

More on tie-points and homeomorphism in \mathbb{N}^*

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Abstract. A point x is a (bow) tie-point of a space X if $X \setminus \{x\}$ can be partitioned into (relatively) clopen sets each with x in its closure. We denote this as $X = A \bowtie_x B$ where A, B are the closed sets which have a unique common accumulation point x . Tie-points have appeared in the construction of non-trivial autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N} = \mathbb{N}^*$ (by Veličković and Shelah & Steprāns) and in the recent study (by Levy and Dow & Techanie) of precisely 2-to-1 maps on \mathbb{N}^* . In these cases the tie-points have been the unique fixed point of an involution on \mathbb{N}^* . One application of the results in this paper is the consistency of there being a 2-to-1 continuous image of \mathbb{N}^* which is not a homeomorph of \mathbb{N}^* .

1. Introduction. A point x is a *tie-point* of a space X if there are closed sets A, B of X such that $X = A \cup B$, $\{x\} = A \cap B$ and x is an adherent point of both A and B . We let $X = A \bowtie_x B$ denote this relation and say that x is a tie-point *as witnessed by* A, B . Let $A \equiv_x B$ mean that there is a homeomorphism from A to B with x as a fixed point. If $X = A \bowtie_x B$ and $A \equiv_x B$, then there is an involution Φ of X (i.e. $\Phi^2 = \text{id}$) such that $\{x\} = \text{fix}(\Phi)$. In this case we will say that x is a *symmetric tie-point of* X .

Let Φ be a continuous function from \mathbb{N}^* into \mathbb{N}^* . Of course, Φ^{-1} can be regarded as a function from the clopen subsets of \mathbb{N}^* into the clopen subsets of \mathbb{N}^* . A function F from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ is a *lifting* of Φ if $F(a)^* = \Phi^{-1}(a^*)$ for all $a \subset \mathbb{N}$. A function h is said to *induce* F (and/or Φ) on I if $F(a) =^* h[a] = \{h(n) : n \in a\}$ for all $a \subset I$. The function F is said to be *trivial* on I if there is such a function h . Since the fixed point set of a trivial autohomeomorphism is clopen, a symmetric tie-point gives rise to a non-trivial autohomeomorphism. An ideal on \mathbb{N} is a *P-ideal* if it is countably directed closed mod finite.

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If A and B are arbitrary compact spaces, and if $x \in A$ and $y \in B$ are accumulation points, then let $A \underset{x=y}{\bowtie} B$ denote the quotient space of $A \oplus B$ obtained by identifying x and y , and let xy denote the collapsed point. Clearly the point xy is a tie-point of this space.

In this paper we establish the following theorem.

THEOREM 1.1. *It is consistent that \mathbb{N}^* has symmetric tie-points x, y as witnessed by A, B and A', B' respectively such that \mathbb{N}^* is not homeomorphic to the space $A \underset{x=y}{\bowtie} A'$.*

COROLLARY 1.2. *It is consistent that there is a 2-to-1 image of \mathbb{N}^* which is not a homeomorph of \mathbb{N}^* .*

One can generalize the notion of tie-point and, for a point $x \in \mathbb{N}^*$, consider how many disjoint clopen subsets of $\mathbb{N}^* \setminus \{x\}$ (each accumulating to x) can be found. Let us say that a tie-point x of \mathbb{N}^* satisfies $\tau(x) \geq n$ if $\mathbb{N}^* \setminus \{x\}$ can be partitioned into n disjoint clopen subsets each accumulating to x . Naturally, we will let $\tau(x) = n$ denote that $\tau(x) \geq n$ and $\tau(x) \not\geq n+1$. It follows easily from [2, 5.1] that each point x of character ω_1 in \mathbb{N}^* is a tie-point and satisfies $\tau(x) \geq n$ for all n . Similarly each P -point of character ω_1 in \mathbb{N}^* is a symmetric tie-point. We list several open questions in the final section.

THEOREM 1.3. *It is consistent that \mathbb{N}^* has a tie-point x such that $\tau(x) = 2$ and $\mathbb{N}^* = A \underset{x}{\bowtie} B$, where neither A nor B is a homeomorph of \mathbb{N}^* . In addition, there are no symmetric tie-points.*

The following theorem of [10] provides an important equivalent condition for the triviality of autohomeomorphisms on \mathbb{N}^* and it will allow us to utilize the results of Steprāns's paper [8].

LEMMA 1.4 (Veličković). *If $F : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ is a lifting of an auto-homeomorphism and there exist Borel functions $\{\psi_n : n \in \omega\}$ and a comeager set $G \subset \mathcal{P}(\mathbb{N})$ such that for every $A \in G$ there is $n \in \omega$ such that $\psi_n(A) =^* F(A)$, then F is trivial.*

This is Theorem 2 of [10] except that the strengthening to the case of a comeager set G is from [8, 2.1]. The topology on $\mathcal{P}(\mathbb{N})$ is the standard one induced by identifying each set $a \subset \mathbb{N}$ with its characteristic function $\chi_a \in 2^{\mathbb{N}}$. For a set $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ and a function F on $\mathcal{P}(\mathbb{N})$, let us say that $F|_{\mathcal{C}}$ is σ -Borel if there is sequence $\{\psi_n : n \in \omega\}$ of Borel functions on $\mathcal{P}(\mathbb{N})$ such that for each $b \in \mathcal{C}$, there is an n such that $F(b) =^* \psi_n(b)$.

The following lemma is also implicit in [10, 1.3]:

LEMMA 1.5. *If F is a lifting of an autohomeomorphism of \mathbb{N}^* and if F is trivial on each member of a P -ideal \mathcal{I} for which $F|_{\mathcal{I}}$ is σ -Borel, then there is a function h which induces F on each member of \mathcal{I} .*

The following partial order \mathbb{P}_2 was introduced by Veličković in [10]. Our need for this poset is articulated in [8] where it is described as a poset which was introduced “to add a non-trivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ while doing as little else as possible—at least assuming PFA”.

DEFINITION 1.6. The partial order \mathbb{P}_2 is defined to consist of all 1-to-1 functions $f : A \rightarrow B$ where

- $A \subseteq \omega$ and $B \subseteq \omega \setminus A$,
- for all $i \in \omega$ and $n \in \omega$, $f(i) \in 2^{n+1} \setminus 2^n$ if and only if $i \in 2^{n+1} \setminus 2^n$,
- $\limsup_{n \rightarrow \omega} |(2^{n+1} \setminus 2^n) \setminus (A \cup B)| = \omega$.

The ordering on \mathbb{P}_2 is \subseteq^* .

We define some trivial generalizations of \mathbb{P}_2 . We use the notation \mathbb{P}_2 to signify that this poset introduces an involution of \mathbb{N}^* because the condition $g = f \cup f^{-1}$ implies that $g^2 = g$. In the definition of \mathbb{P}_2 it is possible to suppress mention of A, B (which we do) and to have the poset \mathbb{P}_2 consist simply of the functions g (and A as $L_g = \{i \in \text{dom}(g) : i < g(i)\}$, and B as $U_g = \{i \in \text{dom}(g) : g(i) < i\}$).

Let \mathbb{P}_1 denote the poset we get if we omit mention of f consisting only of disjoint pairs (A, B) , satisfying the growth condition in Definition 1.6, and extension is coordinatewise mod finite containment. For more consistent notation, we will instead represent the elements of \mathbb{P}_1 as partial functions into 2.

More generally, let \mathbb{P}_l be similar to \mathbb{P}_2 except that we assume that conditions consist of functions g such that $\{i, g(i), g^2(i), \dots, g^l(i)\}$ is contained in $l^{n+1} \setminus l^n$ and has precisely l elements for all $i \in \text{dom}(g) \cap l^{n+1} \setminus l^n$.

The basic properties of \mathbb{P}_2 as defined by Veličković and treated by Shelah and Steprāns are also true of \mathbb{P}_l for all $l \in \mathbb{N}$.

In particular, for example, the following is easily seen:

PROPOSITION 1.7. *If $L \subset \mathbb{N}$ and $\mathbb{P} = \prod_{l \in L} \mathbb{P}_l$ (with full supports) and G is a \mathbb{P} -generic filter, then in $V[G]$, for each $l \in L$, there is a tie-point $x_l \in \mathbb{N}^*$ with $\tau(x_l) \geq l$.*

For the proof of Theorem 1.1 we use $\mathbb{P}_2 \times \mathbb{P}_2$ and for the proof of Theorem 1.3 we use \mathbb{P}_1 . In any such \mathbb{P} and $\vec{f}, \vec{g} \in \mathbb{P}$, we say that $\vec{f} = \langle f_l \rangle$ is an n -preserving extension of $\vec{g} = \langle g_l \rangle$, for an integer n , if for each coordinate l , $f_l \upharpoonright n = g_l \upharpoonright n$ and $f_l \supset g_l$. Also, if $\vec{s} = \langle s_l \rangle$ is a sequence of functions (usually with finite domain), then we define $\vec{s} \sqcup \vec{f}$ to be the sequence $\vec{g} = \langle g_l \rangle$ where, for each coordinate l ,

$$g_l = s_l \sqcup f_l \equiv s_l \cup (f_l \upharpoonright \text{dom}(f_l) \setminus \text{dom}(s_l)).$$

2. Preliminaries. Each poset \mathbb{P} as above is \aleph_1 -closed and, if PFA holds, \aleph_2 -distributive (see [8, p. 4226]). In this paper we will restrict our study to finite products. The following partial order can be used to show that these products are \aleph_2 -distributive.

DEFINITION 2.1. Let \mathbb{P} be a finite product of posets from $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Given $\mathfrak{F} \subset \mathbb{P}$, define $\mathbb{P}(\mathfrak{F})$ to be the partial order consisting of all $g \in \mathbb{P}$ such that there is some $\vec{f} \in \mathfrak{F}$ such that $\vec{g} \equiv^* \vec{f}$. The ordering on $\mathbb{P}(\mathfrak{F})$ is coordinatewise \supseteq as opposed to $^*\supseteq$ in \mathbb{P} .

If \mathfrak{F} is downward directed (in fact it will be a descending sequence), then the forcing $\mathbb{P}(\mathfrak{F})$ introduces a tuple \vec{f} such that $\vec{f} \leq \vec{f}'$ for all $\vec{f}' \in \mathfrak{F}$. Although \vec{f} itself may not be a member of \mathbb{P} , it is simply because the domains of the component functions are too big. Following [6, 2.1], one must then use a σ -centered poset which will choose an appropriate sequence \vec{f}^* of subfunctions of \vec{f} which is a member of \mathbb{P} and which is still below each member of \mathfrak{F} .

A strategic choice of the sequence \mathfrak{F} will ensure that $\mathbb{P}(\mathfrak{F})$ is ccc, but remarkably even more is true. Again we are lifting results from [6, 2.6] and [8, proof of Thm. 3.1] which introduced this innovative factoring of Veličković's original amoeba forcing poset and that it seems to preserve more properties. Let $\omega_2^{<\omega_1}$ denote the standard collapse which introduces a function from ω_1 onto ω_2 .

A poset is said to be ω^ω -*bounding* if every new function in ω^ω is bounded by some ground model function.

LEMMA 2.2. *Let \mathbb{P} be a finite product of posets from $\{\mathbb{P}_l : l \in \mathbb{N}\}$. In the forcing extension, $V[H]$, by $\omega_2^{<\omega_1}$, there is a descending sequence \mathfrak{F} from \mathbb{P} which is \mathbb{P} -generic over V and for which $\mathbb{P}(\mathfrak{F})$ is ccc and ω^ω -bounding.*

It was also shown in [6] that \mathfrak{F} can be chosen so that it, in addition, preserves that $\mathbb{R} \cap V$ is of second category. This is crucial for the proof of Lemma 2.3. We can manage with the ω^ω -bounding property because we are going to use Lemma 2.3. An ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is said to be *dense* if each infinite subset of \mathbb{N} contains a member of \mathcal{I} .

The following main result is extracted from [6] and [8, Theorem 3.3] which we record without proof.

LEMMA 2.3 (PFA). *Let \mathbb{P} be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Let \vec{F} be a \mathbb{P} -name of a lifting of an autohomeomorphism of \mathbb{N}^* . Let \mathfrak{F} and H be as in Lemma 2.2 and let F be the valuation of \vec{F} by \mathfrak{F} . Then F is a lifting of an autohomeomorphism of \mathbb{N}^* (in $V[H]$) and for any dense P -ideal \mathcal{I} on \mathbb{N} and for each $\mathbb{P}(\mathfrak{F})$ -generic filter G , there is an $I \in \mathcal{I}$ such that $F \upharpoonright (V[H] \cap [\mathbb{N} \setminus I]^\omega)$ is σ -Borel in the extension $V[H][G]$.*

Let \mathfrak{F} , \dot{F} , H and F be as in Lemma 2.3 and consider the situation in the forcing extension $V[H]$. Since \mathfrak{F} is generic over V , we will see that for each $Y \in [\mathbb{N}]^\omega$, there is some $f \in \mathfrak{F}$ which decides if $\dot{F} \upharpoonright [Y]^\omega$ is trivial. Also, the genericity of \mathfrak{F} over V , and the fact that no new subsets of \mathbb{N} are added, will ensure that if some f in \mathfrak{F} forces that $\dot{F} \upharpoonright [Y]^\omega$ is not trivial, then $F \upharpoonright [Y]^\omega$ is also not trivial (in $V[H]$). We will assume all these properties of \mathfrak{F} and F throughout the paper.

The following proposition is probably well-known but we do not have a reference.

PROPOSITION 2.4. *Assume that \mathbb{Q} is a ccc ω^ω -bounding poset and that x is an ultrafilter on \mathbb{N} . If G is a \mathbb{Q} -generic filter then there is no set $A \subset \mathbb{N}$ such that $A \setminus Y$ is finite for all $Y \in x$.*

Proof. Assume that $\{\dot{a}_n : n \in \omega\}$ are \mathbb{Q} -names of integers such that $1 \Vdash_{\mathbb{Q}} \text{“}\dot{a}_n \geq n\text{”}$. Let A denote the \mathbb{Q} -name such that $\Vdash_{\mathbb{Q}} \text{“}A = \{\dot{a}_n : n \in \omega\}\text{”}$. Since \mathbb{Q} is ω^ω -bounding, there is some $q \in \mathbb{Q}$ and a sequence $\{n_k : k \in \omega\}$ in V such that $q \Vdash_{\mathbb{Q}} \text{“}n_k \leq \dot{a}_i \leq n_{k+2} \forall i \in [n_k, n_{k+1}]\text{”}$. There is some $l \in \mathbb{3}$ such that $Y = \bigcup_k [n_{3k+l}, n_{3k+l+1})$ is a member of x . On the other hand, for each k , $q \Vdash_{\mathbb{Q}} \text{“}A \cap [n_{3k+l+1}, n_{3k+l+3})$ is not empty”. Therefore $q \not\Vdash_{\mathbb{Q}} \text{“}A \setminus Y$ is finite”. ■

Another interesting and useful general lemma is the following.

LEMMA 2.5. *Let \mathbb{P} be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Let H and \mathfrak{F} be as in Lemma 2.2. Then for each $\mathbb{P}(\mathfrak{F})$ -name $\dot{h} \in \mathbb{N}^{\mathbb{N}}$ there are an increasing sequence $n_0 < n_1 < \dots$ of integers and a condition $\vec{f} \in \mathfrak{F}$ such that either*

- (1) $\vec{f} \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h} \upharpoonright \bigcup\{[n_k, n_{k+1}) : k \in K\} \notin V\text{”}$ for each infinite $K \subset \omega$, or
- (2) for each $i \in [n_k, n_{k+1})$ and each $\vec{g} < \vec{f}$ such that \vec{g} forces a value on $\dot{h}(i)$, $\langle f_l \upharpoonright [n_k, n_{k+1}) \rangle$ also forces a value on $\dot{h}(i)$.

Furthermore, if \vec{f} forces \dot{h} to be finite-to-one, we can arrange that for each k and each $i \in [n_k, n_{k+1})$, \vec{f} forces that $\dot{h}(i) \in [n_{k-1}, n_{k+2})$.

Proof. Fix any $\vec{f} \in \mathbb{P}$. Perform a standard fusion, as in [6, 2.4] or [8, 3.4], to find sequences $\{n_k : k \in \omega\} \subset \omega$ and $\{\vec{f}^k : k \in \omega\} \subset \mathbb{P}$ with the following properties. Each \vec{f}^k is an n_k -preserving extension of \vec{f}^{k-1} . Let $j < n_k$ and let $\vec{s}, \vec{s}^* \in \mathbb{P}$ be such that, for each coordinate l of \mathbb{P} , $s_l \subset s_l^*$ and s_l has domain contained in n_k . If there is some n_k -preserving extension of $\vec{s} \sqcup \vec{f}^k$ which forces a value on $\dot{h}(j)$, then $\vec{s} \sqcup \vec{f}^k$ already does so. Further, if there is some integer $i \geq n_k$ for which $\vec{s}^* \sqcup \vec{f}^k$ has an n_k -preserving extension forcing a value on $\dot{h}(i)$ while $\vec{s} \sqcup \vec{f}^k$ does not, then there is such an integer below n_{k+1} . One also ensures that for each coordinate l there is an m such that

$n_k < 2^m < 2^{m+1} < n_{k+1}$, $[2^m, 2^{m+1}) \setminus \text{dom}(f_l^k)$ has at least k elements (thus ensuring that the end result of the fusion will be a member of \mathbb{P}).

Let \vec{f}^i be the fusion and assume that $\vec{f} < \vec{f}^i$ and $K \in [\omega]^\omega$ are such that \vec{f} forces a value on $\dot{h} \upharpoonright [n_k, n_{k+1})$ for all $k \in K$. We show that the second alternative then holds. By further extending \vec{f} we can assume that if $L = \{m : (\exists l) [n_m, n_{m+1}) \not\subset \text{dom}(f_l)\}$, then $K \cap [m, m']$ is not empty for all $m < m' \in L$.

Let $i \in [n_{m'}, n_{m'+1})$ and let $\vec{g} < \vec{f}$ force a value on $\dot{h}(i)$. Define $\vec{s}' = \langle g_l \upharpoonright [n_{m'}, n_{m'+1}) \rangle$ and note that $\vec{s}' \sqcup \vec{f}$ does force a value on $\dot{h}(i)$ because of the fusion. Assume that $\vec{s}' \sqcup \vec{f}$ does not force a value on $\dot{h}(i)$, and so has no $n_{m'+1}$ -preserving extension which does.

Let m be the maximum member of $L \cap m'$ and choose $k \in K \cap [m, m']$. Set $\vec{s} = \langle f_l' \upharpoonright n_k \rangle$ and $\vec{s}' = \langle \vec{g}_l \upharpoonright n_k \rangle$. We note that i is a witness to the situation that $\vec{s}' \sqcup \vec{f}^{n_k}$ has an n_k -preserving extension to decide, while $\vec{s} \sqcup \vec{f}^{n_k}$ does not. Therefore, by construction, there should be some $j < n_{k+1}$ for which this is true. However, this is not the case since \vec{f} forces a value on $\dot{h} \upharpoonright [n_k, n_{k+1})$.

If \vec{f} forces that \dot{h} is finite-to-one, then \vec{f}^{d} could have been so chosen. In addition, since $\mathbb{P}(\mathfrak{F})$ is ω^ω -bounding we may fix an increasing function $g \in \omega^\omega \cap V$ such that (if \vec{f} forces \dot{h} is finite-to-one) $\vec{f} \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\{i, \dot{h}(i)\} \cup \dot{h}^{-1}(i) \subset g(i)\text{”}$. The only change to the fusion is to additionally demand that n_{k+1} is chosen to be larger than $g(n_k)$ at each stage. ■

3. The trivial ideal. In this section we establish a result that will guarantee that our autohomeomorphisms of \mathbb{N}^* will be trivial of every member of a large P -ideal.

LEMMA 3.1. *Let \mathbb{P} be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$, let H and \mathfrak{F} be as in Lemma 2.2, and let G be $\mathbb{P}(\mathfrak{F})$ -generic over $V[H]$. Assume that $b \in V \cap [\mathbb{N}]^\omega$ is such that $F \upharpoonright [V \cap [b]^\omega]$ is σ -Borel in $V[G]$. Then, in V , there is an increasing sequence $\{n_k : k \in \omega\} \subset \omega$ such that F is trivial on each $a \in [b]^\omega$ for which there is an $r \in \mathfrak{F}$ such that $a \subset \bigcup \{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \text{dom}(r)\}$.*

Proof. For notational convenience we will assume that \mathbb{P} is simply a single member of $\{\mathbb{P}_l : l \in \mathbb{N}\}$. The modifications needed to handle a finite product are completely straightforward and will be omitted.

Fix names $\dot{\psi}_j$ ($j \in \omega$) for the Borel functions. Fix an appropriately large countable elementary submodel $M \prec H(\theta)$. For easier notation, we may just assume that b is actually \mathbb{N} . We will use the notation p with subscripts to refer to members of \mathbb{P} . For a finite set $t \subset \mathbb{N}$ and $n \in \mathbb{N}$, we will use $[t; n]$ to denote the clopen set $\{a \subset \mathbb{N} : a \cap n = t\}$.

We first want to show that we can assume that each $\dot{\psi}_j$ is actually continuous. As is well-known, each Borel function is continuous on a dense G_δ , hence we may fix a sequence $\{\dot{U}_n : n \in \omega\}$ of $\mathbb{P}(\mathfrak{F})$ -names of a descending sequence of dense open sets such that each $\dot{\psi}_j$ is forced to be continuous on the intersection $\bigcap_n \dot{U}_n$. We perform a fusion sequence $\{p_k : k \in \omega\}$ (as in Lemma 3.1) which selects a sequence of intervals $\{[n_k, n_{k+1}) : k \in \omega\}$, and finite sets t_k contained in $[n_k, n_{k+1})$, so that (it is forced by p_k that) for each $s \subset n_k$, $[s \cup t_k; n_{k+1})$ is a subset of U_k . We deal with $F \upharpoonright V \cap [\bigcup_k [n_{2k}, n_{2k+1})]^\omega$ (and by symmetry) with $F \upharpoonright V \cap [\bigcup_k [n_{2k-1}, n_{2k})]^\omega$ by replacing, for $y \subset \bigcup_k [n_{2k}, n_{2k+1})$, $\dot{\psi}_j(y)$ with $\dot{\psi}_j(y \cup \bigcup_k t_{2k}) \setminus F(\bigcup_k t_{2k})$. Thus, we may simply assume that each $\dot{\psi}_j$ is continuous.

We perform another fusion sequence and produce a new sequence $\{n_i : i \in \omega\}$. This also is all done in M . For each i we will select a subset $f_i \subset [n_i, n_{i+1})$ and we are trying to imitate the “forcing a value” idea from [9]. That is, for each i and each $s \in n_i^{n_i}$ and $t \subset n_i$ and each $j \leq i$, we arrange that if $s \sqcup p_i$ has an n_i -preserving extension which is able to force a value on $\dot{\psi}_j[t \cup f_i; n_{i+1}] \upharpoonright n_i$ (meaning all $a \in \dot{\psi}_j[t \cup f_i; n_{i+1}]$ have the same intersection with n_i), then we do so (i.e. by possibly extending f_i or by extending $p_i \upharpoonright [n_i, \infty)$). An additional requirement is to further finitely extend f_i , if possible, so that instead, there is some integer m (which will be made to be less than n_{i+1}) so that $s \sqcup p_i$ has no n_i -preserving extension and $f_i; n_i$ has no further finite extension $h; n'$, which will force a value on $\dot{\psi}_j[t \cup h; n'] \upharpoonright m$. As usual, we also ensure that for each i , there is a unique m_i such that $[2^{m_i}, 2^{m_i+1}) \subset [n_i, n_{i+1})$ and $\text{dom}(p_i) \supset [n_i, n_{i+1}) \setminus [2^{m_i}, 2^{m_i+1})$ and $|[2^{m_i}, 2^{m_i+1}) \setminus \text{dom}(p_i)] \geq i$.

For $e \in \{0, 1, 2\}$, let $f^e = \bigcup_i f_{3i+e}$. Choose a $p \in M$ such that p decides the value of $F(f^e)$ for each such e . We will focus on f^0 but the following argument can be repeated for f^1 and f^2 .

We perform another fusion choosing $\{i_l : l \in \omega\} \subset \{3i : i \in \omega\}$ and conditions r_l . Again, with $n = n_{i_l}$, for each $j \leq i_l$, $s \in n^n$, and $t \subset n$, we choose $r = r_l$ to be an n -preserving extension so that either $s \sqcup r$ has forced a value on $\dot{\psi}_j(t \cup f^0)$, or there is a $3i < i_{l+1}$ such that $s \sqcup r$ has no n -preserving extension which forces a value on $\dot{\psi}_j[(t \cup f^0) \cap n_{3i+1}; n_{3i+1}] \upharpoonright n_{3i}$.

Let $r < p$ extend this final fusion sequence and be an (M, \mathbb{P}) -generic condition. For each s , let $s \sqcup r \in G_s$ be some $\mathbb{P}(\mathfrak{F})$ -filter which is generic over M . This gives us a countable family of Borel functions $\{\text{val}_{G_s}(\dot{\psi}_j) : s \in \omega^{<\omega}, j \in \omega\}$ in $V[H]$ (and in V).

Let $L \subset \omega$ be any set of integers such that $[n_{i_l}, n_{i_{l+1}}) \subset \text{dom}(r)$ for $l \in L$ and $L \cap \{l+1 : l \in L\}$ is empty. Let $Y \subset \bigcup_{l \in L} [n_{i_l}, n_{i_{l+1}})$ be such that, in addition, $Y \cap [n_{3i}, n_{3i+1})$ (since we are using f^0) is empty for all i . To show that F is trivial (using Proposition 1.4) on $[Y]^\omega$, we prove that for each

$y \subset Y$, there are s, j such that

$$s \sqcup r \Vdash_{\mathbb{P}(\mathfrak{F})} "F(y) =^* \dot{\psi}_{j_0}(y \cup f^0) \setminus F(f^0)" .$$

It then follows, since $F(y)$ is an element of V , that $F(y) = \text{val}_{G_s}(\dot{\psi}_j)(y \cup f^0) \setminus F(f^0)$. Since all this is taking place in $V[H]$ we find that $F \upharpoonright [Y]^\omega$ is trivial.

Fix any $r_y < r$ which forces a value on $\dot{F}(y \cup f^0)$ and forces that this is equal to $\dot{\psi}_j(y \cup f^0)$ for some $j \in \omega$. Fix any $l_0 \in L$ such that $j < i_{l_0}$ and set $s = r_y \upharpoonright n_{i_{l_0}}$. More generally, for each $l \in L$, let $s_l = r_y \upharpoonright n_{i_l}$. The next three claims complete the proof that $s = s_0$ and j are as needed above.

CLAIM 1. *For each $l \in \omega \setminus (L \cup l_0)$, $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_l}) \cup f^0)$.*

Proof. Let $t = (y \cup f^0) \cap n_{i_l}$. Note also that $(y \cup f^0) \cap n_{i_{l+1}}$ is equal to $(t \cup f^0) \cap n_{i_{l+1}}$ since $l \notin L$. By assumption and continuity of $\dot{\psi}_j$, r_y forces a value on $\dot{\psi}_j[(y \cup f^0) \cap n_{i_{l+1}}; n_{i_{l+1}}] \upharpoonright n_{3i+1}$ for each $3i < i_{l+1}$. Therefore, $s_l \sqcup r_l$ did (does) have such an n_{i_l} -preserving extension to force values on $\dot{\psi}_j[(t \cup f^0) \cap n_{3i+1}; n_{3i+1}] \upharpoonright n_{3i}$ for each $3i < i_{l+1}$. From this, it follows from the choice of r_l that $s_l \sqcup r$ does force a value on $\dot{\psi}_j(t \cup f^0)$. ■

CLAIM 2. *For each $l \in \omega \setminus l_0$, $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_l}) \cup f^0)$.*

Proof. By Claim 1, we may assume that $l \in L$ and so $l-1 \notin L$. We know by Claim 1 that $s_{l-1} \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_{l-1}}) \cup f^0)$. But since $y \cap n_{i_l}$ is the same as $y \cap n_{i_{l-1}}$, it follows that $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_l}) \cup f^0)$ since $s_{l-1} \sqcup r$ decides it. ■

CLAIM 3. *For each $l \leq l' \in \omega \setminus l_0$, $s_l \sqcup r$ decides $\dot{\psi}_j((y \cap n_{i_{l'}}) \cup f^0)$.*

Proof. We proceed by induction on l' . Assume the claim holds for l' and fails for $l'+1$. Let l be maximal such that it fails for s_l . We know that $l \leq l'$ by Claim 2. It follows that we may assume that $t = y \cap n_{i_{l'+1}} \neq y \cap n_{i_{l'}} = t'$, hence $l \in L$. When $r_{l'+1}$ was defined, it was asked if $((s_l \sqcup r) \upharpoonright n_{i_l}) \sqcup r_{l'+1}$ had an n_{i_l} -preserving extension which forced a value on $\dot{\psi}_j(t \cup f^0)$. Apparently the answer was no. But then, at stage $i = i_l$ in the $\langle p_i : i \in \omega \rangle$ fusion, it was asked if $t' \cup f_i$ had an extension for which $((s_l \sqcup r) \upharpoonright n_{i_l}) \sqcup p_i$ did not have an n_{i_l} -preserving extension to decide arbitrarily far. Well it appears that $t \cup f^0$ is such an extension (note that $f_i \subset f^0$). In this case, $f_i; n_{i_l}$ were chosen so that it has no extension $h; n'$ for which $((s_l \sqcup r) \upharpoonright n_{i_l}) \sqcup p_i$ has an n_{i_l} -preserving extension which will decide $\dot{\psi}_j[t' \cup h; n'] \upharpoonright n_{i_{l+1}}$. But we do know that $s_l \sqcup r$ decides $\dot{\psi}_j(t' \cup f^0) = \dot{\psi}_j((y \cap n_{i_{l'}}) \cup f^0)$. Therefore at stage $i+3$, $(s_l \sqcup r) \upharpoonright n_{i+3}$ would have an n_{i+2} -preserving extension forcing a value on $\dot{\psi}_j[t \cup f_i \cup f_{i+3}; n_{i+3}] \upharpoonright n_{i+2}$ (in fact, it would already do so). In particular, $t \cup f_i; n_{i+1}$ does have an extension, namely $t \cup f_i \cup f_{i+3}; n_{i+4}$, for which $(s_l \sqcup r) \upharpoonright n_{i_l} \cup p_i$ does have an n_{i_l} -preserving extension forcing a value on $\dot{\psi}_j[t \cup f_i \cup f_{i+3}; n_{i+4}] \upharpoonright n_{i+1}$. ■

This ends the proof of Lemma 3.1. ■

COROLLARY 3.2 (PFA). *Let \mathbb{P} be a finite product of posets from the set $\{\mathbb{P}_l : l \in \mathbb{N}\}$. Let \dot{F} be a \mathbb{P} -name of a lifting of an autohomeomorphism of \mathbb{N}^* . Let H, \mathfrak{F}, F, P -ideal \mathcal{I} and $I \in \mathcal{I}$ be as in Lemma 2.2. Then there is an increasing sequence $\{n_k : k \in \omega\} \subset \mathbb{N}$ and a $\mathbb{P}(\mathfrak{F})$ -name \dot{h} for a function on \mathbb{N} such that for each $f \in \mathfrak{F}$ and $a = \bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \text{dom}(f)\}$, F is trivial on $a \setminus I$ and $\mathbb{P}(\mathfrak{F})$ forces that $\dot{h} \upharpoonright (a \setminus I)$ induces F .*

Proof. Let $\{n_k : k \in \omega\}$ be the sequence as constructed in Lemma 3.1. Let \mathcal{J} denote the dense P -ideal consisting of all sets of the form $\bigcup\{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \text{dom}(f)\}$ for some $f \in \mathfrak{F}$. Since there is a natural (and obvious) finite-to-one map sending the ideal \mathcal{J} to an ultrafilter, it follows by Proposition 2.4 that \mathcal{J} generates a dense P -ideal in the forcing extension by $\mathbb{P}(\mathfrak{F})$. By Lemma 2.2, we know that $F \upharpoonright (V[H] \cap [\mathbb{N} \setminus I]^\omega)$ is σ -Borel. Let \mathcal{J}^I be the ideal $\{J \setminus I : J \in \mathcal{J}\}$. By Lemma 3.1, F is trivial on J for each $J \in \mathcal{J}^I$. It then follows easily that, in the forcing extension by $\mathbb{P}(\mathfrak{F})$, $F \upharpoonright \mathcal{J}^I$ is also σ -Borel. Finally, by Lemma 1.5, there is an \dot{h} as required. ■

4. Proof of Theorem 1.1

THEOREM 4.1 (PFA). *In the forcing extension by $\mathbb{P} = \mathbb{P}_2 \times \mathbb{P}_2$, there are symmetric tie-points x, y as witnessed by A, B and C, D respectively such that \mathbb{N}^* is not homeomorphic to the space $A \underset{x=y}{\boxtimes} C$.*

We briefly work in the forcing extension in order to select appropriate names. Let $G \subset \mathbb{P}_2 \times \mathbb{P}_2$ be a generic filter. The tie-point x as witnessed by A, B will be the one given canonically by the \mathbb{P}_2 -generic filter consisting of the first coordinates of G (as per the notation following Definition 1.6). The tie-point y as witnessed by C, D will be given analogously by the second coordinates. More precisely the closed set A will be the closure of the union of the collection $\{L_f^* : (\exists g) (f, g) \in G\}$, while B will be the closure of the union of the collection $\{U_f^* : (\exists g) (f, g) \in G\}$. Of course, x is the ultrafilter (a P_{ω_2} -point) generated by the collection $\{\mathbb{N} \setminus \text{dom}(f) : (\exists g) (f, g) \in G\}$.

Fix any enumeration $\{a_\alpha : \alpha \in \omega_2\}$ of a mod finite increasing cofinal chain in $\{L_f : (\exists g) (f, g) \in G\}$ and similarly $\{c_\alpha : \alpha \in \omega_2\}$ for $\{L_g : (\exists f) (f, g) \in G\}$. We may represent $A \underset{x=y}{\boxtimes} C$ as a quotient of $(\mathbb{N} \times 2)^*$ in which, for each $\alpha \in \omega_2$, $(a_\alpha \times \{0\})^* \cup (c_\alpha \times \{1\})^*$ is mapped canonically to $a_\alpha^* \cup c_\alpha^*$ and the rest of the $(\mathbb{N} \times 2)^*$ is collapsed to a point. Assume there is a homeomorphism from this quotient space to \mathbb{N}^* and let F be any lifting, i.e. we may assume that F is a function from $[\mathbb{N}]^\omega$ into $[\mathbb{N} \times 2]^\omega$ such that if we let $Z_\alpha = F^{-1}(a_\alpha \times \{0\} \cup c_\alpha \times \{1\})$ for each $\alpha \in \omega_2$, then $\{Z_\alpha : \alpha \in \omega_2\}$ forms the dual ideal \mathcal{I} to an ultrafilter z .

Fix \mathbb{P} -names for all the above mentioned objects and apply Corollary 3.2 to find the filter $\mathcal{F} \subset \mathbb{P}$, the \mathbb{P} -name \dot{h} and the sequence $\{n_k : k \in \omega\}$. There is no loss of generality in this proof to assume that the I mentioned in the statement of Corollary 3.2 is the empty set, and let \mathcal{J} be the ideal as defined in the proof of Corollary 3.2. As we are working in $V[H]$, let us use λ to denote the ω_2 from V . For each $J \in \mathcal{J}$, there is a function h_J which induces F on J ; h_J will be a function from J into $(a_\alpha \times \{0\}) \cup (c_\alpha \times \{1\})$ for some $\alpha \in \lambda$.

We finish the proof by showing there is no such \dot{h} .

Since $\mathbb{P}(\mathfrak{F})$ is ω^ω -bounding, we may assume (by selecting a subsequence and renumbering) that the sequence $\{n_k : k \in \omega\}$ and some $\vec{f}_0 = (g_0, g_1) \in \mathfrak{F}$ satisfy:

- (1) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h}(i) \in ([0, n_{k+2}) \times 2\text{”}$,
- (2) for each $i \in [n_k, n_{k+1})$, $\vec{f}_0 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h}^{-1}(\{i\} \times 2) \subset [0, n_{k+2})\text{”}$,
- (3) for each k and each $j \in \{0, 1\}$ there is an m such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \text{dom } g_j$ has at least k elements.

Choose any $(g'_0, g'_1) = \vec{f}_1 < \vec{f}_0$ in \mathfrak{F} such that $\mathbb{N} \setminus \text{dom}(g'_0)$ is contained in $\bigcup_k [n_{6k+1}, n_{6k+2})$ and $\mathbb{N} \setminus \text{dom}(g'_1) \subset \bigcup_k [n_{6k+4}, n_{6k+5})$. Next, choose any $\vec{f}_2 < \vec{f}_1$ in \mathfrak{F} and some $\alpha \in \lambda$ such that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\text{dom}(g'_0) \subset^* a_\alpha \cup g'_0[a_\alpha]$ and $\text{dom}(g'_1) \subset^* c_\alpha \cup g'_1[c_\alpha]\text{”}$. For each $\gamma \in \lambda$, note that $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}a_\gamma \setminus a_\alpha \subset^* \mathbb{N} \setminus \text{dom}(g'_0)\text{”}$ and similarly $\vec{f}_2 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}c_\gamma \setminus c_\alpha \subset^* \mathbb{N} \setminus \text{dom}(g'_1)\text{”}$.

Now consider the two disjoint sets: $Y_0 = \bigcup_k [n_{6k}, n_{6k+3})$ and its complement Y_1 . Since z is the \mathbb{P} -name of an ultrafilter, by possibly extending $\vec{f}_2 = (f_0, f_1)$ even more, we may assume there is some $\beta > \alpha$ such that (by symmetry) $\vec{f}_2 \Vdash_{\mathbb{P}} \text{“}Y_0 \subset^* Z_\beta\text{”}$, in fact we may assume that $\vec{f}_2 \Vdash_{\mathbb{P}} \text{“}F(Y_0) \subset^* (L_{f_0} \times \{0\}) \cup (L_{f_1} \times \{1\})\text{”}$.

Finally, let $\vec{f}_3 = (f'_0, f'_1) < \vec{f}_2$ be chosen so that there is an infinite set $L \subset \mathbb{N}$ such that for $k \in L$, $[n_{6k+1}, n_{6k+2}) \subset \text{dom}(f'_0)$ and $[n_{6k+1}, n_{6k+2}) \not\subset \text{dom}(f_0)$. Set

$$y = \bigcup_{k \in L} [n_{6k+1}, n_{6k+2}) \cap L_{f'_0} \setminus L_{f_0}$$

and choose any $\vec{f}_4 < \vec{f}_3$ and \tilde{y} such that \vec{f}_4 forces that $F(\tilde{y}) = y \times \{0\}$. Since \mathcal{J} is a dense ideal, we may fix any $J \in \mathcal{J}$ such that $J \cap \tilde{y}$ is infinite.

It then follows that $\dot{h}[J \cap \tilde{y}] =^* F(J \cap \tilde{y}) \subset^* y \times \{0\}$, and so $J \cap \tilde{y}$ is forced to be contained in $\bigcup_{k \in L} [n_{6k}, n_{6k+3})$ (by the assumption on the sequence of $\{n_k\}$'s). On the other hand, now that $J \cap \tilde{y} \subset Y_0$, we have

$$F(J \cap \tilde{y}) \subset^* F(Y_0) \cap (\mathbb{N} \times \{0\}) \subset^* L_{f_0} \times \{0\},$$

contradicting the fact that y is disjoint from L_{f_0} .

5. Proof of Theorem 1.3

THEOREM 5.1 (PFA). *In the forcing extension by \mathbb{P}_1 , a tie-point x is introduced such that $\tau(x) = 2$ and with $\mathbb{N}^* = A \times_x B$, neither A nor B is a homeomorph of \mathbb{N}^* . In addition, there is no involution F on \mathbb{N}^* which has a unique fixed point, and so no tie-point is symmetric.*

We begin by proving that neither A nor B can be homeomorphic to \mathbb{N}^* . We proceed much as in the previous section. Let $G \subset \mathbb{P} = \mathbb{P}_1$ be a generic filter. The tie-point x as witnessed by A, B will be the one given canonically by the \mathbb{P} -generic filter. More precisely, the closed set A will be the closure of the union of the collection $\{(f^{-1}(0))^* : f \in G\}$, while B will be the closure of the union of the collection $\{f^{-1}(1) : f \in G\}$. Of course, x is the ultrafilter (a P_{ω_2} -point) generated by the collection $\{\mathbb{N} \setminus \text{dom}(f) : f \in G\}$.

Assume there is an autohomeomorphism from \mathbb{N}^* onto A and let F be any lifting, i.e. we may assume that there is an ultrafilter $z \in \mathbb{N}^*$ with dual ideal \mathcal{I} such that F is a function from $\bigcup_{f \in G} [f^{-1}(0)]^\omega$ onto $\bigcup_{I \in \mathcal{I}} [I]^\omega$ such that for each $f \in G$, $F \upharpoonright [f^{-1}(0)]^\omega$ is a lifting of a homeomorphism from $(f^{-1}(0))^*$ onto I_f^* for some $I_f \in \mathcal{I}$, and for each $I \in \mathcal{I}$, there is an $f \in G$ such that $I \subset^* I_f$.

Fix \mathbb{P} -names for all the above mentioned objects and apply Corollary 3.2 to find the filter $\mathcal{F} \subset \mathbb{P}$, the \mathbb{P} -name \dot{h} and the sequence $\{n_k : k \in \omega\}$. We obtain a contradiction by showing there can be no such \dot{h} .

There is no loss of generality in this proof to assume that the I mentioned in the statement of Corollary 3.2 is the empty set. The ideal denoted \mathcal{J} as defined in the proof of Corollary 3.2 will now be generated by sets of the form $\bigcup \{[n_k, n_{k+1}) \cap f^{-1}(0) : [n_k, n_{k+1}) \subset \text{dom}(f)\}$ for $f \in \mathfrak{F}$. It follows that the ideal $\{F(J) : J \in \mathcal{J}\}$ will be a dense ideal in $[\mathbb{N}]^\omega$. For each $J \in \mathcal{J}$, let h_J denote the function on J for which there is some $f_J \in \mathfrak{F}$ which forces that h_J induces F on J .

Since $\mathbb{P}(\mathfrak{F})$ is ω^ω -bounding, we may assume (by selecting a subsequence and renumbering) that the sequence $\{n_k : k \in \omega\}$ and some $f_0 \in \mathfrak{F}$ satisfy:

- (1) for each $i \in [n_k, n_{k+1})$, $f_0 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h}(i) \in [0, n_{k+2})\text{”}$,
- (2) for each $i \in [n_k, n_{k+1})$, $f_0 \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h}^{-1}(i) \in [0, n_{k+2})\text{”}$,
- (3) for each k there is an m such that $n_k < 2^m < 2^{m+1} < n_{k+1}$, and $[2^m, 2^{m+1}) \setminus \text{dom } f_0$ has at least k elements.

We need a significant strengthening of Lemma 2.5 which holds for $\mathbb{P} = \mathbb{P}_1$.

LEMMA 5.2. *Assume that \dot{h} is a $\mathbb{P}(\mathfrak{F})$ -name of a function from \mathbb{N} into \mathbb{N} . Either there is an $f \in \mathfrak{F}$ such that $f \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h} \upharpoonright \text{dom}(f) \notin V\text{”}$, or there is an $f \in \mathfrak{F}$ and an increasing sequence $m_1 < m_2 < \dots$ of integers such that $\mathbb{N} \setminus \text{dom}(f) = \bigcup_k S_k$ where $S_k \subset 2^{m_{k+1}} \setminus 2^{m_k}$ and for each $i \in S_k$ the*

conditions $f \cup \{(i, 0)\}$ and $f \cup \{(i, 1)\}$ each force a value on $\dot{h}(i)$. Furthermore, if f forces \dot{h} to be finite-to-one, we can arrange that for each k and each $i \in [n_k, n_{k+1})$, either f forces a value on $\dot{h}(i)$, or f forces that $\dot{h}(i) \in [n_k, n_{k+1})$.

Proof. First we choose $f_0 \in \mathfrak{F}$ and some increasing sequence $n_0 < n_1 < \dots$ as in Lemma 2.5. We may choose, for each k , an m_k such that $n_k \leq 2^{m_k} < 2^{m_k+1} < n_{k+1}$ and $\lim_k |2^{m_k+1} \setminus (2^{m_k} \cup \text{dom}(f_0))| = \infty$. For each k , let $S_k^0 = 2^{m_k+1} \setminus (2^{m_k} \cup \text{dom}(f_0))$. By re-indexing we may assume that $|S_k^0| \geq k$, and we may arrange that $\mathbb{N} \setminus \text{dom}(f_0)$ is equal to $\bigcup_k S_k^0$ and set $L_0 = \mathbb{N}$. For each $k \in L_0$, let $i_k^0 = \min S_k^0$ and choose any $f_1' < f_0$ such that (by definition of \mathbb{P}) $I_0 = \{i_k^0 : k \in L_0\} \subset (f_1')^{-1}(0)$ and (by assumption on \dot{h}) f_1' forces a value on $\dot{h}(i_k^0)$ for each $k \in L_0$. Set $f_1 = f_1' \upharpoonright (\mathbb{N} \setminus I_0)$ and for each $k \in L_0$, let $S_k^1 = S_k^0 \setminus (\{i_k^0\} \cup \text{dom}(f_1))$. By further extending f_1 we may also assume that $f_1 \cup \{(i_k^0, 1)\}$ also forces a value on $\dot{h}(i_k^0)$. Choose $L_1 \subset L_0$ such that $\lim_{k \in L_1} |S_k^1| = \infty$. Notice that each i_k^0 is the minimum element of S_k^1 . Again, we may extend f_1 and assume that $\mathbb{N} \setminus \text{dom}(f_1)$ is equal to $\bigcup_{k \in L_1} S_k^1$. Suppose now we have some infinite L_j , some f_j , and for $k \in L_j$, an increasing sequence $\{i_k^0, i_k^1, \dots, i_k^{j-1}\} \subset S_k^0$. Assume further that

$$S_k^j \cup \{i_k^l : l < j\} = S_k^0 \setminus \text{dom}(f_j)$$

and that $\lim_{k \in L_j} |S_k^j \setminus i_k^{j-1}| = \infty$. For each $k \in L_j$, let $i_k^j = \min(S_k^j \setminus \{i_k^l : l < j\})$. By a simple recursion of length 2^j , there is an $f_{j+1} < f_j$ such that, for each $k \in L_j$, $\{i_k^l : l \leq j\} \subset S_k^0 \setminus \text{dom}(f_{j+1})$ and for each function s from $\{i_k^l : l \leq j\}$ into 2, the condition $f_{j+1} \cup s$ forces a value on $\dot{h}(i_k^j)$. Again find $L_{j+1} \subset L_j$ so that $\lim_{k \in L_{j+1}} |S_k^{j+1}| = \infty$ (where $S_k^{j+1} = S_k^0 \setminus \text{dom}(f_{j+1})$) and extend f_{j+1} so that $\mathbb{N} \setminus \text{dom}(f_{j+1})$ is equal to $\bigcup_{k \in L_{j+1}} S_k^{j+1}$.

We are half-way there. At the end of this fusion, the function $\bar{f} = \bigcup_j f_j$ is a member of \mathbb{P} because for each j and $k \in L_{j+1}$, $2^{m_k+1} \setminus (2^{m_k} \cup \text{dom}(\bar{f})) \supset \{i_k^0, \dots, i_k^j\}$. For each k , let $\bar{S}_k = S_k^0 \setminus \text{dom}(\bar{f})$; by possibly extending \bar{f} , we may again assume that there is some L such that $\lim_{k \in L} |\bar{S}_k| = \infty$. What we have proven about \bar{f} is that for each $k \in L$ and each $i \in \bar{S}_k$ and each function s from $i \cap \bar{S}_k$ to 2, $\bar{f} \cup s \cup \{(i, 0)\}$ and $\bar{f} \cup \{(i, 1)\}$ each force a value on $\dot{h}(i)$. By the genericity of \mathfrak{F} , there must be such a condition as f in \mathfrak{F} .

To finish, simply repeat the same process as above except this time choose maximal values and work down the values in \bar{S}_k . That is, there will be an infinite set K and a condition f^\dagger such that for each $k \in K$, there is a decreasing sequence $\{i_k^0, i_k^1, \dots, i_k^{j_k}\} \subset \bar{S}_k \setminus \text{dom}(f^\dagger)$ with $\lim_k \{j_k : k \in K\} = \infty$. These will have the property that for each $k \in K$ and $j \leq j_k$ and each function $s : \{i_k^0, \dots, i_k^{j-1}\} \rightarrow 2$, each of $f^\dagger \cup s \cup \{(i_k^j, 0)\}$ and $f^\dagger \cup s \cup \{(i_k^j, 1)\}$ will force a value on $\dot{h}(i_k^j)$.

Now we show that $f^\dagger \cup \{(i_k^j, e)\}$ ($e \in 2$) forces a value on $\dot{h}(i_k^j)$ as required. If it did not, then we could find extensions f_0, f_1 of $f^\dagger \cup \{(i_k^j, e)\}$ which force different values on $\dot{h}(i_k^j)$. Let $s_0 = f_0 \upharpoonright S_0^k \cap i_k^j$ and $s_1 = f_1 \upharpoonright S_0^k \setminus i_k^j$. Notice that $\bar{f} \cup s_0 \leq f^\dagger \cup s_0$ forces a value (hence the same value as that forced by f_0) on $\dot{h}(i_k^j)$. This is also true for $f^\dagger \cup s_1$ in that it forces the same value on $\dot{h}(i_k^j)$ as that forced by f_1 . The contradiction is that $\bar{f} \cup s_0$ and $f^\dagger \cup s_1$ force distinct values on $\dot{h}(i_k^j)$ although they have the common extension $f^\dagger \cup s_0 \cup s_1$. ■

Returning to the proof of Theorem 5.1, we are ready to use Lemma 5.2 to show that forcing with $\mathbb{P}(\mathfrak{F})$ will not introduce undesirable functions h , analogously to the argument in Theorem 1.1. By Lemma 5.2, we have the condition $f_0 \in \mathfrak{F}$ and the sequence S_k ($k \in \mathbb{N}$) such that $\mathbb{N} \setminus \text{dom}(f_0) = \bigcup_k S_k$ and that for each $i \in \bigcup_k S_k$, $f_0 \cup \{(i, 0)\}$ forces a value (call it $\bar{h}(i)$) on $\dot{h}(i)$. Therefore, \bar{h} is a function with domain $\bigcup_k S_k$ in V . We may assume that $|S_k| \geq k$ for each k . It suffices to find a condition in \mathbb{P} below f_0 which forces that there is some $J \in \mathcal{J}$ such that h_J is not extended by \dot{h} . It is useful to note that if $Y \subset \bigcup_k S_k$ is such that $\limsup |S_k \setminus Y|$ is infinite, then for any function $g \in 2^Y$, $f_0 \cup g \in \mathbb{P}$.

We first check that \bar{h} is 1-to-1 on a cofinite set. If not, there is an infinite set of pairs $E_j \subset \bigcup_k S_k$, $\bar{h}[E_j]$ is a singleton and such that for each k , $S_k \cap \bigcup_j E_j$ has at most two elements. Let L denote the set of k for which S_k meets $\bigcup_j E_j$. By passing to a subcollection of the E_j 's we may assume that L has infinite complement. Let g be the function with domain $\bigcup_j E_j$ which is constantly 0. Then $f_0 \cup g$ forces that \dot{h} agrees with \bar{h} on $\text{dom}(g)$ and so is not 1-to-1. There is a further extension f_1 of $f_0 \cup g$ with the property that $S_k \subset \text{dom}(f_1)$ for all $k \notin L$. Therefore, by virtue of f_1 , there is some $J \in \mathcal{J}$ which contains $\bigcup_j E_j$. However, this is a contradiction, because apparently $h_J = \dot{h} \upharpoonright J$ does not induce a homeomorphism on J^* .

But now that we know that \bar{h} is 1-to-1 we can get a contradiction as follows. Let $f_1 < f_0$ be chosen so as to decide the value of $F(f_0^{-1}(0))$, and let Y denote this value. For each k , let $\bar{S}_k = S_k \setminus \text{dom}(f_1)$ and let L be such that $\{|\bar{S}_k| : k \in L\}$ diverges to infinity. If $Y \cap \bar{h}[\bigcup_{k \in L} \bar{S}_k]$ is infinite, then there is an infinite set $L_0 \subset L$ (with $L \setminus L_0$ also infinite) such that for each $k \in L_0$, there is an $i_k \in \bar{S}_k$ such that $\bar{h}(i_k) \in Y$. Choose any $f_2 < f_1$ such that $f_2(i_k) = 0$ and $S_k \subset \text{dom}(f_2)$ for all $k \in L_0$. It follows that there is a $J \in \mathcal{J}$ with $\bigcup_{k \in L_0} S_k \cap f_2^{-1}(0) \subset J$ and such that

$$f_2 \Vdash_{\mathbb{P}} "F(\{i_k : k \in L_0\}) \cap F(f_0^{-1}(0)) =^* h_J[\{i_k : k \in L_0\}] \cap Y \text{ is infinite}",$$

a contradiction since $\{i_k : k \in L_0\}$ is disjoint from $f_0^{-1}(0)$. On the other hand, let L be as above and L_0 any infinite-cofinite subset. Fix any sequence $\{i_k : k \in L_0\}$ (with each $i_k \in \bar{S}_k$) and select $f_2 < f_1$ so that $f_2(i_k) = 1$ for all $k \in L_0$ and $\bigcup_{k \in L_0} S_k \subset \text{dom}(f_2)$. Set $Y' = \bar{h}[\{i_k : k \in L_0\}]$. Since \bar{h}

is 1-to-1 it follows easily that $f_2 \Vdash_{\mathbb{P}} “(\forall J \in \mathcal{J}) F(J) \cap Y' \text{ is finite}”$. This, of course, is also a contradiction.

Now we consider the possibility that $\tau(x) > 2$. It then follows that one of $A \setminus \{x\}$ or $B \setminus \{x\}$, say the former, can be partitioned into disjoint clopen non-compact sets. Therefore there is some sequence $\{c_\alpha : \alpha \in \omega_2\}$ of \mathbb{P} -names such that for each $\alpha < \beta \in \omega_2$, $c_\beta \subset a_\beta$ and $c_\beta \cap a_\alpha =^* c_\alpha$. In addition, for each $\alpha < \omega_2$ there must be a $\beta \in \omega_2$ such that $c_\beta \setminus a_\alpha$ and $a_\beta \setminus (c_\beta \cup a_\alpha)$ are both infinite.

In this case, we suppose that H and \mathfrak{F} are chosen as in Lemma 2.2, and in the extension by H , let λ denote the ordinal ω_2 from V . In this model we will have a (λ, λ) -gap formed by the families $\{c_\alpha : \alpha \in \lambda\}$ and $\{a_\alpha \setminus c_\alpha : \alpha \in \lambda\}$. Assume that we can show that in the extension obtained by forcing with $\mathbb{P}(\mathfrak{F})$, there is no $C \subset \mathbb{N}$ such that $C \cap a_\alpha =^* c_\alpha$ for all $\alpha \in \lambda$. In other words, for any cofinal sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$, the collections $\{c_{\alpha_\xi}, a_{\alpha_\xi} \setminus c_{\alpha_\xi} : \xi \in \omega_1\}$ form an (ω_1, ω_1) -gap. There are well-known ccc posets Q_1 (see [1, 4.2]) which “freeze” the gap. What we mean here is that there is a family of ω_1 -many dense subsets of the iteration $\omega_2^{<\omega_1} * \mathbb{P}(\mathcal{F}) * Q_1$ such that if a filter meets them all, then the gap will remain a gap in any proper forcing extension. Finally, if we let Q_2 be the σ -centered poset mentioned after Definition 1.6, there is a filter (meeting ω_1 -many dense subsets) on the proper iteration $\omega_2^{<\omega_1} * \mathbb{P}(\mathcal{F}) * Q_1 * Q_2$ which introduces a condition $f \in \mathbb{P}$ which forces that c_λ will not exist.

Thus, we will have shown that $\tau(x) = 2$ once we show that there is no $\mathbb{P}(\mathfrak{F})$ -name for a set C as above. Equivalently, we assume that \dot{h} is a $\mathbb{P}(\mathfrak{F})$ -name for the characteristic function of $\mathbb{N} \setminus C$, and derive a contradiction.

So, given our name \dot{h} , we repeat the steps above up to the point where we have f_0 and the sequence $\{S_k : k \in \mathbb{N}\}$ so that $f_0 \cup \{(i, 0)\}$ forces a value $\bar{h}(i)$ on $\dot{h}(i)$ for each $i \in \bigcup_k S_k$ and $\mathbb{N} \setminus \text{dom}(f_0) = \bigcup_k S_k$.

Let $Y = \bar{h}^{-1}(0)$ and $Z = \bar{h}^{-1}(1)$. Since x is forced to be an ultrafilter, there is an $f_1 < f_0$ such that $\text{dom}(f_1)$ contains one of Y or Z . If $\text{dom}(f_1)$ contains Y , then f_1 forces that $\dot{h}[a_\beta \setminus \text{dom}(f_1)] = 1$, and so $a_\beta \setminus \text{dom}(f_1) \subset^* \mathbb{N} \setminus C$ for all $\beta \in \omega_2$. While if $\text{dom}(f_1)$ contains Z , then f_1 forces that $\dot{h}[a_\beta \setminus \text{dom}(f_1)] = 0$, and so $a_\beta \setminus \text{dom}(f_1) \subset^* C$ for all $\beta \in \omega_2$. However, taking β so large that each of $c_\beta \setminus \text{dom}(f_1)$ and $a_\beta \setminus (c_\beta \cup \text{dom}(f_1))$ are infinite shows that no such \dot{h} exists.

Finally, we show that there are no involutions on \mathbb{N}^* which have a unique fixed point. Assume that Φ is such an involution and that y is the unique fixed point of Φ . Let F be an arbitrary lifting of Φ to $[\mathbb{N}]^\omega$. Let \mathcal{I} denote the dual ideal to y . We first show that \mathcal{I} is a P -ideal (i.e. that y is a P -point). For each $I \in \mathcal{I}$, $F(I)$ is also in \mathcal{I} and $F(I \cup F(I)) =^* I \cup F(I)$. So we may let \mathcal{Z} denote those $I \in \mathcal{I}$ such that $Z =^* F(Z)$. Given $Z \in \mathcal{Z}$, since

$\text{fix}(\Phi) \cap Z^* = \emptyset$, there is a collection $\mathcal{Y} \subset [Z]^\omega$ such that $F(Y) \cap Y =^* \emptyset$ for each $Y \in \mathcal{Y}$, and such that Z^* is covered by $\{Y^* : Y \in \mathcal{Y}\}$. By compactness, we may assume that $\mathcal{Y} = \{Y_0, \dots, Y_n\}$ is finite. Set $Z_0 = Y_0 \cup F(Y_0)$. By induction, replace Y_k by $Y_k \setminus \bigcup_{j < k} Z_j$ and define $Z_k = Y_k \cup F(Y_k)$. Therefore $Y_Z = \bigcup_k Y_k$ satisfies $Y_Z \cap F(Y_Z) =^* \emptyset$ and $Z = Y_Z \cup F(Y_Z)$. This shows that for each $Z \in \mathcal{Z}$ there is a partition of $Z = Z^0 \cup Z^1$ such that $F(Z^0) =^* Z^1$. We can show y is a P -point. Indeed, if $\{Z_n = \overline{Z_n^0 \cup Z_n^1} : n \in \mathbb{N}\} \subset \mathcal{Z}$ are pairwise disjoint, then $y \notin \overline{\bigcup_n Z_n^*}$ since $F(\overline{\bigcup_n (Z_n^0)^*}) = \overline{\bigcup_n (Z_n^1)^*}$ and $\overline{\bigcup_n (Z_n^0)^*}$ is disjoint from $\overline{\bigcup_n (Z_n^1)^*}$.

Fix \mathbb{P} -names for F and the members of \mathcal{I} and let H, \mathcal{F}, F and $\{n_k : k \in \omega\}$ be as given in Lemma 3.1. Also let \mathcal{J} denote the ideal as defined in the proof of Corollary 3.2. Hence $J \in \mathcal{J}$ if there is an $f \in \mathcal{F}$ and $J \subset \bigcup \{[n_k, n_{k+1}) : [n_k, n_{k+1}) \subset \text{dom}(f)\}$. It is again easily argued that the $I \in \mathcal{I}$ as specified in Corollary 3.2 can be assumed to be empty. For each $J \in \mathcal{J}$, let h_J be the function on J such that there is an $f \in \mathcal{F}$ which forces that h_J induces F on J . Let \dot{h} be the $\mathbb{P}(\mathcal{F})$ -name as given in Corollary 3.2. Since F is an involution with a unique fixed point, we may assume that \dot{h} is forced to satisfy that $\dot{h}(\dot{h}(i)) = i \neq \dot{h}(i)$ for all i .

The rest of the proof depends on the following modification of Lemma 5.2.

CLAIM 4. *There is an $f \in \mathfrak{F}$ and a sequence of sets $\{m_k, S_k, T_k : k \in \omega\}$ and mappings $\psi_k : T_k \rightarrow S_k$ such that $S_k \subset 2^{m_k+1} \setminus 2^{m_k} \subset [n_k, n_{k+1})$, $T_k \subset [n_k, n_{k+1})$, $\mathbb{N} \setminus \text{dom}(f) = \bigcup_k S_k$, and for each k and $i \in S_k$ and $f < \bar{f}$, \bar{f} forces a value on $\dot{h} \upharpoonright \psi_k^{-1}(i)$ iff $i \in \text{dom}(\bar{f})$.*

Before proving the claim, let us show how this will finish the proof. For each $i \notin \text{dom}(f)$, there are two functions h_i^0, h_i^1 with domain $\psi^{-1}(i)$ such that $f \cup \{i, e\}$ forces that $h_i^e \subset \dot{h}$. Since, by assumption, for each $i \in \text{dom}(f)$, f does not already force a value on $\dot{h} \upharpoonright \psi^{-1}(i)$, we can choose $j_i \in \psi^{-1}(i)$ such that $v_i = h_i^0(j_i) \neq h_i^1(j_i) = w_i$. Note that, by our assumption on \dot{h} , it also follows that $\psi(v_i) = \psi(w_i) = i$.

Choose $g < f$ which forces a value, Y , on $F(\{j_i\}_{i \notin \text{dom}(f)})$. Assume that $Y \cap \{v_i : i \notin \text{dom}(g)\}$ is infinite. It follows easily that there is some $g^\dagger < g$ such that $Y \cap \{v_i : g^\dagger(i) = 1\}$ is infinite and let $J \subset \{i \in \text{dom}(g^\dagger) : g^\dagger(i) = 1 \text{ and } v_i \in Y\}$ be any infinite set such that $\{j_i\}_{i \in J}$ is in \mathfrak{J} . However, this is a contradiction since

$$g^\dagger \Vdash_{\mathbb{P}} \text{“}F\text{”}(\{j_i : i \in J\}) =^* \dot{h}[\{j_i\}_{i \in J}] = \{w_i\}_{i \in I} \subset^* Y \cap \{v_i\}_{i \in J}.$$

The argument when $\{v_i : i \notin \text{dom}(g)\} \setminus Y$ is infinite is similar.

Now we prove Claim 4. For any k , condition g , and $T \subset [n_k, n_{k+1})$ let $\text{Orb}(T, g)$ denote the set $\{j : (\exists g' < g)(\exists t \in T) g' \Vdash_{\mathbb{P}(\mathfrak{F})} \text{“}\dot{h}(t) = j\text{”}\}$. Fix

any k and the f as above that was selected from Lemma 2.5. Let $g_0 = f \upharpoonright [n_k, n_{k+1})$ and assume, as we may, that $S_0^k = [n_k, n_{k+1}) \setminus \text{dom}(g_0)$ is contained in $[2^m, 2^{m+1})$ for some m . By a simple recursion much as in Lemma 5.2, we can choose increasing sequences $I_l = \{i_0, i_1, \dots, i_{l-1}\} \subset S_0^k$ and extensions $g_l \supset g_{l-1} \supset \dots \supset g_0$ so that $I_l \subset S_l^k = S_0^k \setminus \text{dom}(g_l)$ and $i_l = \min(S_l^k \setminus I_l)$. In addition, select sets $T(i_l) \subset [n_k, n_{k+1}) \setminus \bigcup_{j < l} \text{Orb}(T(i_j), g_l)$ of minimum cardinality (at most 2^l) so that for each $s : S_l^k \rightarrow 2$ there is, if possible, a $t \in T(i_l)$ such that $s \cup g_l \cup \{(i_j, 0)\}$ and $s \cup g_l \cup \{(i_j, 1)\}$ force distinct values on $\dot{h}(t)$. Notice that $\text{Orb}(\{t\}, g_l)$ has cardinality at most 2^l for each $t \in \bigcup_{j < l} T(i_j)$. We also require that for each $s : S_l^k \rightarrow 2$, each of $s \cup \{(i_l, 0)\} \cup g_{l+1}$ and $s \cup \{(i_l, 1)\} \cup g_{l+1}$ force a value on $\dot{h} \upharpoonright \text{Orb}(T(i_l), g_{l+1})$. If $T(i_l)$ is not empty, we can certainly ensure that for at least one $s : S_l^k \rightarrow 2$ and $t \in T(i_l)$, $s \cup \{(i_l, 1)\} \cup g_{l+1}$ forces a distinct value on $\dot{h}(t)$ from that forced by $s \cup \{(i_l, 0)\} \cup g_{l+1}$.

For each successive l , there is a recursion on k so that $f_l = f \cup \bigcup \{g_l^k : k \in \omega\}$ is a condition. If for each k , there is an $s^k : I_l^k \rightarrow 2$ for which no suitable $t \in [n_k, n_{k+1})$ can be chosen, then it is because the condition $g = f_k \cup \bigcup_k (s_k \cup g_j^k)$ forces a value on $\dot{h}(t)$ for all $t \notin \bigcup_k \bigcup_{j < l} \text{Orb}(T(i_k^j), g)$. But if this were the case, then this condition would force a value on $\dot{h}(t)$ for all t .

After infinitely many steps, we may instead assume that (a new choice of) f simply has this property: for each k and each $S^k = [2^{m_k}, 2^{m_k+1}) \setminus \text{dom}(f) = [n_k, n_{k+1}) \setminus \text{dom}(f)$, there is a sequence $\{T(i) : i \in S^k\}$ of pairwise disjoint finite subsets of $[n_k, n_{k+1})$ such that for each $i \in S^k$ and each $s : S^k \cap i \rightarrow 2$, $s \cup \{(i, 0)\} \cup f$ and $s \cup \{(i, 1)\} \cup f$ each force a value on $\dot{h} \upharpoonright T(i)$ while $s \cup f$ does not. (We do not need a superscript on the T 's since they depend only on i and not on k .) We have also ensured that for $i \neq i'$, $\text{Orb}(T(i), f)$ is disjoint from $\text{Orb}(T(i'), f)$.

Now, much as in Lemma 5.2, we repeat the process but rather than choosing minimal members of S^k we choose maximal. A new trouble arises in this proof because of the sizes of the sets $T(i)$, while in Lemma 5.2, each $T(i)$ was just $\{i\}$. To overcome this, we will use the next claim.

CLAIM 5. *For each $f_1 < f$ and infinite $I \subset \mathbb{N} \setminus \text{dom}(f_1)$ and $K \subset \mathbb{N}$ for which $\{|I \cap S^k| : k \in K\}$ diverges to infinity, there is an $f_2 < f_1$, $I' \subset I \setminus \text{dom}(f_2)$, and $K' \subset K$ such that $\{|I' \cap S^k \setminus \text{dom}(f_2)| : k \in K'\}$ diverges to infinity, and for all $i \in I'$, each of $f_2 \cup \{(i, 0)\}$ and $f_2 \cup \{(i, 1)\}$ force a value on $\dot{h} \upharpoonright T(i)$.*

In order to not lose track of our progress, let us again defer the proof of Claim 5 and first finish the proof of Claim 4.

Let $K_0 \subset \omega$ be chosen so that $\{|S^k| : k \in K_0\}$ is strictly increasing. By Claim 5 there is an infinite $K_1 \subset K_0$ and an $f_1 < f$ so that for each $k \in K_1$,

there is an $i_0^k \notin \text{dom}(f_1)$ such that $f_1 \cup \{(i_0^k, 0)\}$ and $f_1 \cup \{(i_0^k, 1)\}$ each force a value on $\dot{h} \upharpoonright T(i_0^k)$, and $|S^k \cap i_0^k| > |S^k|/2$.

By induction on $j > 0$, continue to choose $f_j < f_{j-1}$, $i_j^k \in S^k \cap i_{j-1}^k \setminus \text{dom}(f_j)$ for all k in an infinite set $K_j \subset K_{j-1}$ such that the sequence $\{|(S^k \cap i_j^k) \setminus \text{dom}(f_{j-1})| : k \in K_j\}$ diverges to infinity. We require that for each $k \in K_j$ and $s : \{i_l^k : l \leq j\} \rightarrow 2$, the condition $s \cup f_j$ forces a value on $\dot{h} \upharpoonright T(i_j^k)$.

We find the sequence $\{i_j^k : k \in K_j\}$ by applying Claim 5 as follows. For each function $\psi : j \rightarrow 2$ and each $k \in K_{j-1}$, let s_ψ^k denote the function from $\{i_0^k, \dots, i_{j-1}^k\}$ to 2 such that $s_\psi^k(i_{j'}^k) = \psi(j')$ for each $j' < j$. Start by applying Claim 5 with the f_1 in Claim 5 being $f_{j-1} \cup \bigcup_{k \in K_{j-1}} s_\psi^k$ for some fixed $\psi \in 2^j$, $K = K_{j-1}$ and, $I = \bigcup_{k \in K_{j-1}} S^k \setminus \text{dom}(f_1)$. Simply apply Claim 5 recursively, each time swapping the values of the f_1 used so as to cycle through all the possible $\psi \in 2^j$. After these 2^j steps, each time shrinking the K' and the I' we can let f_j be the final condition denoted f_2 in Claim 5, K_j be the final set K' and let $\{i_j^k : k \in K_j\}$ be any selection from the final I' which has the additional property that $\{|S^k \cap i_j^k| : k \in K_j\}$ diverges to infinity.

What we have now is that for each $k \in K_j$ and each $\psi \in 2^j$, the conditions $s_\psi^k \cup f_j \cup \{(i_j^k, 0)\}$ and $s_\psi^k \cup f_j \cup \{(i_j^k, 1)\}$ each force a value on $\dot{h} \upharpoonright T(i_j^k)$.

When this recursion finishes, let $\{k_j : j \in \omega\}$ be chosen so that $k_j \in K_j$ for each j . Set $\bar{f} \supset \bigcup_j f_j \upharpoonright [n_{k_j}, n_{k_j+1})$. Note that for each j , $k = k_j$ and $l < j$, $i_l^k \notin \text{dom}(\bar{f})$ and \bar{f} is not forcing a value on $\dot{h} \upharpoonright T(i_l^k)$. Assume $g < \bar{f}$, $k = k_j$ and $l < j$. Let $s = g \upharpoonright (S^k \cap i_l^k)$ and $s' = g \upharpoonright (S^k \setminus 1 + i_l^k)$.

There is a $t \in T(i_l^k)$ such that $s \cup f_j \cup \{(i_l^k, 0)\}$ and $s \cup f_j \cup \{(i_l^k, 1)\}$ force different values on $\dot{h}(t)$. Therefore if $i_l^k \notin \text{dom}(g)$, then g cannot decide $\dot{h}(t)$. On the other hand, suppose i_l^k is in $\text{dom}(g)$, let $e = g(i_l^k)$ and assume that there is a $t \in T(i_j^k)$ such that $\dot{h}(t)$ is not decided by g . Fix any $s_1 : \{i_j^k : j < l\} \rightarrow 2$ extending s' such that $s_1 \cup g$ forces a value on $\dot{h}(t)$ and let v be this value. Since g does not decide $\dot{h}(t) = v$, there is some $s_2 : S^k \cap i_j^k \rightarrow 2$ which extends s and forces a value distinct from v on $\dot{h}(t)$. This is a contradiction since $s_1 \cup s_2 \cup g$ is a condition.

We have shown that for each k_j and $l < j$, g forces a value on $\dot{h} \upharpoonright T(i_l^k)$ iff $i_l^k \in \text{dom}(g)$.

It remains to give the following proof.

Proof of Claim 5. Let f_1 and I be as in the statement of the claim and assume there is no such f_2 and I' . Let $\mathfrak{I} = \{I' \in [I]^\omega : \{|I' \cap S^k|_k\} \text{ is bounded}\}$ and $\mathfrak{I}^* = \{I' \in [I]^\omega : \{|I' \cap S^k|_k\} \text{ diverges to infinity}\}$. For each $I' \subset I$, let $K(I') = \{k : I' \cap S^k \neq \emptyset\}$. For any set I , let $\chi_0(I)$ (respectively $\chi_1(I)$) denote the function which is constantly 0 (respectively 1) on I .

Choose, if possible, $e \in \{0, 1\}$ (say $e = 0$) and some pair $f_2 < f_1$ and $I_2 \subset I \setminus \text{dom}(f_2)$ such that $I_2 \in \mathfrak{I}^*$ and $f_2 \cup \chi_0(I_2)$ forces a value on $\dot{h} \upharpoonright T(i)$ for all $i \in I_2$. If no such e exists, then let $f_2 = f_1$ and $I_2 = I$. It now follows (in either case) that for any $f_3 < f_2$ and $I' \subset I_2$, the set of $i \in I'$ for which $f_3 \upharpoonright (\text{dom}(f_3) \setminus I') \cup \chi_1(I')$ forces a value on $\dot{h}(i)$ is a member of \mathfrak{I} .

For each integer k , let \mathcal{S}_k be the set of partial functions from S^k into 2. For integers l, k and condition g , let

$$\mathcal{S}(l, k, g) = \{s \in \mathcal{S}_k : g \upharpoonright S^k \subset s \text{ and } |S^k \setminus \text{dom}(s)| > l\}.$$

For $s \in \mathcal{S}(l, k, f_2 \cup \chi_1(I_2))$, let $I(s)$ be the set of $i \in I_2 \cap S^k$ such that $s \cup f_2$ forces a value on $\dot{h} \upharpoonright T(i)$. Assume that for each l , $\{|I(s)| : s \in \bigcup_{k \in K(I_2)} \mathcal{S}(l, k, f_2 \cup \chi_1(I_2))\}$ is unbounded. We could then find an increasing sequence $\{k_l : l \in \omega\} \subset K(I_2)$ and corresponding $s(k_l) \in \mathcal{S}(l, k_l, f_2 \cup \chi_1(I_2))$ with $\{|I(s(k_l))| : l \in \omega\}$ diverging, in which case the condition $f_2 \cup \chi_1(I_2) \cup \bigcup_l s(k_l)$ would be guilty of forcing a value on $\dot{h} \upharpoonright T(i)$ for each $i \in \bigcup_l I(s(k_l)) \in \mathfrak{I}^*$ —a contradiction.

Therefore there is some l_0 such that for all $k \in K(I_2)$, and $s \in \mathcal{S}(l_0, k, f_2 \cup \chi_1(I_2))$, the set $I(s)$ has cardinality less than l_0 . Now choose an increasing sequence $\{k_l : l \in \omega\} \subset K(I_2)$ so that $|I_2 \cap S^{k_l}|$ has cardinality greater than $l_0 + 2^{2^l}$. Choose any condition $f^\dagger < f_2 \cup \chi_1(I_2)$ so that $\mathbb{N} \setminus \text{dom}(f^\dagger) \subset \bigcup_l S^{k_l}$ and $|S^{k_l} \setminus \text{dom}(f^\dagger)| = l$ for all l . Notice that $\mathcal{S}(l_0, k_l, f^\dagger)$ has cardinality at most 2^{2^l} . For each l and $s \in \mathcal{S}(l_0, k_l, f^\dagger)$, choose an $i_s \in I_2 \cap S^{k_l}$ such that $s \cup f^\dagger$ does not force a value on $\dot{h} \upharpoonright T(i_s)$. Ensure that the selection is such that $i_s \neq i_{s'}$ for distinct $s, s' \in \mathcal{S}(l_0, k_l, f^\dagger)$. Next, for each l and $s \in \mathcal{S}(l_0, k_l, f^\dagger)$, choose $t_s \in T(i_s)$ and distinct u_s, w_s each with the property that there is some extension of $s \cup f^\dagger$ forcing $\dot{h}(t_s)$ to have that value.

We now define an ultrafilter in \mathbb{N}^* . For each g , let $X(g) = \{i_s : s \in \bigcup_l \mathcal{S}(l_0, k_l, g)\}$, $U(g) = \{u_s : i_s \in X(g)\}$, and $W(g) = \{w_s : i_s \in X(g)\}$. Let $z \in \mathbb{N}^*$ be any ultrafilter which extends the family $\{X(g) : g \in \mathfrak{F}, g < f^\dagger\}$.

Since $U(f^\dagger) \cap W(f^\dagger)$ is empty, there is some $g < f^\dagger$ such that either $g \Vdash_{\mathbb{P}} "U(f^\dagger) \notin z"$ or $g \Vdash_{\mathbb{P}} "W(f^\dagger) \notin z"$ (by symmetry assume $U(f^\dagger) \notin z$). By possibly extending g , there is an $X \in x$ such that $g \Vdash_{\mathbb{P}} "F(X) \cap U(f^\dagger) =^* \emptyset"$. Since $X \cap X(g)$ is infinite we can choose an infinite set $L \subset \mathbb{N}$ such that for each $l \in L$, $s_l = g \upharpoonright S^{k_l} \in \mathcal{S}(l_0, k_l, g)$ and $s_l \in X$. For each $l \in L$, let $s_l^* \in \mathcal{S}_l$ be chosen so that $s_l \subset s_l^*$ and $s_l^* \cup f^\dagger$ forces $\dot{h}(i_{s_l}) = u_{s_l}$. By genericity of \mathfrak{F} , there is a $g^\dagger < g$ such that $L' = \{l \in L : S^{k_l} \subset \text{dom}(g^\dagger) \text{ and } s_l^\dagger \subset g^\dagger \upharpoonright S^{k_l}\}$ is infinite. Since

$$g^\dagger \Vdash_{\mathbb{P}} "\{u_{s_l}\}_{l \in L'} = \dot{h}[\{i_{s_l}\}_{l \in L'}] =^* F(\{i_{s_l}\}_{l \in L'}) \subset^* \mathbb{N} \setminus U(f^\dagger)",$$

we have our contradiction.

This completes the proof of Claim 5. ■

6. Questions

QUESTION 6.1. Assume PFA. If G is \mathbb{P}_2 -generic, and $\mathbb{N}^* = A \underset{x}{\bowtie} B$ is the generic tie-point introduced by \mathbb{P}_2 , is it true that A is not homeomorphic to \mathbb{N}^* ? Is it true that $\tau(x) = 2$? Is it true that each tie-point is a symmetric tie-point?

REMARK 1. The tie-point x_3 introduced by \mathbb{P}_3 does not satisfy $\tau(x_3) = 3$. This can be seen as follows. For each $f \in \mathbb{P}_3$, we can partition L_f into $\{i \in \text{dom}(f) : i < f(i) < f^2(i)\}$ and $\{i \in \text{dom}(f) : i < f^2(i) < f(i)\}$.

It seems then that the tie-points x_l introduced by \mathbb{P}_l might be better characterized by the property that there is an autohomeomorphism F_l of \mathbb{N}^* such that $\text{fix}(F_l) = \{x_l\}$, and each $y \in \mathbb{N}^* \setminus \{x_l\}$ has an orbit of size l .

REMARK 2. A small modification to the poset \mathbb{P}_2 will result in a tie-point $\mathbb{N}^* = A \underset{x}{\bowtie} B$ such that A (hence the quotient space by the associated involution) is homeomorphic to \mathbb{N}^* . The modification is to build into the conditions a map from the pairs $\{i, f(i)\}$ into \mathbb{N} . A natural way to do this is to set $f \in \mathbb{P}_2^+$ if f is a 2-to-1 function such that for each n , f maps $\text{dom}(f) \cap (2^{n+1} \setminus 2^n)$ into $2^n \setminus 2^{n-1}$, and again $\limsup_n |2^{n+1} \setminus (\text{dom}(f) \cup 2^n)| = \infty$. \mathbb{P}_2^+ is ordered by almost containment. The generic filter introduces an ω_2 -sequence $\{f_\alpha : \alpha \in \omega_2\}$ and two ultrafilters: $x \supset \{\mathbb{N} \setminus \text{dom}(f_\alpha) : \alpha \in \omega_2\}$ and $z \supset \{\mathbb{N} \setminus \text{range}(f_\alpha) : \alpha \in \omega_2\}$. For each α and $a_\alpha = \{i \in \text{dom}(f_\alpha) : i = \min(f_\alpha^{-1}(f_\alpha(i)))\}$, we set $A = \{x\} \cup \bigcup_\alpha a_\alpha^*$ and $B = \{x\} \cup \bigcup_\alpha (\text{dom}(f_\alpha) \setminus a_\alpha)^*$; then $\mathbb{N}^* = A \underset{x}{\bowtie} B$ is a symmetric tie-point. Finally, the map $F : A \rightarrow \mathbb{N}^*$ defined by $F(x) = z$ and $F \upharpoonright A \setminus \{x\} = \bigcup_\alpha (f_\alpha)^*$ is a homeomorphism.

QUESTION 6.2. Assume PFA. If L is a finite subset of \mathbb{N} and $\mathbb{P}_L = \prod \{\mathbb{P}_l : l \in L\}$, is it true in $V[G]$ that there is a finite upper bound to $\tau(x)$ for the tie-points x ; and if $1 \notin L$, then every tie-point is a symmetric tie-point?

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