Acta Math. Univ. Comenianae
Vol. (submitted to AMUC), pp. 1–13

COMPACT SETS WITHOUT CONVERGING SEQUENCES IN THE RANDOM REAL MODEL

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Abstract. It is shown that in the model obtained by adding any number of random reals to a model of CH, there is a compact Hausdorff space of weight ω_1 which contains no non-trivial converging sequences. It is shown that for certain spaces with no converging sequences, the addition of random reals will not add any converging sequences.

1. Introduction

In this paper we will prove the following result.

Theorem 1.1. *If we add any (cardinal) number of random reals to a model of ZFC + CH, we obtain a model in which there is a compact space of weight ω_1 with no non-trivial converging sequences.*

It is well-known that in any model of Martin's Axiom every compact space of weight less than the continuum is sequentially compact (see [**Fre84**, 24G]).

One motivation for this result is an old question due to Efimov: *Is there an infinite compact Hausdorff space without convergent sequences which does not include a copy of $\beta\mathbb{N}$?*

Fedorchuk [**Fedorchuk 77**] gives a construction of such a space if the splitting number, $\mathfrak{s}$, is ω_1 and $2^{\omega_1} < 2^{\mathfrak{c}}$; his method gives a space of weight $\mathfrak{c}$ and cardinality 2^{ω_1} (see [**Dow 04**] for generalizations to larger values of $\mathfrak{s}$). Any model obtained by adding uncountably many random reals will be a model of $\mathfrak{s} = \omega_1$. However our result provides new situations in which there is an Efimov space (not to mention that it has small weight) because in a model obtained by adding random reals one can arrange that 2^{ω_1} does equal $2^{\mathfrak{c}}$. Talagrand [**Talagrand 80**]

Received . . . .
2000 *Mathematics Subject Classification.* Primary 54A25; Secondary 03E35 54D30.
Key words and phrases. random reals, converging sequences, Efimov problem .
The authors thank the referee for a very careful review. And Carlos Ivorra Castillo for many comments leading to this update.

gives a construction dependent on the continuum hypothesis. Just and Koszmider [**Just-Koszmider**] produce such examples in yet other models.

A space X is **extremally disconnected** if disjoint open sets have disjoint closures (and the closure of every open set is open). A compact space X is an F-space if disjoint cozero sets have disjoint closures. The space $\beta\mathbb{N}$ is extremally disconnected while $\beta\mathbb{N}\setminus\mathbb{N}$ is an F-space that is not extremally disconnected. It is well known that an F-space does not have non-trivial converging sequences.

We will be using the technique of forcing. If $\mathbb{P}$ is a poset in a model V (of ZFC), and if G is a $\mathbb{P}$-generic filter (over V), then $V[G]$ is the forcing extension by $\mathbb{P}$. If K is a compact Hausdorff space in the model V, then there is a natural and canonical compact Hausdorff space $\tilde{K}$ in $V[G]$ definable from K (up to homeomorphism). If $\mathcal{T}$ is the topology on K, then it is well understood that in $V[G]$, $\mathcal{T}$ forms a base for a topology on K, but that, in general, this very natural topology obtained from K in V, is not going to be compact. For example if K is the unit interval, then $\tilde{K}$ is going to be the unit interval including the appropriately valued new reals. The space K with the topology generated by $\mathcal{T}$ will be a dense subset of $\tilde{K}$. If K is zero dimensional, then the most natural method to define $\tilde{K}$ is simply by Stone Duality: in V, $CO(K)$ is the Boolean algebra of clopen sets and K is homeomorphic to the Stone space, $S(CO(K))$. We will let $\tilde{K}$ also be the $S(CO(K))$ as computed in $V[G]$. If U is a closed subset of K (in V), we will usually assume that $\tilde{U}$ is the natural closed subset of $\tilde{K}$.

In the case that K is not zero dimensional, we may assume that K is embedded into $[0,1]^\lambda$ for some ordinal λ. When we pass to $V[G]$, even $[0,1]$ may be larger but we will still have K sitting as a subset of $[0,1]^\lambda$, and we simply take $\tilde{K}$ to be the closure of K in $[0,1]^\lambda$. More details on this idea can be found in [**BG05**] or [**Dow92**, 5.1].

It is helpful to distinguish between $\beta\mathbb{N}$ in $V[G]$ and $\widetilde{\beta\mathbb{N}}$. If new reals are added to $V[G]$, then $\widetilde{\beta\mathbb{N}}$ is not extremally disconnected because the base for the topology is the family $\{\tilde{A} : A \subset \mathbb{N} \text{ and } A \in V\}$.

Our main result is that if K is a compact F-space, then $\tilde{K}$ will not have non-trivial converging sequences. The simplest example then is that $\widetilde{\beta\mathbb{N}}$ will have no non-trivial converging sequences. In the final section we give an example of a compact Hausdorff space K with no non-trivial converging sequences such that $\tilde{K}$ does have non-trivial converging sequences after adding random reals. It follows easily from Koszmider's results in [**Koszmider 90**] that in a Cohen real forcing extension, $\tilde{K}$ will have non-trivial converging sequences for every infinite compact space K.

2. No converging sequences in F-spaces

In the next section we derive Lemma 2.1 as a corollary to a main combinatorial lemma. In this section we establish the main theorem 1.1 as a consequence of Lemma 2.1.

Let κ be any cardinal and consider the probability measure μ on the space 2^κ in which $\mu(b_\alpha) = \mu(\sim b_\alpha) = \frac{1}{2}$ for each $\alpha \in \kappa$, where b_α is the clopen set $\{f \in 2^\kappa : f(\alpha) = 0\}$ and $\sim b_\alpha$ is the complement.

Let $\mathcal{M}$ denote the σ-algebra generated by the basic clopen sets; thus $\mathcal{M}$ is the Baire sets. Random real forcing is the poset $\mathbb{M}$ which is obtained from $\mathcal{M} \setminus \{\emptyset\}$ by identifying two members of $\mathbb{M}$ if the symmetric difference has measure 0 and ordered by inclusion (mod measure 0). The poset $\mathbb{M}$ is ccc and complete (every non-empty subset has a least upper bound).

In our treatment of $\mathbb{M}$, it will be useful to fix a choice of representatives from $\mathbb{M} \cup \{0\}$: for each element b of $\mathbb{M}$, let $\langle b\rangle \in \mathcal{M}$ be chosen from the equivalence class represented by b. For convenience, assume that $\langle 0\rangle$ is the empty set and $\langle 1\rangle$ is the entire measure space 2^κ. We do not, however, make any other assumptions about this assignment, except of course, that $a \leq b \in \mathbb{M}$ will imply that $\langle a\rangle \setminus \langle b\rangle$ will have measure 0.

If $\dot{x}$ is the $\mathbb{M}$-name of an element of $[0,1]^\lambda$ (that is, $1 \Vdash \dot{x} \in [0,1]^\lambda$), then for each basic open subset U of $[0,1]^\lambda$, there is a unique element $b \in \mathbb{M} \cup \{0\}$ such that

$$b \Vdash \dot{x} \in \tilde{U} \quad \text{and} \quad \sim b \Vdash \dot{x} \notin \tilde{U}$$

because $\mathbb{M}$ is complete. It is standard to let $[[\dot{x} \in U]]$ denote this element b. Similarly, if K is zero dimensional and U is a clopen subset of K, we would also have that $[[\dot{x} \in U]]$ is exactly the condition $b \in \mathbb{M}$ such that $b \Vdash \dot{x} \in \tilde{U}$ and $\sim b \Vdash \dot{x} \notin \tilde{U}$. For improved readability, let $\langle[\dot{x} \in U]\rangle$ be an alternate notation for $\langle[[\dot{x} \in U]]\rangle$.

Lemma 2.1. *Let K be a compact zero dimensional metric space, and let $\dot{x}, \{\dot{x}_i : i \in \omega\}$ be $\mathbb{M}$-names of members of $\tilde{K}$ such that*

$$1 \Vdash \dot{x} \in \overline{\{\dot{x}_i : i \in \omega\}} \quad \textit{and} \quad \dot{x}_i \neq \dot{x}_j (i < j)\ .$$

If Y_0 is any clopen subset of K and $X_0 \in \mathcal{M}$ is such that $X_0 \subset \langle[\dot{x} \in Y_0]\rangle$, then for any $\epsilon > 0$, there is a clopen set U_0 of Y_0 such that

$$\mu(X_0 \setminus \bigcup_{i\in\omega} \langle[\dot{x}_i \in U_0]\rangle) < \epsilon \quad \textit{and} \quad \mu(X_0 \cap \langle[\dot{x} \in U_0]\rangle) < \epsilon\ .$$

Proof. See Corollary 3.2. ∎

More informally, the idea is that with sufficiently high probability $\dot{x}$ is not in U_0 while at the same time, with sufficiently high probability at least one of the $\dot{x}_i$'s is in U_0.

The main result of this section is Lemma 2.2, we postpone its proof until we derive some corollaries and Theorem 1.1.

Lemma 2.2. *If K is a compact zero dimensional space, and G is $\mathbb{M}$-generic, then for each infinite set $D \subset \tilde{K}$, there is a sequence, $\{U_n : n \in \omega\} \in V$, of pairwise disjoint clopen subsets of K, such that for each $n \in \omega$, $D \cap \tilde{U}_n$ is not empty.*

Corollary 2.3. *If K is a compact extremally disconnected space, and G is $\mathbb{M}$-generic, then in $V[G]$, every infinite subset D of $\tilde{K}$ has a subset D' such that $\overline{D'}$ maps continuously onto $\widetilde{\beta\mathbb{N}}$.*

Proof. By Lemma 2.2, there is a sequence $\{U_m : m \in \omega\}$ of pairwise disjoint clopen sets with the property that, in $V[G]$, $\tilde{U}_m \cap D$ is not empty for each m. Still in V, the mapping $U_m \mapsto m$, lifts to a mapping g from K onto $\beta\mathbb{N}$. It is routine to check that there is a canonical reinterpretation of g in $V[G]$ which maps $\tilde{K}$ onto $\widetilde{\beta\mathbb{N}}$ and sending $\tilde{U}_m$ to m. It follows then that if $D' \subset D$ is any set such that $D' \cap \tilde{U}_m$ is not empty for each m, then the closure of D' maps continuously onto $\widetilde{\beta\mathbb{N}}$. □

Clearly Theorem 1.1 follows immediately from Corollary 2.3. The following more general statement does not assume that the space is zero dimensional.

Corollary 2.4. *If K is a compact F-space and G is $\mathbb{M}$-generic then, in $V[G]$, every infinite subset of $\tilde{K}$ has an infinite discrete subset whose closure maps onto $\widetilde{\beta\mathbb{N}}$.*

Proof. Let D be any countably infinite subset of $\tilde{K}$ in $V[G]$. By Corollary 2.3, it suffices to show that there is a compact extremally disconnected subset Y of K (in V) such that $D \subset \tilde{Y}$. By a standard technique in forcing, we can fix a countable set $\{\dot{x}_i : i \in \omega\}$ of $\mathbb{M}$-names of elements of $\tilde{K}$ such that $D = \{val_G(\dot{x}_i) : i \in \omega\}$. Working in V, we have K embedded in $[0,1]^\lambda$ for some cardinal λ. Let $\mathcal{I}$ denote the ideal of cozero subsets U of $[0,1]^\lambda$ such that $[[\dot{x}_i \in U]] = 0$ for all $i \in \omega$. Clearly $\tilde{U} \cap D$ is empty for each $U \in \mathcal{I}$. Let Y be the intersection of the family

$$\{[0,1]^\lambda \setminus U : U \in \mathcal{I}\}$$

and notice that $D \subset \tilde{Y}$ in $V[G]$. Every ccc closed subset of a compact F-space is an F-space, and consequently also extremally disconnected. We will be finished by showing that Y is ccc.

Let $\{U_\alpha : \alpha \in \omega_1\}$ be any family of cozero subsets of $[0,1]^\lambda$ such that $\{(Y \cap U_\alpha) : \alpha \in \omega_1\}$ are pairwise disjoint. For each $\alpha < \beta < \omega_1$, we thus have that $U_\alpha \cap U_\beta \in \mathcal{I}$.

It follows then that $[[\dot{x}_i \in U_\alpha \cap U_\beta]]$ is 0, and therefore that $\langle[\dot{x}_i \in U_\alpha]\rangle \cap \langle[\dot{x}_i \in U_\beta]\rangle$ has measure 0. Therefore there is a $\delta < \omega_1$ such that $[[\dot{x}_i \in U_\alpha]] = 0$ for all $\alpha \geq \delta$ and $i \in \omega$. In other words, $U_\alpha \in \mathcal{I}$ for all $\alpha \geq \delta$, proving that $\{Y \cap U_\gamma : \gamma \in \omega_1\}$ is indeed countable. □

Proof of Lemma 2.2. Recall that we have that K is a zero dimensional space, G is an $\mathbb{M}$-generic, and assume that $D \in V[G]$ is an infinite subset of $\tilde{K}$. Let $\{x_i : i \in \omega\}$ be any infinite discrete subset of D and let x be any limit point. By standard forcing arguments, there is a name $\dot{x}$ and a sequence of names $\{\dot{x}_i : i \in \omega\}$ such that $val_G(\dot{x}) = x$ and $val_G(\dot{x}_i) = x_i$ for each $i \in \omega$. In addition, there is some element $b \in G$, such that $b \Vdash \dot{x}$ is a limit of the infinite discrete set $\{\dot{x}_i : i \in \omega\}$ which is contained in $\tilde{K}$. By a possible modification of the names $\dot{x}$ and $\{\dot{x}_i : i \in \omega\}$ we can assume that b is actually 1.

In order to apply Lemma 2.1, we will have to replace K by a suitable metric space Y_0 (in V). To do so identify K with the Stone space of its clopen algebra $CO(K)$. Fix for each $i \in \omega$, a name $\dot{C}_i$ of a member of $CO(K)$ such that $1 \Vdash \dot{C}_i \cap \{\dot{x}_j : j \in \omega\} = \{\dot{x}_i\}$. For each i, there is a countable collection $\mathcal{C}_i \subset CO(K)$ such that $[[U = \dot{C}_i]] = 0$ for all $U \in CO(K) \setminus \mathcal{C}_i$. Let $\mathcal{C}$ be the countable subalgebra of $CO(K)$ that is generated by $\bigcup_{i\in\omega} \mathcal{C}_i$. Let Y_0 be the Stone space of $\mathcal{C}$. For each $i \in \omega$, there is a canonical name $\dot{y}_i$ for a member of $\tilde{Y}_0$ such that $1 \Vdash \dot{y}_i = \dot{x}_i \cap \mathcal{C}$, and similarly a name $\dot{y}$ such that $1 \Vdash \dot{y} = \dot{x} \cap \mathcal{C}$. Clearly for each $U \in \mathcal{C}$ and each $i \in \omega$, $val_G(\dot{x}_i) \in \tilde{U} \subset \tilde{K}$ if and only if $val_G(\dot{y}_i) \in \tilde{U} \subset \tilde{Y}_0$. Of course we are using $\tilde{U}$ in two different senses in the previous sentence.

To show that we have a sequence $\{U_n : n \in \omega\} \in V$, of pairwise disjoint clopen subsets of Y_0 as required, it suffices to show that for any non-zero $a \in \mathbb{M}$, there is such a sequence $\{U_n : n \in \omega\}$ (depending on a) and an $a' < a$ such that, for all n, $a' \Vdash \tilde{U}_n \cap \{\dot{y}_i : i \in \omega\} \neq \emptyset$. Let $a \in \mathbb{M}$ and choose $\epsilon > 0$ so that $\epsilon < \mu(\langle a \rangle)$. We will be done if we now find a sequence $\{U_m : m \in \omega\}$ so that $\bigcup_{i<\omega} \langle[\dot{y}_i \in U_m]\rangle$ has measure greater than $1 - \epsilon$ for all m.

This we do by repeated applications of Lemma 2.1. Set $X_0 = 2^\kappa$ and $\epsilon_0 = \epsilon/2$. By Lemma 2.1, there is a clopen $U_0 \subset Y_0$ such that $\mu(\langle[\dot{y} \in U_0]\rangle) < \epsilon_0$ and $\mu(X_0 \setminus \bigcup_{i\in\omega} \langle[\dot{y}_i \in U_0]\rangle) < \epsilon_0$. Set $Y_1 = Y_0 \setminus U_0$, $X_1 = \langle[\dot{y} \in Y_1]\rangle$, and $\epsilon_1 = \epsilon_0/2$. Observe that X_1 has measure greater than $1 - \epsilon_0$.

By induction on m, we select clopen sets U_m of Y_m and define $Y_{m+1} = Y_m \setminus U_m$, and $X_{m+1} = X_m \setminus \langle[\dot{y} \in U_m]\rangle = \langle[\dot{y} \in Y_{m+1}]\rangle$. The inductive assumptions are that $\mu(\langle[\dot{y} \in U_m]\rangle) < \epsilon/2^{m+1}$ and $\mu(X_m \setminus \bigcup_{i\in\omega} \langle[\dot{y}_i \in U_m]\rangle) < \epsilon/2^{m+1}$. To see that this can continue, we need only observe that $\langle[\dot{y} \in Y_m]\rangle$ has measure greater than $1 - \epsilon + \epsilon/2^m$ since $\langle[\dot{y} \in \bigcup_{n<m} U_m]\rangle$ has measure less than $\sum_{n<m} \epsilon/2^{n+1}$.

Since X_m has measure greater than $1-\epsilon+\epsilon/2^m$ and $\mu(X_m \setminus \bigcup_{i\in\omega}\langle[\dot y_i \in U_m]\rangle) < \epsilon/2^{m+1}$, it follows that $\bigcup_{i\in\omega}\langle[\dot y_i \in U_m]\rangle$ has measure greater than $1-\epsilon$. This completes the proof. $\square$

3. The main combinatorial lemma

In the case that K is a compact metric space and $\dot x$ is the $\mathbb{M}$-name of a member of $\tilde K$, a common approach to random real forcing would be to construct a Borel measurable function f from the measure space $(2^\kappa, \mathcal{M}, \mu)$ into K so that for each open set $U \subset K$, $\langle[\dot x \in U]\rangle = f^{-1}(U)$ (mod measure 0).

If $\dot x$ and $\dot y$ are each names of members of $\tilde K$, then $1 \Vdash \dot x \neq \dot y$ is equivalent to the following statement: there is a countable set $B \subset \mathbb{M}$ such that $\bigcup_{b\in B}\langle b\rangle$ has measure 1, and for each $b\in B$, there are disjoint basic open U, W such that $b \Vdash \dot x \in \tilde U$ and $b \Vdash \dot y \in \tilde W$. This is the same as asserting that $b \leq [[\dot x \in U]] \cap [[\dot y \in W]]$.

Thus, if $\{\dot x_i : i \in \omega\}$ is a sequence of $\mathbb{M}$-names of members of $\tilde K$, then for each i, we would have a Borel measurable function f_i corresponding to $\dot x_i$. If we assume that, for $i<j$, $1 \Vdash \dot x_i \neq \dot x_j$, then we have that $\{x \in 2^\kappa : f_i(x) = f_j(x)\}$ has measure 0. This translation helps explain the interest in the following main combinatorial Lemma. For the other results in this paper, the simplified version in Corollary 3.2 is sufficient and its proof can be read independently. The lemma derives from Shelah via 1G of [**Fremlin N97**] and [**Burke N96**].

Lemma 3.1. *Let (X, Σ, μ) be a probability space and Y a separable metric space. Suppose we have a non-negative finitely additive functional ν defined on the Borel subsets of Y, with $\nu Y = 1$, and a sequence $\langle f_i\rangle_{i\in\mathbb{N}}$ of measurable functions, each defined on a set $X_i \in \Sigma$ and taking values in Y, such that $\{x : x \in X_i \cap X_j, f_i(x) = f_j(x)\}$ is negligible whenever $i \neq j$.*

Set $Z = \bigcap_{j\in\mathbb{N}}\bigcup_{i\geq j} X_i$. Then for any $\epsilon > 0$ there is an open set $H \subset Y$ such that $\nu\overline{H} \leq \epsilon$ and $\mu(Z \setminus \bigcup_{i\in\mathbb{N}} f_i^{-1}[H]) \leq \epsilon$.

Proof. (a) Set $\eta = \frac{1}{4}\epsilon > 0$ and let $M \in \mathbb{N}$ be such that $(1-\eta)^M < \eta$. Let $N \in \mathbb{N}$ be such that $\mu(Z \setminus Z') \leq \eta$, where

$$Z' = \{x \in Z, |\{i : i < N, x \in X_i\}| \geq M\} .$$

(b) Let $\mathcal{A}$ be the ideal of subsets $A \subset Y$ such that $\inf\{\nu G : A \subseteq G, G \text{ open in } Y\}$ equals 0. Let $\mathcal{E}$ be the algebra of $\{E : E \subseteq Y, \partial E \in \mathcal{A}\}$ where $\partial E = \overline{E} \setminus \operatorname{int} E$ is the boundary of E. Then there is a finite partition $\mathcal{D}$ of Y into members of $\mathcal{E}$ such that $\mu B \leq \eta$, where

$$B = \bigcup_{i<j<N}\bigcup_{D\in\mathcal{D}} f_i^{-1}[D] \cap f_j^{-1}[D] .$$

Proof of (b): For any given metric on Y, let $B(y,\alpha)$ denote the open ball of radius α centered at y. Then for any $y_0 \in Y$, the mapping sending a positive real α to $\nu B(y_0,\alpha)$ is non-decreasing, therefore continuous at all but countably many points. This means that $B(y_0,\alpha) \in \mathcal{E}$ for all but countably many α, hence the open sets in $\mathcal{E}$ is a base for Y. There is therefore a sequence $\langle U_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ running over a base for Y. For each n, let $\mathcal{D}_n$ be the partition of Y generated by $\{U_i : i \le n\}$, and set

$$B_n = \bigcup_{i<j<N} \bigcup_{D\in\mathcal{D}_n} f_i^{-1}[D] \cap f_j^{-1}[D] \ .$$

Then $\langle B_n \rangle_{n\in\mathbb{N}}$ is non-increasing and

$$\bigcap_{n\in\mathbb{N}} B_n = \bigcup_{i<j<N} \{x : x \in X_i \cap X_j, f_i(x) = f_j(x)\}$$

is neglible, so there is some n such that $\mu B_n \le \eta$, and we can take $\mathcal{D} = \mathcal{D}_n$. This completes the proof of (b).

(c) Enumerate $\mathcal{D}$ as $\{E_r : r < m\}$. Let $K \subset m$ be a random set, in which each $r < m$ has independently, a probability η of appearing.

That is, we are defining a counting measure ρ on $\mathcal{P}(m)$ by $\rho(\{K\}) = \eta^{|K|}(1-\eta)^{m-|K|} = \eta^{|K|}(1-\eta)^{|m\setminus K|}$.

Set $F_K = \bigcup_{r\in K} E_r$ for each K. Then for any particular $x \in Z' \setminus B$,

$$\Pr(x \notin f_i^{-1}[F_K] \text{ for every } \ i < N) = \Pr(K_x \cap K = \emptyset)$$

(where $K_x = \{r : x \in f_i^{-1}[E_r] \text{ for some } \ i < N\}$)

$$= (1-\eta)^{|K_x|} \le (1-\eta)^M \le \eta$$

because if $x \in Z' \setminus B$ then

$$|K_x| = |\{i : i < N \ , \ x \in X_i\}| \ge M \ .$$

So

$$\Pr\left(\mu\left((Z' \setminus B) \setminus \bigcup_{i<N} f_i^{-1}[F_K]\right) \le 2\eta\mu(Z' \setminus B)\right) \ge \frac{1}{2} \ .$$

A reader with less experience with probability may prefer more basic explanations. Consider the product measure space $\mathcal{P}(m) \times (Z' \setminus B)$. Define the subset S by

$$S = \bigcup_{K\subset m} \left(\{K\} \times \bigcup_{i<N} f_i^{-1}[\bigcup_{r\in K} E_r] \cap (Z' \setminus B)\right) \ .$$

For each $K \subset m$, let S_K denote the vertical section $\bigcup_{i<N} f_i^{-1}[\bigcup_{r\in K} E_r]$ and for each $x \in (Z' \setminus B)$, we let S^x denote the horizontal section at x.

It is easily verified that for each $x \in (Z' \setminus B)$, $S^x = \{K \in \mathcal{P}(m) : K \cap K_x \neq \emptyset\}$; hence $\rho S^x > 1 - \eta$. From this it follows that $(\rho \times \mu)(S) > (1-\eta)\mu(Z' \setminus B)$.

The statement that

$$\Pr\left(\mu\left((Z' \setminus B) \cap \bigcup_{i<N} f_i^{-1}[\bigcup_{r\in K} E_r]\right) \geq (1-2\eta)\mu(Z' \setminus B)\right) \geq \frac{1}{2}$$

is just the statement that

$$\begin{aligned} T =& \{K : \mu((Z' \setminus B) \cap \bigcup_{i<N} f_i^{-1}[\bigcup_{r\in K} E_r]) \geq (1-2\eta)\mu(Z' \setminus B)\} \\ =& \{K : \mu(S_K) \geq (1-2\eta)\mu(Z' \setminus B)\} \end{aligned}$$

has measure greater than $\frac{1}{2}$. To show this observe that

$$\begin{aligned} (1-\eta)\mu(Z' \setminus B) <& (\rho \times \mu)(S) \\ \leq& (\rho \times \mu)(S \cap (T \times (Z' \setminus B)) + (\rho \times \mu)(S \cap (\sim T \times (Z' \setminus B)) \\ \leq& \rho(T)\mu(Z' \setminus B) + \rho(\sim T)(1-2\eta)\mu(Z' \setminus B) \end{aligned}$$

Therefore

$$1 - \eta \leq \rho(T) + \rho(\sim T) - 2\eta\rho(\sim T) = 1 - 2\eta\rho(\sim T),$$

from which it follows that $\rho(\sim T) \leq \frac{1}{2}$.

At the same time,

$$\mathbb{E}(\nu F_K) = \sum_{r=0}^{m-1} \nu E_r \Pr(r \in K) = \eta,$$

so $\Pr(\nu F_K > 2\eta) < \frac{1}{2}$, and there must be some K such that

$$\nu F_K \leq 2\eta,$$
$$\mu((Z' \setminus B) \setminus \bigcup_{i<N} f_i^{-1}[F_K]) \leq 2\eta\nu(Z' \setminus B).$$

But this means that

$$\mu(Z \setminus \bigcup_{i\in\mathbb{N}} f_i^{-1}[F_K]) \leq 2\eta + \mu B + \mu(Z \setminus Z') \leq \epsilon.$$

A similar product measure argument using $\rho \times \nu$ on $\mathcal{P}(m) \times Y$ can be used for more detailed explanations in this case as well. □

Recall we are assuming that we have a sequence $\{\dot{x}_i : i \in \omega\}$ of $\mathbb{M}$-names of members of $\tilde{K}$ for some compact metric K. We will simplify the discussion somewhat and assume that K is also zero dimensional. Let us also assume that $1 \Vdash \{\dot{x}_i : i \in \omega\}$ is discrete (and of course that $1 \Vdash \dot{x}_i \neq \dot{x}_j$ for $i \neq j$). Since $1 \Vdash \tilde{K}$ is compact, we may fix a name $\dot{x}$ of a member of $\tilde{K}$ such that $1 \Vdash \dot{x}$ is a limit point of $\{\dot{x}_i : i \in \omega\}$ (and notice that $1 \Vdash \dot{x} \neq \dot{x}_i$ for each $i \in \omega$).

The following result is really a corollary to the main lemma but for greater clarity for most readers, we will reprove it based on the ideas in the main lemma but using more traditional forcing notation and a direct approach to inductively choose clopen sets D to union up to the desired clopen set U_0. To apply Lemma 3.1, one could set $\nu(U)$ to be $\mu\langle[\dot{x} \in U]\rangle$ for each open $U \subset K$.

Corollary 3.2. *Let K be a compact zero dimensional metric space, and let $\dot{x}, \{\dot{x}_i : i \in \omega\}$ be $\mathbb{M}$-names of members of $\tilde{K}$ such that*

$$1 \Vdash \dot{x} \in \overline{\{\dot{x}_i : i \in \omega\}} \quad \textit{and} \quad \dot{x}_i \neq \dot{x}_j (i < j) \ .$$

If Y_0 is any clopen subset of K and $X_0 \in \mathcal{M}$ is such that $X_0 \subset \langle[\dot{x} \in Y_0]\rangle$, then for any $\epsilon > 0$, there is a clopen set U_0 of Y_0 such that

$$\mu(X_0 \setminus \bigcup_{i\in\omega} \langle[\dot{x}_i \in U_0]\rangle) < \epsilon \quad \textit{and} \quad \mu(X_0 \cap \langle[\dot{x} \in U_0]\rangle) < \epsilon \ .$$

Proof. Let $0 < \eta < \epsilon/10$ and we will identify several subsets of X_0 of measure less than η to ignore or remove from X_0. Since $\langle[\dot{x} = y]\rangle \wedge \langle[\dot{x} = y']\rangle = 0$ for $y \neq y'$, it follows that there is a finite set $F \subset Y_0$ such that $\bigcup_{y \in Y_0 \setminus F} \langle[\dot{x} = y]\rangle \cap X_0$ has measure less than η. Similarly, we may remove a finite subset of $\{\dot{x}_i : i < \omega\}$ so as to ensure that $\langle[\{\dot{x}_i : i < \omega\} \cap F \neq \emptyset]\rangle \cap X_0$ has measure less than η. Let us now assume that these two sets, each of measure less than η, have been removed from X_0.

Let $\mathcal{W}$ be the set of clopen subsets W of Y_0 (if any) satisfying that $\mu(\langle[\dot{x} \in W]\rangle \cap X_0) = 0$ and let $Y_1 = Y_0 \setminus \bigcup \mathcal{W}$. Let $X_0' = X_0 \setminus \bigcup_{i<\omega} \langle[\dot{x}_i \in \bigcup \mathcal{W}]\rangle$. It is a triviality that there is a $W \in \mathcal{W}$ such that the measure of $X_0 \setminus \bigcup_{i<\omega} \langle[\dot{x}_i \in W]\rangle$ is less than $\mu(X_0') + \eta$. If the measure of X_0' is less than $\epsilon/2$, then the proof is complete by setting $U_0 = W$.

Now we assume that $\mu(X_0') \geq \epsilon/2$ and we will choose a clopen U_0 to be added to W. Henceforth we will only remove sets of measure 0 from X_0 since we want to maintain the property that if a clopen set U of Y_0 meets Y_1, then $\langle[\dot{x} \in U]\rangle \cap X_0$ has positive measure. However we will continue to define important subsets of X_0. By possibly removing a set of measure 0, we may assume that $X_0' \subset \langle[\dot{x}_i \notin Y_0 \setminus Y_1]\rangle$ for all $i < \omega$ and that $X_0 \cap \langle[\dot{x} \in U]\rangle$ has positive measure for every clopen subset U of Y_1. Also, if U is any clopen subset of Y_0, then the measure of $X_0 \cap \langle[\dot{x} \in U]\rangle$ is equal to the measure of $X_0 \cap \langle[\dot{x} \in (U \cap Y_1)]\rangle$. This means that for the remainder of the proof we will work with (relatively) clopen subsets of Y_1 rather than Y_0 and will prove that there is such a $U_0 \subset Y_1$ so that

$$\mu(X_0' \setminus \bigcup_{i\in\omega} \langle[\dot{x}_i \in U_0]\rangle) < \epsilon/2 \quad \text{and} \quad \mu(X_0 \cap \langle[\dot{x} \in U_0]\rangle) < \epsilon \ .$$

Let $\{a_n : n \in \omega\}$ be a clopen basis for Y_1. Recall that we are assuming that $X_0 \cap \langle[\dot{x} = y]\rangle$ has measure 0 for all $y \in Y_0 \setminus F$ and that $\langle[\{\dot{x}_i : i < \omega\} \cap F \neq \emptyset]\rangle \cap X_0$ has measure 0.

Fix an integer L so large that $4/\sqrt{L} < \eta$. Since $X_0 \leq \langle[\dot{x} \in Y_0]\rangle$ and $1 \Vdash \dot{x} \in \overline{\{\dot{x}_i : i \in \omega\}}$, $X_0' \setminus \bigcup_{i\in\omega\setminus m} \langle[\dot{x}_i \in Y_1]\rangle \subset X_0' \setminus \bigcup_{i\in\omega\setminus m} \langle[\dot{x}_i \in Y_0]\rangle$ has measure 0 for each integer m. Therefore, for each m, there is an $m' > m$ such that $X_0' \setminus \bigcup_{i\in[m,m')} \langle[\dot{x}_i \in Y_1]\rangle$ has measure less than η/L. Fix an increasing sequence $m_0 < m_1 < \cdots$ such that $X_0' \setminus \bigcup_{i\in[m_j,m_{j+1})} \langle[\dot{x}_i \in Y_1]\rangle$ has measure less than η/L for each j. Let $M = m_L$, and set $X_1' = X_0' \cap \bigcap_{j<L} \bigcup_{i\in[m_j,m_{j+1})} \langle[\dot{x}_i \in Y_1]\rangle$. By construction, $\mu(X_0' \setminus X_1') < \eta$ and for each $j < L$, $X_1' \subset \bigcup_{m_j \leq i < m_{j+1}} \langle[\dot{x}_i \in Y_1]\rangle$.

For each $y \in F$ and $i < M$, we recall that we arranged that $X_0 \cap \langle[\dot{x}_i = y]\rangle$ has measure 0. Thus we may choose $n_y \in \omega$ so that $y \in a_{n_y}$ and, for each $i < M$, $\mu(X_0 \cap \langle[\dot{x}_i \in a_{n_y}]\rangle) < \eta/(2M \cdot |F|)$. Therefore $X_0 \cap \bigcup_{i<M,y\in F} \langle[\dot{x}_i \in a_{n_y}]\rangle$ has measure less than $\eta/2$. Let j_0 be larger than n_y for each $y \in F$. Set

$$X_1 = X_0 \setminus \left((X_0' \setminus X_1') \cup \bigcup_{i<M,y\in F} \langle[\dot{x}_i \in a_{n_y}]\rangle \right) .$$

For each $j \in \omega$, let $\mathcal{D}_j$ be the partition of Y_1 generated by $\{a_n : n < j\}$. That is, for each $D \in \mathcal{D}_j$ and each $n < j$, either $D \subset a_n$ or $D \cap a_n$ is empty. For each j and $i < i' < M$, let $E(j, i, i') = \bigcup_{D\in\mathcal{D}_j} \langle[\dot{x}_i \in D]\rangle \cap \langle[\dot{x}_{i'} \in D]\rangle$. Let $E(\omega, i, i') = \bigcap_{j\in\omega} E(j, i, i')$.

Since $1 \Vdash \dot{x}_i \neq \dot{x}_{i'}$, we have that $E(\omega, i, i')$ has measure 0. Also, we have that $E(j+1, i, i') \subset E(j, i, i')$ for each j. Therefore, there is some j such that $E(j, i, i')$ has measure less than η/M^2. It follows that there is an integer $j > j_0$ such that $E(j, i, i')$ has measure less than η/M^2 for each $i < i' < M$. Let $\mathcal{D} = \mathcal{D}_j$.

Set $X_2 = X_1 \setminus \bigcup_{i<i'<M} E(j, i, i')$ and again we have that $X_1 \setminus X_2$ has measure less than η. Note that for $i < j < M$, $\langle[\dot{x}_i \in D]\rangle \cap \langle[\dot{x}_j \in D]\rangle \cap X_2$ has measure 0 for all $D \in \mathcal{D}_j$. Set $X_2' = X_2 \cap X_1'$ and we still have that for each $x \in X_2'$ and each $i < L$, there is a $k \in [m_i, m_{i+1})$ and a $D \in \mathcal{D}_j$ such that $x \in \langle[\dot{x}_k \in D]\rangle$. Therefore, it follows that for each $x \in X_2'$, $\mathcal{D}_x = \{D \in \mathcal{D} : (\exists i < M)\ x \in \langle[\dot{x}_i \in D]\rangle\}$ has cardinality at least L and that $\mathcal{D}_x \cap \mathcal{D}_F = \emptyset$.

Let J be an integer so large that $9/J < \mu(\langle[\dot{x} \in D]\rangle \cap X_0)$ for each $D \in \mathcal{D}$. Consider any $D \in \mathcal{D} \setminus \mathcal{D}_F$ and $y \in D$. Since $y \notin F$, $\langle[\dot{x} = y]\rangle \cap X_0$ has measure 0, hence there is a clopen set $a_y \subset D$ such that $\langle[\dot{x} \in a_y]\rangle \cap X_0$ has measure less than $1/(8J)$. Therefore there is a finite partition of D by clopen sets E each with the property that $\langle[\dot{x} \in E]\rangle \cap X_0$ has measure less than $1/(8J)$. By taking unions from this partition we find a second parition so that for each member E of this partition the value of $\langle[\dot{x} \in E]\rangle \cap X_0$ either has measure between $1/J$ and $9/(8J)$ or still has measure less than $1/(8J)$. There are at most 8 of the latter sort and at least 8 of the first. It now follows that there is a partition $\mathcal{E}_D$ of D by sets such that each $E \in \mathcal{E}_D$ satisfies that the measure of $\langle[\dot{x} \in E]\rangle \cap X_0$ is greater than $1/J$ and less than $5/(4J)$. By refining $\mathcal{D} \setminus \mathcal{D}_F$, we may assume that for $D \in \mathcal{D} \setminus \mathcal{D}_F$,

$\langle[\dot{x} \in D]\rangle \cap X_0$ has measure greater than $1/J$ and less than $5/(4J)$. It also follows that $|\mathcal{D} \setminus \mathcal{D}_F| < J$.

For each $D \in \mathcal{D}$, let $\nu(D) = \mu(\langle[\dot{x} \in D]\rangle \cap X_0)$. Then we consider the product measure $(\mu \times \nu)$ on $X_2' \times Y_1$.

Define the set $S \subset X_2' \times Y_1$ by

$$S = \bigcup_{D \in \mathcal{D} \setminus \mathcal{D}_F} \bigcup_{i<M} (X_2' \cap \langle[\dot{x}_i \in D]\rangle) \times D$$

and for each $x \in X_2'$ and $D \in \mathcal{D} \setminus \mathcal{D}_F$, let

$$S_x = \bigcup \mathcal{D}_x = \bigcup \{D \in \mathcal{D} : (\exists i < M)\ x \in \langle[\dot{x}_i \in D]\rangle\}$$

$$\text{and} \quad S^D = \bigcup_{i<M} X_2' \cap \langle[\dot{x}_i \in D]\rangle \ .$$

Since $|\mathcal{D}_x| \geq L$, we have that $\nu(S_x) > (L/J)$. Therefore $(\mu \times \nu)(S) > (L/J)\mu(X_2')$.

Also,

$(\mu \times \nu)(S) = \sum_{D \in \mathcal{D} \setminus \mathcal{D}_F} \nu(D)\mu(S^D) < J \cdot (5/(4J)) \cdot \max_{D \in \mathcal{D}}(\mu(S^D))$.

Since $(L/J)\mu(X_2') < J \cdot (5/(4J)) \cdot \max(\mu(S^D))$, we have $(4/5)(L/J)\mu(X_2') < \max_{D \in \mathcal{D}}(\mu(S^D))$. Thus there is a $D_0 \in \mathcal{D} \setminus \mathcal{D}_F$ such that $\mu(S^{D_0}) > (4L/5J)\mu(X_2')$.

Now set $X_3 = X_2' \setminus (\langle[\dot{x} \in D_0]\rangle \cup S^{D_0})$ and $Y_3 = Y_1 \setminus D_0$. Set $S_3 = S \cap (X_3 \times Y_3)$.

Notice that for $x \in X_3$, $D_0 \notin \mathcal{D}_x$, hence we still have that $\nu((S_3)_x) > L/J$. If $\mu(X_3) \leq 1/\sqrt{L}$ we stop, otherwise, arguing exactly as above, there is a $D_1 \in \mathcal{D} \setminus (\{D_0\} \cup \mathcal{D}_F)$, such that $\mu(S^{D_1} \cap X_3) > (4L/5J)\mu(X_3)$.

Inductively we keep choosing such $D_m \in \mathcal{D} \setminus (\mathcal{D}_F \cup \{D_0, \ldots, D_{m-1}\})$, such that $\mu(S^{D_m} \cap X_{2+m}) > (4/5) \cdot (L/J) \cdot \mu(X_{2+m})$ and set $X_{2+m+1} = X_{2+m} \setminus (\langle[\dot{x} \in D_m]\rangle \cup S^{D_m})$. Let ℓ be the first value $m+1$ such that $\mu(X_{2+m+1}) \leq 1/\sqrt{L}$. Now we consider the family of disjoint sets $\{S^{D_m} \cap X_{2+m} : m < \ell\}$ and the fact that $X_2' \setminus X_{2+\ell}$ is covered by this union together with the union of the family $\{\langle[\dot{x} \in D_m]\rangle : m < \ell\}$. We set $U_0 = D_0 \cup \cdots \cup D_{\ell-1}$ and $T = \bigcup\{S^{D_m} \cap X_{2+m} : m < \ell\}$. The relevance of T is that $X_2' \cap \bigcup_{i<\omega} \langle[\dot{x}_i \in U_0]\rangle$ contains T.

For each $m < \ell$, $\mu(S^{D_m} \cap X_{2+m})$ is bigger than $(4/5) \cdot (L/J)\mu(X_{2+m}) > (4/5) \cdot (L/J) \cdot (1/\sqrt{L})$. Therefore, the measure of the union of the family $\{S^{D_m} \cap X_{2+m} : m \leq \ell\}$ is bigger than $\ell \cdot (4L/5J) \cdot (1/\sqrt{L})$. The union of the family $\langle[\dot{x} \in D_m]\rangle \cap X_0$ $(m < \ell)$ has some measure p which is less than $(5/4) \cdot \ell \cdot (1/J)$. Therefore the measure of T is more than $(16/25) \cdot \sqrt{L} \cdot p$.

Now, $1 > (16/25)\sqrt{L} \cdot p$ and $(25/16) \cdot (1/\sqrt{L}) > p$. Also, since $X_2' \setminus X_\ell$ is covered by these sets, we have $\mu(T) + p > \mu(X_2') - 1/\sqrt{L}$; or $\mu(T) > \mu(X_2') - \eta/4 - p >$

$\mu(X_2') - \eta/4 - (25/16) \cdot (1/\sqrt{L})$ which in turn is greater than $\mu(X_2') - \eta/4 - (25/16) \cdot (\eta/4) > \mu(X_2') - (3/4)\eta$. It follows that $\mu(T)$ is greater than $\mu(X_2') - \epsilon/2$ as required.

Finally we compute $\mu(\langle[\dot{x} \in U_0]\rangle \cap X_0)$, it equals $\mu(\langle[\dot{x} \in \bigcup_{m<\ell} D_m]\rangle) = p < (25/16) \cdot (1/\sqrt{L}) < (25/16)(\eta/4) < \epsilon$. □

4. An example that acquires converging sequences

Proposition 4.1. *There is an example of a compact Hausdorff space K which has no non-trivial converging sequences such that $\tilde{K}$ does have non-trivial converging sequences after adding random reals.*

Proof. Recall that $\mathbb{M} \cup \{0\}$ (with measure μ) is the random real forcing algebra obtained from the measure space $X = 2^\kappa$ for some infinite cardinal κ. Let Z denote the Stone space of $\mathbb{M} \cup \{0\}$. We will define a compactification K of $Z \times \mathbb{N}$. K will be the Stone space of a Boolean subalgebra B of the natural algebra $\mathcal{N} = (\mathbb{M} \cup \{0\})^{\mathbb{N}}$ where

$$B = \{\langle b_n \rangle_{n \in \mathbb{N}} : \min\left(\Sigma_{n \in \mathbb{N}}\ \mu(b_n), \Sigma_{n \in \mathbb{N}}\ \mu(X \setminus b_n)\right) \quad \text{is finite } \}.$$

Now

$$\{\langle b_n \rangle_{n \in \mathbb{N}} : \textstyle\sum_{n \in \mathbb{N}} \mu(X \setminus b_n) \text{ is finite}\}$$

is an ultrafilter in B with limit z in K.

It is fairly routine to prove that K contains no non-trivial converging sequences (much like proving that $S(\mathbb{M} \cup \{0\})$ contains none).

Let ν denote the usual product measure on $\mathcal{N}$. We will now force with the random real poset obtained from $\mathcal{N}$ with the measure ν. Let G be a generic filter and for each n, let $g_n \in \tilde{Z} \times \{n\}$ be determined by the (ultra)filter $\{b_n : \langle 1, 1, \ldots, 1, b_n, 1, 1, \ldots \rangle \in G\}$. We show that $\{g_n : n \in \mathbb{N}\}$ converges to z. Assume that $\langle b_n \rangle_{n \in \mathbb{N}} \in B$ is such that $z \notin \tilde{b}$ and suppose that a is in the forcing poset. We show that there is an extension (subset) of a which forces that $\tilde{b} \cap \{g_n : n \in \mathbb{N}\}$ is finite. Let $0 < \epsilon$ be less than $\nu(\langle a \rangle)$. Let, for $n \in \mathbb{N}$, $b^n = \langle 1, 1, \ldots, 1, b_n, 1, \ldots \rangle$ in the measure algebra $\mathcal{N}$. Observe that $\nu(b^n) = \mu(\langle b_n \rangle)$. There is an integer M such that $\sum_{n>M} \mu(\langle b_n \rangle)$ is less than ϵ, hence $\nu(\bigvee_{n>M} b^n) < \epsilon$. This proves that $\nu(a \setminus \bigvee_{n>M} b^n) > 0$. The condition $a' = a \setminus \bigvee_{n>M} b^n$ forces that $g_n \notin \tilde{b}$ for each $n > M$. □

References

BG05. Zoltan Balogh and Gary Gruenhage, **Two more perfectly normal non-metrizable manifolds**, Topology Appl. **151** (2005), no. 1-3, 260–272. MR MR2139756

Burke N96. M.R. Burke, **A theorem of Gitik and Shelah on disoint refinements of sequences of subsets of the real line**, Note of 23.8.96.

Dow92. Alan Dow, **Set theory in topology**, Recent progress in general topology (Prague, 1991), North-Holland, Amsterdam, 1992, pp. 167–197. MR MR1229125

Dow 04. A. Dow, **Efimov spaces and the splitting number**, Topology Proceedings, to appear.

Fedorchuk 77. V.V. Fedorchuk, **A compact space having the cardinality of the continuum with no converging sequences**, Math. Proc. Cambridge Phil. Soc. **81** (1977), 177-181

Fre84. D. H. Fremlin, **Consequences of Martin's axiom**, Cambridge Tracts in Mathematics, vol. 84, Cambridge University Press, Cambridge, 1984. MR **MR780933 (86i:**03001)

Fremlin 93. D.H. Fremlin, **Real-valued-measurable cardinals** pp. 151-304 in Judah 93.

Fremlin N97. D.H. Fremlin, **A theorem of Gitik and Shelah on disjoint refinements of sequences of subsets of the real line**, Note of 4.4.97.

Just-Koszmider. W. Just and P. Koszmider, **Remarks on Cofinalities and Homomorphism Types of Boolean Algebras**, Algebra Universalis 1991, Vol. 28, 138-149.

Juhah 93. H. Judah, **Proceedings of the Bar-Ilan Conference on Set Theory of the Reals 1991**, Amer. Math. Soc. (Israel Mathematics Conference Proceedings 6), 1993.

Koszmider 90. Piotr Koszmider, **The Consistency of the negation of CH and the pseudoaltitude less or equal to omega one**, Algebra Universalis 1990, Vol. 27, 80-87.

Talagrand 80. M. Talagrand, **Un nouveau $\mathcal{C}(K)$ qui possède la propriété de Grothendieck**, Israel J. Math. **37** (1980) 181-191.

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