\omega \setminus (\dot{a}_1 \cup \cdots \cup \dot{a}_{\ell-1})$ is infinite where $\dot{a}_i = \dot{y}_{\tau_i(\alpha_i)}$. Of course $\dot{a}_i$ is an element of M_i for each $1 \leq i < \ell$.

For each $1 \leq i < \ell$, let I_i be the support of $\dot{a}_i$ and, to start, let $I_\ell = \omega$. By enlarging each I_i, we may assume, by Lemma 3.5, that $I_i \cap M_j \subset I_j$ for each $1 \leq i \leq j \leq \ell$. Let $\mathcal{I}_1 \in M_1$ be any finite partition of I_1 satisfying that $I \cap M_j$ is empty or equal to I for each $1 < j \leq \ell$. Similarly, let $\mathcal{I}_2 \in M_2$ be any finite partition of I_2 satisfying that $\mathcal{I}_1 \cap M_2 \subset \mathcal{I}_2$ and also that $I \cap M_j$ is empty or equal to I for each $I \in \mathcal{I}_2$ and $2 < j \leq \ell$. By recursion, we may similarly define $\mathcal{I}_i \in M_i$, for $1 \leq i \leq \ell$, a finite partition of I_i so that $\mathcal{I}_{i'} \cap M_i$ is a subset of $\mathcal{I}_i$ for $1 \leq i' < i$, and so that $I \cap M_j$ is empty or equal to I for all $I \in \mathcal{I}_i$ and $i < j \leq n$. The sequence $\{\mathcal{I}_i : 1 \leq i \leq \ell\}$ satisfies the hypotheses of Lemma 3.6. Now choose any $\lambda \in M_\ell \cap \omega_2$ large enough so that I_ℓ and $\bigcup\{\mathrm{supp}(\dot{y}_{\tau_\ell(\alpha)}) : \alpha \in \omega_1\} \in M_\ell$ are contained in λ. Now choose a bijection ψ on ω_2 that extends the injection in the conclusion of Lemma 3.6. By Proposition 4.4, 1 forces that $\omega \setminus (\widehat{\psi}(\dot{a}_1) \cup \cdots \cup \widehat{\psi}(\dot{a}_{\ell-1}))$ is infinite.

We check that $\widehat{\psi}(\dot{a}_i) \in M_\ell$ for each $1 \leq i < \ell$. Fix $1 \leq i < \ell$, let $m = |\mathcal{I}_i|$ and let $\{I(i,j) : j < m\}$ be any enumeration of $\mathcal{I}_i$. Similar to the proof of Lemma 3.6, let $<_i$ denote the ordering of $\bigcup \mathcal{I}_i$ where $<_i$ agrees with the usual ordering on each $I(i,j)$ $(j < m)$ and every element of $I(i,j)$ is less than each element of $I(i,j+1)$ for $j+1 < m$. It follows that $<_i \in M_i$ is a well-ordering of $\bigcup \mathcal{I}_i$ and the transitive collapse, $\sigma_i \in M_i$, of $<_i$ sends $\bigcup \mathcal{I}_i$ to an ordinal $\gamma_i < \delta_i \leq \delta_\ell$. Now we have that $\widehat{\sigma_i}(\dot{a}_i)$ (with the obvious meaning of $\widehat{\sigma_i}$) is a $\mathrm{Fn}(\gamma_i, 2)$-name in $M_i \cap H(\aleph_1)$ and so there is $\xi < \delta_i$ such that $g_0(\xi) = \widehat{\sigma_i}(\dot{a}_i)$. Since $g_0 \in M_i \cap M_\ell$, it now follows that $\dot{c}_i = \widehat{\sigma_i}(\dot{a}_i) \in M_\ell$. Although the mapping ψ is not an element of M_ℓ, the mapping $\psi \circ \sigma_i^{-1}$ is an element of M_ℓ, and it follows that $(\widehat{\psi \circ \sigma_i^{-1}})(\dot{c}_i)$ is an element of M_ℓ. Since $\widehat{\sigma_i^{-1}}(\dot{c}_i)$ is equal to $\dot{a}_i$, it follows that $\widehat{\psi}(\dot{a}_i) \in M_\ell$, since $\widehat{\psi}(\dot{a}_i) = (\widehat{\psi \circ \sigma_i^{-1}})(\dot{c}_i)$.

Now we have, by elementarity, that there is a $\xi \in M_\ell \cap \omega_2$ such that 1 forces that $\dot{y}_\xi$ is equal to $\omega \setminus (\widehat{\psi}(\dot{a}_1) \cup \cdots \cup \widehat{\psi}(\dot{a}_{\ell-1}))$. By Claim 3, there is an $\alpha_\ell < \delta_\ell$ such that 1 forces that $\dot{y}_\xi \setminus \dot{y}_{\tau_\ell(\alpha_\ell)}$ is infinite. With $\dot{a}_\ell = \dot{y}_{\tau_\ell(\alpha_\ell)}$, we have that 1 forces that $\omega \setminus (\widehat{\psi}(\dot{a}_1) \cup \cdots \cup \widehat{\psi}(\dot{a}_{\ell-1}) \cup \dot{a}_\ell)$ is infinite. Let $J = \mathrm{supp}(\dot{a}_\ell) \setminus \bigcup \mathcal{I}_n$. We note that $J \subset \lambda$ and $\psi[J] \cap \psi[I] = \emptyset$ for all $I \in \bigcup\{\mathcal{I}_i : 1 \leq i \leq \ell\}$. Therefore, there is another permuation φ of ω_2 such that $\varphi \restriction J$ is the identity, and $\psi \restriction I \subset \varphi$ for all $I \in \bigcup\{\mathcal{I}_i : 1 \leq i \leq \ell\}$. It follows that $\widehat{\varphi}(\dot{a}_i) = \widehat{\psi}(\dot{a}_i)$ for $1 \leq i < \ell$ and that $\widehat{\varphi}(\dot{a}_\ell) = \dot{a}_\ell$. By Proposition 4.4, we have that 1 forces that $\omega \setminus (\dot{a}_1 \cup \cdots \cup \dot{a}_\ell)$ is infinite, as required. □

We finish the proof of the theorem. By assumption (2) on $\tau \in \mathcal{T}$, we have that 1 forces that $\omega \setminus \dot{y}_{\tau(\alpha)} \subset^* \omega \setminus \dot{y}_{\zeta_\tau}$ for all $\tau \in \mathcal{T}$ and $\alpha < \delta_\tau$. Therefore, by Claim 4, 1 forces that the family $\{\omega \setminus \dot{y}_{\zeta_\tau} : \tau \in \mathcal{T}\}$ has the property that finite intersections are infinite. Therefore there is a $\mathrm{Fn}(\omega_2, 2)$-name, $\dot{\mathcal{U}}$, of a ultrafilter that 1 forces $\dot{U}$ includes the family $\{\omega \setminus \dot{y}_{\zeta_\tau} : \tau \in \mathcal{T}\}$. Evidently, 1 forces that, for each $\tau \in \mathcal{T}$, $\mathcal{F}_\tau$ is not a subfilter of $\dot{\mathcal{U}}$. By Claim 2, it follows that 1 forces that $\dot{U}$ does not have a nowhere dense P-subfilter. □

5. P-points in the Random model

In this section we prove the result about the existence of P-points in the random real model.

Definition 5.1. *We define the radom real poset $\mathcal{R}_{\omega_2}$ as follows.*

1. *For any countable subset I of ω_2, we let $\mathcal{R}_I$ denote the random real poset of Borel subsets of 2^I. The ordering is subset modulo measure 0. For convenience we view $\mathcal{R}_I$ as a subposet of $\mathcal{R}_J$ whenever $I \subset J \subset \omega_2$; in particular, when J is uncountable, $\mathcal{R}_J$ is the union of the family $\{\mathcal{R}_I : I \in [J]^{\aleph_0}\}$. For each $b \in \mathcal{R}_{\omega_2}$, we assume there is a canonical countable support of b, $\mathrm{supp}(b)$, such that $b \in \mathcal{R}_{\mathrm{supp}(b)}$, satisfying that $\mathrm{supp}(b) \subset I$ if $b \in \mathcal{R}_I$.*
2. *If ψ is a bijection between sets $I, J \subset \omega_2$, then we let $\widehat{\psi}$ denote the canonical lifting of ψ to an isomorphism from $\mathcal{R}_I$ to $\mathcal{R}_J$ and of canonical $\mathcal{R}_I$-names of ω to $\mathcal{R}_J$-names.*
3. *An $\mathcal{R}_{\omega_2}$-name, $\dot{a}$, is a canonical name of a subset of ω if, for each $n \in \omega$, there is an antichain $b_n(\dot{a}) \in \mathcal{R}_{\omega_2}$ such that $\dot{a}$ is equal to $\{(b_n(\dot{a}), \check{n}) : n \in \omega\}$. (Shelah's definition of name). The support of a canonical name $\dot{a}$, $\mathrm{supp}(\dot{a})$, is defined as $\bigcup\{\mathrm{supp}(b_n(\dot{a})) : n \in \omega\}$. As is customary, we may also use $[[n \in \dot{a}]]$ to denote $b_n(\dot{a})$.*

Lemma 5.2. *Let $M_1, \ldots, M_n$ be in $\mathcal{M}$ and enumerated so that $\delta_{M_i} \leq \delta_{M_{i+1}}$ for each $1 \leq i \leq n$. Suppose that, for each $1 \leq i \leq n$, $I_i \in M_i \cap [\omega_2]^{\aleph_0}$. Then there is a permuation ψ on ω_2 such that if, for each $1 \leq i \leq n$, $\{\dot{a}(i,k) : k \in \omega\} \in M_i$ is a set of canonical $\mathcal{R}_{I_i}$-names of subsets of ω, then $\widehat{\psi}(\dot{a}(n,k)) = \dot{a}(n,k)$ for all $k \in \omega$ and $\{\widehat{\psi}(\dot{a}(i,k)) : k \in \omega\} \in M_n$ for each $i < n$.*

Proof. Let $\{I_i : 1 \leq i \leq n\}$ be given. There is no loss of generality if we enlarge each I_i. By Lemma 3.5, we may assume that $I_i \cap M_j \subset I_j$, for each $1 \leq i \leq j \leq n$. Let $\mathcal{I}_1 \in M_1$ be any finite partition of I_1 satisfying that $I \cap M_j$ is empty or equal to I for each $1 < j \leq n$. Similarly, let $\mathcal{I}_2 \in M_2$ be any finite partition of I_2 satisfying that $\mathcal{I}_1 \cap M_2 \subset \mathcal{I}_2$ and also that $I \cap M_j$ is empty or equal to I for each $I \in \mathcal{I}_2$ and $2 < j \leq n$. By recursion, we may similarly define $\mathcal{I}_i \in M_i$, for $1 < i \leq n$, a finite partition of I_i so that for $1 \leq i' < i$, $\mathcal{I}_{i'} \cap M_i \subset \mathcal{I}_i$, and $I \cap M_j$ is empty or equal to I for all $I \in \mathcal{I}_i$ and $i < j \leq n$. The sequence $\{\mathcal{I}_i : 1 \leq i \leq n\}$ satisfies the hypotheses of Lemma 3.6, and so we may choose a bijection ψ on ω_2 that extends the injection in the conclusion of Lemma 3.6. For each $1 \leq i \leq n$, let $\{I(i,k) : k < m_i\}$ be an enumeration of $\mathcal{I}_i$ and let ρ_i be the bijection from I_i onto an ordinal $\beta_i \in \omega_1$ satisfying that ρ_i maps each $I(i,k)$ $(i < m_i)$ onto an interval in an order-preserving fashion. It follows that $\rho_i \in M_i$ and that for each canonical $\mathcal{R}_{I_i}$-name $\dot{a}$ of a subset of ω in M_i, $\widehat{\rho_i}(\dot{a})$ is also in M_i.

Now let, for each $1 \leq i \leq n$, $\{\dot{a}(i,k) : k \in \omega\} \in M_i$ be a sequence of canonical $\mathcal{R}_{I_i}$-names of subsets of ω. Since $\{\widehat{\rho_i}(\dot{a}(i,k))) : k \in \omega\}$ is also in M_i, it follows that there is a $\xi_i \in \delta_i$ such that $g_0(\xi_i) = \{\widehat{(\rho_i}(\dot{a}(i,k))) : k \in \omega\}$. Since $\delta_i \leq \delta_n$, $\{\widehat{(\rho_i}(\dot{a}(i,k))) : k \in \omega\}$ is in M_n. Even though ρ_i itself may not be in M_n, the function $(\rho_i \circ \psi^{-1}) \restriction \psi[I_i]$ is in M_n since the partition $\{\psi[I(i,k)] : k < m_i\}$ is in M_n. Simple diagram chasing shows that $(\widehat{\rho_i \circ \psi^{-1}})$ sends the sequence $\{\widehat{\psi}(\dot{a}(i,k)) : k \in \omega\}$ to $\{\widehat{\rho_i}(\dot{a}(i,k))) : k \in \omega\}$, allowing us to conclude that $\{\widehat{\psi}(\dot{a}(i,k)) : k \in \omega\}$ is in M_n. □

Definition 5.3. *For a monotone increasing unbounded function $h \in \omega^\omega$, let $r_h \in \omega^\omega$ denote the function satisfying that $h(r_h(k)) \leq k < h(1 + r_h(k))$ for all $k \in \omega$.*

If $h \in \omega^\omega$ is a monotone unbounded function, then r_h is a finite-to-one function.

Definition 5.4. *If $\vec{a} = \langle a_n : n \in \omega \rangle$ is a sequence of subsets of ω and if $r \in \omega^\omega$, then $\vec{a}_r$ is defined as the set $\{k : k \in a_{r(k)}\}$.*

Proposition 5.5. *Let $\vec{a} = \langle a_n : n \in \omega \rangle$ and $\vec{u} = \langle u_n : n \in \omega \rangle$ each be descending sequences of infinite subsets of ω such that $u_n \subset a_n$ for all n. Then there is a monotone increasing unbounded function $f \in \omega^\omega$ such that $f \leq^* h \in \omega^\omega$ where h is strictly increasing,*

(1) *$\vec{u}_{r_h}$ is infinite and $\vec{u}_{r_h} \subset \vec{a}_{r_h}$,*
(2) *$\vec{u}_{r_h} \setminus u_n$ is finite for all $n \in \omega$.*

Proof. It is immediate from the definition that $\vec{u}_r \subset \vec{a}_r$ for all $r \in \omega^\omega$. Similarly, if $r \in \omega^\omega$ and $r \leq^* r_f$, then $\vec{u}_r \supset^* \vec{u}_{r_f}$. If r is a finite-to-one function then $\vec{u}_r \setminus u_n$ is finite because $r(k) \geq n$ implies that $k \in u_n$.

Assume that $f, h \in \omega^\omega$ are strictly increasing functions with $f \leq^* h$. We check that $r_h \leq^* r_f$. Fix any n_0 so that $f(n) \leq h(n)$ for all $n \geq n_0$. Since each of r_f and r_h are finite-to-one, there is a k_0 such that each of $r_f(k_0)$ and $f_h(k_0)$ are greater than n_0. Consider any $k \geq k_0$ and recall that we have that $f(r_f(k)) \leq k < f(1 + r_f(k))$

and $h(r_h(k)) \leq k < h(1 + r_h(k))$. Since $f(r_h(k)) \leq h(r_h(k)) \leq k$, we have that $r_h(k) < 1 + r_f(k)$. Therefore to complete the proof of the proposition, we simply have to prove there is a strictly increasing $f \in \omega^\omega$ so that $\vec{u}_{r_f}$ is infinite. Define $f(n)$ by induction on n so that, for each n, there is a $k \in u_n$ such that $f(n) < k < f(n+1)$. It follows that $r_f(k) = n$ and that $k \in \vec{u}_{r_f}$. $\square$

If $\vec{a}$ is a sequence $\langle \dot{a}_m : m \in \omega \rangle$ of canonical $\mathcal{R}_{\omega_2}$-names of subsets of ω that is forced by 1 to be a descending sequence of infinite sets and if $r \in \omega^\omega$, then we let $\dot{\vec{a}}_r$ denote the canonical name for the set $\{k : k \in \dot{a}_{r(k)}\}$.

Lemma 5.6. *Let $\delta_1 \leq \cdots \leq \delta_n$ be members of D, and for each $1 \leq i \leq n$, let $M_i \in \mathbb{M}_{\delta_i}$. For all $1 \leq i \leq n$, let $\vec{a}^{\,i} = \langle \dot{a}(i,m) : m \in \omega \rangle \in M_i$ be a sequence of canonical $\mathbb{R}_{\omega_2}$-names of subsets of ω. For $1 \leq i \leq n$, let h_i denote h_{δ_i} and let $r_i = r_{h_i}$. Let $1 \leq n' \leq n$ be minimal such that $\delta_{n'} = \delta_n$.*
For each $m \in \omega$, assume that 1 forces that

(1) *$\dot{a}(i, m+1) \subset \dot{a}(i,m)$ for each $1 \leq i \leq n$, and*
(2) *$(\bigcap_{1 \leq i < n'} \dot{\vec{a}}^{i}_{r_i}) \cap \dot{a}(n', m) \cap \cdots \cap \dot{a}(n,m)$ is infinite.*

Then 1 also forces that $\vec{a}^{\,1}_{r_1} \cap \cdots \cap \dot{\vec{a}}^{n}_{r_n}$ is infinite.

Proof. For each $1 \leq i \leq n$, let $I_i \in M_i$ be chosen so that, for each $m \in \omega$, $\mathrm{supp}(\dot{a}(i,m)) \subset I_i$. Choose a permutation ψ as in Lemma 5.2. Since ψ is a permutation, we have that

$$\dot{w}_m = \Big(\bigcap_{1 \leq i < n'} \widehat{\psi}(\dot{\vec{a}}^{i}_{r_i}) \Big) \cap \widehat{\psi}(\dot{a}(n',m)) \cap \cdots \cap \widehat{\psi}(\dot{a}(n,m))$$

is forced by 1 to be infinite. Similarly, $\vec{w} = \langle \dot{w}_m : m \in \omega \rangle$ is forced by 1 to be a descending sequence. Now we argue that $\vec{w} \in M_n$. It follows from Lemma 5.2 that $\vec{u}_i = \langle \widehat{\psi}(\dot{a}(i,m)) : m \in \omega \rangle$ is in M_n for all $1 \leq i \leq n$. It will thus suffice to prove that $\widehat{\psi}(\dot{\vec{a}}^{i}_{r_i})$ is in M_n for each $1 \leq i < n'$. If $1 \leq i < n'$, then $\dot{\vec{u}}^{i}_{r_i}$ is in M_n since $\delta_i < \delta_n$ and, by Proposition 3.2, $r_i \in M_n$. Routine checking shows that $\widehat{\psi}(\dot{\vec{a}}^{i}_{r_i})$ is equal to $\dot{\vec{u}}^{i}_{r_i}$ for each $1 \leq i < n'$.

By applying Proposition 5.5 in the forcing extension by $\mathcal{R}_{\omega_2}$ we can choose an $\mathcal{R}_{\omega_2}$-name, $\dot{f}$, of a strictly increasing function in ω^ω so that 1 forces that $\dot{\vec{w}}_{r_h}$ is an infinite set for all strictly increasing $h \in \omega^\omega$ satisfying that $\dot{f} \leq h$. By elementarity, there is such a name $\dot{f}$ in M_n and using that $\mathcal{R}_{\omega_2}$ adds no unbounded reals, there is function $\tilde{h} \in M_n \cap \omega^\omega$ such that 1 forces that $\dot{f} <^* \tilde{h}$. Now it follows that 1 forces that $\dot{\vec{w}}_{r_h}$ is infinite for all strictly increasing $h \in \omega^\omega$ such that $\tilde{h} \leq^* h$. Let $r = r_{\tilde{h}}$ and note that $r \in M_i$ for all $n' \leq i \leq n$.

For each $n' \leq i \leq n$, 1 forces, by Proposition 5.5, that $\dot{\vec{u}}^{i}_{r} \supset \dot{\vec{w}}_r$. Similarly to the case for values of i less than n' as above, we also have that $\widehat{\psi}(\dot{\vec{a}}^{i}_{r})$ is equal to $\dot{\vec{u}}^{i}_{r}$ for all $n' \leq i \leq n$. Since ψ is a permutation, we now have that 1 forces that $(\bigcap_{1 \leq i < n'} \dot{\vec{a}}^{i}_{r_i}) \cap \dot{\vec{a}}^{n'}_{r} \cap \cdots \cap \dot{\vec{a}}^{n}_{r}$ is infinite. Since $\tilde{h} <^* h_i$ for all $n' \leq i \leq n$, the proof is now finished by the final appeal to Proposition 5.5 to confirm that 1 forces that, for all $n' \leq i \leq n$, $\dot{\vec{a}}^{i}_{r_i}$ mod finite contains $\dot{\vec{a}}^{i}_{r}$. $\square$

Theorem 5.7. *There is a collection $\mathcal{A}$ of canonical $\mathcal{R}_{\omega_2}$-names of subsets of ω such that for each generic filter $G \subset \mathcal{R}_{\omega_2}$, the set $\{\mathrm{val}_G[\dot{a}] : \dot{a} \in \mathcal{A}\}$ is a P-point ultrafilter on ω.*

Proof. For each $\delta \in D$, let r_δ be equal to r_{h_δ}. Let $\mathcal{N}$ denote the set of all canonical $\mathbb{R}_{\omega_2}$-names of infinite subsets of ω. Also let $\mathcal{N}^{\omega\downarrow} \subset \mathcal{N}^\omega$ denote the infinite sequences that are forced by 1 to be descending. For $\delta \in D$, let $\mathcal{N}_\delta$ be the union of $\mathcal{N} \cap \bigcup \mathbb{M}_\delta$ and $\bigcup\{\mathcal{N}_\gamma : \gamma \in D \cap \delta\}$. By induction on $\delta \in D$ we define families $\mathcal{A}_\delta, \mathcal{U}_\delta, \mathcal{P}_\delta$ with the inductive hypotheses: for all $\gamma, \delta \in D$ with $\gamma < \delta$

(1) $\mathcal{U}_\gamma \subset \mathcal{U}_\delta$ and $\mathcal{A}_\gamma \subset \mathcal{A}_\delta \subset \bigcup\{\mathcal{N}_\xi : \xi \in D \cap (\delta+1)\}$,
(2) $\mathcal{P}_\delta = \{\dot{\vec{a}}_{r_\delta} : (\exists \zeta \in D \cap \delta)(\exists M \in \mathbb{M}_\delta)\ \vec{a} \in M \cap \mathcal{A}_\zeta^\omega \cap \mathcal{N}^{\omega\downarrow}\}$,
(3) $\mathcal{U}_\delta = \bigcup\{\mathcal{P}_\xi : \xi \in D \cap (\delta+1)\}$,
(4) for any finite subset $\mathcal{A}'$ of $\mathcal{A}_\delta \cup \mathcal{U}_\delta$, 1 forces that $\bigcap \mathcal{A}'$ is infinite,
(5) $\mathcal{A}_\delta$ is maximal with respect to properties (1) and (4),

Let $\delta \in D$ and assume that we have defined $\mathcal{A}_\gamma, \mathcal{U}_\gamma, \mathcal{P}_\gamma$ for all $\gamma \in D \cap \delta$. Let $\mathcal{P}_\delta$ be defined as in condition (2), and let $\mathcal{U}_\delta$ be defined as in (3). Let $\mathcal{A}'_\delta = \bigcup_{\gamma \in D\cap\delta} \mathcal{A}_\gamma$.

Claim 5. 1 forces that $\bigcap \mathcal{A}'$ is infinite for all finite $\mathcal{A}' \subset \mathcal{U}_\delta \cup \mathcal{A}'_\delta$.

Proof of Claim. Let $\mathcal{A}'$ be a finite subset of $\mathcal{U}_\delta \cup \mathcal{A}'_\delta$. Let $n = |\mathcal{A}'|$ and let $n' \leq n$ be chosen so that $|\mathcal{A}' \setminus \mathcal{P}_\delta|$ has cardinality $n'-1$. We may choose $\{M_{n'}, \ldots, M_n\} \subset \mathbb{M}_\delta$ and a sequence $\{\vec{a}^{n'}, \cdots, \vec{a}^n\}$ so that for each $n' \leq i \leq n$, the pair $M_i, \vec{a}^i$ satisfies the criteria in the definition of $\mathcal{P}_\delta$ and such that $\mathcal{A}' \cap \mathcal{P}_\delta$ is equal to $\{\dot{\vec{a}}^{n'}_{r_\delta}, \cdots, \dot{\vec{a}}^n_{r_\delta}\}$.

In order to uniformize notation, we will treat members of $\mathcal{A}' \cap \mathcal{A}'_\delta$ as $\dot{\vec{a}}_r$ for some constant ω-sequences $\vec{a}$ in $(\mathcal{A}'_\delta)^\omega$. In this way, we can similarly choose sequences $\{M_i : 1 \leq i < n'\} \subset \mathbb{M}$ and $\{\vec{a}^i : 1 \leq i < n'\}$ such that, for each $1 \leq i < n'$:

(1) $\delta_i = \delta_{M_i} < \delta$,
(2) $\delta_i \leq \delta_{i+1}$ for $1 < i+1 < n'$,
(3) $\vec{a}^i \in M_i \cap \mathcal{A}^\omega_{\xi_i}$ for some $\xi_i \leq \delta_{M_i}$
(4) $\mathcal{A}' \setminus \mathcal{P}_\delta$ is equal to $\{\dot{\vec{a}}^i_{r_{\delta_i}} : i < n'\}$.

It now follows from Lemma 5.6 that 1 forces that $\bigcap \mathcal{A}'$ is infinite. □

Now use Zorn's Lemma to extend $\mathcal{A}'_\delta$ to our set $\mathcal{A}_\delta$ so that it is maximal with respect to condition (4) (as required in (5)). This completes the inductive construction and naturally we let $\mathcal{A}$ be defined as the union of the family $\{\mathcal{A}_\delta : \delta \in D\}$. We finish by proving that if $G \subset \mathcal{R}_{\omega_2}$ is a generic filter, then $\mathcal{U}_G = \{\mathrm{val}_G[\dot{a}] : \dot{a} \in \mathcal{A}\}$ is a P-point. We do so with a some easy to prove claims.

Claim 6. For each $\vec{u} \in (\mathcal{A})^\omega$, there is a $\delta \in D$ and a $\dot{a} \in \mathcal{U}_\delta$ such that 1 forces that $\dot{a}$ is a pseudointersection for the sequence $\vec{u}$.

Proof of Claim. By a simple modification of $\vec{u}$, we can assume that 1 forces that it is a descending sequence. Choose any $\gamma \in D$ large enough so that $\vec{u} \in \mathcal{A}^\omega_\gamma$. By Lemma 3.4, there is an $M \in \mathbb{M}$ such that $\{\gamma, \vec{u}\} \in M$. Let $\delta = \delta_M$. Of course we then have that $\dot{\vec{u}}_{r_\delta} \in \mathcal{U}_\delta$. □

Claim 7. For each $\delta \in D, \mathcal{U}_\delta \subset \mathcal{A}$.

Proof of Claim. Let $\delta \in D$ and $\dot{u} \in \mathcal{U}_\delta$. By Lemma 3.4, we may choose $M \in \mathbb{M}$ such that $\dot{u} \in M$. It then follows from induction assumption (5) that $\dot{u} \in \mathcal{A}_{\delta_M}$. □

Claim 8. For each $\dot{u} \in \mathcal{N}$, there is an $\dot{a} \in \mathcal{A}$ such that 1 forces that $\dot{u} \in \{\dot{a}, \omega \setminus \dot{a}\}$.

Proof of Claim. If 1 forces that every finite subset of $\{\dot{u}\} \cup \mathcal{A}$ has infinite intersection, then it follows from Lemma 3.4 and inductive condition (5) that $\dot{a} = \dot{u}$

would be as required. Otherwise, fix any antichain $\{b_n : n \in \omega\}$ of $\mathcal{R}_{\omega_2}$ maximal with respect to the property that, for each n, there is a finite $\mathcal{A}_n \subset \mathcal{A}$ such that, either b_n forces that $\dot{u} \cap \bigcap \mathcal{A}_n$ is finite. We omit the easy argument that we can, by possibly enlarging $\mathcal{A}_n$, assume that, for each $n \in \omega$, b_n forces that $\dot{u} \cap \bigcap \mathcal{A}_n$ is empty. Let b be the condition $\bigcup_n b_n \in \mathcal{R}_{\omega_2}$. By the maximality of the antichain $\{b_n : n \in \omega\}$, it follows that $(1-b)$ forces that every finite subset of $\{\dot{u}\} \cap \mathcal{A}$ has infinite intersection. We define a name $\dot{a}$. For each $m \in \omega$, let $[[m \in \dot{a}]] = ([[m \in \dot{u}]] \cap (1-b)) \cup (\bigcup_n (b_n \setminus [[m \in \dot{u}]]))$. Note that $1-b$ forces that $\dot{a} = \dot{u}$ and, for each n, b_n forces that $\dot{u} = \omega \setminus \dot{a}$. For each $n \in \omega$, b_n forces that $\dot{a}$ contains $\bigcap \mathcal{A}_n$. Then $\dot{a} \in \mathcal{A}$ since it easily follows that 1 forces that every finite subset of $\{\dot{a}\} \cup \mathcal{A}$ has infinite intersection. □

It follows from induction condition (4) that $\mathcal{U}_G$ has the finite intersection property. It follows from Claims 2 and 3 that $\mathcal{U}_G$ is a base for a P-filter. It follows from Claim 4 that it is an ultrafilter. This completes the proof. □

6. Scales and nowhere dense P-filters

In this section when we use the letter $\mathcal{S}$ it refers to a scale in ω^ω, i.e. a $<^*$-unbounded set of monotone increasing functions in ω^ω that is well-ordered by $<^*$. As usual, the letter $\mathcal{U}$ will refer to an ultrafilter on ω. An ultrafilter $\mathcal{U}$ gives rise to the pre-order $<_{\mathcal{U}}$ on ω^ω where $f <_{\mathcal{U}} g$ providing $\{n : f(n) < g(n)\} \in \mathcal{U}$.

A filter $\mathcal{F}$ on ω is non-meager if for each strictly increasing $f \in \omega^\omega$, there is an $F \in \mathcal{F}$ such that $\{n : F \cap [n, f(n)] = \emptyset\}$ is infinite ([24] and [16]). Clearly a non-meager P-filter is a nowhere dense P-filter.

A P-point $\mathcal{U}$ is non-meager P-filter and for any finite-to-one function, φ, from ω onto ω, the filter $\phi^{-1}[\mathcal{U}] = \{\phi^{-1}(U) : U \in \mathcal{U}\}$ is also a non-meager P-filter. It follows then that NCF implies that every ultrafilter on ω has a non-meager P-subfilter. However this is not the first model known to have this property.

Proposition 6.1 ([1])**.** *If a scale $\mathcal{S}$ is not $<_{\mathcal{U}}$-unbounded, then $\mathcal{U}$ has a P-subfilter with a base that is a tower.*

Lemma 6.2. *If a scale $\mathcal{S}$ is not $<_{\mathcal{U}}$-unbounded, then $\mathcal{U}$ has a non-meager P-subfilter with a base that is a tower.*

Proof. Assume that $\mathcal{S}$ is not $<_{\mathcal{U}}$-unbounded. Choose any strictly increasing $g \in \omega^\omega$ such that $f <_{\mathcal{U}} g$ for all $f \in \mathfrak{S}$. For each $f \in \mathcal{S}$, let $a_f = \{n \in \omega : f(n) < g(n)\} \in \mathcal{U}$. Since $\mathcal{S}$ is well-ordered by $<^*$, it is immediate that $\{a_f : f \in \mathcal{S}\}$ is descending when ordered by ${}^*\supset$. To show that it is a tower, it suffices to prove that the filter it generates is non-meager. Consider any strictly increasing $h \in \omega^\omega$. We must produce an $f \in \mathcal{S}$ such that $\{k \in \omega : a_f \cap [k, h(k)] = \emptyset\}$ is infinite. Since $\mathcal{S}$ is $<^*$-unbounded, there is an $f \in \mathcal{S}$ such that the set $\{k \in \omega : f(k) > g(h(k))\}$ is infinite. Choose k so that $f(k) > g(h(k))$ and let $k \le n \le h(k)$. It follows that $g(n) \le g(h(k)) < f(k) \le f(n)$. □

Corollary 6.3 ([1])**.** *If there are scales $\mathcal{S}_1, \mathcal{S}_2$ that have different cofinalities, then every ultrafilter on ω has a non-meager P-subfilter with a base that is a tower.*

Proof. Fix any ultrafilter $\mathcal{U}$. Assume that $\mathcal{S}_1$ is unbounded with respect to $<_{\mathcal{U}}$ and let κ_1 be the order-type of $\mathcal{S}_1$ with respect to the $<^*$-ordering. It follows that $\mathfrak{S}_1$ is cofinal in $<_{\mathcal{U}}$ and therefore, that $\mathcal{S}_2$ can not be cofinal in $<_{\mathcal{U}}$. This implies that $\mathcal{S}_2$ is bounded in $<_{\mathcal{U}}$ and the statement now follows from Lemma 6.2. □

Next we discuss the status of two statements that have been extensively investigated by Peter Nyikos (see [20]).

Axiom 2: $(\exists\mathcal{S})(\forall\mathcal{U})(\mathcal{S}$ is $<_{\mathcal{U}}$-unbounded) (equivalently $\mathfrak{b} = \mathfrak{d}$ [20])

Axiom 3: $(\exists\mathcal{U})(\forall\mathcal{S})(\mathcal{S}$ is $<_{\mathcal{U}}$-unbounded)

Question: does Axiom 2 (i.e. $\mathfrak{b} = \mathfrak{d}$) imply Axiom 3? ([20, Problem 5])

Corollary 6.4. *If there is an ultrafilter on ω that has no non-meager P-subfilter with a base that is a tower, then Axiom 3 holds.*

Corollary 6.5 ($\square_{\omega_1}$ and CH)**.** *In the forcing extension by $Fn(\omega_2, 2)$, Axiom 3 holds.*

Theorem 6.6. *If $\mathfrak{b} = \mathfrak{d}$, then there is an ultrafilter that has no non-meager P-subfilter that has a base that is a tower.*

Proof. By the assumption $\mathfrak{b} = \mathfrak{d}$, there is scale $\{f_\alpha : \alpha < \mathfrak{b}\}$ that is a dominating family consisting of strictly increasing functions. For each tower, $\mathcal{A} \subset [\omega]^{\aleph_0}$ that generates a non-meager filter, choose an elementary submodel $M_{\mathcal{A}}$ of $H(\mathfrak{c}^+)$ such that $\mathcal{A}$ and $\{f_\alpha : \alpha < \mathfrak{b}\}$ are elements of $M_{\mathcal{A}}$ and such that $M_{\mathcal{A}} \cap \mathfrak{b}$ is an initial segment of $\mathfrak{b}$. Since $\mathfrak{b}$ is a regular cardinal, the Lowenheim-Skolem theorem implies that such a set $M_{\mathcal{A}}$ exists. Also choose an element $a(\mathcal{A}) \in \mathcal{A}$ that is mod finite contained in every member of $M \cap \mathcal{A}$. Let $\mathfrak{A}$ be the set of all such towers $\mathcal{A}$. We will prove that $\mathcal{W} = \{\omega \setminus a(\mathcal{A}) : \mathcal{A} \in \mathfrak{A}\}$. The theorem then follows since any ultrafilter $\mathcal{U}$ that contains $\mathcal{W}$ does not contain a tower that generates a non-meager P-filter.

Let $\{\mathcal{A}_1, \ldots, \mathcal{A}_\ell\}$ be a subset of $\mathfrak{A}$ and assume that it is enumerated so that, for $1 \leq i < j \leq \ell$, $M_{\mathcal{A}_i} \cap \mathfrak{b} \subset M_{\mathcal{A}_j} \cap \mathfrak{b}$. For each $1 \leq i \leq \ell$, let $M_i = M_{\mathcal{A}_i}$ and $\eta_i = \sup(M_i \cap \mathfrak{b})$. Choose $\alpha_0 < \eta_1$ arbitrarily and $a_1 \in \mathcal{A}_1 \cap M_1$ such that $J_1 = \{n : a_1 \cap [f^n_{\alpha_0}(0), f^{n+1}_{\alpha_0}(0)] = \emptyset\}$ is infinite. Since $J_1 \in M_1$, there is a $\alpha_1 < \eta_1$ such that for all $k \in \omega$, there is an $n \in J_1$ such that $[f^n_{\alpha_0}(0), f^{n+1}_{\alpha_0}(0)] \subset [k, f_{\alpha_1}(k)]$. Now $\alpha_1 < \eta_2$, hence there is an $a_2 \in \mathcal{A}_2 \cap M_2$ such that $J_2 = \{n : a_2 \cap [f^n_{\alpha_1}(0), f^{n+1}_{\alpha_1}(0)] = \emptyset\}$ is infinite. Again, there is an $\alpha_2 < \eta_2$ such that for each $k \in \omega$, there is an $n \in J_2$ such that $[f^n_{\alpha_1}(0), f^{n+1}_{\alpha_1}(0)] \subset [k, f_{\alpha_2}(k)]$. We continue this induction, choosing, for $1 \leq i \leq \ell$, $a_i \in \mathcal{A}_i \cap M_i$ and $\alpha_i < \eta_i$ such that $J_i = \{n : a_i \cap [f^n_{\alpha_{i-1}}(0), f^{n+1}_{\alpha_{i-1}}(0)] = \emptyset\}$ is infinite, and, for each $k \in \omega$, there is an $n \in J_i$, such that $[f^n_{\alpha_{i-1}}(0), f^{n+1}_{\alpha_{i-1}}(0)] \subset [k, f_{\alpha_i}(k)]$.

Now, for each $1 \leq i \leq \ell$, $a_i \subset^* a(\mathcal{A}_i)$, hence, to prove that $\bigcap\{\omega \setminus a(\mathcal{A}_i) : 1 \leq i \leq \ell\}$ is infinite, it suffices to prove that $\omega \setminus (a_1 \cup \cdots \cup a_\ell)$ is infinite. Fix any $k \in \omega$ and choose $n_\ell \in J_\ell$ such that $k < f^{n_\ell}_{\alpha_{\ell-1}}(0)$ and a_ℓ is disjoint from $[f^{n_\ell}_{\alpha_{\ell-1}}(0), f^{n_\ell+1}_{\alpha_{\ell-1}}(0)]$. By the definition of $\alpha_{\ell-1}$, there is an $n_{\ell-1} \in J_{\ell-1}$ such that $[f^{n_{\ell-1}}_{\alpha_{\ell-2}}(0), f^{n_{\ell-1}+1}_{\alpha_{\ell-2}}(0)]$ is contained in $[f^{n_\ell}_{\alpha_{\ell-1}}(0), f^{n_\ell+1}_{\alpha_{\ell-1}}(0)]$. Continuing this descending recursion, there is a sequence $\{n_1, \cdots, n_\ell\}$ such that, for each $1 \leq i \leq \ell$, $n_i \in J_i$ and $[f^{n_i}_{\alpha_{i-1}}(0), f^{n_i+1}_{\alpha_{i-1}}(0)]$ is contained in $[f^{n_{i+1}}_{\alpha_i}(0), f^{n_{i+1}+1}_{\alpha_i}(0)]$. Since $n_1 \in J_1$, there is an non-empty interval $I \subset [f^{n_2}_{\alpha_1}(0), f^{n_2+1}_{\alpha_1}(0)]$ that is disjoint from a_1. Now proceedings upwards, it follows that, for $1 \leq i < \ell$, I is contained in $[f^{n_{i+1}}_{\alpha_i}(0), f^{n_{i+1}+1}_{\alpha_i}(0)]$, and so is disjoint from a_{i+1}. ∎

This answers Problem 5 in [20].

Corollary 6.7. *Axiom 2 implies Axiom 3*

Problem 8 in [20] asks if Axiom 3 holds in models obtained by adding random reals over models of CH. Since these are models of $\mathfrak{b} = \mathfrak{d}$, this also answers that problem.

7. Laver model and nowhere dense P-filters

Nyikos asked in [20, Problem 7] if, in the Laver model, every ultrafilter has a nowhere P-subfilter. In this section we show that this is the case. Having discovered the method for Laver forcing, we then show that it generalizes, along the lines of Theorem 2.5.

Conditions $T \in \mathbb{L}$ are infinite downward closed subtrees of $\omega^{<\omega}$ with the property that there is a stem, $\mathrm{stem}(T)$, such that no predecessor is branching and each $\mathrm{stem}(T) \leq t \in T$ has infinitely many immediate successors. For $t \in \omega^{<\omega}$, it is convenient to note that $|t|$ is the domain of the function t. We will say that $T \in \mathbb{L}$ is increasing if for each $\mathrm{stem}(T) < t \in T$, $t(i) < t(j)$ for all $|\mathrm{stem}(T)| \leq i < j < |t|$. We use $\dot{f}_{\mathbb{L}}$ to denote the generic function in ω^ω added by $\mathbb{L}$ where each $T \in \mathbb{L}$ forces that $\mathrm{stem}(T) \subset \dot{f}_{\mathbb{L}}$.

For $T \in \mathbb{L}$, $k \in \omega$, and branching node $t \in T$, let

$$T_{t,k} = \{s \in T : s \subset t \text{ or } (t \subset s \text{ and } s(|t|) \geq k)\} \ .$$

Let τ be an $\mathbb{L}$-name for an integer. We use T_t for $T_{t,0}$ and say that T_t weakly forces $\tau = m$ (for $m \in \omega$) if $T_{t,k} \Vdash \tau = m$ for some $k \in \omega$. We may say that T_t weakly decides τ if, for some m, T_t weakly forces that $\tau = m$. We use the notation $T_t \Vdash_w \tau = m$ to indicate that T_t weakly forces that $\tau = m$. Similarly, $T_t \Vdash_w \tau > m$ will abbreviate that there is a $k \in \omega$ such that $T_{t,k} \Vdash \tau > m$. Observe that if $T_t \Vdash_w \tau > m$, then for all $T' <_0 T_t$, $T' \Vdash_w \tau > m$. Let us note that for all $\mathrm{stem}(T) \leq t \in T$ and $m \in \omega$, $T_{t,m+1} \Vdash \dot{f}_{\mathbb{L}}(|t|) > m$.

Following Laver [17], $<_0$ is used to denote the usual root preserving extension relation on $\mathbb{L}$. The poset $\mathbb{L}$ has the pure-decision property [17] meaning, for example, that for any $T \in \mathbb{L}$, $n, m \in \omega$ and $\mathbb{L}$-name τ of an integer, there is a $T' <_0 T$ such that $T' \Vdash \tau = m$ or $T' \Vdash \tau \neq m$.

Proposition 7.1. *For each $T_0 \in \mathbb{L}$ and $\mathbb{L}$-name τ of an integer, there is a $T <_0 T_0$ such that either $T \Vdash \tau = m$ for some $m \in \omega$, or $T \Vdash_w \tau > m$ for each $m \in \omega$.*

Proof. Assume that, for each $m \in \omega$, there is no $T <_0 T_0$ such that $T \Vdash \tau = m$. Of course this means that for each $T <_0 T_0$ and each $m \in \omega$, there is a $T' <_0 T$ such that $T \Vdash \tau \neq m$. Let $t = \mathrm{stem}(T_0)$. By induction on $m \in \omega$, choose $T_{m+1} <_0 T_m$ such that $T_{m+1} \Vdash \tau \neq m$. Choose any strictly increasing sequence $\{k_m : m \in \omega\}$ so that $t_m = t^\frown k_m \in T_{m+1}$. Let $T = \bigcup\{(T_{m+1})_{t_m} : m \in \omega\}$. It is easily checked that $T' \in \mathbb{L}$ and that $T' <_0 T$. Now let $m \in \omega$ and we show that $T_{t,k_m} \Vdash \tau > m$. Choose any $\tilde{T} < T_{t,k_m}$ such that $\mathrm{stem}(T_0) < \mathrm{stem}(\tilde{T})$. It follows that there is an $\ell \geq m$ such that $\tilde{T} < (T_{\ell+1})_{t_\ell}$. Since $T_{\ell+1} \Vdash \tau > \ell$, we have that $\tilde{T} \Vdash \tau > \ell$. □

Proposition 7.2. *If τ is an $\mathbb{L}$-name of an integer then there is a $T <_0 T_0$ such that, for $t = \mathrm{stem}(T_0)$,*

(1) *for each $t^\frown k \in T$, there is an m such that $T_{t^\frown k} \Vdash \tau = m$, or*
(2) *for each $t^\frown k \in T$, $T_{t^\frown k} \Vdash_w \tau > m$ for all $m \in \omega$.*

Proof. Let $t = \mathrm{stem}(T_0)$. We first apply Proposition 7.1 as follows. For each $k > 0$ such that $t^\frown k \in T_0$, choose $T_k <_0 (T_0)_{t^\frown k}$ such that, either there in an m_k such that $T_k \Vdash \tau = m_k$, or $T_k \Vdash_w \tau > m$ for each $m \in \omega$. Let K be the set of k such that $t^\frown k \in T_0$ and $T_k \Vdash \tau = m_k$ for some m_k. If K is infinite, then $T = \bigcup\{T_k : k \in K\}$. If K is finite, then $T = \bigcup\{T_k : k \notin k$ and $t^\frown k \in T_0\}$. $\square$

Let us say that T is τ-trimmed if either (1) or (2) of Proposition 7.2 holds.

The following is proven in [9, 2.13].

Proposition 7.3. *Let $\dot g$ be an $\mathbb{L}$-name of a strictly increasing function in ω^ω and let $T \in \mathbb{L}$ force that $\dot f_{\mathbb{L}} \leq \dot g$. Then there are disjoint sets $\{A_n : n \in \omega\}$ and $T' < T$ such that $T' \Vdash (\forall n \in \omega)\dot g(n) \in \check A_n$.*

This can be significantly improved. The extra conditions (1)-(3) in this Lemma are there so that we may deduce that $\mathbb{L}$ is tower preserving.

Lemma 7.4. *Let $\dot g$ be an $\mathbb{L}$-name of a strictly increasing function in ω^ω and let $T_0 \in \mathbb{L}$ force that $\dot f_{\mathbb{L}} \leq \dot g$. Then there are $T <_0 T_0$ and pairwise disjoint finite sets $\{F_t : t \in T\}$, such that, $T' \subset T$ in $\mathbb{L}$, T' forces that the range of $\dot g$ is contained in $\bigcup\{F_t : t \in T'\}$. We may also ensure that, for all $t \in T$:*

1. *$|F_t| \leq |t|$,*
2. *if $F_t \neq \emptyset$, then for some n, $T_t \Vdash \dot g(n) \in F_t$,*
3. *if $F_{t^\frown k} = \emptyset$ for some $t^\frown k \in T$ then $F_{t^\frown k} = \emptyset$ for all $t^\frown k \in T$.*

To prove this lemma we will need the ideas from [17] for performing standard fusion arguments in $\mathbb{L}$. Let $\prec$ denote the lexicographic ordering on $\omega^{<\omega}$, i.e. $t \prec t^\frown \ell$ for all $t \in \omega^{<\omega}$ and $\ell \in \omega$. Fix any enumeration $\{\bar t_i : i \in \omega\}$ of $\omega^{<\omega}$ (the maximal element of $\mathbb{L}$) with the property that $\bar t_i \prec \bar t_j$ implies $i < j$. Give any $T \in \mathbb{L}$, let $\{t_i^T : i \in \omega\}$ be the induced enumeration of the branching nodes of T where $t_0^T = \mathrm{stem}(T)$ and for each $\mathrm{stem}(T) \leq t \in T$, if $t = t_i^T$, then $\{t^\frown \ell : \ell \in \omega\} \cap T$ is enumerated (in an increasing fashion) by $\{t_j^T : \bar t_j \in \{t^\frown k : k \in \omega\}\}$. For $\mathrm{stem}(T) \leq t \in T \in \mathbb{L}$, let $\rho(T, t) = i$ where $t = t_i^T$. We can now define, for each $0 < n \in \omega$, the relation $<_n$ on $\mathbb{L}$ by $T_2 <_n T_1$ providing $T_2 < T_1$ and $t_i^{T_2} = t_i^{T_1}$ for $i \leq n$. Each $<_n$ is a transitive relation and if $\{T_n : n \in \omega\} \subset \mathbb{L}$ satisfies that $T_{n+1} <_n T_n$ for all n, then $T_\omega = \bigcap_n T_n = \{t_n^{T_n} : n \in \omega\} \cup \{t \in T_0 : t < \mathrm{stem}(T_0)\}$ is a member of $\mathbb{L}$ satisfying that $T_\omega <_n T_n$ for all n.

Now we prove Lemma 7.4. The proof is an easy modification of a similar result in [18] (as was [9, 2.13]).

Proof. We have that $T_0 \Vdash \dot g(n) \geq \dot f_{\mathbb{L}}(n)$ for all $n \in \omega$. Consider any $\mathrm{stem}(T_0) \leq t \in T_0$ and note that this means that, for each $m \in \omega$, $(T_0)_t \Vdash_w \dot g(|t|) > m$. Let $t_0 = \mathrm{stem}(T_0)$ and let $j_0 \leq |t_0|$ be minimal such that there exists $T_0' <_0 T_0$ such that $T_0' \Vdash_w \dot g(j_0) > m$ for all $m \in \omega$. Choose $T_1 <_0 T_0$ and $L_0 \in \omega$ so that $T_1 \Vdash \dot g(j_0) > L_0$ and there is a set $F_{t_0} \subset L_0$ with $|F_0| \leq |t_0|$ such that $T_1 \Vdash \dot g(j) \in F_{t_0}$ for all $j < j_0$. Note that for all $T' <_0 T_1$, $T' \Vdash_w \dot g(j_0) > m$ for all $m \in \omega$. We can recursively construct a sequence $\{T_\ell, j_\ell, t_\ell, L_\ell\}$ so that the following holds:

1. $j_\ell < L_\ell \in \omega$, $T_\ell \in \mathbb{L}$, $t_\ell = t_\ell^{T_\ell}$, and $T_{\ell+1} <_\ell T_\ell$,
2. $F_{t_\ell} \subset L_\ell \setminus L_{\ell-1}$ and $|F_{t_\ell}| \leq |t_\ell|$,
3. $j_\ell \leq |t_\ell|$ and $(T_{\ell+1})_{t_\ell} \Vdash_w \dot g(j_\ell) > m$ for all $m \in \omega$,
4. $(T_{\ell+1})_{t_\ell} \Vdash F_{t_\ell} = \{\dot g(j) : j_{\ell'} \leq j < j_\ell\}$ where $t_{\ell'}$ is the predecessor of t_ℓ,

(5) $t_\ell \notin (T_{\ell+1})_{t_i,L_\ell}$ and $(T_{\ell+1})_{t_i,L_\ell} \Vdash \dot g(j_i) > L_\ell$ for all $i \leq \ell$,
(6) $(T_{\ell+1})_{t_\ell}$ is $\dot g(j_\ell)$-trimmed.

The choice of $t_\ell = t_\ell^{T_\ell}$ is clear. Let $\tilde T_\ell$ denote $(T_\ell)_{t_\ell}$. Let $\ell' < \ell$ be chosen so that $t_{\ell'}$ is the immediate predecessor of t_ℓ. Choose j_ℓ minimal so that there exists $T'_\ell <_0 \tilde T_\ell$ satisfying that $T'_\ell \Vdash_w \dot g(j_\ell) > m$ for all $m \in \omega$. By further strengthening, we may also assume that for all $j < j_\ell$, there is an m such that $T'_\ell \Vdash \dot g(j) = m$, and that T'_ℓ is $\dot g(j_\ell)$-trimmed. By induction hypothesis (5), $\tilde T <_0 (T_\ell)_{t_{\ell'},L_{\ell-1}}$ which, if $j_{\ell'} < j_\ell$, implies that $T'_\ell \Vdash \dot g(j) > L_{\ell-1}$ for all $j_{\ell'} \leq j < j_\ell$. Let $F_{t_\ell} = \{m : (\exists j \geq j_{\ell'})\ T'_\ell \Vdash \dot g(j) = m\}$. It follows that $L_{\ell-1} < \min(F_{t_\ell})$. Choose L_ℓ large enough so that $F_{t_\ell} \subset L_\ell$ and $t_\ell(|t_{\ell'}|) < L_\ell$. Choose any $L_\ell < k_\ell \in \omega$ large enough so that $(T_\ell)_{t_i,k_\ell} \Vdash \dot g(j_i) > L_\ell$ for all $i < \ell$ and so that $(T'_\ell)_{t_\ell,k_\ell} \Vdash \dot g(j_\ell) > L_\ell$. Set $T_{\ell+1} = (T'_\ell)_{t_\ell,k_\ell} \cup \bigcup\{(T_\ell)_{t_i,k_\ell} : i < \ell\}$. Since $t_\ell \notin \bigcup\{(T_\ell)_{t_i,k_\ell} : i < \ell\}$, it follows that $(T_{\ell+1})_{t_\ell} <_0 T'_\ell$ and so condition (4) holds. For $i < \ell$, $t_\ell \notin (T_\ell)_{t_i,k_\ell}$ and so it follows from condition (5) as an induction hypothesis that, $(T_{\ell+1})_{t_i,L_\ell}$ is equal to $(T_\ell)_{t_i,k_\ell}$. This verifies that (5) holds at stage ℓ. The conditions (1)-(4) and (6) are routine to verify.

The required condition T for the statement of the lemma is $\bigcap\{T_n : n \in \omega\}$ and it should be clear that $t_\ell^T = t_\ell$ for all $\ell \in \omega$. The family of finite sets $\{F_t : t \in T\}$ (where $F_t = \emptyset$ for $t < t_0$) is a pairwise disjoint family since $L_{\ell-1} < \min(F_{t_\ell}) < L_\ell$. Let $n, m \in \omega$ and assume that $T' < T$ forces that $\dot g(n) = m$. We show there is a $t \in T'$ such that $m \in F_t$. If $n < j_0$, then it is clear that $T' \Vdash \dot g(n) \in F_{t_0}$, so we assume that $j_0 \leq n$. Let $t_\ell = \mathrm{stem}(T')$. Since $T_{t_\ell} \Vdash_w \dot g(j_\ell) > m$, it follows that $n < j_\ell$. Choose $0 < i < \ell$ be minimal so that $t_i \subset t_\ell$ and $n < j_i$. Then, by (1), $T' < (T_{i+1})_{t_i}$, and by (4), $m \in F_{t_i}$.

Now we verify the extra conditions (2) and (3) in the statement of the Lemma. Condition (2) follows from induction hypothesis (4) and condition (3) follows from induction hypothesis (6). □

Corollary 7.5. $\mathbb{L}$ *is tower preserving.*

Proof. Let $\mathcal{A}$ be a tower on ω and let $\dot Y$ be the $\mathbb{L}$-name of an infinite subset of ω. Assume that $T_0 \in \mathbb{L}$ forces that $\dot Y \setminus a$ is finite for each $a \in \mathcal{A}$. Choose any $\mathbb{L}$-name of a strictly increasing function $\dot g \in \dot Y^\omega$ satisfying that $\dot f_{\mathbb{L}} \leq \dot g$. Let $\dot Y, \dot g$ and $\mathcal{A}$ be members of a countable elementary submodel M of $H(\mathfrak{c}^+)$. By elementarity, there are $T \leq_0 T_0$ and $\{F_t : t \in T\}$ in M that satisfying the conclusion of Lemma 7.4. Fix any $a \in \mathcal{A}$ such that $b \setminus a$ is infinite for all $b \in M \cap [\omega]^{\aleph_0}$ and $F_t \cap a = \emptyset$ for $t \leq \mathrm{stem}(T)$. Set $T_a = \{t \in T : F_t \cap a = \emptyset\}$. It suffices to prove that $T_a \in \mathbb{L}$ since, by Lemma 7.4, T_a forces that the range of $\dot g$ is disjoint from a. Let $\mathrm{stem}(T) \leq t \in T$ and we prove that there is an infinite set of k such that $t^\frown k \in T$ and $F_{t^\frown k} \cap a = \emptyset$. Choose the minimal $\ell \leq |t| + 1$ such that $K_0 = \{k : t^\frown k \in T \text{ and } |F_{t^\frown k}| = \ell\}$ is infinite. If $\ell = 0$, we are done. Otherwise, let $\{v(k,j) : j < \ell\}$ be an increasing enumeration of $F_{t^\frown k}$ for each $k \in K_0$. Note that K_0 and the sequence of enumerations are elements of M. Recursively choose $a_0 \supset a_1 \supset \cdots \supset a_\ell$ from $\mathcal{A} \cap M$ so that, for each $j < \ell$, $K_{j+1} = \{k \in K_0 : v(k,j) \notin a_j\}$ is infinite. Therefore K_ℓ is infinite and, for all but finitely many $k \in K_\ell$, $a \cap \{v(k,j) : j < \ell\}$ is empty. □

We use the properties from Lemma 7.4 to define a family of $\mathbb{L}$-names that is forced to generate a dense P-ideal.

Definition 7.6. *If $T \in \mathbb{L}$ a T-sequence is a function $\vec{F}$ from T into $[\omega]^{<\aleph_0}$ consisting of pairwise disjoint sets. We use $\vec{F}_t$ (for $t \in T$) rather than $\vec{F}(t)$. If $\vec{F}$ is a T-sequence, we let $\dot{Y}(\vec{F})$ denotes the $\mathbb{L}$-name $\{(T_t, \check{m}) : t \in T \text{ and } m \in \vec{F}_t\}$.*

Proposition 7.7. *For a T-sequence $\vec{F}$ and $a \in [\omega]^{\aleph_0}$, $T \Vdash a \not\subset^* \dot{Y}(\vec{F}) = \bigcup_{n\in\omega} \vec{F}_{\dot{f}_{\mathbb{L}}\restriction n}$.*

Proof. The fact that $T \Vdash \bigcup_{n\in\omega} \vec{F}_{\dot{f}_{\mathbb{L}}\restriction n} = \dot{Y}(\vec{F})$ is immediate from the definitions of $\dot{f}_{\mathbb{L}}$ and $\dot{Y}(\vec{F})$. Let $T' < T$ and let t_1, t_2 be distinct immediate successors of $\mathrm{stem}(T')$. Therefore $Y_1 = \bigcup\{\vec{F}_t : t_1 \leq t \in T'\}$ and $Y_2 = \bigcup\{\vec{F}_t : t_2 \leq t \in T'\}$ are disjoint. Without loss of generality, we may assume that $a \setminus Y_1$ is infinite. Since $(T')_{t_1} \Vdash \dot{Y}(\vec{F})$ is contained in Y_1, it follows that T' does not force that $a \subset^* \dot{Y}(\vec{F})$. □

With this notation, we record the following corollary of Lemma 7.4.

Corollary 7.8. *For each $\mathbb{L}$-name $\dot{Y}$ and $T_0 \in \mathbb{L}$ such that $T_0 \Vdash \dot{Y} \in [\omega]^{\aleph_0}$, there is a $T <_0 T_0$ and a T-sequence $\vec{F}$ such that $T \Vdash \dot{Y}(\vec{F}) \subset \dot{Y}$.*

Definition 7.9. *Set $\dot{\mathcal{Y}}_{\mathbb{L}} = \{(T, \dot{Y}(\vec{F})) : T \in \mathbb{L} \text{ and } \vec{F} \text{ is a } T\text{-sequence}\}$.*

Lemma 7.10. *The name $\dot{\mathcal{Y}}_{\mathbb{L}}$ is forced to generate a dense P-ideal.*

Proof. We already noted in Corollary 7.8 that it is forced to be dense. Since $\mathbb{L}$ is proper it suffices to prove that for any countable subset $\{(T_n, \dot{Y}(\vec{F}_n)) : n \in \omega\}$ of $\dot{\mathcal{Y}}$, there is $T < T_0$ and a T-sequence $\vec{F}$ such that $T \Vdash \dot{Y}(\vec{F}_n) \subset^* \dot{Y}(\vec{F})$ for all $n \in \omega$. Notice that if $T' < T_0$ is such that $T' \cap T_n$ is finite for some n, then $T' \Vdash \dot{Y}(\vec{F}_n)$ is finite. For each $t \in T_0$ and $\ell \in \omega$, we can define $F(t,\ell)$ to be $\bigcup\{(\vec{F}_n)_t : n \leq \ell \text{ and } \mathrm{stem}(T_n) < t \in T_n\}$. We note that for each $t \in T_0$ and $\ell \in \omega$, the sequence $\{F(t^\frown k, \ell) : t^\frown k \in T_0\}$ is a point-finite family of finite sets.

We recursively choose a fusion sequence $\{T_{0,\ell} : \ell \in \omega\}$ where $T_{0,0} = T_0$ and $T_{0,\ell+1} <_\ell T_{0,\ell}$. We define t_ℓ to be $t_\ell^{T_{0,\ell}}$, and we set $F_\ell = F(t_\ell, \ell)$. We make the following simple demand that when choosing $T_{0,\ell+1}$ (and thereby $t_{\ell+1}$): if t' is the immediate predecessor of $t_\ell^{T_0}$ in T_0, then, for all immediate successors t of t' in $T_{0,\ell}$, if $t \notin \{t_i : i \leq \ell\}$, then $F(t, \ell+1)$ is disjoint from $\bigcup\{F_i : i \leq \ell\}$. Since the family $\{F(t'^\frown k, \ell) : t'^\frown k \in T_\ell\}$ is point-finite, there is a sufficiently large integer k so that every immediate successor t of t' in $(T_\ell)_{t',k}$ will have this property, and so the fusion can proceed. Let $T = \bigcap\{T_{0,\ell} : \ell \in \omega\}$ and again note that $t_\ell^T = t_\ell$ for all $\ell \in \omega$. By the construction, $\vec{F}$ is a T-sequence where $\vec{F}_{t_\ell} = F_\ell$ for all $\ell \in \omega$. Consider any $n \in \omega$ and assume there is a $T' < T, T_n$. By possibly extending T' further, we may assume that $\mathrm{stem}(T_n) < \mathrm{stem}(T') = t_m^T$ for some $m > n$. For each $\mathrm{stem}(T') \leq t' = t_\ell^T \in T'$, it follows that $\ell > n$ and that $(\vec{F}_n)_{t'} \subset F_\ell = \vec{F}_{t'}$. This of course ensures that $T' \Vdash \dot{Y}(\vec{F}_n) \subset^* \dot{Y}(\vec{F})$. □

Since Lemma 7.10 shows that $\mathbb{L}$ is a tower preserving forcing we can formulate the main theorem of this section as follows.

Theorem 7.11 (CH). *If $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ is a countable support iteration of proper tower preserving posets of cardinality $\aleph_1$ and satisfies that $P_\lambda \Vdash \dot{Q}_\alpha = \mathbb{L}$ for all λ in a stationary subset of $\{\lambda \in \omega_2 : cf(\lambda) = \omega_1\}$, then P_{ω_2} forces that every ultrafilter on ω extends a tower of cofinality ω_1.*

Proof. Let S denote the stationary set of $\lambda \in \omega_2$ that have cofinality ω_1 and satisfy that $P_\lambda \Vdash \dot{Q}_\lambda = \mathbb{L}$. For each $\lambda \in S$, let $\dot{\mathcal{Y}}_\lambda$ denote a $P_{\lambda+1}$-name such that P_λ forces that $\dot{\mathcal{Y}}_\lambda = \dot{\mathcal{Y}}_{\mathbb{L}}$ from Definition 7.9. It follows from Proposition 2.1, that P_{ω_2} forces that $\dot{\mathcal{Y}}_\lambda$ is a dense P-ideal for each $\lambda < \omega_2$.

Let $\dot{\mathcal{U}}$ be a P_{ω_2}-name and let $p \in P_{\omega_2}$ be any condition that forces $\dot{\mathcal{U}}$ is an ultrafilter on ω. We prove that there is a $\lambda \in S$ such that $p \Vdash \dot{\mathcal{U}}$ is disjoint from $\dot{\mathcal{Y}}_\lambda$. This implies that p forces that the nowhere dense P-filter dual to $\dot{\mathcal{Y}}_\lambda$ is a subfilter of $\dot{\mathcal{U}}$.

Let $2^{\aleph_2} < \theta$ be a regular cardinal, and let M be an elementary submodel of $H(\theta)$ such that $\{p, \dot{\mathcal{U}}, P_{\omega_2}\} \in M$, $M^\omega \subset M$, and $|M| = \aleph_1$. Let $\lambda = M \cap \omega_2$ and $\dot{\mathcal{U}}_\lambda = M \cap \dot{\mathcal{U}}$.

Let $p \in G_\lambda$ be any P_λ-generic filter. By Proposition 2.4, $\mathcal{U}_\lambda = \mathrm{val}_{G_\lambda}(\dot{\mathcal{U}}_\lambda)$ is an ultrafilter on ω in the model $V[G_\lambda]$. Furthermore, since $M^\omega \subset M$ and P_{ω_2} is proper, if A is any countable subset of $H(\omega_1)$ in $V[G_\lambda]$, there is a P_λ-name $\dot{A} \in M$ such that $\mathrm{val}_{G_\lambda}(\dot{A}) = A$ (see [11, 4.5]).

Fix any P_λ-names $\dot{T}$, $\dot{\vec{F}}$ and let $p' \in G_\lambda$ force that $\dot{T} \in \mathbb{L}$ and $\dot{\vec{F}}$ is a $\dot{T}$-sequence. Let $T = \mathrm{val}_{G_\lambda}(\dot{T})$ and $\vec{F} = \mathrm{val}_{G_\lambda}(\dot{\vec{F}})$, i.e. $\dot{Y}(\vec{F})$ is an arbitrary member of the $\mathbb{L}$-name $\dot{\mathcal{Y}}_{\mathbb{L}}$. If there is any $T' < T$ such that $\bigcup\{\vec{F}_t : t \in T'\} \notin \mathcal{U}_\lambda$, then there is an P_λ-name $\dot{T}'$ and a $q \in G_\lambda$ forcing that $\dot{T}' < \dot{T}$ and $\bigcup\{\dot{\vec{F}}_t : t \in \dot{T}'\}$ is not in $\dot{\mathcal{U}}$. This means that the condition $q \cup \{\langle \lambda, \dot{T}' \rangle\} \in P_{\lambda+1}$ forces that $\dot{Y}(\vec{F}) \notin \dot{\mathcal{U}}$. Now suppose there is no such T' as above. We may choose $q \in G_\lambda$ that forces $\bigcup\{\dot{\vec{F}}_t : t \in \dot{T}'\}$ is in $\dot{\mathcal{U}}$ for all $\dot{T}' < \dot{T}$. For each $t \in T$, let $A_t = \bigcup\{F_{t^\frown k} : t^\frown k \in T\}$. We may also let $\dot{A}_t \in M$ denote a P_λ-name for each such A_t (although A_t is definable from $\dot{\vec{F}}$). Notice now that $q \cup \{\langle \lambda, \dot{T}' \rangle\} \in P_{\lambda+1}$ forces that $\dot{Y}(\dot{\vec{F}})$ meets each $\dot{A}_t$ in a finite set. To show that $q \cup \{\langle \lambda, \dot{T} \rangle\} \in P_{\lambda+1}$ forces that $\dot{Y}(\dot{\vec{F}}) \notin \dot{\mathcal{U}}$ we prove q forces that every element of $\dot{\mathcal{U}}$ meets some $\dot{A}_t$ in an infinite set. To do so, we invoke elementarity and simply show, by contradiction, that this is is true for $U \in \mathcal{U}_\lambda$. Assume there is a $U \in \mathcal{U}_\lambda$ such that $U \cap A_t$ is finite for all t. We may then choose an $h \in \omega^\omega$ so that for all $\ell \in \omega$ and $t = t^T_\ell$, $U \cap \vec{F}_{t^\frown k} = \emptyset$ for all $t^\frown k \in T$ with $h(\ell) < k$. Now choose $T' < T$ and m so that $T' \Vdash h(n) < \dot{f}_{\mathbb{L}}(n)$ for all $n > m$. It then follows that $U \cap \bigcup\{\vec{F}_t : t \in T'\}$ is contained in the finite union $\bigcup\{\vec{F}_t : t \in \{t^T_i : i \leq m\}\}$ – which contradicts our assumption on $\vec{F}$ and $\mathcal{U}_\lambda$. □

Question 7.1. Can either Mathias or Hechler forcing replace Laver forcing in the statement of Theorem 7.11?

Question 7.2. Does PFA imply that every ultrafilter on ω has a nowhere dense P-subfilter?

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