PFA(S)[S] and countably compact spaces

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Abstract

We show a number of undecidable assertions concerning countably compact spaces hold under PFA(S)[S]. We also show the consistency without large cardinals of *every locally compact, perfectly normal space is paracompact.*

1 Introduction

This note is a sequel to [5]. As shown in [14], [17], [13], and [5], forcing with a coherent Souslin tree over a model of an iteration axiom such as PFA produces a model with many of the consequences of the iteration axiom plus some useful consequences of V = L. As in [17], it is useful to catalog the former consequences for future use, especially when their proofs are non-trivial. As in [5], it is also useful to distinguish consequences of the method which do not require large cardinals. For a discussion of what we call PFA(S)[S], see e.g. [13] and [4]. Our main results here are a proof that PFA(S)[S] implies countably tight perfect pre-images of ω_1 include copies of ω_1, a simplification of the proof in [13] that PFA(S)[S] implies locally compact, perfectly normal

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spaces are paracompact, and a demonstration that this last conclusion can be obtained without the use of large cardinals.

In [5] we proved

Theorem 1.1. *PFA$(S)[S]$ implies every sequentially compact, non-compact regular space includes an uncountable free sequence. If the space has character $\leq \aleph_1$, then it includes a copy of ω_1.*

Corollary 1.2. *PFA$(S)[S]$ implies* (**PPI**)*: Every first countable perfect pre-image of ω_1 includes a copy of ω_1.*

Although this paper is about PFA$(S)[S]$, many of the results follow from just PFA. Some are new, because, although **PPI** was known to follow from PFA, the stronger version in Theorem 1.1 was not formulated and proven from PFA earlier. A simpler version of the proof in [5] proves it from PFA.

2 Countably compact, perfectly normal spaces

Theorem 2.1. *PFA$(S)[S]$ implies every countably compact regular space with closed sets G_δ's is compact.*

The conclusion was proved from MA_{ω_1} by W. Weiss [19]. It is not at all obvious that it can be obtained from PFA$(S)[S]$, since Weiss' proof uses MA_{ω_1}(σ-centred) essentially. The perfectly normal version of Theorem 2.1 was proved by S. Todorcevic several years ago with a non-trivial stand-alone proof. We now can easily obtain the stronger version from 1.1.

Proof. Since open sets are F_σ's, discrete subspaces of such a space are σ-closed-discrete and hence countable. Countably compact regular spaces with points G_δ are first countable and hence sequentially compact, so the result follows from Theorem 1.1. □

In [13] it was shown that

Theorem 2.2. *There is a model of PFA$(S)[S]$ in which every locally compact perfectly normal space is paracompact.*

The proof depended on $\sum^-$:

> *In a compact T_2, countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.*

The proof of $\sum^-$ (see [8]) depended on the following result, due to Todorcevic:

Lemma 2.3 [17]**.** *PFA(S)[S] implies every compact countably tight space is sequential.*

Lemma 2.3 has taken a very long time to appear. It therefore may be of interest that we can avoid using it to prove Theorem 2.2. We first note that in [8], $\sum^-$ is proved for compact sequential spaces, and Lemma 2.3 is then appealed to. Let us therefore show that that restricted version of $\sum^-$ suffices.

Theorem 2.4. *Assume:*

1. *Countably compact, perfectly normal spaces are compact;*
2. $\sum^-$ *for compact sequential spaces;*
3. *Normal first countable spaces are collectionwise Hausdorff.*

Then locally compact, perfectly normal spaces are paracompact.

Proof. Given a locally compact, perfectly normal space X that is not compact, consider its one-point compactification $X^* = X \cup \{*\}$. To show X^* is sequential, take a non-closed subspace Y of X^*. If $Y \subseteq X$, Y is not countably compact. Then it has a countably infinite closed discrete subspace D. Viewed as a sequence, D converges to the point at infinity. Now suppose $*$, the point at infinity, is in $Y \subseteq X^*$. Let z be a limit point of Y which is not in Y. Then z is a limit point of $Y - \{*\}$, and so there is a sequence from Y converging to z, since X is first countable. □

We now can show the closure of a Lindelöf subspace Z of X is Lindelöf. $\overline{Z}^*$, the one-point compactification of $\overline{Z}$, is sequential. We claim it is hereditarily Lindelöf. If not, it has a right-separated subspace R of size $\aleph_1$, which by $\sum^-$ for sequential spaces is σ-discrete. Let R' be an uncountable discrete subspace of $R \cap \overline{Z}$. By closed sets G_δ, R' is σ-closed-discrete, so let R'' be a closed discrete subspace of R of size $\aleph_1$. By $\aleph_1$-collectionwise Hausdorffness and normality, expand R'' to a discrete collection of open sets. The traces of those open sets on Z form an uncountable discrete collection, contradicting Lindelöfness. □

Continuing with the proof of Theorem 2.4, we have improved [13] by not needing that there are no first countable L-spaces. Instead we use:

Lemma 2.5. *Locally compact, perfectly normal, collectionwise Hausdorff spaces are the topological sum of subspaces with Lindelöf number* $\leq \aleph_1$*.*

This was proved by G. Gruenhage in [11], with the slightly stronger assumption of "collectionwise normality with respect to compact sets". This was later improved by H.J.K. Junnila (unpublished) to just use collectionwise Hausdorffness. Since a sum of paracompact spaces is paracompact, we have reduced the problem to showing:

> (†) *Locally compact, perfectly normal spaces with Lindelöf number* $\leq \aleph_1$ *are paracompact.*

Because such a space has countable tightness and – under our assumptions – has closures of Lindelöf subspaces Lindelöf, it can be written as an increasing union of $\{X_\alpha\}_{\alpha<\omega_1}$, where X_α is Lindelöf and open, $\overline{X}_\alpha \subseteq X_{\alpha+1}$, and for limit λ, $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$. If the space were not paracompact, stationarily often $\overline{X}_\alpha - X_\alpha$ would be non-empty. Picking a point from each such boundary, one obtains a locally countable subspace of size $\aleph_1$. Applying $\sum^-$ for sequential spaces and closed sets G_δ, that subspace is σ-closed-discrete. Via pressing down, normality, and $\aleph_1$-collectionwise Hausdorffness, we obtain an uncountable discrete collection of open sets tracing onto some X_α, contradicting Lindelöfness. □

Instead of relying on Junnila's unpublished work, we could have used:

Lemma 2.6. *Suppose every locally compact, perfectly normal space is collectionwise Hausdorff. Then every locally compact, perfectly normal space is collectionwise normal with respect to compact sets.*

Proof. Let $\mathcal{K}$ be a discrete collection of compact sets in a locally compact, perfectly normal space X. Consider the quotient space $X/\sim$ obtained by collapsing each member of $\mathcal{K}$ to a point. It is routine to verify that $X/\sim$ is locally compact, perfectly normal, and that if $X/\sim$ is collectionwise Hausdorff, then $\mathcal{K}$ is separated in X. □

Following the scheme in [5], we shall also prove:

Theorem 2.7. *If ZFC is consistent then so is ZFC plus "locally compact perfectly normal spaces are paracompact".*

Proof. First, as in [13], we perform a preliminary forcing to obtain $\diamondsuit$ for stationary systems on all regular uncountable cardinals. This will give us a coherent Souslin tree S, the form of $\diamondsuit$ on ω_2 used in [5], and after we force with an $\aleph_2$-c.c. proper S-preserving iteration of size $\aleph_2$, that

> $(*)$ *normal first countable $\aleph_1$-collectionwise Hausdorff spaces are collectionwise Hausdorff.*

Following the scheme in [5] (previously seen in [18] and [3]) we form an $\aleph_2$-length countable support iteration of $\aleph_2$-p.i.c. S-preserving proper posets establishing **PPI** and $\sum^-$, after which we then force with S. The last forcing was shown in [13] to produce:

> (**CW**) *normal first countable spaces are $\aleph_1$-collectionwise Hausdorff.*

As noted in [13], the other forcings will preserve $\diamondsuit$ for stationary systems on ω_2, so combining $(*)$ and **CW**, we obtain (3).

The only new point to observe is that each stage of the iteration producing $\sum^-$ for compact sequential spaces is proper and S-preserving, and has cardinality $\aleph_1$, by CH. "Proper and S-preserving" was shown in [8]. As in the **PPI** without large cardinals proof in [5], we can code up the necessary information about locally countable sets of size $\aleph_1$ so as to get a poset of size $\aleph_1$ for establishing an instance of $\sum^-$. We can then iterate $\aleph_2$ times, alternately forcing with the **PPI** and $\sum^-$ posets, before forcing with S. Since, as noted in [5], $\sum^-$ implies $\mathfrak{b} > \aleph_1$, we don't need to add dominating reals to obtain this as in [5]. As was done in [5], one uses the diamond to show that the iteration is long enough. $\square$

If one is not trying to avoid Moore-Mrówka (i.e. the conclusion of 2.3), it is not hard to prove:

Theorem 2.8. *Assume $\sum^-$ and that normal first countable spaces are collectionwise Hausdorff. Then locally compact, perfectly normal spaces are paracompact.*

Proof. We only used "countably compact perfectly normal spaces are compact" in order to prove that the space's one-point compactification was sequential. Instead we shall prove just that it is countably tight. By [1], it suffices to prove our space includes no perfect pre-image of ω_1. But that follows easily from perfect normality. $\square$

Here is a variation of Theorem 2.1.

Theorem 2.9. *PFA$(S)[S]$ implies that if X is countably compact, locally compact, and does not include a perfect pre-image of ω_1, then X is compact.*

Proof. By [1], the one-point compactification X^* is countably tight. By PFA$(S)[S]$ and [17] (also 2.3), X^* is sequential and hence sequentially compact. (Dow [4] obtains sequential compactness via PFA$(S)[S]$ without using 2.3.) Then X is sequentially compact, for take a sequence. It has a convergent subsequence in X^*. The limit of that sequence cannot be the point at infinity, $*$, else the sequence would be closed discrete, violating countable compactness. By 1.1 it suffices to show X has no uncountable free sequence. Suppose it did. Then that free sequence has $*$ in its closure, else it would be free in X^*, contradicting countable tightness. Since the closure of the free sequence in X is not closed in X^*, it is not sequentially closed there, so there is a sequence in the closure of the free sequence which converges to $*$, since there is nowhere else for it to converge to. But then, again there is an infinite closed discrete subset of X, contradiction. □

The following corollary is of interest with regard to the question of whether there exist large countably compact, locally countable spaces.

Corollary 2.10. *PFA$(S)[S]$ implies that if X is uncountable, countably compact, locally countable, and T_3, then X includes a copy of ω_1.*

Proof. Since X is locally countable, it cannot be compact. It is locally compact, so by the Theorem, it must include a perfect pre-image of ω_1. It is first countable, so then must include a copy of ω_1. □

3 ω-bounded spaces

Here are some more applications of PFA$(S)[S]$ to countably compact spaces. Recall:

Definition. *A space is ω-**bounded** if every countable subset of it has compact closure.*

It is known that:

Lemma 3.1 [10]**.** *An ω-bounded space which is not compact includes a perfect pre-image of ω_1.*

For hereditarily normal spaces, under PFA$(S)[S]$ we can improve "perfect pre-image" to copy:

Theorem 3.2. *PFA$(S)[S]$ implies every hereditarily normal, ω-bounded, non-compact space includes a copy of ω_1.*

Proof. Apply Lemma 3.1, **PPI**, and Lemma 3.3. □

Lemma 3.3 [6]**.** *PFA$(S)[S]$ implies every hereditarily normal perfect pre-image of ω_1 includes a first countable perfect pre-image of ω_1 and hence a copy of ω_1.*

Surprisingly, we can do considerably better:

Theorem 3.4. *PFA$(S)[S]$ implies every hereditarily normal, countably compact, non-compact space includes a copy of ω_1.*

Proof. The first step is to prove:

Theorem 3.5. *PFA$(S)[S]$ implies separable, hereditarily normal, countably compact spaces are compact.*

Proof. Let $\dot{X}$ be our S-name of a hereditarily normal separable countably compact space. We first explain that it suffices to show that $\dot{X}$ is forced to be sequentially compact.

The argument is by contradiction and goes as follows. If $\dot{X}$ is forced to be sequentially compact but not compact, then by Theorem 1.1 $\dot{X}$ is forced to contain an uncountable free sequence. We then invoke [16] which tells us that under $\mathfrak{q} = \aleph_1$ – and hence under **CW** – separable, hereditarily normal, countably compact spaces do not have uncountable free sequences.

Now we show that $\dot{X}$ is sequentially compact. Suppose that ω is any infinite discrete subset of $\dot{X}$, and assume that ω is forced to have no converging subsequence.

If we consult the proof of Theorem 3.3 in [4] we find in the proof that there must be a condition $s \in S$, an infinite subset $b \subset \omega$, and a family F_s of functions from ω into $[0,1]$ such that s forces each $f \in F_s$ has a continuous extension to all of $\dot{X}$, and for each infinite $a \subset b$, there is an $f \in F_s$ such that s forces that $f[a]$ is not a converging sequence in $[0,1]$. Since the proof is short we provide it here.

For each $t \in S$, let F_t denote the set of all $f \in [0,1]^\omega$ such that t forces that f has a continuous extension to all of X. Say that $a \subset \omega$ is split by F_t

if there is an $f \in F_t$ such that t forces that the set $\{f(n) : n \in a\}$ does not converge. Fix any well-ordering of S in order type ω_1 and recursively choose, if possible, a mod finite, length ω_1 chain of infinite subsets $\{a_t : t \in S\}$ of ω so that a_t is not split by F_t. We are working in PFA(S) (a model of $\mathfrak{p} = \mathfrak{c}$) so we may choose an a which is mod finite contained in a_t for all $t \in S$. It is easy to see that 1 forces that a is a converging sequence in the countably compact space X. Therefore, this induction must have stopped at some $s \in S$ and if we choose any infinite a_s mod finite contained in a_t for each t coming before s in the well-ordering, we have have the desired pair b, F_s.

Now F_s is a family of actual functions, not just names of functions. This means there is an embedding e of ω into $[0,1]^{F_s}$ where $e(n) = e_n$ is defined by $e_n(f) = f(n)$. It follows that $\{e_n : n \in b\}$ is a completely divergent sequence in $[0,1]^{F_s}$. This of course means that the closure K of $\{e_n : n \in b\}$ is a compact non-sequential space. This is happening in a model of PFA(S), and we prove in [5] that there is an uncountable free sequence in K. It follows easily that there is a $\aleph_1$-sized subset F'_s of F_s satisfying that $\{e_n \restriction F'_s : n \in b\}$ also has an uncountable free sequence in its closure. Let $\{y_\alpha : \alpha \in \omega_1\} \subset [0,1]^{F'_s}$ denote this free sequence. Since the PFA(S) model is a model of $\mathfrak{p} > \omega_1$, we may fix, for each $\alpha \in \omega_1$, a subset a_α of b so that $\{e_n \restriction F'_s : n \in a_\alpha\}$ converges to y_α.

Now we pass to the PFA(S)[S] extension and have a look at X. Clearly the product of the family of continuous extensions of the functions in F'_s, call this φ, is a continuous function from X into $[0,1]^{F'_s}$. For each $\alpha \in \omega_1$, let x_α be any limit point of a_α. For each $n \in b$, $\varphi(n)$ is equal to $x_n \restriction F'_s$, and therefore, $\varphi(x_\alpha)$ is equal to y_α. It follows immediately that $\{y_\alpha : \alpha \in \omega_1\}$ is a free sequence. □

Theorem 3.5 gives:

Corollary 3.6. *PFA(S)[S] implies hereditarily normal, countably compact spaces are ω-bounded.*

We then apply 3.2 to obtain 3.4. □

Remark. Under PFA, one can replace the "hereditarily" in Theorem 3.5 by "first countable", but not under PFA(S)[S]. PFA(S)[S] implies $\mathfrak{p} = \aleph_1$ [17] and under that hypothesis, there is a locally compact, locally countable, separable, normal, countably compact space which is not compact. This space is due to Franklin and Rajagopalan [9]. See discussion in Section 7 of [2] and Section 2 of [15].

The PFA version of this next result was proven by Eisworth [7].

Theorem 3.7. *PFA$(S)[S]$ implies a countably tight perfect pre-image of ω_1 includes a copy of ω_1.*

Proof. By Lemma 2.3 (as shown in [17]) PFA$(S)[S]$ implies that all countably tight compact spaces are sequential. Since a perfect pre-image of ω_1 is locally compact it will suffice to prove this theorem for sequential perfect pre-images of ω_1. The proof is technical and requires familiarity with [5] so is postponed to the end of the paper. □

Theorem 3.7 will be used in [6] to prove:

Proposition 3.8. *There is a model of form PFA$(S)[S]$ in which a locally compact, normal, countably tight space is paracompact if and only if its separable closed subspaces are Lindelöf, and it does not include a copy of ω_1.*

Just as for hereditarily normal, one can vary 3.1 to get

Theorem 3.9. *PFA$(S)[S]$ implies a countably tight, ω-bounded space which is not compact includes a copy of ω_1.*

Proof. Immediate from 3.1 and 3.7. □

Problem 1. *Does PFA$(S)[S]$ imply every non-compact, countably tight, countably compact space includes a perfect pre-image of ω_1? If so, by 3.7 it would include a copy of ω_1.*

4 Some problems of Nyikos

In [16] Peter Nyikos raises a number of questions about hereditarily normal countably compact spaces and settles some of them under PFA. We can obtain similar results under PFA$(S)[S]$ and also answer some questions he left open. We shall use his numbering, for easy reference.

Statement A. *Every compact space of countable tightness is sequential.* Follows from PFA$(S)[S]$ [17].

Statement 2 (= 1.3(2)). *Every separable, T_5, countably compact space is compact.*

Follows from PFA(S)[S]: Theorem 3.5.

Statement 3 *Every countably compact T_5 space is either compact or includes a copy of ω_1.*
Follows from PFA(S)[S]: Theorem 3.4.

1.3(1) *Every free sequence in a separable, countably compact T_5 space is countable.*
Follows from $\mathfrak{q} = \aleph_1$, and hence from PFA(S)[S].

1.3(3) *Every countably compact T_5 space is sequentially compact.*
Follows from PFA(S)[S].

Proof. The proof of 3.5 establishes this in the separable case. Apply the separable case to the closure of a given sequence. □

We thus have PFA(S)[S] implies

Statement 1 (1.3(4)) *Every compact T_5 space is sequentially compact.*

1.4 *In a countably compact T_5 space, every countable subset has compact, Fréchet-Urysohn closure.*

We can do better under PFA(S)[S]. We actually get first countable closure. By 3.5 we get compact; by [17] we get hereditarily Lindelöf, hence first countable, since it is shown there that PFA(S)[S] implies compact, hereditarily normal, separable spaces are hereditarily Lindelöf.

Nyikos' Problem 2 asks if Statement A is compatible with $\mathfrak{q} = \aleph_1$. It is.

He asks whether the following is consistent:

Statement 5. *Every compact separable T_5 space is of character $< \mathfrak{p}$.*

Since PFA(S)[S] implies such spaces are first countable, the answer is trivially "yes".

Nyikos' Problem 5 asks if there is a ZFC example of a separable, T_5, locally compact space of cardinality $\aleph_1$. He points out that if there are no locally compact S-spaces and $\mathfrak{q} = \aleph_1$, then there is a negative answer. Since the one-point compactification of an S-space is an S-space, PFA(S)[S] yields a negative answer.

5 Countably tight perfect pre-images of ω_1

We now complete the proof of Theorem 3.7. The reader is referred to [5] for the final stages of the proof. As noted above in the first steps of the proof of Theorem 3.7, we may assume, for the remainder of the section, that we have an S-name $\dot{X}$ of a sequential space with a perfect mapping $\dot{f}$ onto the ordinal ω_1. Note that $\dot{X}$ is forced to be locally compact. By passing to a subspace we can assume that the preimage of each successor ordinal is a singleton. Then, by simple renaming, we may assume that $\{\alpha+1 : \alpha \in \omega_1\}$ is a dense subset of $\dot{X}$. We can now assume that the base set for $\dot{X}$ is the ordinal ω_2 (since PFA(S) implies that $\mathfrak{c} = \omega_2$, and the cardinality of a sequential space of density at most ω_1 is at most $\mathfrak{c}$). Next we fix an assignment of S-names of open neighborhood bases $\{\dot{U}(x,\xi) : \xi \in \omega_2\}$, for each $x \in \omega_2$. Obviously repetitions are allowed. We may assume, by the continuity of $\dot{f}$, that 1 forces that $\dot{f}[\dot{U}(x,\xi)] \subset [0, \dot{f}(x)]$ for all $x, \xi \in \omega_2$.

Now we discuss the special forcing properties that a coherent Souslin tree will have. Assume that g is a generic filter on S viewed as a cofinal branch. For each $s \in S$, $o(s)$ is the level (order-type of domain) of s in S. For any $t \in S$, define $s \oplus t$ to be the function $s \cup t \restriction [o(s), o(t))$. Of course when $o(t) \leq o(s)$, $s \oplus t$ is simply s. One of the properties of S ensures that $s \oplus t \in S$ for all $s, t \in S$. We similarly define $s \oplus g$ to be the branch $\{s \oplus t : t \in g\}$. We let $\dot{X}[g]$ (or even $X[g]$) denote the space obtained by evaluating the topology using g, so $\dot{X}[s \oplus g]$ will be a different perfect pre-image of ω_1 also existing in the model $V[g]$. More generally for an S-name $\dot{A}$, we will let $\dot{A}[g]$ denote the standard evaluation of $\dot{A}$ by g.

We recall a useful method of calculating closure in sequential spaces. A tree $T \subset \omega^{<\omega}$ is said to be well-founded if it contains no no infinite branch. Let **WF** denote all downward closed well-founded trees $T \subset \omega^{<\omega}$ with the property that each non-maximal node has a full set of immediate successors. Such a tree has an associated rank function, rk_T which maps elements of T into ω_1. If $t \in T$ is a maximal node, then $\mathrm{rk}_t(t) = 0$, and otherwise, $\mathrm{rk}_T(t)$ is equal to $\sup\{\mathrm{rk}_T(t') + 1 : t < t' \in T_t\}$. The rank of T itself will be $\mathrm{rk}_T(\emptyset)$ and we let $\mathbf{WF}(\alpha)$ denote the set of trees of rank less than α. Suppose that σ is any function from $\max(T) = \{t \in T : \mathrm{rk}_T(t) = 0\}$ into ω_1 as a subset of a space X. By induction on rank of $t \in T$, define an evaluation $e(\sigma, t)$ to be the limit (if it exists) of the sequence $\{e(\sigma, t^\frown n) : n \in \omega\}$. We will say that σ is X-converging if $e(\sigma, t)$ exists for all $t \in T$. It is well-known that every point in the sequential closure of $\omega_1 \subset X$ will equal $e(\sigma, \emptyset)$ for some such σ.

Now, given our S-name $\dot{X}$, we will define Λ to be the set of all σ as above (the underlying $T \in \mathbf{WF}$ is simply the downward closure of the domain of σ) with the propery that 1 forces that σ is converging. Since S has the ccc property and adds no new subsets of ω, it follows that if $s \in S$ forces that an ordinal $\zeta \in \omega_2$ is in the (sequential) closure of some δ, then there is a $\sigma \in \Lambda$ such that s forces that $e(\sigma, \emptyset) = \zeta$. However, even though 1 forces that σ converges, it is not true that 1 forces that $e(\sigma, \emptyset) = \zeta$. Nevertheless the set Λ makes for a very useful substitute for S-names of members of $\dot{X}$. Now that the actual ordinal value associated to λ depends on the generic, we will use $e_g(\sigma)$ (and suppress the second coordinate) to refer to the ordinal ζ in $X[g]$ that is equal to $e(\sigma, \emptyset)$. Similarly, we use $e_s(\sigma)$ if s decides this value.

Definition 5.1. *For any sequence $\langle \sigma_n : n \in \omega \rangle$ of members of Λ, say that σ is constructed from $\langle \sigma_n : n \in \omega \rangle$ if for each $n \in \omega$ and each node $t \in \mathrm{dom}(\sigma)$ (a maximal node of the associated tree) with $t(0) = n$, and each node $t_1 \in \mathrm{dom}(\sigma_n)$,*

1. *there a node $t_2 \in \mathrm{dom}(\sigma)$ such that $t_2(0) = n$, $\sigma(t_2) = \sigma_n(t_1)$, $\mathrm{dom}(t_2) = 1 + \mathrm{dom}(t_1)$, and $t_2(1 + j) = t_1(j)$ for all $j \in \mathrm{dom}(t_1)$,*
2. *there is a node $t_3 \in \mathrm{dom}(\sigma_n)$ such that $\sigma(t) = \sigma_n(t_3)$, $\mathrm{dom}(t) = 1 + \mathrm{dom}(t_3)$, and $t_3(j) = t(1 + j)$ for all $j \in \mathrm{dom}(t_3)$,*

Definition 5.2. *For each integer $n > 0$, and subset B of Λ^n we define the hierarchy $\{B^{(\alpha)} : \alpha \in \omega_1\}$ by recursion. For a limit α, $B^{(\alpha)}$ (which could also be denoted as $B^{(<\alpha)}$) will equal $\bigcup_{\beta<\alpha} B^{(\beta)}$. The members of $B^{(\alpha+1)}$ for any α, will consist of $B^{(\alpha)}$ together with all $\vec{b} \in \Lambda^n$ with the property that there is a sequence $\langle \vec{b}_\ell : \ell \in \omega \rangle$ consisting of members of $B^{(\alpha)}$ such that, for each $i < n$, $\vec{b}(i)$ is built from the sequence $\{\vec{b}_\ell(i) : \ell \in \omega\}$.*

A subset B of Λ^n will be said to be S-sequentially closed if $B^{(\omega_1)} = B$.

The next lemma should be obvious.

Lemma 5.3. *For each $A \subset \Lambda$, 1 forces that $e[A^{(\omega_1)}]$ is a sequentially compact subset of $\dot{X}$.*

Definition 5.4. *For each S-name $\dot{A}$ and $s \Vdash \dot{A} \subset \Lambda^n$, we define the S-name $(\dot{A})^{(\omega_1)}$ according to the property that for each $s < t$ and $t \Vdash \vec{b} \in (\dot{A})^{(<\omega_1)}$, there is a countable $B \subset Y^n$ such that $t \Vdash B \subset \dot{A}$ and $\vec{b} \in B^{(<\omega_1)}$.*

By our assumption that ω_1 has no complete accumulation points, the family $\{(\omega_1 \setminus \delta)^{(\omega_1)} : \delta \in \omega_1\}$ is a free filter of S-sequentially closed subsets of Λ. By Zorn's Lemma, we can extend it to a maximal free filter, $\mathcal{F}_0$, of S-sequentially closed subsets of Λ. To apply PFA(S) we require that we have a maximal filter in the forcing extension by S. The filter $\mathcal{F}_0$ may not generate a maximal filter in the extension $V[g]$ and so we will have to extend it. We will also need there to be a close connection between the behavior of our chosen maximal filter in $V[g]$ and in $V[s \oplus g]$ for all $s \in S$. We refer to this as "symmetry".

We introduce some notational conventions. Let $S^{<\omega}$ denote the set of finite tuples $\langle s_i : i < n\rangle$ (ordered lexicographically) for which there is a δ such that each $s_i \in S_\delta$. Our convention will be that they are distinct elements. We let bS denote the collection $\{s \oplus g : s \in S\}$ (technically this is an S-name for the set of all ω_1-branches of S in $V[g]$). We will be working in the product structure Λ^{bS} and we let $\Pi_{\langle s_i : i<n\rangle}$ denote the projection from Λ^{bS} to the product $\Lambda^{\{s_i \oplus g \ : \ i<n\}}$, and we let $\tilde{\Pi}_{\langle s_i \ : \ i<n\rangle}$ be the projection onto Λ^n. The notation $\Pi^X_{\langle s_i \ : \ i<n\rangle}$ will be used as the notation for the projection (in the extension $V[g]$) from the product space $\mathbf{\Pi}\{X[s \oplus g] \ : \ s \in S\}$ (we ignore repetitions) onto $\mathbf{\Pi}\{X[s_i \oplus g] \ : \ i < n\}$. In case of possible confusion, we adopt a standard convention that for a singleton $s \in S$, we identify $\Lambda^{\{s\}}$ with Λ and $\mathbf{\Pi}\{X[s \oplus g]\}$ with $X[s \oplus g]$; and similarly Λ^n with $n = 1$ is treated as simply being Λ.

Definition 5.5. *For $\vec{\sigma} \in \Lambda^{\langle s_i \ : \ i<n\rangle}$ and generic g, we intend that $e_g(\vec{\sigma})$ should equal the vector $\langle e_{s_i \oplus g}(\vec{\sigma}_i) \ : \ i < n\rangle$. Similarly, for $\dot{A}$ a name of a subset of $\Lambda^{\langle s_i \ : \ i<n\rangle}$, if a condition s forces a value on $\dot{A}$, then we can use $e_s(\dot{A})$ as an abbreviation for $e_{s\oplus g}(\dot{A}) = \{e_{s\oplus g}(\vec{\sigma}) \ : \ \vec{\sigma} \in \dot{A}[s \oplus g]\}$.*

Definition 5.6. *Suppose that $\dot{A}$ is an S-name of a subset of Λ^n for some n, in particular, that some s forces this. Let s' be any other member of S with $o(s') = o(s)$. We define a new name $\dot{A}^s_{s'}$ (the (s, s')-transfer perhaps) which is defined by the property that for all $\langle \sigma_i\rangle_{i<n} \in \Lambda^n$ and $s < t \in S$ such that $t \Vdash \langle \sigma_i\rangle_{i<n} \in \dot{A}$, we have that $s' \oplus t \Vdash \langle \sigma_i\rangle_{i<n} \in \dot{A}^s_{s'}$.*

Lemma 5.7. *For any generic $g \subset S$, $\dot{A}[s \oplus g] = (\dot{A}^s_{s'})[s' \oplus g]$.*

Theorem 5.8. *There is a family $\mathcal{F} = \{(s^\alpha, \{s^\alpha_i \ : \ i < n_\alpha\}, \dot{F}_\alpha) \ : \ \alpha \in \lambda\}$ where,*

1. *for each $\alpha \in \lambda$, $\{s^\alpha_i \ : \ i < n_\alpha\} \in S^{<\omega}$, $s^\alpha \in S$, $o(s^\alpha_0) \leq o(s^\alpha)$,*

2. $\dot{F}_\alpha$ *is an* S*-name such that* $s^\alpha \Vdash \dot{F}_\alpha = (\dot{F}_\alpha)^{(\omega_1)} \subset \Lambda^{n_\alpha}$

3. *for each* $s \in S$ *and* $F \in \mathcal{F}_0$, $(s, \{s\}, \check{F}) \in \mathcal{F}$,

4. *for each* $s \in S_{o(s^\alpha)}$, $(s, \{s_i^\alpha : i < n_\alpha\}, (\dot{F}_\alpha)_s^{s^\alpha}) \in \mathcal{F}$,

5. *for each generic* $g \subset S$, *the family* $\{\tilde{\Pi}^{-1}_{\langle s_i^\alpha : i<n_\alpha\rangle}((\dot{F}_\alpha)[g]) : s^\alpha \in g\}$ *is finitely directed; we let* $\dot{\mathcal{F}}_1$ *be the* S*-name for the filter base it generates.*

6. *For each generic* $g \subset S$ *and each* $\langle s_i : i < n\rangle \in S^{<\omega}$, *the family* $\{(\dot{F}_\alpha)[g] : s^\alpha \in g \text{ and } \{s_i \oplus g : i < n\} = \{s_i^\alpha \oplus g : i < n_\alpha\}\}$ *is a maximal filter on the family of* S*-sequentially closed subsets of* Λ^n.

Proof. Straightforward recursion or Zorn's Lemma argument over the family of "symmetric" filters (those satisfying (1)-(5)). □

Definition 5.9. *For any* $\langle s_i : i < \ell\rangle \in S^{<\omega}$, *let* $\dot{\mathcal{F}}_{\langle s_i : i<\ell\rangle}$, *respectively* $\dot{\mathcal{F}}^{\sim}_{\langle s_i : i<\ell\rangle}$, *denote the filter on* $\Lambda^{\langle s_i \oplus g : i<\ell\rangle}$, *respectively* Λ^ℓ, *induced by* $\Pi_{\langle s_i : i<\ell\rangle}(\dot{\mathcal{F}}_1)$, *respectively* $\tilde{\Pi}_{\langle s_i : i<\ell\rangle}(\dot{\mathcal{F}}_1)$. *Except for re-naming of the index set, these are the same.*

Definition 5.10. *Let* $\mathcal{A}$ *denote the family of all* $(s, \langle s_i : i < \ell\rangle, \dot{A})$ *satisfying that* $o(s) \geq o(s_0)$, $\langle s_i : i < \ell\rangle \in S^{<\omega}$, *and* $s \Vdash \dot{A} \in \dot{\mathcal{F}}^+_{\langle s_i : i<\ell\rangle}$. *As usual, for a family* $\mathcal{G}$ *of sets,* $\mathcal{G}^+$ *denotes the family of sets that meet each member of* $\mathcal{G}$.

Lemma 5.11. *For each* $(s, \langle s_i : i < n\rangle, \dot{A}) \in \mathcal{A}$, *the object* $(s, \langle s_i : i < n\rangle, \dot{A}^{(\omega_1)})$ *is in the list* $\mathcal{F}$.

5.1 S-preserving proper forcing

This next statement was a lemma in the proof of PPI in [5], we change it to a definition.

Definition 5.12. *Suppose that* $M \prec H(\kappa)$ *(for suitably big* κ*) is a countable elementary submodel containing* $\Lambda, \mathcal{A}$. *Let* $M \cap \omega_1 = \delta$. *Say that the sequence* $\langle y^M(s) : s \in S_\delta\rangle$ *is an* M*-acceptable sequence providing that for every* $(\bar{s}, \{s_i : i < n\}, \dot{A}) \in \mathcal{A} \cap M$, *and every* $s \in S_\delta$ *with* $\bar{s} < s$, *there is a* $B \subset \Lambda^n \cap M$ *such that* $s \Vdash B \subset \dot{A}$ *and* $s \Vdash \langle y^M(s_i \oplus s) : i < n\rangle \in B^{(\delta+1)}$.

We must give a new proof that there is an acceptable sequence consisting of points with a special countable sequence of neighborhoods (see Lemma 5.14). The complicated condition (3) is capturing the net effect of all the convergence and symmetry requirements that will emerge in the definition of our poset.

Definition 5.13. *An M-acceptable sequence $\langle y^M(s) : s \in S_\delta\rangle$ ($M \cap \omega_1 = \delta$) is (M, ω)-acceptable providing there is a countable set $T \subset \omega_2$ and a $\gamma > \delta = M \cap \omega_1$ such that, for each $s \in S_\gamma$,*

1. *s forces an ordinal value x on $e_g(y^M(g \cap S_\delta))$,*
2. *for each $\xi \in T$, s forces a value, denoted $U_s(x, \xi)$, on $\dot{U}(x, \xi) \cap M$ which is a subset of $X[s \oplus g]$ (which is a set of ordinals),*
3. *s forces that x is the only point that is in the closure of each of the sets*

$$\Pi^X_{\langle s\oplus g\rangle}\left((\Pi^X_{\langle s_i \ : \ i<n\rangle})^{-1}\left(\mathbf{\Pi}\{U_{s_i\oplus s'}(\xi_i) \ : \ i < n\} \cap e_s[F \cap M]\right)\right)$$

where $\langle \xi_i : i < n\rangle \in T^n$, $s' \in S_\gamma$, $s' \restriction \delta = s \restriction \delta$, and s forces that F is a member of $M \cap \mathcal{F}_{\langle s_i \ : \ i<n\rangle}$ for some $\langle s_i : i \in n\rangle \in S^{<\omega} \cap M$. Note that s does force a value on $e_g[F \cap M]$, and so this value is $e_s[F \cap M]$.

It may help to unravel condition (3) a little. The set $e_s[F \cap M]$ is the ordinal evaluation of a set that s has forced to be in our filter. Then we use an open set intersected with M, namely $\mathbf{\Pi}\{U_{s_i\oplus s'}(\xi_i) : i < n\}$, that comes from sets that s' forces are in the topology. The connection here is that $e_s[F \cap M]$ will be equal to $e_{s'}[F' \cap M]$ for some other F' forced by s' to be in the filter. Then the map $(\Pi^X_{\langle s_i \ : \ i<n\rangle})^{-1}$ pulls this intersection back into the full product $\mathbf{\Pi}\{X[s \oplus g] : s \in S\}$ and, finally, $\Pi^X_{\langle s\oplus g\rangle}$ is just the projection to the single coordinate $s \oplus g$. Now we are asking that s will force that x will (still) be the limit even though s' may well have assigned a different and disjoint set of ordinals to be limits of any of these sets. This aspect is necessary for the poset to be proper and is already handled (can be shown to be) by M-acceptability. The additional requirement that s forces that x is the only such limit is to ensure we get a copy of ω_1.

Lemma 5.14. *For each countable (suitable) $M \prec H(\theta)$ (i.e. with $\dot{\mathcal{F}}_1, \dot{X}, S$ all in M) there is an (M, ω)-acceptable sequence.*

We postpone the proof. But henceforth we adopt the notation that for a countable suitable $M \prec H(\theta)$, the sequence $\langle y^M(s) \ : \ s \in S_{M\cap\omega_1}\rangle$ denotes the $\prec$-least (M,ω)-suitable sequence. Also let $\{T_M(n) \ : \ n \in \omega\}$ denote a countable subset of ω_2 witnessing that the sequence is (M,ω)-acceptable, and not just M-acceptable. We can arrange that $\{T_M(n) \ : \ n \in \omega\}$ is forced to form a regular filter base.

Now we are ready to define our poset $\mathcal{P}$. Another change we make from [5] is that we no longer have the assumption of countable bases of neighborhoods, and so we use finite subsets of ω_2 rather than of ω as side conditions. We will be able to weaken the character assumption (through Lemma 5.12) by an appeal to [17, 8.5] which showed that there will be a rich supply of relative G_δ-points.

Definition 5.15. *A condition $p \in \mathcal{P}$ consists of $(\mathcal{M}_p, S_p, H_p)$ where $\mathcal{M}_p$ is a finite $\in$-chain of countable suitable elementary submodels of $H(\theta)$ and H_p is a finite subset of ω_2. We let M_p denote the maximal element of $\mathcal{M}_p$ and let $\delta_p = M_p \cap \omega_1$. We require that S_p is a finite subset of S_{δ_p}. For $s \in S_p$ and $M \in \mathcal{M}_p$, we use both $s \restriction M$ and $s \cap M$ to denote $s \restriction (M \cap \omega_1)$. Note that the sequence $\{y^M(s) \ : \ s \in S_{M\cap\omega_1}\}$ is in each M' whenever $M \in M'$ are both in $\mathcal{M}_p$.*

It is helpful to simultaneously think of p as inducing a finite subtree, $S_p^{\downarrow}$, of S equal to $\{s \restriction M \ : \ s \in S_p, \text{ and } M \in \mathcal{M}_p\}$.

For each $s \in S_p$ and each $M \in \mathcal{M}_p \setminus M_p$ we define an S-name $\dot{W}_p(s \restriction M)$ of a neighborhood of $e_s(y^M(s \restriction M))$. It is defined as the name of the intersection of all sets of the form $\dot{U}(e_{s'}(y^{M'}(s' \restriction M')),\xi)$ where $s' \in S_p$, $\xi \in H_p$, $M' \in \mathcal{M}_p \cap M_p$, and $s \restriction M \subset s' \restriction M'$ and $e_{s'}(y^M(s \restriction M)) \in \dot{U}(e_{s'}(y^{M'}(s' \restriction M')),\xi)$. We adopt the convention that $\dot{W}_p(s\cap M)$ is all of X if $s \cap M \notin S_p^{\downarrow}$.

The definition of $p < q$ is that $\mathcal{M}_q \subset \mathcal{M}_p$, $H_q \subset H_p$, $S_q \subset S_p^{\downarrow}$, and for each $s' \in S_p$ and $s \in S_q$ below s', we have that s' forces that $e(y^M(s \restriction M)) \in \dot{W}_q(s \restriction M')$ whenever $M \in \mathcal{M}_p \setminus \mathcal{M}_q$ and M' is the minimal member of $M_q \cap (\mathcal{M}_q \setminus M)$. Another trivial change that we list separately for emphasis is that for each $M \in \mathcal{M}_q$ and each $k < |H_q|$, we require that $T_M(k) \in H_p$.

It is proven in [5] that a version of $\mathcal{P}$ is proper and S-preserving. The superficial distinction between the two posets can be handled in either of two ways. We can re-enumerate each $\{\dot{U}(x,n) \ : \ n \in \omega\}$ so as to be an enumeration of the sequence $\{\dot U(x, T_M(n)) \ : \ n \in \omega\}$, or we can examine the proof in [5] and notice that the proof did not in any way use that we were restricting to countably many neighborhoods of each point.

Before we prove Lemma 5.14, let us prove that this works.

Lemma 5.16. *If $\mathcal{P}$ is proper and S-preserving, then PFA(S) implies that S forces that $\dot X$ contains a copy of ω_1.*

Proof. For any condition $q \in \mathcal{P}$, let $\mathcal{M}(q)$ denote the collection of all M such that there exists a $p < q$ such that $M \in \mathcal{M}_p$. For each $\beta < \alpha \in \omega_1$, $s \in S_\alpha$, $m \in \omega$, and $\xi \in \omega_2$, let

$$\begin{aligned} D(\beta, \alpha, s, \xi, m) = \{ & p \in \mathcal{P} \ : \ \xi \in H_p \ , |H_p| \geq m, (\exists \bar s \in S_p) \ s < \bar s, \text{and} \\ & (\exists M \in \mathcal{M}_p) \ (\beta \in M, \alpha \notin M) \ \text{ or} \\ & (\forall M \in \mathcal{M}(p))(\beta \in M \Rightarrow \alpha \in M)\} \ . \end{aligned}$$

It is easily shown that each $D(\beta, \alpha, s, \xi, m)$ is a dense open subset of $\mathcal{P}$. Consider the family $\mathcal{D}$ of all such $D(\beta, \alpha, s, \xi, m)$, where $\beta, \alpha, \xi \in \omega_1$ and $m \in \omega$, and let G be a $\mathcal{D}$-generic filter. Let $\mathcal{M}_G = \{M \ : \ (\exists p \in G) \ M \in \mathcal{M}_p\}$ and let $C = \{M \cap \omega_1 \ : \ M \in \mathcal{M}_G\}$. For each $\delta \in C$, let M_δ denote the member of $\mathcal{M}_G$ such that $M_\delta \cap \omega_1 = \delta$ (we omit the trivial proof that there is exactly one such M for each $\gamma \in C$). Let $g \subset S$ be a generic filter and for each $\delta \in C$, let s_δ be the element of $g \cap S_\delta$. Also for each $\delta \in C$, let $x_\delta = e_g(y^{M_\delta}(s_\delta))$. Let us also note that for any $\beta < \delta$ both in C, x_β is equal to $e_{s_\delta}(y^{M_\beta}(s_\beta))$ because of the fact that y^{M_β} is an element of M_δ.

We show that the set $W = \{x_\gamma \ : \ \gamma \in C\}$ is homeomorphic to the ordinal ω_1. Indeed, the map $f = \dot f[g]$ is a homeomorphism onto the cub C. It is certainly 1-to-1 and onto. Let us show that it is a closed map. Let $\{\delta_\ell \ : \ \ell \in \omega\} \subset C$ be strictly increasing with supremum δ. We simply have to prove that x_δ is a limit of the sequence $\{x_{\delta_\ell} \ : \ \ell \in \omega\}$. To do so, by Definition 5.13, it suffices to prove that if $\gamma \in C \setminus \delta + 1$, T, $\langle \xi_i \ : \ i < n\rangle \in T^n$, $s' \in S_\gamma$, $s' \restriction \delta = s_\delta$, and s_γ forces that $\dot F$ is a member of $M_\delta \cap \mathcal{F}_{\langle s_i \ : \ i<n\rangle}$ for some $\langle s_i \ : \ i \in n\rangle \in S^{<\omega} \cap M_\delta$ (as in condition (3)), then $\{x_{\delta_\ell} \ : \ \ell \in \omega\}$ meets

$$\Pi^X_{\langle s_\gamma \oplus g\rangle} \left((\Pi^X_{\langle s_i \ : \ i<n\rangle})^{-1} \left(\mathbf{\Pi}\{U_{s_i \oplus s'}(\xi_i) \ : \ i < n\} \cap e_{s_\gamma}[F \cap M] \right) \right)$$

Fix any $m \in \omega$ large enough so that $\{\xi_i \ : \ i < n\} \subset \{T(\ell) \ : \ \ell < m\}$ and choose $p \in G \cap D(0, \delta, s_\gamma, m, m)$ so that $\{M_\delta, M_\gamma\} \subset \mathcal{M}_p$. Let β be the maximum element of $C \cap \delta$ such that $M_\beta \in \mathcal{M}_p$. By strengthening p and possibly raising β, we can assume that $\{s_i : i < n\}, \dot{F}$ are in M_β, and that $\langle s_\beta, \{s_i : i < n\}, \dot{F}\rangle$ is in $\mathcal{F}$. Choose $\ell_0 \geq m$ such that $\delta_\ell > \beta$ for all $\ell > \ell_0$. Additionally choose $r < p$ with $r \in G$ so that $s' \in S_r^{\downarrow}$, and for each $i < n$, $\{s_i \oplus s_\gamma, s_i \oplus s'\} \subset S_r^{\downarrow}$. Choose ℓ large enough so that $M \cap \omega_1 < \delta_\ell$ for all $M \cap \mathcal{M}_r \cap M_\delta$.

We note, in $V[g]$, that $x_{\delta_\ell} = e_{s_\delta}(y^{M_{\delta_\ell}}(s_0 \oplus s_{\delta_\ell}))$ and $e_{s_\delta}(y^{M_{\delta_\ell}}(s_0 \oplus s_{\delta_\ell}))$ is clearly an element of $\Pi^X_{\langle s_\gamma \oplus g\rangle}((\Pi^X_{\langle s_i \oplus g \,:\, i<n\rangle})^{-1}(\langle e_{s_\delta}(y^{M_{\delta_\ell}}(s_i \oplus s_{\delta_\ell})) : i < n\rangle))$. Therefore it suffices to prove that $\langle e_{s_\delta}(y^{M_{\delta_\ell}}(s_i \oplus s_{\delta_\ell})) : i < n\rangle$ is in each of $\mathbf{\Pi}\{U_{s_i \oplus s'}(\xi_i) : i < n\}$ and $e_{s_\gamma}[F \cap M]$. Choose any $\ell' > \ell$. We note that $\langle e_{s_\delta}(y^{M_{\delta_\ell}}(s_i \oplus s_{\delta_\ell})) : i < n\rangle$ is equal to $\langle e_{s_{\delta_{\ell'}}}(y^{M_{\delta_\ell}}(s_i \oplus s_{\delta_\ell})) : i < n\rangle$. It follows from the definition of each $\dot{W}_r$ that $s_i \oplus s'$ forces that $\dot{W}_r(s_i \oplus s_\delta) \cap M_\delta \subset U_{s_i \oplus s'}(\xi_i)$ for each $i < n$. It therefore follows, from the ordering on $\mathcal{P}$, that $\langle e_{s_{\delta_{\ell'}}}(y^{M_{\delta_\ell}}(s_i \oplus s_{\delta_\ell})) : i < n\rangle \in \mathbf{\Pi}\{U_{s_i \oplus s'}(\xi_i) : i < n\}$. Finally, since $\langle y^{M_{\delta_\ell}}(\bar{s}) : \bar{s} \in S_{\delta_\ell}\rangle$ is M_{δ_ℓ}-acceptable, we have that $\langle e_{s_{\delta_{\ell'}}}(y^{M_{\delta_\ell}}(s_i \oplus s_{\delta_\ell})) : i < n\rangle$ is a member of $e(B^{(\delta_\ell+1)})$ for some B that is forced by s_δ to be a subset of $\dot{F}$. That is, we have that $\langle e_{s_{\delta_{\ell'}}}(y^{M_{\delta_\ell}}(s_i \oplus s_{\delta_\ell})) : i < n\rangle$ is a member of $e_{s_\gamma}[F \cap M]$. $\square$

Now we prove Lemma 5.14.

Proof of Lemma 5.14. Fix M etc. as in the hypothesis of the Lemma. For each generic filter g, let $X[g](\delta)$ denote the compact pre-image of $[0, \delta]$ by the function $\dot{f}$. Let us note that the countable product space $\Pi\{X[s \oplus g_0](\delta) : s \in S_\delta\}$ is compact and sequential ([12, 2.5]); and therefore we know from [17, 8.5] that every closed subset has a relative G_δ-point.

Let $\{s_{\delta,j} : j \in \omega\}$ be the $\prec$-least enumeration of S_δ ($\delta = M \cap \omega_1$). We will also use s_δ to denote $s_{\delta,0}$. We establish some notation. Given any generic $g \subset S$, and working in the extension $V[g]$, let $\Pi_{M,g}$ denote the projection map from the product $\mathbf{\Pi}\{X[s \oplus g] : s \in S\}$ onto the countable product $\mathbf{\Pi}\{X[s_{\delta,j} \oplus g] : j \in \omega\}$. Let us note that, for each j, $X[s_{\delta,j} \oplus g] \cap M$ as a set is simply equal to $\omega_2 \cap M$. Now let Π_g denote the canonical isomorphism from $\mathbf{\Pi}\{X[s_{\delta,j} \oplus g] \cap M : j \in \omega\}$ to $(\omega_2 \cap M)^\omega$ which is induced by the bijection on the indices sending $s_{\delta,j}$ to j.

We adopt one more notational convention. Consider a vector $\vec{x}$ that is a function from S_δ to ω_2, i.e. $\vec{x} \in \omega_2^{S_\delta}$. When given any generic filter g, we will regard $\vec{x}$ (without using any notational device) as also being a function from $\{s \oplus g \ : \ s \in S_\delta\}$ into ω_2. Thus, given any g, we see $\vec{x}$ as representing a point in $\mathbf{\Pi}\{X[s_{\delta,j} \oplus g] \ : \ j \in \omega\}$.

Now fix any generic g_0 with $s_\delta \in g_0$. Recall the notation that $\mathcal{F}$ is the enumerated family $\{(s^\alpha, \{s_i^\alpha \ : \ i < n_\alpha\}, \dot{F}_\alpha) \ : \ \alpha \in \lambda\}$. Let $\Lambda_\delta = \{\alpha \in M \cap \lambda \ : \ s^\alpha < s_\delta\}$. We have that, for each $\alpha \in \Lambda_\delta$, s_δ forces a value on $\dot{F}_\alpha \cap M \subset Y^n$. It follows also that s_δ forces a value on $\tilde{\Pi}^{-1}_{\langle s_i^\alpha \ : \ i<n\rangle}(\dot{F}_\alpha \cap M)$. Let $e_\delta(\dot{F}_\alpha)$ denote the resulting subset $\Pi_{M,g_0}\left(\tilde{\Pi}^{-1}_{\langle s_i^\alpha \ : \ i<n\rangle}(\dot{F} \cap M)\right)$ of $\mathbf{\Pi}\{X[s_{\delta,j} \oplus g_0] \ : \ j < \omega\}$. Let $\mathcal{H}_0$ denote the countable collection, which is in V, $\{\Pi_g[e_\delta(\dot{F}_\alpha)] \ : \ \alpha \in \Lambda_\delta\}$, of subsets of $(\omega_2 \cap M)^\omega$.

Before continuing, let us observe that for each $\beta \in M \cap \lambda$ and each $j \in \omega$ such that $s^\beta < s_{\delta,j}$, there is an $\alpha \in \Lambda_\delta$ such that s_δ forces that $\dot{F}_\alpha \cap M$ is equal to the same set that $s_{\delta,j}$ forces that $\dot{F}_\beta \cap M$ to be. This is by property (4) of Theorem 5.8 and the fact that S is a coherent tree. Indeed, it is immediate that for each $\xi \in \delta$ with $o(s^\beta) < \xi$, we also have that $(s_{\delta,j} \restriction \xi, \{s_i^\beta \ : \ i < n_\beta\}, \dot{F}_\beta)$ is in the list $\{(s^\alpha, \{s_i^\alpha \ : \ i < n_\alpha\}, \dot{F}_\beta) \ : \ \alpha \in M \cap \lambda\}$. Therefore, by increasing $o(s^\beta)$ we can assume that $s^\beta \oplus s_\delta$ is equal to s_j^δ. Then property (4) of Theorem 5.8 says that there is an $\alpha \in \Lambda_\delta$ so that $(s^\alpha, \{s_i^\alpha \ : \ i < n_\alpha\}, \dot{F}_\alpha)$ satisfies that $\{s_i^\alpha \ : \ i < n_\alpha\}$ is equal to $\{s_i^\beta \ : \ i < n_\alpha\}$ and $\dot{F}_\alpha$ is equal to $(\dot{F}_\beta)^{s^\beta}_{s^\alpha}$. What the current paragraph has shown is that if we repeat this same process with any other generic g and any other member of S_δ (in place of s_δ), then we still end up with the same collection $\mathcal{H}_0$. Stated another way, each $s \in S \setminus M$ forces that, letting g denote the generic, the collection $\{(\Pi_{M,g} \circ \Pi_g)^{-1}(H) \ : \ H \in \mathcal{H}_0\}$ is equal to the collection of all sets of the form $(\Pi^X_{\langle s_i \ : \ i<n\rangle})^{-1}(e_s(M \cap \dot{F}))$ that we have to consider in the statement of our Lemma.

Continuing in $V[g_0]$, we work in the space $\mathbf{\Pi}\{X[s \oplus g_0](\delta) \ : \ s \in S_\delta\}$. Since $\Pi_{g_0}^{-1}[\mathcal{H}_0]$ has the finite intersection property, and letting K_0 denote the intersection of the closures, we may choose some $\vec{x}_0 \in \omega_2^{S_\delta}$ (following our convention) in K_0 with relative countable character. Again note that $\vec{x}_0$ is actually a sequence of ordinals in ω_2 in that, for each $s \in S_\delta$, $\vec{x}_0(s \oplus g_0) \in \omega_2$. Choose a countable subset $T_0 \subset \omega_2$ so as to generate a relative neighborhood base for the point. Finally, choose $s_0 \in g_0$ (with $s_\delta \leq s_0$) so that s_0 forces all these properties. Here is how much progress we have made in the proof :

Claim 1. *For each $j \in \omega$, each extension s of $s_{\delta,j} \oplus s_0$ forces that $\vec{x}_0(s_{\delta,j})$ is the only point that is in the closure of each of the sets*

$$\Pi^X_{\langle s \oplus g\rangle}\left((\Pi^X_{\langle s_i \ : \ i<n\rangle})^{-1}\left(\mathbf{\Pi}\{U_{s_i \oplus s'}(\xi_i) \ : \ i < n\} \cap e_s[M \cap F]\right)\right)$$

where $\langle \xi_i \ : \ i < n\rangle \in T_0^n$*,* $s' \supset s_{\delta,j}$ *is sufficiently large, and* s_0 *forces that* F *a member of* $M \cap \mathcal{F}_{\langle s_i \ : \ i<n\rangle}$ *for some* $\langle s_i \ : \ i \in n\rangle \in S^{<\omega} \cap M$*.*

Thus, so long as our final choice for the countable set T contains T_0 and our choice for $\{y^M(s) \ : \ s \in S_\delta\}$ is forced by s_0 to satisfy that $e(y^M(s_{\delta,j})) = \vec{x}_0(s_{\delta,j})$ for each $j \in \omega$, then we will have satisfied condition (3) for each s that extends one of the elements of the antichain $\{s_{\delta,j} \oplus s_0 \ : \ j \in \omega\}$.

This was the first step of an induction. We enlarge the family $\mathcal{H}_0$ to the family $\mathcal{H}_1$ that also has the finite intersection property. Namely, for each $n \in \omega$, and each $\langle \xi_i \ : \ i < n\rangle \in T_0^n$, the set

$$\Pi_{g_0}\left(\mathbf{\Pi}\{U_{s_{\delta,i}\oplus s_0}(\xi_i) \ : \ i < n\} \times \mathbf{\Pi}\{M \cap X_{s_{\delta,i}\oplus g_0} \ : \ n \le i \in \omega\}\right)$$

is also in $\mathcal{H}_1$. Each member of $\mathcal{H}_1$ is a subset of $(\omega_2 \cap M)^\omega$. Let g_1 be any other generic with $s_0 \notin g_1$. Replacing $\mathcal{H}_0$ in the above argument by $\mathcal{H}_1$, we work in the space $\mathbf{\Pi}\{X_{s\oplus g_1}(\delta) \ : \ s \in S_\delta\}$, and let K_1 be the intersection of the closures of all members of $\Pi_{g_1}^{-1}[\mathcal{H}_1]$ and we again choose a point $\vec{x}_1 \in K_1$ with relative countable character. Next, choose a countable set $T_0 \subset T_1 \subset \omega_2$ witnessing that $\vec{x}_1$ has countable character in K_1. Choose any $s_1 \in g_1$ (incomparable with s_0) that forces all of the above properties. Again expand $\mathcal{H}_1$ to $\mathcal{H}_2$ by adding all sets of the form

$$\Pi_{g_1}\left(\mathbf{\Pi}\{U_{s_{\delta,i}\oplus s_1}(\xi_i) \ : \ i < n\} \times \mathbf{\Pi}\{M \cap X_{s_{\delta,i}\oplus g_1} \ : \ n \le i \in \omega\}\right)$$

where $n \in \omega$ and $\langle \xi_i \ : \ i < n\rangle \in T_1^n$. It is immediate by the construction that the analogue of Claim 1 holds when s_0 is replaced by s_1 and $\vec{x}_0$ is replaced by $\vec{x}_1$. In fact, we formulate a stronger statement for our inductive hypothesis. Suppose that $\beta \in \omega_1$ and that we have chosen $\{s_\alpha \ : \ \alpha < \beta\}$, $\{T_\alpha \ : \ \alpha < \beta\}$, $\{\mathcal{H}_\alpha \ : \ \alpha < \beta\}$, and $\{\vec{x}_\alpha \ : \ \alpha < \beta\}$ such that

$\mathbf{IH}_\beta$: for each $\alpha < \beta$ and each $j \in \omega$

1. $\{s_\alpha \ : \ \alpha < \beta\} \subset S \setminus M$ is an antichain,
2. $\{T_\alpha \ : \ \alpha \in \beta\}$ is an increasing chain of countable subsets of ω_2,
3. $\{\mathcal{H}_\alpha \ : \ \alpha < \beta\}$ is an increasing chain and $\mathcal{H}_\alpha$ is a family of subsets of $(M \cap \omega_2)^\omega$ that has the finite intersection property,
4. $\{\vec{x}_\alpha \ : \ \alpha < \beta\} \subset \omega_2^{S_\delta}$
5. s_α forces that $\vec{x}_\alpha$ is a point in $\mathbf{\Pi}\{X_{s_{\delta,i}\oplus g}(\delta) \ : \ i \in \omega\}$

6. for each $\xi \in T_\alpha$, $s_{\delta,j} \oplus s_\alpha$ forces a value, $U_{s_{\delta,j}\oplus s_\alpha}$, on $\dot{U}(\vec{x}_\alpha(s_{\delta,j}), \xi) \cap M$,

7. $\mathcal{H}_\alpha$ is equal to $\bigcup_{\eta<\alpha} \mathcal{H}_\eta$ together with all sets of the form

$$\Pi_{g_\alpha} \left(\mathbf{\Pi}\{U_{s_{\delta,i}\oplus s_\alpha}(\xi_i) \ : \ i < n\} \times \mathbf{\Pi}\{M \cap X_{s_{\delta,i}\oplus g_\alpha} \ : \ n \le i \in \omega\} \right)$$

where $n \in \omega$, $\langle \xi_i \ : \ i < n\rangle \in T^n_\alpha$, and g_α is any generic with $s_\alpha \in g_\alpha$,

8. for any $\alpha \le \eta < \beta$, s_α forces that $\vec{x}_\alpha$ is the only point that is in the closure of each of the sets in the collection $\{\Pi^{-1}_{g_\alpha}(H) \ : \ H \in \mathcal{H}_\eta\}$.

Having defined the family as above, we stop the construction if $\{s_\alpha \ : \ \alpha \in \beta\}$ is a maximal antichain. Otherwise, we choose a generic g_β disjoint from $\{s_\alpha \ : \ \alpha < \beta\}$ and work in $V[g_\beta]$. The construction of $\vec{x}_\beta$, $T_\beta \supset \bigcup\{T_\alpha \ : \ \alpha < \beta\}$, and $\mathcal{H}_\beta$ proceeds as in the selection for $\beta = 1$. For any $\alpha < \beta$, item (8) appears to be getting stronger but only in the sense that we are preserving that $\vec{x}_\alpha$ is in the closure of $\Pi^{-1}_{g_\alpha}(H)$ for $H \in \mathcal{H}_\beta$. The fact that $\mathcal{H}_\beta$ has the finite intersection property guarantees that there is some point in all of the closures, and by the uniqueness guaranteed by $\mathbf{IH}_\beta$ we have that $\vec{x}_\alpha$ is that point.

Now we assume that $\{s_\alpha \ : \ \alpha < \beta\}$ is a maximal antichain. For the statement of the Lemma we can let γ be any large enough ordinal such that $o(s_\alpha) \le \gamma$ for all $\alpha < \beta$. We want to define our M-acceptable sequence $\langle y^M(s) \ : \ s \in S_\delta\rangle$ so that each s_α forces $\vec{x}_\alpha$ to equal $\langle e(y^M(s)) \ : \ s \in S_\delta\rangle$. We skip the proof that $\mathbf{IH}_\beta$ ensures that if we succeed, then this is also an (M, ω)-acceptable sequence. Let $\mathcal{H}_\beta$ be the filter base generated by all finite intersections from the family $\bigcup\{\mathcal{H}_\alpha \ : \ \alpha < \beta\}$. $\mathcal{H}_\beta$ is a filter base on the product set $(M \cap \omega_2)^\omega$. Let us note that for each $H \in \mathcal{H}_\beta$, there is an $j = j_H \in \omega$ such that H can be factored as $\mathbf{\Pi}\{H(i) \ : \ i < j\} \times (M \cap \omega_2)^{\omega\setminus j}$. Since $\mathbf{\Pi}\{H(i) \ : \ i < j_H\}$ is a subset of M, it then follows that $H \cap M$ is not empty for each $H \in \mathcal{H}_\beta$.

Let $\{\alpha_k \ : \ k \in \omega\}$ be an enumeration of Λ_δ. Let $\{H_m \ : \ m \in \omega\}$ enumerate a descending base for $\mathcal{H}_\beta$. Choose the sequence $\{H_m \ : \ m \in \omega\}$ so that the element $\Pi_g[e_\delta(\dot{F}_{\alpha_m})]$ of $\mathcal{H}_0$ contains H_m. For each m let $j_m \ge m, j_{H_m}$. For each m, choose $\vec{\xi}_m \in M \cap H_m$. Choose any $\eta_m \in M \cap \delta$ large enough so that for each $i < j_m$, $(s_{\delta,i} \restriction \eta_m) \oplus s_\delta$ is equal to $s_{\delta,i}$. If needed, we can increase η_m so that, for each $i < j_m$, $s_{\delta,i}$ forces that the ordinal $\vec{\xi}_m(i)$ is in the sequential closure of $\eta_m \subset \omega_1$. Since s_δ forces that

$\mathbf{\Pi}\{X_{s_{\delta,i}\oplus g}(\eta_m) \ : \ i < j_m\}$ is sequential, there is an $\bar{s}_m < s_\delta$ (in M) and a $\vec{y}_m \in Y^{\{s_{\delta,i}\restriction\eta_m\oplus\bar{s}_m \ : \ i<j_m\}} \cap M$ so that $\bar{s}_m$ forces that $e_g(\vec{y}_m)$ is equal to $\dot{\vec{\xi}}_m \restriction j_m$. We can, and do, require a little more of each such witness $\vec{y}_m$. There are many members $\vec{y}$ of $\Lambda^{\{s_{\delta,i} \ : \ i<m\}}$ that satisfy that s_δ forces that $e_g(\vec{y})$ is equal to $\dot{\vec{\xi}}_m \restriction \eta_m$. It is a simple matter to merge finitely many of them so as to ensure that for each $\alpha \in \{\alpha_k \ : \ k \leq m\}$, $\vec{y}_m \restriction \{s^\alpha_i \oplus \bar{s} \ : \ i < n_\alpha\}$ is a member of $e_g(\dot{F}_\alpha)$. This means that if $(s^\alpha, \{s^\alpha_i \ : \ i < n_\alpha\}, \dot{A})$ is a member of $\mathcal{A} \cap M$ satisfying that $\dot{F}_\alpha = (\dot{A})^{(\omega_1)}$, (for some $\alpha \in \{\alpha_k \ : \ k \leq m\}$) then $\vec{y}_m \restriction \{s^\alpha_i \oplus \bar{s} \ : \ i < n_\alpha\}$ is a member of $(\dot{A})^{(<\delta)}$.

We are ready to define $y^M(s_{\delta,i})$ as in Definition 5.2. It should be clear that each member of the maximal antichain $\{s_\alpha \ : \ \alpha < \beta\}$ forces that $\{\vec{y}_m(i) \ : \ i < m \in \omega\}$ is a sequence that converges, with s_α forcing that it converges to $\vec{x}_\alpha(i)$. Thus, we define $y^M(s_{\delta,i})$ as being built from $\{\vec{y}_m(i) \ : \ i < m \in \omega\}$. This shows that $y^M(s_{\delta,i})$ is a member of Λ. Finally we check that the sequence $\{y^M(s) \ : \ s \in S_\delta\}$ is M-acceptable. Following Definition 5.12, we take any $(\bar{s}, \{\bar{s}_i \ : \ i < n\}, \dot{A})$ in $\mathcal{A} \cap M$. We also fix any $j < \omega$ and letting $s_{\delta,j} > \bar{s}$ be the s in the statement in Definition 5.12. Choose any $m > j$ large enough so that $\{\bar{s}_i \oplus s_\delta \ : \ i < n\}$ is a subset of $\{s_{\delta,i} \ : \ i < j_m\}$. By possibly increasing $\bar{s}$, we can assume that there is an $\alpha \in \Lambda_\delta$ so that $s^\alpha_i = \bar{s}_i \oplus s^\alpha_0$ $(i < n_\alpha)$ and $\dot{F}_\alpha$ is equal to $\left(\dot{A}^{\bar{s}}_{s^\alpha}\right)^{(\omega_1)}$. So long as m is also large enough that this α is in the set $\{\alpha_k \ : \ k \leq m\}$ we have that s_δ forces that $\vec{y}_m \restriction \{s^\alpha_i \oplus g \ : \ i < n_\alpha\}$ is in $\dot{F}_\alpha$. This is equivalent to the fact that $s_{\delta,j}$ forces that $\vec{y}_m \restriction \{s^\alpha_i \oplus g \ : \ i < n_\alpha\}$ (which equals $\vec{y}_m \restriction \{\bar{s}_i \oplus g \ : \ i < n_\alpha\}$) is in $(\dot{A})^{(<\delta)}$. Let $J \in \omega$ be chosen so that each $m > J$ is sufficiently large as above. Well now, for each $m > J$, let $\vec{b}_m \in Y^n$ be simply $\vec{y}_m \restriction \{\bar{s}_i \oplus s_\delta\}$ with the indices relabelled. Then it follows that $s_{\delta,j}$ forces that $\langle y^M(\bar{s}_i \oplus s_{\delta,j}) \ : \ i < n\rangle$ is in $B^{(\delta+1)}$ where $B = \{\vec{b}_m \ : \ J < m \in \omega\}$ is forced by $s_{\delta,j}$ to be a subset of $\dot{A} \cap M$. $\square$

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