HEREDITARILY NORMAL MANIFOLDS OF DIMENSION GREATER THAN ONE MAY ALL BE METRIZABLE

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ABSTRACT. P. J. Nyikos has asked whether it is consistent that every hereditarily normal manifold of dimension greater than one is metrizable, and proved it is if one assumes the consistency of a supercompact cardinal, and, in addition, that the manifolds are hereditarily collectionwise Hausdorff. We are able to omit these extra assumptions.

1. NYIKOS' MANIFOLD PROBLEM

For us, a *manifold* is a locally Euclidean topological space in which every non-empty open set has the same dimension. Lindelöf subsets of a manifold are separable and metrizable ([17, 2.6]). Mary Ellen Rudin proved that MA + $\sim$CH implies every perfectly normal manifold is metrizable [23]. Hereditary normality (T_5) is a natural weakening of perfect normality; Peter Nyikos noticed that, although the Long Line and Long Ray are hereditarily normal non-metrizable manifolds, and indeed the only 1-dimensional non-metrizable connected manifolds [18], it is difficult to find examples of dimension greater than 1 (although one can do so with $\diamondsuit$ [23] or CH [24]). He therefore raised the problem of whether it was consistent that there weren't any [16], [18]. In a series of papers [19, 20, 21, 22] he was finally able to prove this from the consistency of a supercompact cardinal, if he also assumed that the manifolds were hereditarily collectionwise Hausdorff. We will demonstrate that neither of these extra assumptions is necessary:

[1]Research supported by NSF grant DMS-1501506.
[2]Research supported by NSERC grant A-7354.

Date: July 24, 2018.
2000 Math. Subj. Class. Primary 54A35, 54D15, 54D45, 54E35, 03E05, 03E35, 03E65; Secondary 54D20, 03E55.

Key words and phrases: hereditarily normal, manifold, metrizable, coherent Souslin tree, proper forcing, PFA(S)[S], locally compact, P-ideal, perfect pre-image of ω_1, sequentially compact.

Theorem 1.1. *It is consistent relative to ZFC that every hereditarily normal manifold of dimension greater than* 1 *is metrizable.*

The proof consists of two main steps. The first step is to isolate a few consequences of the recently introduced forcing axiom PFA(S)[S] and to prove, in Theorem 1.4 below, that the main result is a consequence of the conjunction of these. Given the high level of interest in the ongoing exploration of PFA(S)[S], it is of independent interest to note, Theorem 1.2 below, that the main result is, in fact, a consequence of PFA(S)[S]. The second step of the proof of Theorem 1.1 is to prove that the large cardinal aspect of PFA(S)[S] is not needed to establish the consistency of the hypothesis of Theorem 1.4.

For a coherent Souslin tree S (see §2), PFA(S) is the statement [33, §4]: If $\mathcal{P}$ is a proper poset that preserves S and if $\mathcal{D}_\alpha(\alpha < \omega_1)$ is a sequence of dense open subsets of $\mathcal{P}$ there is a filter $\mathcal{G} \subset \mathcal{P}$ such that $\mathcal{G} \cap \mathcal{D}_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. The notation PFA(S)[S] is adopted in [13] to abbreviate that the universe is a forcing extension by S of a model in which S was a coherent Souslin tree and in which PFA(S) held. The analysis of forcing axioms relative to a coherent Souslin tree, and resulting model after forcing with this tree, emerged from a series of papers including [15, 8, 26, 14, 12, 33]. This last reference is the first of these to specifically investigate consequences of PFA(S) and to incorporate the method of using countable elementary submodels as side conditions to build proper posets that preserve S.

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Theorem 1.2. *It is a consequence of PFA(S)[S] that every hereditarily normal manifold of dimension greater than 1 is metrizable.*

We will isolate some known (quotable) consequences of PFA(S)[S]. The bounding number $\mathfrak{b}$ is the minimum cardinal such that there is a subset of ω^ω with that cardinality that is unbounded in the mod finite ordering; our first consequence of PFA(S)[S] is that $\mathfrak{b}$ is greater than $\aleph_1$ [12]. Another is that the value of the continuum $\mathfrak{c}$ is $\aleph_2$ [33, 4.4]. The next is the important P-ideal dichotomy. For a set X and cardinal κ, $[X]^\kappa$ and $[X]^{\leq\kappa}$ denote the sets $\{Y \subset X : |Y| = \kappa\}$ and $\{Y \subset X : |Y| \leq \kappa\}$, respectively.

Definition 1.3. *An ideal on a set X is a collection of subsets of X that includes all the finite subsets of X and is closed under subsets and finite unions. An ideal on X is a proper ideal if the set X is not an element. If $\mathcal{J}$ is an ideal on a set X, then $\mathcal{J}^\perp_X$ denotes the ideal of subsets of X that have finite intersection with every member of $\mathcal{J}$.*

An ideal $\mathcal{I}$ of countable subsets of a set X is a **P-ideal** *if whenever $\{I_n \colon n \in \omega\} \subseteq \mathcal{I}$, there is a $J \in \mathcal{I}$ such that $I_n - J$ is finite for all n.*

> **PID** ***is the statement:*** *For every P-ideal $\mathcal{I}$ of countable subsets of some uncountable set A either*
> (i) *there is an uncountable $B \subset A$ such that $[B]^{\aleph_0} \subset \mathcal{I}$, or else*
> (ii) *the set A can be decomposed into countably many sets, $\{B_n : n \in \omega\}$, such that $[B_n]^{\aleph_0} \cap \mathcal{I} = \emptyset$ for each $n \in \omega$.*

The consistency of **PID** does have large cardinal strength but for P-ideals on ω_1 it does not – see the discussion at the bottom of page 6 in [33]. A statement similar to the **PID** for ideals on ω_1 is the one we need; it also does not have large cardinal strength and is weaker than the ω_1 version of the statement in [33, 6.2]. The statement $\mathbf{P}_{22}$ was introduced in [6]. For completeness, and to introduce the ideas we will need for another consequence of PFA(S)[S], we include a proof in §2 that it is a consequence of PFA(S)[S].

> $\mathbf{P_{22}}$ **is the statement:** Suppose $\mathcal{I}$ is a P-ideal on a stationary subset B of ω_1. Then either
> (i) there is a stationary $E \subseteq B$ such that every countable subset of E is in $\mathcal{I}$,
> or (ii) there is a stationary $D \subseteq B$ such that $[D]^{\aleph_0} \cap \mathcal{I}$ is empty.

The ω_1 version of Theorem 6.2 in [33] replaces requirement (i) in P_{22} with the requirement that E should be $B \cap C$ for some closed and unbounded subset C of ω_1.

A space X is said to be $\aleph_1$-collectionwise Hausdorff if the points of any closed discrete subset of cardinality at most $\aleph_1$ can be surrounded by pairwise disjoint open sets (separated). If a separable space is hereditarily $\aleph_1$-collectionwise Hausdorff, then it can have no uncountable discrete subsets (i.e. it has countable spread).

The next consequence of PFA(S)[S] is:

> **CW*:*** *Normal, first countable spaces are $\aleph_1$-collectionwise Hausdorff.*

CW was first shown to be consistent in [28]; it was derived from $V = L$ in [10], and was shown to be a consequence of PFA(S)[S] in [13]. In fact, it is shown in [13] that simply forcing with any Souslin tree will produce a model of **CW**. Let us note now that **CW** implies that any hereditarily normal manifold is hereditarily $\aleph_1$-collectionwise Hausdorff. Therefore **CW** implies that each separable hereditarily normal manifold has countable spread.

Our next axiom is our crucial new additional consequence of PFA(S)[S]:

> **PPI$^+$**: *every regular first countable countably compact non-compact space includes a copy of the ordinal space ω_1.*

A space is countably compact if every countable open cover has a finite subcover. A first countable countably compact space is sequentially compact; a space is sequentially compact if every infinite sequence has a convergent subsequence. An important component of our proof will be to also show that it is a consequence of PFA(S)[S] that every non-compact sequentially compact space includes an uncountable free sequence. A free sequence in a space X is a well-ordered sequence of points with the property that each initial segment has closure disjoint from the closure of its complementary final segment. We will say that a point x is a complete accumulation point of a set A if for every neighborhood U of x, $U \cap A$ has cardinality equal to the cardinality of A.

Let **GA** denote the conjunction of hypotheses: $\mathfrak{b} > \aleph_1$, **CW**, **PPI$^+$** and $\mathbf{P}_{22}$. We show that **GA** is a consequence of PFA(S)[S], and also establish the following desired theorem. We show in §4 that **GA** is consistent relative to ZFC.

theorem1.1 **Theorem 1.4. GA** *implies that all hereditarily normal manifolds of dimension greater than one are metrizable.*

We acknowledge some other historical connections.

The statement **PPI$^+$** is a strengthening of

> **PPI**: *Every first countable perfect pre-image of ω_1 includes a copy of ω_1.*

PPI was proved from PFA by Fremlin [11], see also e.g. [5]. Another consequence of PFA(S)[S] relevant to this proof is

> $\sum^-$: *In a compact T_2, countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.*

$\sum^-$ was proved from MA$+\sim$CH by Balogh [2], extending work of [27]. $\sum^-$ was shown to be a consequence of PFA(S)[S] in [9]. The close connection of $\sum^-$ to the work in this paper is apparent in [29]. The following proposition, in one form or another, is well-known. It is relevant to our work because we can use $\mathfrak{b} > \aleph_1$ in connection with $\mathbf{P}_{22}$.

isP **Proposition 1.5.** *If $\mathcal{J}$ is an ideal of subsets of a set X such that $\mathcal{J}$ is generated by fewer than $\mathfrak{b}$ many sets, then the ideal $\mathcal{I} = [X]^{\leq\aleph_0} \cap \mathcal{J}^{\perp}_X$ is a P-ideal.*

If $B \subset X$ satisfies that $[B]^{\aleph_0} \cap \mathcal{I}$ is empty, then for each sequence $\{B_n : n \in \omega\}$ of infinite subsets of B, there is a $J \in \mathcal{J}$ such that $\{n \in \omega : J \cap B_n$ is infinite $\}$ is infinite.

Proof. We assume that $X \notin \mathcal{J}$ since otherwise $\mathcal{I}$ is the P-ideal of finite subsets of X. Let $\kappa < \mathfrak{b}$ and let $\{J_\alpha : \alpha < \kappa\}$ be a subset of $\mathcal{J}$. Suppose that each $J \in \mathcal{J}$ is included in J_α for some $\alpha < \kappa$. Now let $\{I_n : n \in \omega\}$ be any subset of $\mathcal{I}$. For each n, fix an enumerating function e_n from ω onto I_n. For each $\alpha \in \kappa$, there is a function $f_\alpha \in \omega^\omega$ so that, for each $n \in \omega$ and each $m > f_\alpha(n)$, $e_n(m) \notin J_\alpha$. Using $\mathfrak{b} > \kappa$, there is an $f \in \omega^\omega$ such that $f_\alpha <^* f$ for each $\alpha \in \kappa$. For each n, let $F_n = \{e_n(m) : m \leq f(n)\}$. It follows that $I = \bigcup\{I_n \setminus F_n : n \in \omega\}$ meets each J_α in a finite set. Thus $I \in \mathcal{I}$ and mod finite includes I_n for each n.

Now suppose that $[B]^{\aleph_0} \cap \mathcal{I}$ is empty and let $\{B_n : n \in \omega\}$ be infinite subsets of B. For each n, $B_n \notin \mathcal{I}$ so we can choose an $\alpha_n < \kappa$ so that $B_n \cap J_{\alpha_n}$ is infinite. For each n, fix an enumerating function e_n from ω onto $B_n \cap J_{\alpha_n}$. Assume that for each $J \in \mathcal{J}$, $\{n \in \omega : |J \cap B_n| = \aleph_0\}$ is finite. Then, for each $\alpha \in \kappa$, there is a function $g_\alpha \in \omega^\omega$ such that, for all but finitely n, $J_\alpha \cap (B_n \cap J_{\alpha_n}) \subset \{e_n(k) : k < g_\alpha(n)\}$. Let $g \in \omega^\omega$ be chosen so that $g_\alpha <^* g$ for all $\alpha \in \kappa$. Set $I = \{e_n(g(n)) : n \in \omega\}$. For each $\alpha \in \kappa$, $I \cap J_\alpha$ is finite, hence $I \in \mathcal{I}$. The map sending n to $e_n(g(n))$ is finite-to-one, hence I is an infinite subset of B, and this contradicts that $[B]^{\aleph_0}$ is disjoint from $\mathcal{I}$. □

We will need the following consequence of **GA** which is a weaker statement than $\sum^-$. The key fact that PFA(S)[S] implies compact, separable, hereditarily normal spaces are hereditarily Lindelöf was first proven in [33, 10.6].

Lemma 1.6. GA *implies that if X is a hereditarily normal manifold then separable subsets of X are Lindelöf.* metric

Proof. Let Y be any separable subset of X and assume that Y is not Lindelöf. Recursively choose, for $\alpha \in \omega_1$, points y_α, together with open sets U_α, so that $y_\alpha \in Y \setminus \bigcup_{\beta<\alpha} U_\beta$, $y_\alpha \in U_\alpha$, and $\overline{U_\alpha}$ is separable and compact. We work with ideals on the set $\tilde{Y} = \{y_\alpha : \alpha \in \omega_1\}$. The collection $\{U_\alpha \cap \tilde{Y} : \alpha \in \omega_1\}$ generates an ideal $\mathcal{J}$ on $\tilde{Y}$. Let $\mathcal{I}$ denote the ideal $\mathcal{J}^\perp_{\tilde{Y}}$. By Proposition 1.5, $\mathcal{I}$ is a P-ideal

If B is any subset of $\tilde{Y}$ such that $[B]^{\aleph_0} \subset \mathcal{I}$, then B is discrete since, for each $y_\beta \in B$, $B \cap U_\beta$ is finite. However, since X is first countable and hereditarily normal, it follows from **CW** that separable subsets of X do not include uncountable discrete subsets (as per the discussion following the introduction of **CW**). Therefore, by $\mathbf{P}_{22}$, we must then have that there is an uncountable $B \subset \tilde{Y}$ satisfying that $[B]^{\aleph_0} \cap \mathcal{I}$ is empty. Now let A be the closure (in X) of B. We check that A is sequentially compact. Let $\{x_n : n \in \omega\}$ be any infinite subset of A; we show that there is an $\alpha \in \omega_1$ such that $\{x_n : n \in \omega\} \cap \overline{U_\alpha}$ is infinite. Since $\overline{U_\alpha}$ is first countable and compact, this will show that A is sequentially compact. If $\{x_n : n \in \omega\} \cap B = \{y_\beta : \beta \in b\}$ is infinite, then $\{y_\beta : \beta \in b\} \notin \mathcal{I}$, and so there is an $\alpha \in \omega_1$ such that $\{y_\beta : \beta \in b\} \cap U_\alpha$ is infinite. Of course this means that $\{x_n : n \in \omega\} \cap \overline{U_\alpha}$ is also infinite. Otherwise we may suppose that $\{x_n : n \in \omega\}$ is disjoint from B. This means that each x_n is a limit point of B and so we may choose, for each n, an infinite $B_n \subset B$ such that B_n converges to x_n. By the second assertion in Proposition 1.5, there is an $\alpha \in \omega_1$ such that $U_\alpha \cap B_n$ is infinite for infinitely many n. Since $x_n \in \overline{U_\alpha}$ for each n such that $U_\alpha \cap B_n$ is infinite, it follows that $\{x_n : n \in \omega\} \cap \overline{U_\alpha}$ is infinite as required.

To finish the proof, we apply $\mathbf{PPI}^+$ to conclude that either A is compact or it includes a copy of ω_1. Since ω_1 includes uncountable discrete sets and Y, by assumption, is separable, we must have that A is compact. However, the final contradiction is that A has the non-Lindelöf subset B and so A cannot be covered by finitely many Euclidean open subsets of X. □

Given an integer n, a manifold of dimension n is often called an n-manifold. An alternative definition of *manifold* is that the space be locally Euclidean. Since each component of a locally Euclidean space is an n-manifold, for some integer n, and such a space is metrizable if each of its components is metrizable, we prefer to avoid this unnecessary generality. For an n-manifold X, let $\mathcal{B}_X$ denote the collection of compact subsets of X that are homeomorphic to the closed Euclidean n-ball $\mathbb{B}^n$. Brouwer's Invariance of Domain theorem states that each member of $\mathcal{B}_X$ has dense interior in X. We will use the notation $\mathrm{int}(B)$ to denote the interior of any set $B \subset X$.

The literature on non-metrizable manifolds has identified two main types of non-Lindelöf manifolds, literally called Type I and Type II. A manifold is Type II if it is separable and non-Lindelöf. Lemma 1.6 shows that there are no hereditarily normal Type II manifolds if **GA** holds. A manifold is said to be Type I, e.g. the Long Line, if it can be written as an increasing ω_1-chain, $\{Y_\alpha : \alpha \in \omega_1\}$, where each Y_α is Lindelöf, open, and includes the closure of each Y_β with $\beta < \alpha$. In this next definition, we use the set-theoretic notion of countable elementary submodels to help make a more strategic choice of a representation of our Type I manifolds. For a cardinal θ, the notation $H(\theta)$ denotes the standard set-theoretic notion of the set of all sets that are hereditarily of cardinality less than θ. These are commonly used as stand-ins for the entire set-theoretic universe to avoid issues with Gödel's famous incompleteness theorems in arguments and constructions using elementary submodels. We refer the reader to any advanced book on set theory for information about the properties of $H(\theta)$. The reader unfamiliar with elementary submodels may find [4] useful.

Definition 1.7. *Suppose that X is a non-metrizable n-manifold. Let θ be a regular uncountable cardinal with $\{X, \mathcal{B}_X\} \in H(\theta)$.*

1. *If M is an elementary submodel of $H(\theta)$ and $\{X, \mathcal{B}_x\} \in M$, then $X(M)$ will denote the union of the collection $\mathcal{B}_X \cap M$.*
2. *A family $\{M_\alpha : \alpha \in \omega_1\}$ is an elementary chain for X if there is a regular cardinal θ with $X \in H(\theta)$ so that for each $\alpha \in \omega_1$, M_α is a countable elementary submodel of $H(\theta)$ such that X and each M_β ($\beta < \alpha$) are members of M_α. The chain is said to be a continuous chain if for each limit $\alpha \in \omega_1$, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$.*

Here is the main reason for our preference to use elementary submodels in this proof. Again the main ideas are from [22], but the proof using elementary submodels is much simpler.

Throughout the paper the term *component* refers to a maximal nonempty connected set. The *quasicomponent* of a point is equal to the intersection of all clopen sets containing the point. In a compact space, components and quasicomponents coincide ([7, 6.1.25]).

Lemma 1.8. *Suppose that X is a non-metrizable hereditarily normal n-manifold with $n > 1$ and let θ be a regular uncountable cardinal. Suppose that M is a countable elementary submodel of $H(\theta)$ and that* elementary *$\{X, \mathcal{B}_X\}$ is in M. Then $X(M)$ is an open Lindelöf subspace of X with the property that its boundary, $\partial X(M)$, is not empty and is covered by non-trivial connected compact subsets.*

Proof. Since X is an n-manifold the family $\mathcal{B}_X$ has the property that whenever O is open in X and $x \in O$ there is a $B \in \mathcal{B}_X$ such that x is in the interior of B and $B \subseteq O$. Since $n > 1$, $\mathrm{int}(B) \setminus \{x\}$ is connected for all $B \in \mathcal{B}_X$ and $x \in X$. Similarly, if U is a connected open subset of X, then $U \setminus \{x\}$ is connected for any $x \in X$. The boundary, ∂B, of any $B \in \mathcal{B}_X$ is itself an $n-1$-manifold and is therefore path-connected.

Let Y denote the set $X(M)$. Since M is countable, Y is equal to a countable union of compact subsets of X. Since Y is metrizable, and X is not, Y is a proper subset of X. Each member of $\mathcal{B}_X \cap M$ is separable and hence $B \cap M$ is dense in B whenever $B \in \mathcal{B}_X \cap M$; it follows that $Y \cap M$ is a dense subset of Y.

We also note that Y is open since if $B \in \mathcal{B}_X \cap M$, then B is compact and so is included in the interior of a finite union of members of $\mathcal{B}_X$. By elementarity, there is such a finite set in $\mathcal{B}_X \cap M$. Similarly we have the following fact.

componentClaim **Claim 1.** *For each finite subset $\mathcal{B}'$ of $\mathcal{B}_X \cap M$, each Lindelöf component of $X \setminus \bigcup \mathcal{B}'$ that meets M will be a subset of Y.*

This again follows by elementarity: if C is such a component and if $y \in C \cap M$, then M will witness that there is a countable collection of members of $\mathcal{B}_X$ that covers the component of y in $X \setminus \bigcup \mathcal{B}'$.

Each manifold is the free union of its components and so a non-metrizable manifold must have a non-Lindelöf component. By elementarity there is a non-Lindelöf component of X that meets M. Since this component is not included in the open set Y, it follows that the boundary $\partial Y = \partial X(M) = \overline{Y} \setminus Y$ is not empty. Now let x be any point in ∂Y. Take any $B \in \mathcal{B}_X$ with x in its interior, and let K denote the component of x in the compact set $B \cap \partial Y$. We must simply prove that $K \setminus \{x\}$ is not empty, so we can certainly assume that K does not meet the boundary of B. Therefore the quasicomponent of x in $\partial Y \cap B$ also does not meet the boundary of B. Since the boundary of B is compact there is a relatively clopen subset D of $\partial Y \cap B$ that contains x and is disjoint from the boundary of B. Since D is relatively clopen in $\partial Y \cap B$, K is a subset of D. The relative complement, C, of D in $\partial Y \cap B$ includes the union of all components of $\partial Y \cap B$ that do meet the boundary of B. Since D and $C \cup \partial B$ are disjoint compact subsets of B, we can choose a relatively open subset of B, W, with $D \subseteq W$ and $\overline{W} \cap (C \cup \partial B) = \emptyset$. Since W is an open subset of $\mathrm{int}(B)$, we have that W is an open subset of X whose closure is a subset of the interior of B and is disjoint from C. Since $D \subset W$, we also have that ∂W is disjoint from $C \cup D$.

Since $C \cup D$ includes $\partial Y \cap B$, we therefore have that ∂W is disjoint from ∂Y. From this we conclude that $\partial W \cap Y$ is closed and compact since it is equal to the closed subset, $\partial W \cap B \cap \overline{Y}$, of B. There is a finite subfamily $\mathcal{B}_1$ of $M \cap \mathcal{B}_X$ such that $\partial W \cap Y \subset \bigcup \mathcal{B}_1 \subset \mathrm{int}(B)$. Since $\bigcup \mathcal{B}_1 \subset Y$, x is an element of the open set $W \setminus \bigcup \mathcal{B}_1$. Let E be the component of x in $W \setminus \bigcup \mathcal{B}_1$. Since $Y \cap M$ is dense in Y, E meets $Y \cap M$. Since E is Lindelöf and not included in Y it follows by Claim 1, that E is not a component of $X \setminus \bigcup \mathcal{B}_1$. This means that the closure of E must meet the boundary of W. Since E is disjoint from $\bigcup \mathcal{B}_1$, $\overline{E} \cap \partial W$ is disjoint from Y, from which it follows that the closure of E

must meet $\partial W \cap Y$. Since $\partial W \cap Y$ is equal to $\partial W \cap \overline{Y}$ and is included in $\bigcup \mathcal{B}_1$, the closure of E must in fact meet $\partial W \setminus \overline{Y}$. It follows that $E \setminus \overline{Y}$ and $E \cap Y$ are not empty, and neither contain x. Since $E \setminus \{x\}$ is also connected, we may choose a point $z \neq x$ in $E \cap \partial Y$. We note that $K \cup \{z\}$ is included in the closure of $E \cap Y$ and we now show that $z \in K$.

Let $\{B_k : k \in \omega\}$ enumerate all those members of $\mathcal{B}_X \cap M$ that are included in the interior of B. Since these sets are all closed, it follows that $K \cup \{z\}$ is contained in the closure of $W \cap Y \setminus (B_0 \cup \cdots \cup B_k)$ for all $k \in \omega$. Choose any k_0 large enough so that $\mathcal{B}_1 \subset \{B_0, \ldots, B_{k_0}\}$. Note that each point of $\overline{W} \cap Y$ is contained in the interior of B_k for some $k \in \omega$. For each k, let O_k denote the component of x in $\overline{W \cap Y \setminus (B_0 \cup \cdots \cup B_k)}$. Since $x \in K$ is connected, $K \subset O_k$ for each $k \in \omega$. An intersection of a countable descending family of connected compact sets is connected ([7, 6.1.19]). Since $\overline{W \cap Y}$ is compact, the intersection of the family $\{O_k : k \in \omega\}$ is connected and is included in $B \cap \partial Y$. Therefore this intersection is equal to K. We finish by proving that $z \in O_k$ for each $k \in \omega$.

Towards a contradiction, assume that $z \notin O_k$ for some k. Since the family is descending, we may assume that $k \geq k_0$. Again using that components are equal to quasicomponents in compact spaces, we may choose a closed partition F_1, F_2 of $\overline{W \cap Y \setminus (B_0 \cup \cdots \cup B_k)}$ such that $x \in F_1$ and $z \in F_2$. Since $W \setminus (B_0 \cup \cdots \cup B_k \cup F_2)$ is a neighborhood of x, we may choose $y_1 \in Y \cap M$ in the component of x in $W \setminus (B_0 \cup \cdots \cup B_k \cup F_2)$. Similarly $W \setminus (B_0 \cup \cdots \cup B_k \cup F_1)$ is a neighborhood of z, so choose $y_2 \in Y \cap M$ that is an element of the component of z in $W \setminus (B_0 \cup \cdots \cup B_k \cup F_1)$. By Claim 1, the components of y_1 and y_2, respectively, in the subspace $X \setminus (B_0 \cup \cdots \cup B_k)$ are not Lindelöf. Therefore their respective components meet the boundary of B and, since the boundary of B is path-connected, there is a path in $X \setminus (B_0 \cup \cdots \cup B_k)$ containing y_1 and y_2. That the boundary is path-connected is the only place in the entire proof that we use the dimension being greater than 1. By elementarity, there is a path $P \in M$ included in $X \setminus (B_0 \cup \cdots \cup B_k)$ joining y_1 and y_2. Since P is compact it is covered by a finite subcollection of $\mathcal{B}_X$. Again by elementarity, P is covered by a finite subcollection of $M \cap \mathcal{B}_X$ and so P is included in $Y \setminus (B_0 \cup \cdots \cup B_k)$. Since $\partial W \cap Y$ is included in $\bigcup \mathcal{B}_1$ and $k \geq k_0$, the set $W \cap (Y \setminus (B_0 \cup \cdots \cup B_k))$ is a relatively clopen subset of $Y \setminus (B_0 \cup \cdots \cup B_k)$ and so $P \subset W \cap Y$. Of course it then follows that y_1 and y_2 are in the same component of $\overline{W \cap Y \setminus (B_0 \cup \cdots \cup B_k)}$. This is our desired contradiction since $y_1 \in F_1$, $y_2 \in F_2$, and $P \subset F_1 \cup F_2$. □

We now obtain our preferred representation of X as a Type I submanifold.

component

Corollary 1.9 (GA). *Suppose that X is a non-metrizable hereditarily normal n-manifold of dimension greater than 1. Then there is an increasing chain $\{Y_\alpha : \alpha \in \omega_1\}$ satisfying that, for each $\alpha \in \omega_1$,*

1. *Y_α is an open Lindelöf subset of X whose the boundary ∂Y_α is non-empty and included in $Y_{\alpha+1}$,*
2. *each point of ∂Y_α is contained in an infinite compact connnected subset of ∂Y_α,*
3. *if α is a limit, then $Y_\alpha = \bigcup\{Y_\beta : \beta \in \alpha\}$.*

For any such chain $\{Y_\alpha : \alpha \in \omega_1\}$, $\bigcup\{Y_\alpha : \alpha \in \omega_1\}$ is both closed and open in X and, for each closed subset C of ω_1, $\bigcup\{\partial Y_\delta : \delta \in C\}$ is closed in X.

Proof. Fix a continuous elementary chain $\{M_\alpha : \alpha \in \omega_1\}$ for X. Fix any $\alpha \in \omega_1$. By Lemma 1.8, $Y_\alpha = X(M_\alpha)$ is Lindelöf with non-empty boundary, $\partial X(M_\alpha)$, and each component in $\partial X(M_\alpha)$ is non-trivial. By Lemma 1.6, $\overline{X(M_\alpha)}$ is Lindelöf, and so by elementarity, $M_{\alpha+1} \cap \mathcal{B}_X$ is a cover of $\overline{X(M_\alpha)}$. Since X is first countable, $\bigcup\{Y_\alpha : \alpha \in \omega_1\}$ is closed because any $x \in X$ that is in the closure will be in $\overline{Y_\alpha} \subset Y_{\alpha+1}$ for some $\alpha \in \omega_1$.

Now let $\{Y_\alpha : \alpha \in \omega_1\}$ be a chain satisfying items (1)-(3) and let C be a closed subset of ω_1. To show that $\bigcup\{\partial Y_\delta : \delta \in C\}$ is closed, we show that if $\{y_n : n \in \omega\}$ is a subset that converges to a point y, then y is also in $\bigcup\{\partial Y_\delta : \delta \in C\}$. For each $n \in \omega$, choose $\delta_n \in C$ such that $y_n \in \partial Y_{\delta_n}$. If the set $\{\delta_n : n \in \omega\}$ is finite, then let δ be chosen so that $\{n \in \omega : \delta = \delta_n\}$ is infinite. Then y is an element of the closed set ∂Y_δ. Otherwise, choose $\delta \in C$ so that the set $\{\delta_n : n \in \omega \text{ and } \delta_n < \delta\}$ is cofinal in δ. Then we have that $\{y_n : n \in \omega\} \cap (Y_\delta \setminus Y_\beta)$ is infinite for all $\beta < \delta$. Since each Y_β is open, this means that $y \in \overline{Y_\delta} \setminus \bigcup\{Y_\beta : \beta < \delta\}$. By item (3), we have that $y \in \overline{Y_\delta} \setminus Y_\delta = \partial Y_\delta$. □

copyomega1

Lemma 1.10 (GA). *Suppose X is a non-metrizable hereditary normal manifold of dimension greater than 1 and that $\{Y_\alpha : \alpha \in \omega_1\}$ is an increasing chain as in Corollary 1.9. Let A be a stationary subset of ω_1. If, for each $\alpha \in A$, we have $U_\alpha \in \mathcal{B}_X$ and $y_\alpha \in \operatorname{int}(U_\alpha) \cap \partial Y_\alpha$, then there is a stationary set $A_1 \subset A$ such that the closure of $\{y_\alpha : \alpha \in A_1\}$ is included in $\bigcup\{U_\alpha : \alpha \in A\}$ and is sequentially compact.*

Proof. It is immediate from the properties in Corollary 1.9 that the set $\{y_\alpha : \alpha \in A \text{ and } y_\alpha \notin \overline{\{y_\beta : \beta \in A \cap \alpha\}}\ \}$ is a dense discrete subset of $\{y_\alpha : \alpha \in A\}$. This implies that $\overline{\{y_\alpha : \alpha \in A\}}$ is nowhere dense in X.

We first prove that if $E \subset A$ is stationary, then $D = \{y_\alpha : \alpha \in E\}$ is not discrete. If D were discrete, then by hypothesis **CW** there should be a separation $\{W_\alpha : \alpha \in E\} \subset \mathcal{B}_X$; namely that the members of $\{W_\alpha : \alpha \in E\}$ are pairwise disjoint and $y_\alpha \in \operatorname{int}(W_\alpha)$ for each $\alpha \in E$. Assume that $\{W_\alpha : \alpha \in E\} \subset \mathcal{B}_X$ and that $y_\alpha \in \operatorname{int}(W_\alpha)$ for each $\alpha \in E$. For each limit $\alpha \in E$, using item (3) of Corollary 1.9, there is a $\beta_\alpha < \alpha$ such that $W_\alpha \cap Y_{\beta_\alpha}$ is not empty. By the pressing down lemma, there is a fixed β such that $\beta = \beta_\alpha$ for uncountably many $\alpha \in E$. Since Y_β is separable, there are $\alpha, \alpha' \in E$ such that $W_\alpha \cap W_{\alpha'} \cap Y_\beta$ is not empty. This shows that D is not discrete.

Define the ideal $\mathcal{J}$ generated by the family of countable $b \subset A$ such that $\{y_\alpha : \alpha \in b\} \subset U_\beta$ for some $\beta \in A$. Then $a \in \mathcal{I} = \mathcal{J}_A^\perp$ if $a \in [A]^{\aleph_0}$ and, for all $\beta \in A$, $\{y_\alpha : \alpha \in a\} \cap U_\beta$ is finite. It follows from Proposition 1.5 that $\mathcal{I}$ is a P-ideal on A. If $E \subset A$ satisfies that $[E]^{\aleph_0} \subset \mathcal{I}$, then $D = \{y_\alpha : \alpha \in E\}$ is discrete. Therefore there is no such stationary E, and so by $\mathbf{P}_{22}$, there is a stationary subset A_1 of A such that $[A_1]^{\aleph_0} \cap \mathcal{I}$ is empty. Using Proposition 1.5, we verify that every infinite subset of $X_{A_1} = \overline{\{y_\alpha : \alpha \in A_1\}}$ meets some U_β in an infinite set. Since each U_β is compact and metrizable, this will show that X_{A_1} is sequentially compact. If b is any infinite subset of A_1, then $b \notin \mathcal{I}$ and so there is a $\beta \in A$ such that $\{x_\alpha : \alpha \in b\} \cap U_\beta$ is infinite. Now suppose that $\{x_n : n \in \omega\}$ is any infinite subset of X_{A_1} that is disjoint from $\{y_\alpha : \alpha \in A_1\}$. For each $n \in \omega$ we may choose an infinite $b_n \subset A_1$ such that $\{y_\alpha : \alpha \in b_n\}$ converges to x_n. By Proposition 1.5, there is a $\beta \in A$ such that $U_\beta \cap \{y_\alpha : \alpha \in b_n\}$ is infinite for each n in an infinite set $L \subset \omega$. Now we have our U_β meeting $\{x_n : n \in \omega\}$ in an infinite set since by the compactness of U_β we have that $x_n \in U_\beta$ for each $n \in L$.

Let U_A denote the union $\bigcup\{U_\alpha : \alpha \in A\}$. Finally we prove that $X_{A_1} \subset U_A$. Since $y_\alpha \in U_\alpha$ for each $\alpha \in A_1$, we have that $\{y_\alpha : \alpha \in A_1\} \subset U_A$. If x is any other point of X_{A_1}, then there is an infinite set $b \subset A_1$ such that $\{x_\alpha : \alpha \in b\}$ converges to x. Since there is a $\beta \in A$ such that $U_\beta \cap \{y_\alpha : \alpha \in b\}$ is infinite, we have that $x \in \overline{U_\beta} = U_\beta$. □

Now we are ready to give a proof of the main theorem. The clever topological ideas of the proof are taken from [19, p. 189]. A sketch of this proof (minus the elementary submodels) appears in [29]. The main idea of the proof is to use $\mathbf{PPI}^+$ to find copies of ω_1 and, combined with Corollary 1.9, to show that, in fact, there is a copy of the dense subset, $\omega_1 \times (\omega + 1)$, of the Tychonoff plank in the space. We then show that such a copy has a non-normal subspace.

Proof of Theorem 1.4. Let X be a non-metrizable manifold of dimension greater than 1. We assume that X is hereditarily normal but then obtain a contradiction by producing a non-normal subspace. Let $\{Y_\alpha : \alpha \in \omega_1\}$ be chosen as in Corollary 1.9. For each $\alpha \in \omega_1$, choose any point $x_\alpha \in \partial Y_\alpha$. For each $\alpha \in \omega_1$ choose $U_\alpha \in \mathbb{B}_X$ so that $x_\alpha \in \mathrm{int}(U_\alpha)$. By Lemma 1.10, let A be a stationary subset of ω_1 such that $X_A = \overline{\{x_\alpha : \alpha \in A\}}$ is sequentially compact. Then we may apply the hypothesis $\mathbf{PPI}^+$ and choose a copy W of ω_1 included in X_A. Since countable subsets of W have compact closure included in W, W is a closed subset in X. W has no non-trivial connected subsets, which also implies that the interior of W is empty.

Let $W = \{w_\xi : \xi \in \omega_1\}$ be the homeomorphic indexing of W. For each $\alpha \in \omega_1$, $\overline{Y_\alpha} = Y_\alpha \cup \partial Y_\alpha$ is Lindelöf since, by Corollary 1.9, it is included in the Lindelöf set $Y_{\alpha+1}$. Therefore, we have that, for each α, $W \cap Y_\alpha$ is countable, and its closure is included in $Y_{\alpha+1}$. It follows that there is a cub $C \subset \omega_1$ satisfying that for each $\gamma < \delta$, both in C, the set $\{w_\beta : \gamma \le \beta < \delta\}$ is included in $Y_\delta \setminus Y_\gamma$. Therefore $\{w_\gamma : \gamma \in C\}$ is another copy of ω_1 with the property that $w_\delta \in \partial Y_\delta$ for each $\delta \in C$.

For each $\gamma \in C$, let $K_\gamma \subset \partial Y_\gamma$ be the component of w_γ in K_γ. By Corollary 1.9, K_γ is a non-trivial compact connected set. Since W has no non-trivial connected subsets, we can make another selection $y_\gamma \in K_\gamma \setminus W$. Now choose, for each $\gamma \in C$, a basic set $V_\gamma \in \mathcal{B}_X$ so that y_γ is in the interior of V_γ and $V_\gamma \subset X \setminus W$. We again apply Lemma 1.10 and choose a stationary set $A_1 \subset C$ so that the closure of $\{y_\alpha : \alpha \in A_1\}$ is sequentially compact and included in $\bigcup\{V_\gamma : \gamma \in C\}$. In particular then, the closure of $\{y_\alpha : \alpha \in A_1\}$ is disjoint from W. For each $\gamma \in C$, the closure of $\{y_\alpha : \alpha \in A_1 \cap \gamma\}$ is compact because it is a sequentially compact closed subset of the Lindelöf space $\overline{Y_\gamma}$.

Since X is normal, there is a continuous function f from X into $[0,1]$ such that $f[W] = \{1\}$ and $f(y_\alpha) = 0$ for all $\alpha \in A_1$. Note that $f[K_\alpha] = [0,1]$ for each $\alpha \in A_1$. Finally we are ready to produce our non-normal subspace for our contradiction. For each $\alpha \in A_1$, choose yet another point $z_\alpha \in K_\alpha$, in such a way that the map f restricted to $\{z_\alpha : \alpha \in A_1\}$ is one-to-one. For each $\alpha \in A_1$, choose any $B_\alpha \in \mathcal{B}_X$ so that $z_\alpha \in \mathrm{int}(B_\alpha)$. By Lemma 1.10, we choose stationary $A_2 \subset A_1$ so that the closure of $\{z_\alpha : \alpha \in A_2\}$ is sequentially compact. We again have that for each $\gamma \in C$, the closure of $\{z_\alpha : \alpha \in A_2 \cap \gamma\}$ is sequentially compact and included in the Lindelöf set $\overline{Y_\gamma}$ and so the closure of each countable subset of $\{z_\alpha : \alpha \in A_2\}$ is compact. Let Z denote the closure of the set $\{z_\alpha : \alpha \in A_2\}$, and for each $r \in [0,1]$, let $Z_r = f^{-1}(r) \cap Z$. We will use the following property of these subsets of Z. Consider any open set U of X that includes $Z_r \cap \partial Y_\gamma$ for any $r \in [0,1]$ and $\gamma \in C$. Since $Z_r \cap Y_\gamma$ has compact closure, there is a $\beta < \gamma$ such that $(Z_r \cap Y_\gamma) \setminus Y_\beta$ is included in U. Therefore the pressing down lemma implies that if U is an open set that includes $Z_r \cap \partial Y_\gamma$ for a stationary set of $\gamma \in C$, there is a $\beta \in \omega_1$ such that $Z_r \setminus Y_\beta$ is included in U.

Choose any $r \in [0,1]$ such that r is a complete accumulation point of $\{f(z_\alpha) : \alpha \in A_2\}$. Choose any sequence $\{r_n : n \in \omega\} \subset [0,1] \setminus \{r\}$ converging to r so that each r_n is also a complete accumulation point of $\{f(z_\alpha) : \alpha \in A_2\}$. We now prove there is a cub $C_\omega \subset C$ such that $Z_{r_n} \cap \partial Y_\gamma$ and $Z_r \cap \partial Y_\gamma$ are not empty for each $n \in \omega$ and $\gamma \in C_\omega$. For each $\beta \in \omega_1$, there is a value $g(\beta) \in C \setminus \beta$ such that the closure of the set $\{f(z_\alpha) : \beta < \alpha \in A_2 \cap g(\beta)\}$ includes $\{r_n : n \in \omega\} \cup \{r\}$. Let C_ω denote the set of $\delta \in C \setminus \omega$ satisfying that $g(\beta) < \delta$ for all $\beta < \delta$. If $C_\omega \cap \gamma$ is cofinal in γ and $\beta < \gamma$, then then there is a $\delta \in C_\omega \cap \gamma$ such that $\beta < \delta$. Since $g(\beta) \leq \delta$, we have that $g(\beta) < \gamma$. This shows that C_ω is a closed subset of C. To see that C_ω is unbounded, fix any $\gamma_0 \in \omega_1$. By recursion on $n \in \omega$, set γ_{n+1} to be the supremum of the countable set $\{g(\beta) : \beta \leq \gamma_n\}$. The sequence $\{\gamma_n : n \in \omega\}$ is strictly increasing and it is easy to check that the supremum is in C_ω. Now we prove that C_ω is as claimed by showing that if $\delta \in C_\omega$ and s is any element of $\{r_n : n \in \omega\} \cup \{r\}$, then $Z_s \cap \partial Y_\delta$ is not empty. Choose a strictly increasing sequence $\{\beta_m : m \in \omega\}$ cofinal in δ with the property that $g(\beta_m) \leq \beta_{m+1}$. For each $m \in \omega$, s is in the closure of $\{f(z_\alpha) : \beta_m < \alpha < g(\beta_m)\}$ so we may choose α_m in the interval $(\beta_m, g(\beta_m))$ so that $|s - f(z_{\alpha_m})| < \frac{1}{m+1}$. The set $\{\delta\} \cup \{\alpha_m : m \in \omega\}$ is a closed subset of ω_1 and so, by Corollary 1.9, $\partial Y_\delta \cup \bigcup \{\partial Y_{\alpha_m} : m \in \omega\}$ is closed. Since Z is sequentially compact, the sequence $\{z_{\alpha_m} : m \in \omega\}$ has a limit point z that is in ∂Y_δ. By continuity of f, $f(z)$ is a limit of the set $\{f(z_{\alpha_m}) : m \in \omega\}$ and so $s = f(z)$. This implies that $z \in Z_s$ and proves that $Z_s \cap \partial Y_\delta$ is not empty.

Let C'_ω be the set of relative limit points of C_ω, and set $Z_r(C'_\omega) = \bigcup \{Z_r \cap \partial Y_\gamma : \gamma \in C'_\omega\}$. Since Z_r is closed in Z, it follows that $Z_r \setminus Z_r(C'_\omega)$ is a closed subset of $Z \setminus Z_r(C'_\omega)$. By Corollary 1.9, $\bigcup \{\partial Y_\gamma : \gamma \in C'_\omega\}$ is a closed set. Therefore $H = Z \cap \bigcup \{\partial Y_\gamma : \gamma \in C'_\omega\}$ is a closed subset of Z and $H \setminus Z_r(C'_\omega)$ is a closed subset of $Z \setminus Z_r(C'_\omega)$. We show that $Z_r \setminus Z_r(C'_\omega)$ and $H \setminus Z_r(C'_\omega)$ cannot be separated by disjoint open subsets of $Z \setminus Z_r(C'_\omega)$. Since $Z_r \setminus Z_r(C'_\omega)$ and $H \setminus Z_r(C'_\omega)$ are disjoint, this will complete the proof. Suppose that U is an open subset of $Z \setminus Z_r(C'_\omega)$ that includes $H \setminus Z_r(C'_\omega)$. For each $n \in \omega$, we showed in the third paragraph of the proof that there is a $\beta \in \omega_1$ such that $Z_{r_n} \setminus Y_\beta$ is included in U for each $n \in \omega$. Choose any $\gamma \in C_\omega \setminus C'_\omega$ with $\beta < \gamma$.

For each n, choose $z'_n \in Z_{r_n} \cap \partial Y_\gamma$. Since $Z \cap \partial Y_\gamma$ is compact, let z be any limit point of $\{z'_n : n \in \omega\}$. By the continuity of f, $f(z) = r$ and so $z \in Z_r \cap \partial Y_\gamma$. In other words, $z \in Z_r \setminus Z_r(C'_\omega)$, completing the proof that $H \setminus Z_r(C'_\omega)$ and $Z_r \setminus Z_r(C'_\omega)$ cannot be separated by open sets. □

2. ON $\mathbf{P}_{22}$

As usual, S is a coherent Souslin tree. For us, it will be a full branching downward closed subtree of $2^{<\omega_1} = \{0,1\}^{<\omega_1}$. Naturally it is a Souslin tree (no uncountable antichains) and has the additional property

> for each $s \in S$ and $t \in 2^{<\omega_1}$ with $\mathrm{dom}(t) = \mathrm{dom}(s)$, t is in S if and only if $\{\xi \in \mathrm{dom}(s) \colon s(\xi) \neq t(\xi)\}$ is finite.

The diamond principle $\diamondsuit$ implies the existence of coherent Souslin trees and they exist in any single Cohen real forcing extension (see [33, 3.1,3.2], [3], [31]). In a forcing argument using S as the forcing poset, we will still use $s' < s$ to mean that $s' \subset s$, and so, s is a stronger condition. We will also use the more compact notation $o(s)$ to denote the order-type of $\mathrm{dom}(s)$ for $s \in S$. For each $\alpha \in \omega_1$, $S_\alpha = \{s \in S \colon o(s) = \alpha\}$. In this section we give a proof, following [33, 6.1], that our statement $\mathbf{P}_{22}$ is a consequence of PFA(S)[S].

Here are some simple facts about forcing with a Souslin tree that we will need repeatedly. The first is that a Souslin tree is a ccc forcing notion, so each cub subset of ω_1 in the forcing extension will include a cub from the ground model ([25, III 1.8]). Also, it follows from item (4) in the next result that forcing with a Souslin tree will not add any countable subsets of the ground model.

Lemma 2.1. basiclemma *Suppose that S is a Souslin tree and $S \in M$ for some countable elementary submodel M of any $H(\theta)$ ($\theta \geq \omega_2$). If $\dot{x} \in M$ is an S-name, and $s \in S \setminus M$, then there are $a, \alpha, s' \in M$ with $s' < s$ and $\alpha \in \omega_1$ such that*

1. *$s \Vdash \dot{x} \in \theta$ if and only if $s' \Vdash \dot{x} = \alpha$*
2. *$s \Vdash \dot{x} = \emptyset$ if and only if $s' \Vdash \dot{x} = \emptyset$,*
3. *if $s \Vdash M \cap \omega_1 \in \dot{x}$, then $s' \Vdash (\dot{x} \cap \omega_1)$ is stationary,*
4. *if $s \Vdash \dot{x} \in [\omega_1]^{\aleph_0}$, then $s' \Vdash \dot{x} = a$.*

Proof. The second item is just a (useful) special case of the first. So we consider $\dot{x}$ to be any S-name and assume that $s \Vdash \dot{x} \in \theta$. The set of conditions that force a value on $\dot{x}$ is a dense and open subset of S, and since $\dot{x}$ is in M, so too is this subset of S. Since S is a ccc forcing there is a $\gamma \in M \cap \omega_1$ such that each element of S_γ forces a specific ordinal value on $\dot{x}$. Therefore, for some $\alpha \in M \cap \theta$, $s' = s \restriction \gamma$ forces that $\dot{x} = \alpha$. Conversely, if for some $\alpha \in \theta$ and $s' < s$ we have that $s' \Vdash \dot{x} = \alpha$, then s clearly forces that $\dot{x} \in \theta$.

Now assume that $s \Vdash M \cap \omega_1 \in \dot{x}$. Let us assume, for a contradiction, that s forces that $\dot{x} \cap \omega_1$ is not stationary. Let $\dot{X}$ denote an S-name of the set of ground model cub subsets of ω_1 that are disjoint from $\dot{x}$, hence $s \Vdash \dot{X}$ is not empty. So, now apply item (2) to $\dot{X}$ for some $s' < s$ in M. Since $s' \Vdash \dot{X}$ is not empty, there is, by elementarity, a cub C in M such that $s' \Vdash C \in \dot{X}$. Of course this means that $s \Vdash \dot{x} \cap C$ is empty. However $C \cap M$ is, by elementarity, unbounded in $M \cap \omega_1$, and, since C is closed, $M \cap \omega_1 \in C$, which s forces is in $\dot{x}$.

Item (4) is proven by a repeated application of item (1). □

In the proof of Theorem 1.2, the only poset that we will actually force with is the coherent Souslin tree S and we will soon need some additional standard notation for doing so. If $\dot{a}$ is an S-name and if $g \subset S$ is a generic filter, then $val_g(\dot{a})$ denotes the evaluation of $\dot{a}$ in the generic extension. If x is any set (in the ground model), the canonical name for x is denoted $\check{x}$. When there is no risk of confusion we routinely simply use x to refer to the set in the extension. By the well-known Forcing Lemma, and for a name $\dot{a}$ of a subset of some ground model $H(\kappa)$, we get by with the definition that $val_g(\dot{a}) = \{x \in H(\kappa) \colon (\exists s \in g)\ s \Vdash \check{x} \in \dot{a}\}$. We will say that a condition $s \in S$ forces a value on such a name if there is a set b such that $s \Vdash \dot{a} = \check{b}$. If M is an elementary submodel of any such $H(\kappa)$ with $S \in M$, then one uses $M[g]$ to denote the set $\{val_g(\dot{x}) : x \in M,\ \text{and } \dot{x} \text{ is an } S\text{-name}\}$. In particular, $H(\kappa)[g]$ is such an instance and is equal to the new $H(\kappa)$ as calculated in the forcing extension $V[g]$. We also have (and will use) that $M[g]$ is an elementary submodel of $H(\kappa)[g]$ (see [25, III 2.11]). Oddly enough, the generic filter itself has a canonical name, $\dot{g}$, which we must use in statements of the forcing language. With this notation, $1 \Vdash \check{M} \subset \check{M}[\dot{g}]$, and as with any $\check{x}$, we may use $M[\dot{g}]$ in a forcing statement rather than $\check{M}[\dot{g}]$.

When applying PFA(S) from the definition, we will need the following basic facts about proper forcing.

Definition 2.2. [25] *If M is a countable elementary submodel of $H(\kappa)$ for some regular κ and $\mathcal{P}$ is a poset in M, then a condition p is an M-generic condition for $\mathcal{P}$ providing that for each dense open $D \subset \mathcal{P}$ that is in M, p forces that the generic filter meets $D \cap M$.*

A poset $\mathcal{P}$ is proper if for each regular κ such that the power set of $\mathcal{P}$ is an element of $H(\kappa)$ and each countable elementary submodel M of $H(\kappa)$ with $\mathcal{P} \in M$, each member of $\mathcal{P} \cap M$ has an extension that is an M-generic condition for $\mathcal{P}$. defproper

Remark 2.3. *For a dense set $D \in M$ for a poset $\mathcal{P} \in M$, the statement* alsoMgeneric *that p forces the generic filter meets $D \cap M$ is equivalent to the statement that each extension $r \in D$ of p is compatible with some member of $D \cap M$.*

Proposition 2.4. [15] *A poset $\mathcal{P}$ is proper and preserves S if for each κ as in Definition 2.2, and each countable submodel M of $H(\kappa)$ with* miyamoto *$\mathcal{P}, S \in M$, each condition in $M \cap \mathcal{P}$ has an extension p satisfying that (s,p) is M-generic for $S \times \mathcal{P}$ for each $s \in S \setminus M$.*

Evidently the assumptions of Proposition 2.4 show that $\mathcal{P}$ is proper. For completeness we explain how to conclude that forcing with $\mathcal{P}$ preserves that S is Souslin. Consider a $\mathcal{P}$-name $\dot{A} \in M$ that is forced to be a maximal antichain of S. Let $\delta = M \cap \omega_1$; we prove that p (as in Proposition 2.4) forces that every member of S_δ is above some member of $\dot{A}$, thereby proving that $\dot{A} \cap M$ is a maximal antichain. Let $G \subset \mathcal{P}$ be any generic filter with $p \in G$. Let $g \subset S$ be any generic filter for S and choose any $s^\dagger \in S_\delta \cap g$.

Just reversing the coordinates, we define $D \subset S \times \mathcal{P}$ (again in M) where $(s,q) \in D$ if q forces that s is above some member of $\dot{A}$. Since $\dot{A}$ is forced by $\mathcal{P}$ to be a maximal antichain of S, it is a standard fact about product forcing that D is a dense subset of $S \times \mathcal{P}$ (e.g. see [25, II 1.5]). Since $(s^\dagger, p)$ forces that the generic filter for $S \times \mathcal{P}$ meets $D \cap M$, and $g \times G$ is a generic filter for $S \times \mathcal{P}$, we may choose $(s,r) \in D \cap M \cap (g \times G)$. It follows that $s < s^\dagger$ and that r forces that s, hence $s^\dagger$, is above some member of $\dot{A}$.

Proposition 2.4 from [15] actually applies to all Souslin trees. It is useful to have a simpler sufficient condition that a poset $\mathcal{P}$ is proper and S-preserving in the case that S is a coherent Souslin tree. We begin with this definition that we will need repeatedly.

Definition 2.5. *For each $s,t \in S$ with $o(s) \le o(t)$, let $s \oplus t$ denote the function $s \cup (t \restriction [o(s), o(t))\ )$. Since S is a coherent Souslin tree, $s \oplus t \in S$.*

Suppose that $\dot{A}$ is an S-name and that $s \in S$ forces that $\dot{A}$ is a subset of some ground model set B. If s' is any other member of S transfer *with $o(s') = o(s)$, we define a new name $\dot{A}^s_{s'}$ (the (s,s')-transfer) which is defined by the property that for all $b \in B$ and $t \in S$,*

$$s' \oplus t \Vdash \check{b} \in \dot{A}^s_{s'} \text{ if and only if } \quad s \oplus t \Vdash \check{b} \in \dot{A}$$

and, for all $\bar{s} \in S_{o(s)} \setminus \{s'\}$, $\bar{s} \Vdash \dot{A}^s_{s'} = \emptyset$.

We frequently use that for $\dot{A}$ and $s, s' \in S$ as in Definition 2.5, $(\dot{A}^s_{s'})^{s'}_s$ is equal to $\dot{A}^s_s$ and $s \Vdash \dot{A} = \dot{A}^s_s$.

Lemma 2.6. *If S is a coherent Souslin tree, then a poset $\mathcal{P}$ is proper and S-preserving if and only if $S \times \mathcal{P}$ is proper.* Spreserve

Proof. If $\mathcal{P}$ is proper and S-preserving, then it follows from Proposition 2.4 that $S \times \mathcal{P}$ is proper. Now assume that $S \times \mathcal{P}$ is proper. The original formulation of properness is that the stationarity of certain types of sets are preserved. Clearly if $S \times \mathcal{P}$ preserves these, then so does $\mathcal{P}$. Thus, $\mathcal{P}$ is proper. We now prove, by contradiction, that forcing with $\mathcal{P}$ preserves that S is ccc. Suppose that $\dot{A}$ is a $\mathcal{P}$-name of a maximal antichain of S and that some $p_0 \in \mathcal{P}$ forces that $\dot{A}$ is uncountable. Let M be a countable elementary submodel of $H(\kappa)$ for a κ as in Definition 2.2 with $p_0, \dot{A}, \mathcal{P}, S$ all in M. Since we are assuming that $S \times \mathcal{P}$ is proper, each condition in $M \cap (S \times \mathcal{P})$ has an extension that is M-generic for $S \times \mathcal{P}$ so let (s_1, p_1) be an M-generic condition for $S \times \mathcal{P}$ with $p_1 < p_0$ and $\delta \leq o(s_1)$. We may now choose a condition $p_2 < p_1$ so that there is an $s^* \in S \setminus M$ such that $p_2 \Vdash s^* \in \dot{A}$.

Since S is coherent, we can choose $\alpha \in \delta$ so that s_1 and s^* agree on $[\alpha, \delta)$. Set $t_1 = s_1 \restriction \alpha \in M$ and $t^* = s^* \restriction \alpha \in M$. Now define a $\mathcal{P}$-name $\dot{B}$ in M by the rule that a condition $p \in \mathcal{P}$ forces that some $s \in S$ is in $\dot{B}$ if either s is incompatible with t_1 or p forces that $t^* \oplus s$ is above some member of $\dot{A}$. Let $D \subset S \times \mathcal{P}$ be defined to be the set of all pairs (s, p) such that p forces that $s \in \dot{B}$. Since every element of $\mathcal{P}$ forces that $\dot{A}$ is a maximal antichain, one shows as in the paragraphs following Proposition 2.4 that the set $D \in M$ is a dense open subset of $S \times \mathcal{P}$. Using that (s_1, p_1) is an M-generic condition and that $(s^*, p_2) \in D$, we can, by Remark 2.3, choose $(s, p) \in D \cap M$ compatible with (s_1, p_2). Let p_3 be any condition in $\mathcal{P}$ stronger than each of p and p_2. Since s is compatible with s_1 it is also compatible with t_1. Therefore, since p forces that $s \in \dot{B}$, we have that p forces that $t^* \oplus s$ is above some member of $\dot{A}$. We have achieved our contradiction since p_3 forces that $s^* \in \dot{A}$ and that $t^* \oplus s < s^*$ is above some member of $\dot{A}$.

□

Proposition 2.7. *Assume PFA(S), then S forces that $\mathbf{P}_{22}$ holds.* P22holds

Proof. Let $\dot{B}$ be an S-name such that the root of S forces that $\dot{B}$ is a stationary subset of ω_1. Suppose that $\dot{\mathcal{I}}$ is an S-name of a P-ideal of countable subsets of $\dot{B}$. To prove $\mathbf{P}_{22}$, we may assume that some condition in S forces that $\dot{\mathcal{I}} \cap [E]^{\aleph_0} \neq \emptyset$ for all stationary subsets E of $\dot{B}$. By virtue of Definition 2.5, we can assume for convenience that the root of S is that condition. We will prove, Claim 6, that there is an S-name $\dot{E}$ and a condition $s_0 \in S$ that forces $\dot{E}$ to be a stationary subset of $\dot{B}$ that satisfies that $[\dot{E}]^{\aleph_0} \subset \dot{\mathcal{I}}$. This will complete the proof that PFA(S)[S] implies $\mathbf{P}_{22}$.

Fix any well-ordering $\prec$ of $H(\aleph_2)$.

claima **Claim 2.** *For each countable elementary submodel M of $(H(\aleph_2), \dot{\mathcal{I}}, \prec)$ and each $s \in S_{M\cap\omega_1}$, there is a set $a(s,M)$ such that*

$$s \Vdash a(s,M) \in \dot{\mathcal{I}} \textit{ and } (\forall a \in \dot{\mathcal{I}} \cap M[\dot{g}])(a \setminus a(s,M) \textit{ is finite}) .$$

Proof of Claim 2: Let s be any element of $S_{M\cap\omega_1}$. Since s forces that $\dot{\mathcal{I}}$ is a P-ideal, there is a $\prec$-minimal S-name $\dot{a}$ such that 1 forces that each member of $M[\dot{g}] \cap \dot{\mathcal{I}}$ is a subset mod finite of $\dot{a}$. Choose M' to be any countable elementary submodel of $H(\kappa)$ with $M, S, s, \dot{a} \in M'$. Since forcing with S adds no countable sets of ordinals ((4) of Lemma 2.1), we know that $\dot{a}$ is forced to be a ground model subset of ω_1. Moreover, if $\{s_n : n \in \omega\}$ is an enumeration of all the elements of level $S_{M'\cap\omega_1}$ that are extensions of s, then there is a countable family $\{a_n : n \in \omega\}$ of countable subsets of ω_1 such that, for each n, $s_n \Vdash \dot{a} = a_n$. Furthermore, again by Lemma 2.1, s forces a value on each $\dot{a} \in M$ such that $s \Vdash \dot{a} \in \dot{\mathcal{I}}$. Let $\mathcal{J}$ denote the countable family of sets forced by s to be members of $M[\dot{g}] \cap \dot{\mathcal{I}}$. Note that every member of $\mathcal{J}$ is mod finite included in every member of $\{a_n : n \in \omega\}$. We may choose $a(s,M)$ to be the $\prec$-minimal set that splits this (ω,ω)-gap, i.e. $a(s,M)$ includes mod finite each member of $\mathcal{J}$ and is included mod finite in each element of $\{a_n : n \in \omega\}$.

For each countable elementary submodel M of $H(\aleph_2, \dot{\mathcal{I}}, \prec)$, and each $s \in S \setminus M$, we let $a(s,M)$ equal the $\prec$-minimal set $a(s \restriction \delta, M)$ as in Claim 2. We will now consider the name $\dot{B}$. Let B_1 equal the set of $\delta \in \omega_1$ for which there is an $s \in S$ forcing that $\delta \in \dot{B}$. B_1 is a stationary set since the root of S forces that $\dot{B}$ is a subset of B_1.

In order to apply PFA(S), we let $\mathcal{P}$ be the collection of all functions of the form $p : \mathcal{M}_p \to S$, where

(1) $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $(H(\aleph_2), \prec)$
(2) $\{\dot{\mathcal{I}}, B_1\} \in M$ for each $M \in \mathcal{M}_p$,
(3) for each $M \in \mathcal{M}_p$, $p(M) \in S \setminus M$ and if $\delta = M \cap \omega_1 \in B_1$ then $p(M)$ forces that $\delta \in \dot{B}$,
(4) $p(N)$ is in M whenever $N, M \in \mathcal{M}_p$ with $N \in M$.

We let $p \leq q$ if,

(5) $\mathcal{M}_p \supset \mathcal{M}_q$ and $q = p \restriction \mathcal{M}_q$,
(6) if $M \in \mathcal{M}_q$ and $N \in M \cap \mathcal{M}_p \setminus \mathcal{M}_q$, then $N \cap \omega_1 \in a(q(M), M)$ providing $N \cap \omega_1 \in B_1$ and $p(N) < q(M)$.

Claim 3. *For each $\delta \in \omega_1$, D_δ is a dense subset of $\mathcal{P}$, where* cub

$$D_\delta = \{p \in \mathcal{P} \colon \delta \in \bigcup \mathcal{M}_p \ \ \textit{and } (\exists M \in \mathcal{M}_p)(\textit{either } M \cap \omega_1 = \delta$$
$$\textit{or } \ (\delta \in M \ \textit{and } (\forall q < p) \ (\bigcup (M \cap \mathcal{M}_q) = \bigcup (M \cap \mathcal{M}_p) \ )\} \ .$$

Proof of Claim 3: Let $p_1 \in \mathcal{P}$ be arbitrary. We first show that p_1 has an extension p_2 with $\delta \in \bigcup \mathcal{M}_{p_2}$. Choose any countable elementary submodel M_2 of $(H(\aleph_2), \prec)$ such that p_1, δ and $\{\dot{\mathcal{I}}, \dot{B}\}$ are in M_2. Since p_1 is finite it is a subset of M_2. Set $\mathcal{M}_{p_2}$ equal to $\mathcal{M}_{p_1} \cup \{M_2\}$ and choose $p_2(M_2) \in S \setminus M_2$ so that, if $M_2 \cap \omega_1 \in B_1$, then $p_2(M_2)$ forces $M_2 \cap \omega_1 \in \dot{B}$. Also, for each $N \in \mathcal{M}_{p_1}$, $a(p(N), N)$ is chosen as the $\prec$-least set satisfying Claim 2, and so it is in M_2. We have constructed $p_2 \in \mathcal{P}$, and $p_1 = p_2 \restriction \mathcal{M}_{p_1}$. Condition (6) in the definition of $p_2 < p_1$ is vacuous, and so we have constructed our desired extension $p_2 < p_1$ with $\delta \in \bigcup \mathcal{M}_{p_2}$.

Let δ^* be the minimum of the set $\{M \cap \omega_1 : (\exists p^* < p_2) \ M \in \mathcal{M}_{p^*}$ and $\delta \leq M \cap \omega_1\}$. Choose $p_2^* < p_2$ such that $\delta^* \in \{M \cap \omega_1 : M \in \mathcal{M}_{p_2^*}\}$. If $\delta^* = \delta$, then p_2^* is in D_δ. If p_2^* is not in D_δ, then let M be the element of $\mathcal{M}_{p_2^*}$ with $M \cap \omega_1 = \delta^* > \delta$ and note that there is a $q < p_2^*$ such that $\bigcup (M \cap \mathcal{M}_q) \neq (M \cap \mathcal{M}_{p_2^*})$. Choose such a q and let $\bar{M}$ be the maximum element of $M \cap \mathcal{M}_q$. Choose any extension $s_{\bar{M}}$ of $q(\bar{M})$ in $S \cap M$ so that $\delta \leq o(s_{\bar{M}})$. We now define an extension p of p_2^* that is in D_δ. Set $\mathcal{M}_p$ equal to $\mathcal{M}_{p_2^*} \cup \{\bar{M}\}$ and set $p = p_2^* \cup \{(\bar{M}, s_{\bar{M}})\}$. Since we already have that q is an extension of p_2^*, it is routine to check that p is also an extension of p_2. It is immediate that $p \in D_\delta$ because if $q < p$ and $\bar{M} \in M' \in \mathcal{M}_q \cap M$, then $p(\bar{M}) \in M'$, implying that $\delta \in M'$, and this contradicts the assumption on p_2.

Claim 4. *If $G \subset \mathcal{P}$ is a filter that meets D_δ for all $\beta < \delta \in \omega_1$, then $C = \{M \cap \omega_1 : (\exists p \in G) \ M \in \mathcal{M}_p\}$ is a closed and unbounded subset* isacub *of ω_1.*

Proof of Claim 4: For $\delta \in \omega_1$, the fact that G meets D_δ implies that $C \setminus \delta$ is not empty. To show that C is closed we assume $\delta \notin C$ and show that it is not a limit point of C by showing that $C \cap \delta$ has a maximum element. Choose $p \in G \cap D_\delta$ and let $M \in \mathcal{M}_p$ be as in the definition of D_δ. Since δ is not in C, $\delta \in M$. If $\mathcal{M}_p \cap M$ is empty, let $\beta = 0$, otherwise let $\bar{M}$ be the maximum element of $\mathcal{M}_p \cap M$, and let $\beta = \bar{M} \cap \omega_1$. It now follows that for all $q < p$ in G and $M' \in \mathcal{M}_q \cap M$, then $M' \cap \omega_1$ is less than or equal to β. It thus follows that C is disjoint from the interval (β, δ).

In order to apply PFA(S) to $\mathcal{P}$, we have to show that $\mathcal{P}$ is a proper poset that preserves that S is Souslin. For the moment, assume that $S \times \mathcal{P}$ is proper, and by PFA(S), choose a filter G on $\mathcal{P}$ as in Claim 4 and let C_G be the corresponding cub, also as in Claim 4. Let $\dot{E}_G$ be an S-name having the property that for each $p \in G$ and $M \in \mathcal{M}_p$ with $\delta = M \cap \omega_1 \in B_1$, the condition $p(M)$ forces that δ is in $\dot{E}_G$.

claims0 **Claim 5.** *There is an $s_0 \in S$ that forces that $\dot{E}_G$ is a stationary subset of $\dot{B}$.*

Proof of Claim 5: Item (3) in the definition of $\mathcal{P}$ ensures that every member of S forces that $\dot{E}_G$ is a subset of $\dot{B}$. Now we prove that some s_0 forces that $\dot{E}_G$ is stationary. Let D denote the set of $s \in S$ such that there is a cub $C_s \subset C_G$ such that s forces that $C_s \cap \dot{E}_G$ is empty. If D is dense there is $\gamma \in \omega_1$ such that $S_\gamma \subset D$. Let C^γ denote the cub $\bigcap\{C_s : s \in S_\gamma\}$. Since B_1 is stationary, there is a $\delta \in C^\gamma \cap B_1$ above γ. Choose $p \in G$ and $M_\delta \in \mathcal{M}_p$ so that $\delta = M_\delta \cap \omega_1$. Therefore $p(M_\delta)$ forces that $\delta \in \dot{E}_G$. But this contradicts that $\delta \in C_s$ where s is the unique element of S_γ that is below $p(M_\delta)$. Therefore D is not dense and our desired value s_0 is any element with no extension in D.

p22 **Claim 6.** *s_0 forces that $[\dot{E}]^{\aleph_0} \subset \dot{\mathcal{I}}$, where s_0 and $\dot{E}$ are chosen as in Claim 5.*

Proof of Claim 6: It suffices to show that if $s_0 < s \in S$ and $\gamma = o(s)$, then s has an extension forcing that $\dot{E} \cap \gamma \in \dot{\mathcal{I}}$. Since $s \Vdash \dot{E}$ is stationary, there is a $\delta \in C \cap B_1$ and a $p \in G$, with $\delta = M \cap \omega_1$ for some $M \in \mathcal{M}_p$, such that $s < p(M_\delta)$. Therefore, by condition (6) of the definition of $\mathcal{P}$, $p(M_\delta)$ forces that $\dot{E} \cap \delta \in \dot{\mathcal{I}}$.

We finish the proof of the Proposition by proving that $\mathcal{P}$ is proper and preserves that S is Souslin. Let M be any countable elementary submodel of $H(\kappa)$ for some regular $\kappa > \omega_2$ with $S \times \mathcal{P} \in M$. By Proposition 2.4, it will suffice to show that any pair $(s^\dagger, q)$ where $s^\dagger \in S \setminus M$ and $M \cap H(\aleph_2) \in \mathcal{M}_q$ is an M-generic condition for $S \times \mathcal{P}$.

Consider any dense open set D of $S \times \mathcal{P}$ that is a member of M. By Remark 2.3 it suffices to show that any extension of $(s^\dagger, q)$ that is in D is compatible with an element of $D \cap M$. For simplicity we rename $(s^\dagger, q)$ to be such an extension that is in D. We make two more extensions of $(s^\dagger, q)$. With the first, using that $s^\dagger$ forces that $\dot{B}$ is stationary, we can extend so as to assume that there is an $M^\dagger \in \mathcal{M}_q \setminus M$ such that $q(M^\dagger) < s^\dagger$. The second of these is to simply extend $s^\dagger$ so that there is a countable elementary submodel of $H(\kappa)$ containing q but not $s^\dagger$.

It is useful to regard D as an S-name $\dot{D}$ of a dense open subset of $\mathcal{P}$ in the sense that if $(t,p) \in D$, then we interpret this as $t \Vdash p \in \dot{D}$. It is evident from conditions (4) and (5) of the definition of $\mathcal{P}$ that $q_0 = q \restriction M$ is in M and that q is an extension of q_0. Let $\delta = M \cap \omega_1$. Let $\{M_1, \ldots, M_\ell\}$ be an increasing enumeration of $\mathcal{M}_q \setminus M$. Of course $M_1 = M \cap H(\aleph_2)$. Let $\{s_0, \ldots, s_m\}$ be any one-to-one list of the set $\{s^\dagger \restriction \delta, q(M_1) \restriction \delta, \ldots, q(M_\ell) \restriction \delta\}$ so that $s_0 = s^\dagger \restriction \delta$. For each $1 \leq j \leq \ell$, let m_j denote the value such that $s_{m_j} = q(M_j) \restriction \delta$. For each $1 \leq j \leq \ell$, $q(M_j)$ forces that $a(q(M_j), M_j)$ includes, mod finite, each member of $M[\dot{g}] \cap \dot{\mathcal{I}}$. For each $0 \leq k \leq m$, let a_k be the intersection of all $a(q(M_j), M_j)$ such that $1 \leq j \leq \ell$ and $m_j = k$, i.e. $s_k = q(M_j) \restriction \delta$.

Let J denote those $1 \leq j \leq \ell$ such that $q(M_j)(\xi) = s^\dagger(\xi)$ for all $\delta \leq \xi \in \mathrm{dom}(q(M_j))$. Note that the assumption above on $M^\dagger \in \mathcal{M}_q \setminus M$ ensures that J is not empty. Since S is a coherent Souslin tree, there is a $\bar{\delta} \in M$ such that $s_0 \restriction [\bar{\delta}, \delta) = s_i \restriction [\bar{\delta}, \delta)$ for each $i \leq m$. By increasing $\bar{\delta}$ we can also ensure that $\bar{M} \cap \omega_1 < \bar{\delta}$ for each $\bar{M} \in \mathcal{M}_{q_0}$. Let $\bar{s}_i = s_i \restriction \bar{\delta}$ for $i \leq m$, and notice that $\{\bar{s}_0, \ldots, \bar{s}_m\} \in M_1$. Note that $J = \{j : 1 \leq j \leq \ell \text{ and } \bar{s}_0 \oplus p(M_j) < s^\dagger\}$. Also, define J_B to be the set $\{j \in J : M_j \cap \omega_1 \in B_1\}$.

Say that $(t,p) \in D$ is <u>like</u> $(s^\dagger, q)$ providing

(1) there is a $M_1^p \in \mathcal{M}_p$ such that $\bar{\delta} \in M_1^p$ and $q_0 = p \restriction M_1^p$,
(2) $\mathcal{M}_p \setminus M_1^p$ has size ℓ, enumerated as $\{M_1^p, \ldots M_\ell^p\}$ in increasing order,
(3) $\bar{s}_{m_j} < p(M_j^p)$ for $1 \leq j \leq \ell$
(4) $J = \{j : 1 \leq j \leq \ell \text{ and } \bar{s}_0 \oplus p(M_j^p) < t\}$,
(5) $J_B = \{j \in J : M_j^p \cap \omega_1 \in B_1\}$.

Our proof that $S \times \mathcal{P}$ is proper will depend on finding some $(t,p) \in D \cap M$ that is <u>like</u> $(s^\dagger, q)$ and, in addition, is compatible with $(s^\dagger, q)$. Of course this requires that $t < s^\dagger$, but what else? Since $\mathcal{M}_p \in M_1$ and $p < q_0$ we automatically would have that $\mathcal{M}_p \cup \mathcal{M}_q$ is an $\in$-chain. The most difficult (and remaining) requirement is to ensure that if $p(M_j^p) < q(M_k)$ and $M_j^p \cap \omega_1 \in B_1$, then $M_j^p \cap \omega_1$ must be in $a(q(M_k), M_k)$. Since $\bar{s}_{m_j} < p(M_j^p) < q(M_k)$, it follows that $m_k = m_j$ and so a_{m_j} is a subset of $a(q(M_k), M_k)$, and we will ensure that $M_j^p \cap \omega_1$ is in a_{m_j}.

The set $\mathbf{L} \subset D$ consisting of those pairs (t,p) that are <u>like</u> $(s^\dagger, q)$ is an element of M. For each $p \in \mathcal{P}$ such that there is a $t \in S$ with $(t,p) \in \mathbf{L}$, let

$$\Delta_p = \langle M_1^p \cap \omega_1, M_2^p \cap \omega_1, \ldots, M_\ell^p \cap \omega_1 \rangle \ .$$

Set
$$\dot F_\ell = \{(t,\Delta_p) : (t,p)\in \mathbf{L}\}\,,$$
which is an S-name of a subset of ω_1^ℓ that is in M and in $H(\aleph_2)$, so it follows that $\dot F_\ell \in M_1$. By reverse induction on $\ell > k \geq 1$, we define $\dot F_k$. Having defined $\dot F_{k+1}$, we define for any increasing sequence $\delta_1 < \cdots < \delta_k < \omega_1$, the S-name
$$\dot F_{k+1}(\langle \delta_1,\dots,\delta_k\rangle) = \{(\bar t_0,\gamma) : \bar t_0\in S \text{ and } (\bar t_0, \langle\delta_1,\dots,\delta_k,\gamma\rangle)\in \dot F_{k+1}\}$$
and then we put $(t_0,\langle\delta_1,\dots,\delta_k\rangle)$ in $\dot F_k$ providing t_0 forces that $\dot F_{k+1}(\langle\delta_1,\dots,\delta_k\rangle)$ is stationary.

The next step is to prove that, for each $k<\ell$, $s^\dagger \Vdash \Delta_q\restriction k \in \dot F_k$. Again, this is by reverse induction on $\ell > k \geq 0$. Let $\vec\gamma = \Delta_q = \langle\gamma_1,\dots,\gamma_\ell\rangle$. Certainly, $s^\dagger \Vdash \vec\gamma\in \dot F_\ell$. We again take note of the fact that $\dot F_k\in M_1$ for each $0\leq k\leq \ell$. Now let $j=k+1$ and assume that $s^\dagger\Vdash \vec\gamma\restriction j\in \dot F_j$. Observe that $\dot F_j(\vec\gamma\restriction k)$ is a member of the model M_j, and that $\gamma_j = M_j\cap\omega_1$ is forced by $s^\dagger$ to be an element of $\dot F_j(\vec\gamma\restriction k)$. By Lemma 2.1 (3), this means that $s^\dagger$ forces that $\dot F_j(\vec\gamma\restriction k)$ is stationary. This completes the inductive step that $s^\dagger$ forces that $\vec\gamma\restriction k$ is in $\dot F_k$. For the last case, when $k=0$, we have proven that $s^\dagger$ forces that $\dot F_0$ includes the empty sequence.

To complete the proof, we work our way back up from $k=0$ to $k=\ell$ in order to pick a suitable $(t,p)\in D\cap M$ that is compatible with $(s^\dagger,q)$. Recall that the main requirement, once we know that $(t,p)\in\mathbf{L}\cap M$, is to have that $\delta_j\in a_{m_j}$ for each $j\in J_B$ where $\Delta_p=\langle\delta_1,\dots,\delta_\ell\rangle$. We begin with the fact that $s^\dagger$ forces that $\dot F_0\in M_1$ is non-empty. By Lemma 2.1, there is an $t_0\in M\cap S$ with $t_0<s^\dagger$ that also forces the empty sequence is in $\dot F_0$. By definition, $t_0\Vdash \dot F_1(\emptyset)$ is stationary.

If 1 is not in J_B, then we simply apply Lemma 2.1 to choose $t_0\leq t_1 < s^\dagger$ in M_1 and a $\delta_1\in M\cap\omega_1$ such that $t_1\Vdash \delta_1\in \dot F_1(\emptyset)$. We now handle the more difficult case when 1 is in J_B. Here we use our assumptions on $\dot{\mathcal I}$ in order to similarly find the pair t_1 and δ_1, but this time with $\delta_1\in a_{m_1}$, so that t_1 forces that $\delta_1\in\dot F_1(\emptyset)$.

We have that $t_0\Vdash \dot F_1(\emptyset)$ is stationary. Let g be a generic filter for S with $s^\dagger\in g$. Since S is coherent, $\bar s_{m_1}\oplus g$ is also a generic filter for S. Since we are assuming that $1\in J_B$, we check that $val_g(\dot F_1(\emptyset))$ is a subset of $val_{\bar s_{m_1}\oplus g}(\dot B)$. Fix any (t_1,γ_1) in the name $\dot F_1(\emptyset)$ so that $t_1\in g$. This means that $(t_1,\langle\gamma_1\rangle)$ is in the S-name $\dot F_1$. By recursion, continue to extend until we obtain $(t_\ell,\langle\gamma_1,\dots,\gamma_\ell\rangle)$ in the S-name $\dot F_\ell$ with $t_\ell\in g$. Choose $\bar p\in\mathcal P$ so that $(t_\ell,\bar p)\in\mathbf{L}$ and $\langle\gamma_1,\dots,\gamma_\ell\rangle$ is equal to $\Delta_{\bar p}$. By condition (3) of the definition of $\mathcal P$, $\bar p(M_1^{\bar p})$ forces that $\gamma_1\in\dot B$. Since

$1 \in J$, we have that $\bar{s}_0 \oplus \bar{p}(M_1^{\bar{p}}) < t_\ell < s^\dagger$, and so $\bar{p}(M_1^{\bar{p}})$ is in the filter $\bar{s}_{m_1} \oplus g$. The ideal $\mathcal{I}(\bar{s}_{m_1})$ we get by interpreting the name $\dot{\mathcal{I}}$ using the filter $\bar{s}_{m_1} \oplus g$ is a P-ideal satisfying that $[E]^{\aleph_0} \cap \mathcal{I}(\bar{s}_{m_1})$ is non-empty for all stationary sets $E \subset \dot{B}$. Since $E = val_g(\dot{F}_1(\emptyset))$ is such a stationary set and is in the model $M_1[g]$, there is an infinite set $a \in M_1[g]$ such that $a \in [E]^{\aleph_0}$ and $a \in \mathcal{I}(\bar{s}_{m_1})$. Again by elementarity, and Lemma 2.1, there is a condition $t_1 \in M \cap g$ extending t_0 and satisfying that $t_1 \Vdash a \subset \dot{F}_1(\emptyset)$ and $\bar{s}_{m_1} \oplus t_1 \Vdash a \in \dot{\mathcal{I}}$. We have noted that a_{m_1} includes mod finite a and so we may choose a $\delta_1 \in a \cap a_{m_1}$.

We now have that $(t_1, \langle \delta_1 \rangle)$ is in the name $\dot{F}_1 \cap M$, $t_1 < s^\dagger$, and if $1 \in J_B$, then $\delta_1 \in a_{m_1}$. We proceed recursively to choose $\delta_j \in M_1$ (for $1 < j \leq \ell$) and an extension $t_j \in M_1 \cap S$ of t_{j-1} so that $t_j < s^\dagger$, the pair $(t_j, \langle \delta_1, \ldots, \delta_j \rangle)$ is in the S-name $\dot{F}_j$, and so that, if $j \in J_B$, then $\delta_j \in a_{m_j}$. Now choose $(t_\ell, p) \in \mathbf{L} \cap M$ so that $\Delta_p = \langle \delta_1, \ldots, \delta_\ell \rangle$, and that $\delta_j \in a_{m_j}$ for each $j \in J_B$. Of course this means that $(t_{\bar{\ell}}, p) \in D \cap M$ and $(t_\ell, p) \not\perp (s^\dagger, q)$ as required. □

3. ON $\mathbf{PPI}^+$

In this section we prove that PFA(S)[S] implies $\mathbf{PPI}^+$. As mentioned in the introduction we will work with general sequentially compact regular spaces and not just spaces that are first countable. Our approach requires this anyway, since we will be constructing a more general sequential structure in the ground model arising from an S-name of a sequentially compact space. As a first step we prove that we can work in the sequential closure of a subset of cardinality $\aleph_1$. A subset E of a space X is sequentially closed if every converging sequence from E converges to a point of E. The sequential closure of a set is the intersection of all sequentially closed sets that include it. It will be useful to have an internal description of the sequential closure of a set. There is a well-known space in the study of sequential spaces, namely the space S_ω from [1]. This is the strongest sequential topology on the set of finite sequences of integers, $\omega^{<\omega}$, in which, for each $t \in \omega^{<\omega}$, the set of immediate successors, $\{t^\frown n : n \in \omega\}$, converges to t. If T is any subtree of $\omega^{<\omega}$, we will consider T to be topologized as a subspace of S_ω. To avoid confusion with the Souslin tree, we will call this the Seq_ω-topology. As usual, for $t \in T$, T_t will denote the subtree of T consisting of all $t' \in T$ which are comparable with t.

Of particular use will be those $T \subset \omega^{<\omega}$ that are well-founded (that is, contain no infinite branch). Let $\mathbf{WF}$ denote those downward closed well-founded trees T with the property that every branching node has a full set of immediate successors. Such a tree will have a root, $\mathrm{root}(T)$

(which need not be the root of $\omega^{<\omega}$) which is either the minimal branching node or, if there are no branching nodes, the maximum member of T. For $T \in \mathbf{WF}$, we let $\max(T)$ denote the maximal elements of T and let $Br(T) = \{t \in T \setminus \max(T) : \mathrm{root}(T) \leq t\}$, which is the set of branching nodes of T. When discussing the topology on $T \in \mathbf{WF}$ we ignore the nodes strictly below the root of T. There is a natural notion of the rank of each $T \in \mathbf{WF}$. The rank of T will really be the rank of T_t where t is the root of T. We use $\mathrm{rk}(T)$ to denote the ordinal $\alpha \in \omega_1$ which is the rank of T. If $t \in T$ is a maximal node, then $\mathrm{rk}(T_t) = 0$, and if $\mathrm{root}(T) \subset t \in T$, then $\mathrm{rk}(T_t) = \sup\{\mathrm{rk}(T_{t'})+1 : t < t' \in T_t\}$. We let $\mathbf{WF}(\alpha) = \{T \in \mathbf{WF} : \mathrm{rk}(T) \leq \alpha\}$ and $\mathbf{WF}(<\alpha) = \bigcup_{\beta<\alpha} \mathbf{WF}(\beta)$.

Definition 3.1. *Suppose that X is the base set for a topology and let A be a subset of X. Let $\Sigma(A, X)$ denote the set of functions σ such that*

(1) *the domain of σ is some $T \in \mathbf{WF}$,*
(2) *$\sigma(t) \in A$ for each maximal node t of T,*
(3) *σ is a continuous function from T into X.*

If $\sigma \in \Sigma(A, X)$, we will say that σ converges to x if $\sigma(\mathrm{root}(T)) = x$.

seqlcl **Proposition 3.2.** *If X is a Hausdorff space and $A \subset X$, then the sequential closure of A in X is equal to $\{\sigma(\mathrm{root}(T)) : \sigma \in \Sigma(A, X)$ and $T = \mathrm{dom}(\sigma)\}$ and has cardinality at most $|A|^{\aleph_0}$.*

Proof. Let $\sigma \in \Sigma(A, X)$ and let $T \in \mathbf{WF}$ be the domain of σ. There is no loss of generality in assuming that the empty sequence is the root of T. For each $t \in T$, $\mathrm{rk}(T_t)$ is a countable ordinal. By induction on $\mathrm{rk}(T_t)$, it follows that $\sigma(t)$ is in the sequential closure of A. Now to prove that $\{\sigma(\mathrm{root}(T)) : \sigma \in \Sigma(A, X)$ and $T = \mathrm{dom}(\sigma)\}$ is the sequential closure of A, it suffices to prove that $\{\sigma(\mathrm{root}(T)) : \sigma \in \Sigma(A, X)$ and $T = \mathrm{dom}(\sigma)\}$ is sequentially closed. Suppose that $\{\sigma_n : n \in \omega\} \subset \Sigma(A, X)$ and that $\{x_n : n \in \omega\}$ converges to a point $x \in X$, where for each $n \in \omega$, σ_n converges to x_n. For each n, let T_n denote the domain of σ_n. There is no loss in assuming, or arranging, that the root of T_n is the immediate successor of the empty function and that $\mathrm{root}(T_n)(0)$ is equal to n. With this assumption, it follows that $T = \bigcup_n T_n$ is an element of $\mathbf{WF}$. Similarly, the function σ with domain T, such that $\sigma(\mathrm{root}(T)) = x$ and $\bigcup_n \sigma_n \subset \sigma$, is an element of $\Sigma(A, X)$ that converges to x. This shows that $\{\sigma(\mathrm{root}(T)) : \sigma \in \Sigma(A, X)$ and $T = \mathrm{dom}(\sigma)\}$ is sequentially closed in X.

If A is finite, then it is sequentially closed. The cardinality of $\mathbf{WF}$ is $\mathfrak{c}$ because every $T \in \mathbf{WF}$ is a subset of the countable set $\omega^{<\omega}$. Similarly, for each $T \in \mathbf{WF}$, the set $\{\sigma \in \Sigma(A, X) : T = \mathrm{dom}(\sigma)\}$ is bounded by $|A|^{\aleph_0}$. Since a union of $\mathfrak{c}$ many sets of cardinality at most $|A|^{\aleph_0}$ has cardinality at most $|A|^{\aleph_0}$, this shows that the sequential closure of an infinite set A has cardinality at most $|A|^{\aleph_0}$. □

Proposition 3.3. *Every non-compact regular space in which countable subsets have compact sequential closure has a subset of cardinality $\aleph_1$ which has no complete accumulation points in its sequential closure.* smallsubspace

Proof. Let $\mathcal{U}$ be an open cover of a space X that has no finite subcover. Since X is countably compact, $\mathcal{U}$ has no countable subcover. For convenience we assume that $\mathcal{U}$ is closed under finite unions. Choose $x_0 \in X$ and $x_0 \in U_0 \in \mathcal{U}$. We continue the recursion for all $\alpha \in \omega_1$ so that, having chosen $\{x_\beta, U_\beta : \beta < \alpha\}$, we choose an $x_\alpha \notin \bigcup_{\beta<\alpha} U_\beta$ and then a $U_\alpha \in \mathcal{U}$ so that the (compact) sequential closure of $\{x_\beta : \beta \leq \alpha\}$ is included in U_α. It then follows that the sequential closure of $\{x_\alpha : \alpha < \omega_1\}$ is included in the union of the open collection $\{U_\alpha : \alpha \in \omega_1\}$ and no complete accumulation point can be in any of the U_α's. □

We will need a stronger version of Proposition 3.3.

Theorem 3.4. *It is a consequence of PFA(S)[S] that each sequentially compact non-compact regular space includes a subset of cardinality $\aleph_1$ that has no complete accumulation point in its sequential closure.* itsomega1

Proof. We work in a model of PFA(S) and prove that if we force with S the conclusion of the theorem holds. Let $g \subset S$ be any generic filter and we begin by making a reduction in the forcing extension $V[g]$. Suppose that X is a sequentially compact non-compact space. If the sequential closure of every countable subset of X is compact, then the conclusion of the Theorem holds by Proposition 3.3. Therefore we may assume there is a countable set whose sequential closure, Y, is not compact. We choose such a Y and fix a maximal free filter $\mathcal{F}$ of closed subsets of Y. Since $\mathcal{F}$ is a free filter, its members are not compact sets. Since X is sequentially compact, Y is also sequentially compact and countably compact. It now follows that $\mathcal{F}$ is closed under countable intersections. By re-labelling, we assume that the ordinal ω is dense in Y, and by Proposition 3.2, we have that Y has cardinality at most $\mathfrak{c}$. Say that $H \in \mathcal{F}^+$ providing $H \cap F$ is not empty for all $F \in \mathcal{F}$.

We break into three cases. For the remainder of the proof, we work in the subspace Y and the closure operation refers to the closure of a subset of Y in Y. In the first case, assume that there is a descending collection $\{F_\alpha : \alpha \in \omega_1\} \subset \mathcal{F}$ such that $\bigcap\{F_\alpha : \alpha \in \omega_1\}$ is empty. For each $\alpha \in \omega_1$, choose any point $x_\alpha \in F_\alpha$. The sequence $\{x_\alpha : \alpha \in \omega_1\}$ has no complete accumulation point in Y since such a point would be in the closure of $\{x_\beta : \alpha < \beta \in \omega_1\} \subset F_\alpha$ for each α. This completes the proof in this first case. The second case is that there is some set $H \in \mathcal{F}^+$ with the property that $\overline{H_0} \cap Y \notin \mathcal{F}$ for all countable $H_0 \subset H$. Since $\mathcal{F}$ is maximal, if $\overline{H_0} \notin \mathcal{F}$, there is an $F \in \mathcal{F}$ that is disjoint from $\overline{H_0}$. Following [11], we recursively construct an uncountable sequence $\{h_\alpha : \alpha \in \omega_1\} \subset H$ together with a descending family $\{F_\alpha : \alpha \in \omega_1\} \subset \mathcal{F}$. Having chosen $\{h_\beta : \beta < \alpha\}$ we choose $F_\alpha \subset \bigcap\{F_\beta : \beta < \alpha\}$ so that F_α is disjoint from $\overline{\{h_\beta : \beta < \alpha\}}$. Then we choose $h_\alpha \in F_\alpha$. It follows that the ω_1-sequence $\{h_\alpha : \alpha \in \omega_1\}$ is a free sequence in Y since, for each $\alpha \in \omega_1$, F_α is disjoint from the closure (in Y) of $\{h_\beta : \beta < \alpha\}$ and includes the closure of $\{h_\beta : \alpha \leq \beta\}$. Each complete accumulation point (if any) of $\{h_\alpha : \alpha \in \omega_1\}$ is in $\bigcap\{F_\alpha : \alpha \in \omega_1\}$, and so $\{h_\alpha : \alpha \in \omega_1\}$ is our desired ω_1-sequence.

Now we consider the final case. In this case we have that $\mathcal{F}$ is a maximal free filter of closed subsets of Y with the property that every intersection of at most $\aleph_1$ members of $\mathcal{F}$ is again in $\mathcal{F}$. In addition, $\mathcal{F}$ satisfies that for each $H \in \mathcal{F}^+$, there is a countable $H_0 \subset H$ such that $\overline{H_0} \in \mathcal{F}$. As mentioned above, the cardinality of Y is at most $\mathfrak{c}$, and since $\mathcal{F}$ is generated by its separable members, it has a base of cardinality at most $\mathfrak{c}$. Since PFA(S)[S] implies that $\mathfrak{c} = \aleph_2$, we may fix a strictly descending chain $\{F_\alpha : \alpha \in \omega_2\} \subset \mathcal{F}$ that is a base for $\mathcal{F}$. For each $\alpha \in \omega_2$, choose a point $x_\alpha \in F_\alpha \setminus F_{\alpha+1}$ together with a continuous bounded real-valued function f_α on Y satisfying that $f_\alpha[F_{\alpha+1}] = 1$ and $f_\alpha(x_\alpha) < 0$. Choose any $\sigma_\alpha \in \Sigma(\omega, Y)$ so that σ_α converges to x_α. It follows that $f_\alpha \circ \sigma_\alpha$ is a member of $\Sigma(f_\alpha[\omega], \mathbb{R})$ that converges to a negative real. Let T_α denote the domain of σ_α and, with no loss of generality, assume that $\emptyset = \mathrm{root}(T_\alpha)$. Recall that $f_\beta \circ \sigma_\alpha(\emptyset)$ denotes the real number to which $f_\beta \circ \sigma_\alpha$ converges. Let us note that for all $\beta < \alpha < \omega_1$, $f_\beta \circ \sigma_\alpha(\emptyset) = 1$ and $f_\alpha \circ \sigma_\alpha(\emptyset) < 0$. Also note that if $H \in [\omega_2]^{\aleph_2}$, then $\{x_\alpha : \alpha \in H\} \in \mathcal{F}^+$ and so there is a countable $H_0 \subset H$ and an $\eta \in \omega_2$ such that the closure of $\{x_\alpha : \alpha \in H_0\}$ contains F_η. In particular, if $\eta < \alpha$ and if L is a finite subset of $\omega_2 \setminus \alpha$ with $f_\gamma \circ \sigma_\alpha(\emptyset) < 0$ for each $\gamma \in L$, then there is a $\beta \in H_0$ such that $f_\gamma \circ \sigma_\beta(\emptyset) < 0$ for each $\gamma \in L$.

Now we pass back to the ground model. Let $\dot{Y}$ be an S-name for the topological space Y under discussion with $1 \Vdash \omega \subset \dot{Y}$. We can assume that the base set for the sequential closure of ω is ω_2. For each $\alpha \in \omega_2$, fix S-names $\dot{x}_\alpha, \dot{\sigma}_\alpha, \dot{T}_\alpha$ and $\dot{f}_\alpha$ for $x_\alpha, \sigma_\alpha, T_\alpha$ and f_α respectively as defined in the previous paragraph. Choose a condition $\tilde{s} \in S$ that forces the following properties for each $\alpha, \beta \in \omega_2$:

(1) $\dot\sigma_\alpha$ is an element of $\Sigma(\omega, \dot Y)$, $\dot T_\alpha = \mathrm{dom}(\dot\sigma_\alpha)$, and $\dot\sigma_\alpha(\emptyset)$ equals $\dot x_\alpha$,
(2) $\dot f_\alpha$ is a continuous real-valued function on $\dot Y$ and $\dot f_\alpha \circ \dot\sigma_\alpha(\emptyset) < 0$,
(3) if $\beta < \alpha$, $\dot f_\beta \circ \dot\sigma_\alpha$ converges to 1
(4) if $H \subset \omega_2$ has cardinality $\aleph_2$, there is a countable $H_0 \subset H$ and an $\eta \in \omega_2$ such that for all $\eta < \alpha$ and finite $L \subset \omega_2 \setminus \alpha$, if $\dot f_\gamma \circ \dot\sigma_\alpha(\emptyset) < 0$ for all $\gamma \in L$, then there is a $\beta \in H_0$ such that $\dot f_\gamma \circ \dot\sigma_\beta(\emptyset) < 0$ for all $\gamma \in L$.

Property (4) corresponds to the condition that x_α is in the closure of $\{x_\xi : \xi \in H_0\}$ for all $\alpha > \eta$ as discussed above. By Lemma 2.1, for each $\alpha \in \omega_2$, pick an $s_\alpha \in S$ extending $\tilde s$ such that s_α forces a value on each of $\dot x_\alpha, \dot T_\alpha, \dot\sigma_\alpha$ and $\dot f_\alpha \restriction \omega$. That is, there is a $T_\alpha \in \mathbf{WF}$ which is the domain for a function σ_α with domain T_α and range included in ω_2. We can assume that s_α forces that $\dot\sigma_\alpha$ is equal to σ_α, hence $\dot x_\alpha$ is forced to equal $\sigma_\alpha(\emptyset)$, and that there is a function f_α with domain equal to ω union the range of σ_α, such that s_α forces that $\dot f_\alpha$ agrees with f_α on all these values. Choose any $\bar s \in S$ so that $\Gamma = \{\alpha \in \omega_2 : s_\alpha = \bar s\}$ has cardinality $\aleph_2$. Let $\mathcal{X}$ denote the indexed family $\{\langle T_\alpha, \sigma_\alpha, f_\alpha \restriction \omega\rangle : \alpha \in \Gamma\}$.

Now we define a poset $\mathcal{P}$ in a similar (but simpler) fashion as in the proof of $\mathbf{P}_{22}$. Our reduction from S-names to the family of values that the names are forced to equal allows us to ignore S when defining the poset $\mathcal{P}$. We let $\mathcal{P}$ be the set of all functions p of the form $p : \mathcal{M}_p \to \Gamma$, where

(1) $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $H(\aleph_3)$,
(2) $\mathcal{X} \in M$ for each $M \in \mathcal{M}_p$,
(3) for $M_2 \in \mathcal{M}_p$ and $M_1 \in \mathcal{M}_p \cap M_2$, $\sup(M_1 \cap \omega_2) \le p(M_1) \in \Gamma \cap M_2$.

We let $p \le q$ for $p, q \in \mathcal{P}$, if

(4) the function p includes the function q,
(5) if $M_2 \in \mathcal{M}_q$ and if $M_1 \in \mathcal{M}_p \cap M_2$ is such that $\mathcal{M}_q \cap M_2 \in M_1$, then $f_{q(M)} \circ \sigma_{p(M_1)}$ extends continuously to $T_{\sigma_{p(M_1)}}$ and $f_{q(M)} \circ \sigma_{p(M_1)}(\emptyset) < 0$, for each $M \in \mathcal{M}_q$ such that $f_{q(M)} \circ \sigma_{q(M_2)}(\emptyset) < 0$.

Note that in condition (5), it follows from condition (3) that $f_{q(M)} \circ \sigma_{q(M_2)}$ converges to 1 for each $M \in \mathcal{M}_q \cap M_2$. To see that $\le$ is a transitive ordering on $\mathcal{P}$, assume that $r \le p$ and $p \le q$. Let $M_2 \in \mathcal{M}_q$ and suppose there is $M_1 \in \mathcal{M}_r \cap M_2$ such that $\mathcal{M}_q \cap M_2 \in M_1$. Also let $M \in \mathcal{M}_q$ be such that $f_{q(M)} \circ \sigma_{q(M_2)}(\emptyset) < 0$. We have to establish that $f_{q(M)} \circ \sigma_{r(M_1)}(\emptyset) < 0$. If $M_1 \in \mathcal{M}_p$, then this follows by the assumption that $p \le q$. If $\mathcal{M}_p \cap M_2 \in M_1$, then it follows from the assumption that $r \leq p$. Otherwise, choose the minimal $M_3 \in \mathcal{M}_p \cap M_2 \setminus M_1$ such that $\mathcal{M}_p \cap M_3 \in M_1$. Then $\mathcal{M}_q \cap M_2 \in M_3$ and so $p \leq q$ implies that $f_{q(M)} \circ \sigma_{p(M_3)}(\emptyset) = f_{p(M)} \circ \sigma_{p(M_3)}(\emptyset) < 0$. Then $f_{p(M)} \circ \sigma_{r(M_1)}(\emptyset) < 0$ since $r \leq p$ and $f_{p(M)} \circ \sigma_{p(M_3)}(\emptyset) < 0$.

We will prove that $\mathcal{P}$ is proper and preserves that S is Souslin. Before doing so, we show that this means that $\bar{s}$ forces that $\dot{X}$ has an ω_1-sequence with no complete accumulation point in its sequential closure. It is immediate from the definition of the ordering on $\mathcal{P}$, that for each $\delta \in \omega_1$, the set $D_\delta = \{q \in \mathcal{P} : \delta \in \bigcup \mathcal{M}_q\}$ is a dense open subset of $\mathcal{P}$. Let $G \subset \mathcal{P}$ be a filter such that $G \cap D_\delta$ is not empty for all $\delta \in \omega_1$. Set $C = \{\delta \in \omega_1 : (\exists q \in G)\ (\exists M \in \mathcal{M}_q)\ \delta = M \cap \omega_1\}$. For each $\delta \in C$, choose $q_\delta \in G$ and $M_\delta \in \mathcal{M}_{q_\delta}$ such that $\delta = M_\delta \cap \omega_1$. Let $\alpha_\delta \in \Gamma$ be the value of $q_\delta(M_\delta)$. We prove, by induction on $\delta \in C$, that $\bar{s}$ forces that the closure of $\{\dot{x}_{\alpha_\beta} : \beta \in C \cap \delta{+}1\}$ is disjoint from the closure of $\{\dot{x}_{\alpha_\gamma} : \delta < \gamma \in C\}$. Given $\delta \in C$, let ξ be the maximum element of $\{0\} \cup \{M' \cap \omega_1 : M' \in \mathcal{M}_{q_\delta} \cap M_\delta\}$. By the induction hypothesis, it follows that $\bar{s}$ forces that $\{\dot{x}_{\alpha_\beta} : \beta \in C \cap \xi{+}1\}$ and $\{\dot{x}_{\alpha_\gamma} : \delta < \gamma \in C\}$ have disjoint closures. Now we prove that $\bar{s}$ forces that $\{\dot{x}_{\alpha_\beta} : \xi < \beta \in C \cap \delta{+}1\}$ and $\{\dot{x}_{\alpha_\gamma} : \delta < \gamma \in C\}$ have disjoint closures. In fact we show that $\dot{f}_{\alpha_\delta}$ is forced to be a continuous function that separates them in the sense that $\dot{f}_{\alpha_\delta}\left(\{\dot{x}_{\alpha_\beta} : \xi < \beta \in C \cap \delta{+}1\}\right) \subset (-\infty, 0)$ and $\dot{f}_{\alpha_\delta}\left(\{\dot{x}_{\alpha_\gamma} : \gamma \in C \setminus \delta\}\right) = \{1\}$.

For $\delta < \gamma \in C$, $q_\delta, q_\gamma \in G$ are compatible, and so, by condition (3) of $\mathcal{P}$ $\alpha_\delta < \alpha_\gamma$ and, by property (3) of the construction, $\bar{s}$ forces that $\dot{f}_{\alpha_\delta}(\dot{x}_{\alpha_\gamma}) = 1$ and $f_{\alpha_\delta} \circ \sigma_{\alpha_\delta}(\emptyset) < 0$. Now consider any β with $\xi < \beta \in C \cap \delta$. Since q_β and q_δ are in the filter G, we may choose $\bar{q} \in G$ below each of them. Then $M_\beta \in \mathcal{M}_{\bar{q}} \cap M_\delta$ and, by the definition of ξ, $\mathcal{M}_{q_\delta} \cap M_\delta \in M_\beta$. Therefore, since $\bar{q} < q_\delta$, it follows from condition (5) in the definition of $\mathcal{P}$ that $f_{\alpha_\delta} \circ \sigma_{\alpha_\beta}$ extends continuously to T_{α_β} and that $f_{\alpha_\delta} \circ \sigma_{\alpha_\beta}(\emptyset) < 0$. This means that $\bar{s}$ forces that $\dot{f}_{\alpha_\delta}(\dot{x}_{\alpha_\beta}) < 0$ as required.

Now we return to the proof that $\mathcal{P}$ is proper and preserves that S is Souslin. Since S is a coherent Souslin tree it suffices to show that $\mathcal{P}$ preserves that $\{s \in S : \bar{s} \subseteq s\}$ is Souslin. We will construct a pair $(s, p) \in S \times \mathcal{P}$ (with $\bar{s} \leq s$) that is M-generic for $S \times \mathcal{P}$. Let κ and countable $M \subset H(\kappa)$ be as in Definition 2.2 with, additionally, $\bar{s}, S \in M$. For any $q \in \mathcal{P} \cap M$, $p = q \cup \{(M \cap H(\aleph_3), \min(\Gamma \setminus \sup(M \cap \omega_2))\}$ is in $\mathcal{P}$ and so we may choose $p \in \mathcal{P}$ to be any condition such that $M \cap H(\aleph_3) \in \mathcal{M}_p$. Let s be any member of $S \setminus M$ with $\bar{s} \subset s$. By Proposition 2.4 it will suffice to prove that (s, p) is M-generic for $S \times \mathcal{P}$.

Let $D \in M$ be any dense open subset of $S \times \mathcal{P}$. By Definition 2.2, we have to prove that if $(\tilde{s}, q)$ is any extension of (s, p), then there is a compatible condition in $D \cap M$. Since D is dense, we may assume that $(\tilde{s}, q) \in D$. Let $\bar{\mathcal{M}}_0 = \mathcal{M}_q \cap M$ and choose any $\bar{M} \in M \cap H(\aleph_3)$ such that $\bar{M}$ is a countable elementary submodel of $H(\aleph_3)$ and $\bar{\mathcal{M}}_0 \in \bar{M}$.

Now let $g \subset S$ be any generic filter for S with $\tilde{s} \in g$. We will argue in the forcing extension $V[g]$. The poset $\mathcal{P}$ with its same ordering is a poset in $V[g]$. Again we use the basic fact about product forcing (e.g. see [25, II 1.5]).

Claim 7. *The set $val_g(D) = \{r \in \mathcal{P} : (\exists t \in g)\ (t, r) \in D\}$ is a dense subset of $\mathcal{P}$.* inVg

Similarly, for each $r \in \mathcal{P}$ and each $\tilde{M} \in \mathcal{M}_r$, we again use that the set $\tilde{M}[g] = \{val_g(\tau) : \tau$ is an S-name in $\tilde{M}\}$ is an elementary submodel of $H(\aleph_3)$ in $V[g]$ (see [25, III 2.11]). Of course q is an element of $val_g(D)$. To finish the proof that (s, p) is an M-generic condition for $S \times M$, it suffices to show there is an $r \in val_g(D) \cap M$ that is compatible with q. Let $\{M_0, \ldots, M_{\ell-1}\}$ enumerate $\mathcal{M}_q \setminus M = \mathcal{M}_q \setminus \bar{M}$ in increasing order. For each $i < \ell$, let $\alpha_i \in \Gamma$ denote $q(M_i)$.

Say that $r \in val_g(D)$ is <u>like</u> q providing

(1) $q \restriction \bar{\mathcal{M}}_0 \subset r$ and $\mathcal{M}_r \cap \bar{M} = \bar{\mathcal{M}}_0$,
(2) $\mathcal{M}_r \setminus \bar{M}$ has size ℓ, enumerated as $\{M^r_0, \ldots, M^r_{\ell-1}\}$ in increasing order.

For each r that is <u>like</u> q and each $i < \ell$, let α^r_i denote $r(M^r_i)$. Set $\mathbf{L}_\ell$ equal to the collection $\{\langle \alpha^r_0, \ldots, \alpha^r_{\ell-1}\rangle : r$ is <u>like</u> $q\}$. For any tuple $\langle \beta_j : j < i\rangle \in \omega_2^i$ and $\beta \in \omega_2$, we use $\langle \beta_j : j < i\rangle^\frown\langle\beta\rangle$ to denote the tuple $\langle \beta_j : j \leq i\rangle$ where $\beta_i = \beta$. By reverse induction on $0 \leq i < \ell$, we define a set $\mathbf{L}_i$. If $\mathbf{L}_{i+1}$ has been defined, then for any i-tuple $\langle \beta_j : j < i\rangle \in \Gamma^i$, $\mathbf{L}_{i+1}(\langle \beta_j : j < i\rangle)$ is equal to the set $\{\beta : \langle \beta_j : j < i\rangle^\frown\langle\beta\rangle \in \mathbf{L}_{i+1}\}$. Then we define $\langle \beta_j : j < i\rangle$ to be in $\mathbf{L}_i$ providing $\mathbf{L}_{i+1}(\langle \beta_j : j < i\rangle)$ has cardinality $\aleph_2$. Using the facts that $\langle \alpha_0, \ldots, \alpha_{\ell-1}\rangle \in \mathbf{L}_\ell$, and that $M_i[g]$ is an elementary submodel of $H(\aleph_3)$ with $\mathbf{L}_j \in M_i[g]$ for each $j \leq \ell$ and $i < \ell$, it follows by reverse induction on $i < \ell$, that $\langle \alpha_j : j < i\rangle$ is in $\mathbf{L}_i$. In particular, we have that the empty sequence is an element of $\mathbf{L}_0$.

Now we note that the following claim holds in $V[g]$ because, as shown above, it is forced by $\bar{s}$.

Claim 8. *If $H \subset \Gamma$ has cardinality $\aleph_2$ then there is an $\eta \in \omega_2$ and a countable $H_0 \subset H \cap \eta$ such that for all $\alpha \in \Gamma \setminus \eta$ and finite $L \subset \Gamma \setminus \alpha$, if $f_\gamma \circ \sigma_\alpha(\emptyset) < 0$ for all $\gamma \in L$, then there is a $\beta \in H_0$ such that $f_\gamma \circ \sigma_\beta(\emptyset) < 0$ for all $\gamma \in L$.* sepble

Let $L = \{\alpha_i : i < \ell \text{ and } f_{\alpha_i} \circ \sigma_{\alpha_0}(\emptyset) < 0\}$. Applying Claim 8 to the set $H = \mathbf{L}_0(\langle\ \rangle)$, there is an $\eta \in M$ and a $\beta_0 \in \mathbf{L}_0(\langle\ \rangle) \cap M$ such that $f_\gamma \circ \sigma_{\beta_0}(\emptyset) < 0$ for all $\gamma \in L$. Similarly, there is a $\beta_1 \in \mathbf{L}_1(\langle\beta_0\rangle) \cap M$ such that $f_\gamma \circ \sigma_{\beta_1}(\emptyset) < 0$ for all $\gamma \in L$. Repeating this step ℓ times, we establish that there is a $\langle\beta_0, \dots, \beta_{\ell-1}\rangle \in \mathbf{L}_\ell \cap M$ such that $f_\gamma \circ \sigma_{\beta_i}(\emptyset) < 0$ for each $i < \ell$ and each $\gamma \in L$. By elementarity, there is an $r \in M[g] \cap \mathcal{P}$ that is <u>like</u> q such that $\langle\beta_i : i < \ell\rangle = \langle\alpha^r_i : i < \ell\rangle$. Of course $r \in M$ since, by Lemma 2.1, $\mathcal{P} \cap M[g] \subset M$.

Since $r \in val_g(D) \cap M$, we complete the proof by showing that $r \cup q$ is an extension in $\mathcal{P}$ of each of r and q. Conditions (1) and (2) are immediate for $\mathcal{M}_{r\cup q}$ and (3) follows from the fact that $r \in M_0$. Condition (5) holds vacuously for the relation $q \cup r < r$ while the previous paragraph established that (5) holds for the relation $r \cup q < q$. □

We return to our proof of $\mathbf{PPI}^+$. For the remainder of the section we assume that we have an S-name of a sequentially compact non-compact regular space $\dot{X}$ that is equal to the sequential closure of the set of countable ordinals, ω_1, and that every point in the sequential closure of the set ω_1 is forced (by 1) to have a neighborhood meeting ω_1 in a countable set. We note that by Theorem 3.4, PFA(S) implies that any S-name of a sequentially compact non-compact space has to include such a subspace, but we emphasize (for use in §4) that the results in the remainder of this section do not require the assumption that PFA(S) holds. We do apply PFA(S) for our main result in Proposition 3.27, but it is explicitly added as a hypothesis in that statement.

The cardinality of the sequential closure of a set of size $\aleph_1$ in a regular space is at most $\mathfrak{c}$, hence we may assume that the base set is the ordinal $\mathfrak{c}$. To ensure that the cardinality of $\dot{X}$ is at least $\mathfrak{c}$, we can replace $\dot{X}$ by the free union of $\dot{X}$ and the unit interval (with a suitable re-indexing so that ω_1 is still dense). Next we choose an assignment of S-names of neighborhoods $\{\dot{U}(x,n) : x \in \mathfrak{c}, n \in \omega\}$. We assume that 1 forces that the closure of $\dot{U}(x, n+1)$ is included in $\dot{U}(x,n)$ for each $x \in \mathfrak{c}$ and $n \in \omega$. We also assume that 1 forces that the closure of any finite union from the collection $\{\dot{U}(x,n) : x \in \mathfrak{c}, n \in \omega\}$ meets ω_1 in a countable set. If we are also assuming that $\dot{X}$ is forced to be first countable, then we assume that $\{\dot{U}(x,n) : n \in \omega\}$ is forced to form a neighborhood base for x.

Fix a regular cardinal κ large enough so that the second power set (i.e. the power set of the power set) of $\mathfrak{c}$ has cardinality less than κ. Let $\prec$ be a well-ordering of $H(\kappa)$. Since S is ccc and has cardinality $\aleph_1$, it is also true in the forcing extension by S that κ is a regular cardinal and the cardinality of the second power set of $\mathfrak{c}$ is less than κ. Moreover, every element of $H(\kappa)$ in the forcing extension has an S-name in the $H(\kappa)$ of the ground model. Therefore whenever we choose an S-name for a subset of $\mathfrak{c}$ or a subset of its power set, we may choose such a name from $H(\kappa)$. In order to avoid getting more technical about S-names, we will occasionally choose an S-name by describing some properties that it must satisfy and to then choose the $\prec$-minimal name fulfilling those properties. Throughout this section we will assume without mention that any S-names under discussion are elements of $H(\kappa)$.

3.1. **The sequential structure.** Since S is ccc, it follows that if $\{\dot{x}_n : n \in \omega\}$ is a sequence of S-names and $1 \Vdash \dot{x}_n \in \dot{X}$ for each n, then there is an infinite $L \subset \omega$ such that $1 \Vdash \{\dot{x}_n : n \in L\}$ is a converging sequence in $\dot{X}$. To see this, assume that the root of S does not force that $\{\dot{x}_n : n \in \omega\}$ is a converging sequence and let $L_0 = \omega$. Recursively choose, by Lemma 2.1 (4), a mod finite descending sequence $\{L_\alpha : \alpha \in \gamma\}$ and conditions $\{s_\alpha : \alpha \in \gamma\}$ satisfying that s_α forces that $\{\dot{x}_n : n \in L_\beta\}$ (for $\beta < \alpha$) is not converging, while $\{\dot{x}_n : n \in L_\alpha\}$ is converging. Since the family $\{s_\alpha : \alpha \in \gamma\}$ is an antichain, this process must end for some $\gamma < \omega_1$. Finally L is any infinite set that is mod finite included in each member of $\{L_\alpha : \alpha \in \gamma\}$.

Definition 3.5. *Say that a sequence $\{\dot{x}_n : n \in L\}$ is an S-converging sequence in $\dot{X}$ providing $1 \Vdash \{\dot{x}_n : n \in L\}$ is a converging sequence (which includes, for example, constant sequences).*

With this new terminology we have proven above that

Lemma 3.6. *For any $\{\dot{x}_n : n \in \omega\}$ of S-names of ordinals in $\mathfrak{c}$, there is an infinite $L \subset \omega$ such that $\{\dot{x}_n : n \in L\}$ is an S-converging sequence.* Sconverge

Of course our space $\dot{X}$ is forced to be sequentially compact and equal to the sequential closure of ω_1. We again work with trees $T \in \mathbf{WF}$ and accompanying functions y but with a new twist. We define a set Y of such functions and interpret a (ground model) sequential topology on that set.

Definition 3.7. *A function $y \in Y_\alpha$ providing there is a T_y in $\mathbf{WF}(\alpha)$ such that* doty

1. *the domain of y is $\max(T_y)$ (we will refer to T_y as the domain of y),*
2. *the range of y is included in ω_1,*
3. *the condition 1 of S forces that y extends continuously into $\dot{X}$ to all of $Br(T_y)$.*

For any such $y \in Y_\alpha$ and branching node $t \in T_y$, we associate an S-name by recursion on the rank of $(T_y)_t$: $\dot{y}(t)$ denotes the canonical S-name such that $s \in S$ forces an ordinal value x on $\dot{y}(t)$ providing s forces such a value on $\dot{y}(t')$ for all branching nodes $t' \in (T_y)_t$ above t and s forces that the continuous extension of y to all of $(T_y)_t$ takes value x at t. For maximal nodes $t \in T_y$, $\dot{y}(t)$ will just denote $y(t)$.

We let $Y = \bigcup_{\alpha\in\omega_1} Y_\alpha$.

This next lemma is a straightforward reformulation of the property of a potentially suitable function y being in Y.

Lemma 3.8. *Each function y with $T_y \in \mathbf{WF}(0)$ and unique maximal node t is in Y_0 so long as $y(t) \in \omega_1$. A function y with domain $T_y \in \mathbf{WF}(1)$ is in Y_1 if and only if $\{y(\mathrm{root}(T)^\frown n) : n \in \omega\}$ is an S-converging sequence. For each $T_y \in \mathbf{WF}$, and function y from $\max(T_y)$ into ω_1, by induction on the rank of T_y, y is in Y if and only if for each branching $t \in T_y$, each $y \restriction \max(\ (T_y)_{t^\frown n}\ )$ is in Y, and $\{\dot{y}(\ t^\frown n\ ) : n \in \omega\}$ is an S-converging sequence.*

Definition 3.9. *Say that y_1 and y_2 in Y are congruent, denoted $y_1 \approx y_2$, providing there is a tree $T \in \mathbf{WF}$ such that*

1. *the set $Br(T_{y_1})$ equals $\{\mathrm{root}(T_{y_1})^\frown t : t \in Br(T)\}$,*
2. *the set $Br(T_{y_2})$ equals $\{\mathrm{root}(T_{y_2})^\frown t : t \in Br(T)\}$,*
3. *and for each maximal $t \in T$, $y_1(\mathrm{root}(T_{y_1})^\frown t)$ equals $y_2(\mathrm{root}(T_{y_2})^\frown t)$.*

Clearly if $y_1 \approx y_2$, then $\dot{y}_1(\mathrm{root}(T_{y_1}))$ names the same element as $\dot{y}_2(\mathrm{root}(T_{y_2}))$. Now that we have identified our structure Y we extend the notion to define a closure operator on any given finite power of Y which will help us understand points in the sequential closure of ω_1 in $\dot{X}$. If $y \in Y$, we will also use $e(y)$ as a more compact notation for $\dot{y}(\mathrm{root}(T_y))$. Similarly, if $\vec{y} \in Y^n$ (for some $n \in \omega$), we will use $e(\vec{y})$ to denote the point $\langle e(\vec{y}_0), e(\vec{y}_1), \ldots, e(\vec{y}_{n-1})\rangle$.

seql **Definition 3.10.** *For each countable index set H, and subset B of Y^H we similarly define the hierarchy $\{B^{(\alpha)} : \alpha \in \omega_1\}$ by recursion. For limit α, $B^{(\alpha)}$ equals $\bigcup_{\beta<\alpha} B^{(\beta)}$ and $B^{(0)} = B$. The members of $B^{(\alpha+1)}$ for any α, consist of the union of $B^{(\alpha)}$ together with all those $\vec{b} = \langle y_h : h \in H\rangle \in Y^H$ such that there is a sequence $\{\vec{b}_k : k \in \omega\}$ of members of $B^{(\alpha)} \cap (Y_\alpha)^H$ such that*

$$(\forall h \in H)(\exists m \in \omega)(\forall k \in \omega \setminus m)\ (\vec{b}_k)_h \approx y_h \restriction (T_{y_h})_{t_h^\frown k} \ \textit{ where } t_h = \mathrm{root}(T_{y_h})\ .$$

When $\vec{b}$ satisfies this definition with respect to $\{\vec{b}_k : k \in \omega\}$, we abbreviate this by saying that $\{\vec{b}_k : k \in \omega\}$ Y-converges to $\vec{b}$. Also if we say that $\{\vec{b}_k : k \in L\}$ Y-converges to $\vec{b}$ for some infinite set L, we just *mean that for any bijection $f : \omega \to L$, the sequence $\{\vec{b}_{f(k)} : k \in \omega\}$ Y-converges to $\vec{b}$.*

For any countable index set H, we may view Y^H as an S-sequential structure and for any $A \subset Y^H$, we say that $A^{(\omega_1)}$ is the sequential closure and is sequentially closed. Notice that this S-sequential structure on Y^H is defined in the ground model.

Definition 3.11. *For any countable set H and S-name $\dot{A}$ such that $1 \Vdash \dot{A} \subset Y^H$, we define, for each $\delta \in \omega_1$, the S-name $(\dot{A})^{(\delta)}$ to be the set of pairs $(s, \vec{y}) \in S \times Y^H$ such that there is a countable $B \subset Y^H$ with $s \Vdash B \subset \dot{A}$ and $\vec{y} \in B^{(\delta)}$.* seqldef

The next lemma captures how the topology on Y will be key to the definition of our proper S-preserving poset in the next subsection.

Lemma 3.12. *Suppose that $B \subset Y^n$ and $\vec{y} = \langle y_i : i < n\rangle \in B^{(\alpha)}$ for some $\alpha \in \omega_1$. If, for each $i < n$, O_i is an Seq_ω-open subset of T_{y_i} with $\mathrm{root}(T_{y_i}) \in O_i$, then there is a sequence $\langle t_i : i < n\rangle$ and an element $\langle b_i : i < n\rangle \in B$ such that* seqopen

(1) *for each $i < n$, $t_i \in O_i$,*
(2) *for each $i < n$, b_i is congruent to $y_i \restriction (T_{y_i})_{t_i}$.*

Proof. We proceed by induction on α. There is nothing to prove if α is a limit, since there is then a $\beta < \alpha$ such that $\vec{y} \in B^{(\beta)}$. Otherwise, let $\alpha = \beta + 1$ and, by Definition 3.10, there is a sequence $\{\vec{b}_k : k \in \omega\}$ of elements of $B^{(\beta)} \cap (Y_\beta)^n$ such that, for each $k \in \omega$ and each $i < n$, $(\vec{b}_k)_i$ is congruent to $y_i \restriction (T_{y_i})_{\mathrm{root}(T_{y_i})^\frown k}$. Choose a $k \in \omega$ so that $\mathrm{root}(T_{y_i})^\frown k$ is in O_i for each $i < n$. For each $i < n$, let T'_i denote the domain of $(\vec{b}_k)_i$ and let t'_i denote the root of T'_i. For each $i < n$, let O'_i denote the open subset of T'_i consisting of those $(t'_i)^\frown t \in \omega^{<\omega}$ such that $\mathrm{root}(T_{y_i})^\frown k^\frown t \in O_i$. Now apply the induction hypothesis to $\vec{b}_k$. □

Note that the members of Y_0 have a singleton domain and for each $\alpha \in \omega_1$, $Y_0 \cap e^{-1}(\alpha)$ is the set of members y of Y_0 such that $e(y) = y(\max(T_y)) = \alpha$. To see this, let $y \in Y$ and fix any $\gamma \in \omega_1$ so that each $s \in S_\gamma$ decides the ordinal value for $e(y)$. Next, fix a possibly larger $\delta \in \omega_1$ so that for each $s \in S_\delta$, s forces that $\dot{U}(e(y), 0) \cap \omega_1 \subset \delta$. Since $s \in S_\delta$ also forces that $\dot{U}(e(y), 1)$ is included in the closure of $\dot{U}(e(y), 0) \cap \omega_1$, s forces that $e(y)$ is not in the sequential closure of $\omega_1 \setminus \delta$. On the other hand, s forces that for all $y' \in \left(\bigcup_{\delta \leq \alpha} Y_0 \cap e^{-1}(\alpha)\right)^{(\omega_1)}$, $e(y')$ is in the sequential closure of $\omega_1 \setminus \delta$. Therefore there is a free filter of S-sequentially closed subsets of Y including $\left(\bigcup_{\alpha > \delta} Y_0 \cap e^{-1}(\alpha)\right)^{(\omega_1)}$

for each $\delta \in \omega_1$. By Zorn's Lemma, we can extend it to a maximal free filter, $\mathcal{F}_0$, of S-sequentially closed subsets of Y.

3.2. **A new application of PFA(S).** The filter $\mathcal{F}_0$ may not generate a maximal filter in the extension $V[g]$ and so, for the purposes of using such a filter to define a proper poset adding a copy of ω_1, we will have to extend it to some S-name of a maximal filter $\dot{\mathcal{F}}_0$. Looking ahead to the PFA(S) step requiring that this poset also be S-preserving, we then run into severe problems caused by the fact that, for some s, we obtain a different filter in $V[s \oplus g]$ than in $V[g]$. Recall that $s \oplus g$ is the (generic) filter generated by $\{s \oplus t : t \in g\}$. The key innovation to overcome this problem is to work in a much larger product structure. We adopt a new approach by extending this filter coherently to a directed family of finite powers of Y. See the definition of a symmetric $\mathcal{F}_0$-filter base below. It will be convenient to also use the notation $s \oplus g$ to denote the element $\bigcup\{s \oplus t : t \in g\} \in 2^{\omega_1}$.

We introduce some more notational conventions. Let $S^{\rightarrow}$ denote the set of non-empty finite sets $\{s_i : i < n\}$ for which there is a δ such that each $s_i \in S_\delta$. The index ordering on the tuples in $S^{\rightarrow}$ will always be the lexicographic ordering of S as a subtree of $2^{<\omega_1}$. We will be using the (s, s')-transfer construction from Definition 2.5 when s forces that $\dot{A}$ is a subset of some power of Y. Here is another such definition. It is simply a suggestive terminology for transfering a subset of $Y^{\{s_i : i<n\}}$ to a subset of $Y^{\{s_i \oplus s : i<n\}}$.

Definition 3.13. *If $\{s_i : i < n\} \in S^{\rightarrow}$ and $s \in S$ are such that $o(s_0) \leq o(s)$, then for $\vec{y} \in Y^{\{s_i : i<n\}}$, let $(\vec{y})^{\oplus s}$ be the element $\vec{z}$ of $Y^{\{s_i \oplus s : i<n\}}$ defined by $\vec{z}(s_i \oplus s) = \vec{y}(s_i)$ for each $i < n$. Similarly, for an S-name*
goup *$\dot{A}$ such that $s \Vdash \dot{A} \subset Y^{\{s_i : i<n\}}$, the name $\dot{A}^{\oplus s}$ is the name defined by the property that: for each $s < s'$ and $\vec{y} = \langle y_i : i < n\rangle \in Y^{\{s_i : i<n\}}$, $s' \Vdash \vec{y} \in \dot{A}$ if and only if $s' \Vdash (\vec{y})^{\oplus s} \in \dot{A}^{\oplus s}$.*

If g is a generic filter for S, $(\vec{y})^{\oplus g} \in Y^{\{s_i \oplus g : i<n\}}$ and $\dot{A}^{\oplus g}$ are defined analogously for $\{s_i : i < n\}$, $\vec{y}$, and $\dot{A}$ as above.

Suppose that $\dot{A}$, $\{s_i : i < n\}$, and $s \in S$ are as in the definition. If $B \subset Y^{\{s_i : i<n\}}$ is a set in the ground model, then $B^{\oplus s}$ is similarly defined. Then $B^{\oplus s} \subset Y^{\{s_i \oplus s : i<n\}}$ and $s \Vdash \dot{A}^{\oplus s} \subset Y^{\{s_i \oplus s : i<n\}}$. Also, $\dot{A}^{\oplus s}$ means the same thing as $\dot{A}^{\oplus \check{s}}$ in the forcing language. Since Y and S are elements of $H(\aleph_2)$, for any $\{s_i : i < n\} \in S^{\rightarrow}$ and S-name $\dot{A}$ of a subset of $Y^{\{s_i : i<n\}}$, there is an S-name $\dot{B}$ in $H(\aleph_2)$ such that $1 \Vdash \dot{B} = \dot{A}$.

Definition 3.14. *Let $\mathcal{V}$ denote the set of all tuples $(s, \{s_i : i < n\}, \dot{F})$ where*

(1) $s \in S$, $\{s_i : i < n\} \in S^{\to}$, $o(s_0) \le o(s)$,
(2) $\dot{F} \in H(\aleph_2)$ *is an S-name and* $s \Vdash \dot{F} \ne \emptyset$, *and*
(3) 1 *forces that* $\dot{F} = \dot{F}^{(\omega_1)} \subset Y^{\{s_i : i<n\}}$.

For each $\alpha \in \omega_1$, let $\mathcal{V}_\alpha$ be the set of $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{V}$ satisfying that $o(s) \le \alpha$.

For this next definition we remind the reader that the (s, s')-transfer $(\dot{A})^s_{s'}$ was defined in Definition 2.5 and that $\emptyset$ is the root of S.

Definition 3.15. *A family $\mathcal{SB} \subset \mathcal{V}$ is a symmetric $\mathcal{F}_0$-filter base if it includes the family $\{(\emptyset, \{\emptyset\}, \check{F}) : F \in \mathcal{F}_0\}$ and for each $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}$,* symmet

(1) $(\tilde{s}, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}$ *for each* $s < \tilde{s} \in S$,
(2) *for each* $\tilde{s} \in S_{o(s)}$, $(\tilde{s}, \{s_i \oplus s : i < n\}, (\dot{F}^{\oplus s})^s_{\tilde{s}}) \in \mathcal{SB}$,
(3) *for each* $\ell_0 < \omega$ *and* $\{(s, \{s^\ell_i : i < n_\ell\}, \dot{F}_\ell) : \ell < \ell_0\} \subset \mathcal{SB}$, *there is an* $(s, \{\tilde{s}_i : i < m\}, \dot{H}) \in \mathcal{SB}$ *such that for each* $s < s' \in S$ *and* $\vec{y} \in Y^{\{\tilde{s}_i : i<m\}}$,
  (a) *for each* $\ell < \ell_0$, $\{s^\ell_i \oplus s : i < n_\ell\} \subset \{\tilde{s}_i : i < m\}$, *and*
  (b) *if* $s' \Vdash \vec{y} \in \dot{H}$, *then, for each* $\ell < \ell_0$, $s' \Vdash \vec{y} \restriction \{s^\ell_i \oplus s : i < n_\ell\} \in \dot{F}^{\oplus s}_\ell$.

The intuition behind Definition 3.15 is that we are actually building a filter in $V[g]$ (g any S-generic branch) on the product space Y^{bS} where bS is the set of all ω_1-branches of S in $V[g]$. An element $(s, \{s_i : i < n\}, \dot{F})$ of $\mathcal{SB}$ with $s \in g$ corresponds to the set of elements $\vec{y} \in Y^{bS}$ satisfying that $\vec{y} \restriction \{s_i \oplus g : i < n\}$ is in $val_g(\dot{F}^{\oplus g})$. Conditions (1) and (2) combine to ensure that, for all $s \le \tilde{s} \in g$, we have $(\tilde{s}, \{s_i \oplus \tilde{s} : i < n\}, \dot{F}^{\oplus s})$ in $\mathcal{SB}$. More roughly speaking, $(val_g(\dot{F}_\alpha))^{\oplus g}$ is a projection of a member of that filter. Condition (3) is the finite directedness property for the above described filter on Y^{bS}. Condition (2) is the symmetry condition which is designed to ensure that the filter described above in $V[g]$ is the same filter as we will get in $V[s \oplus g]$ for any $s \in S$. The connection to (2) is that if s is in g and $g' = \tilde{s} \oplus g$ then $val_g(\dot{F}^{\oplus g})$ is equal to $val_{g'}(\ (\dot{F}^{\oplus s})^s_{\tilde{s}} \ )^{\oplus g'}$. We again mention the fact that if $s < \tilde{s}$ and $s' \in S_{o(\tilde{s})}$, it may happen that $\tilde{s}$ forces that $\dot{F}$ is disjoint from $(\dot{F})^{s'}_{\tilde{s}}$. Now each of $(\tilde{s}, \{s_i \oplus \tilde{s} : i < n\}, \dot{F}^{\oplus s})$ and $(\tilde{s}, \{s_i \oplus s' : i < n\}, (\dot{F}^{\oplus s'})^{s'}_{\tilde{s}} = ((\dot{F})^{s'}_{\tilde{s}})^{\oplus s'})$ will be in $\mathcal{SB}$, but fortunately the index sets $\{s_i \oplus \tilde{s} : i < n\}$ and $\{s_i \oplus s' : i < n\}$ are disjoint.

Lemma 3.16. *There is a symmetric $\mathcal{F}_0$-filter $\mathcal{SB}_{max}$ satisfying the* ismaxl *additional maximality condition*

(4) *for each* $s \in S$ *and* $\{s_i : i < n\} \in S^{\to}$ *with* $o(s_0) = o(s)$, s *forces that* $\mathcal{F}_{s,\{s_i:i<n\}}$ *is a maximal filter of* S*-sequentially closed subsets of* $Y^{\{s_i:i<n\}}$, *where* $\mathcal{F}_{s,\{s_i:i<n\}}$ *is defined to be the set* $\{\dot{F} : (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_{max}\}$.

Proof. For any family $\mathcal{G} \subset \mathcal{V}$, define the upward closure of $\mathcal{G}$ to be

$$\{(s, \{s_i : i < n\}, \dot{H}) \in \mathcal{V} : (\exists \dot{F})\, (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{G} \text{ and } s \Vdash \dot{H} \supset \dot{F}\}$$

and let $\mathcal{G}^{\dagger} = \{(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{G} : o(s_0) = o(s)\}$. For each $\alpha \in \omega_1$ and each $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{V}^{\dagger}_{\alpha}$, let $\dot{F}(S_\alpha)$ be an S-name for the set $\{\vec{y} \in Y^{S_\alpha} : \vec{y} \restriction \{s_i : i < n\} \in \dot{F}\}$. We say that the support of $\dot{F}(S_\alpha)$ is $\{s_i : i < n\}$. Just for clarity we note that for any $s < t \in S$ and $\vec{y} \in Y^{S_\alpha}$,

$$t \Vdash \vec{y} \in \dot{F}(S_\alpha) \text{ if and only if } t \Vdash \vec{y} \restriction \{s_i : i < n\} \in \dot{F} .$$

For any $\beta < \alpha < \omega_1$ and $\mathcal{G} \subset \mathcal{V}_\beta$, let

$$\mathcal{G}^{\uparrow\alpha} = \{(s \oplus \tilde{s}, \{s_i \oplus \tilde{s} : i < n\}, \dot{F}^{\oplus\tilde{s}}) : \tilde{s} \in S_\alpha,\ (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}^{\dagger}\} .$$

We inductively construct an increasing chain $\{\mathcal{SB}_\alpha : \alpha \in \omega_1\}$ where $\mathcal{SB}_\alpha \subset \mathcal{V}_\alpha$ and then prove that $\mathcal{SB}_{max} = \bigcup\{\mathcal{SB}_\alpha : \alpha \in \omega_1\}$ has the properties (1)-(4). We start the induction by choosing any S-name $\dot{\mathcal{F}}_0$ of a maximal filter of S-sequentially closed subsets of Y that extends $\mathcal{F}_0$. We set

$$\mathcal{SB}_0 = \{(\emptyset, \{\emptyset\}, \dot{F}) \in \mathcal{V}_0 : 1 \Vdash \dot{F} \in \dot{\mathcal{F}}_0\} .$$

Suppose $0 < \alpha \in \omega_1$ and that we have constructed the increasing sequence $\{\mathcal{SB}_\beta : \beta < \alpha\}$ satisfying the following inductive hypotheses for each $\gamma < \beta < \alpha$:

(1) $\mathcal{SB}_\beta \subset \mathcal{V}_\beta$, $(\mathcal{SB}_\gamma \cup \mathcal{SB}_\gamma^{\uparrow\beta}) \subset \mathcal{SB}_\beta$,

bind2 (2) $(s \oplus \tilde{s}, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_\beta$ if $\tilde{s} \in S_\beta$ and $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_\gamma$,

bind3 (3) $(\tilde{s}, \{s_i : i < n\}, (\dot{F})^{s}_{\tilde{s}}) \in \mathcal{SB}_\beta$ if $s, \tilde{s} \in S_\beta$, and $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}^{\dagger}_\beta$,

bind4 (4) $\mathcal{SB}^{\dagger}_\beta$ is finitely directed: for any $\{(s, \{s^\ell_i : i < n_\ell\}, \dot{F}_\ell) : \ell < \ell_0\} \subset \mathcal{SB}^{\dagger}_\beta$ ($\ell_0 \in \omega$), there is an $\dot{H}$ such that $(s, \bigcup\{\{s^\ell_i : i < n_\ell\} : \ell < \ell_0\}, \dot{H}) \in \mathcal{SB}^{\dagger}_\beta$ and s forces that $\vec{y} \restriction \{s^\ell_i : i < n_\ell\} \in \dot{F}_\ell$ for each $\vec{y} \in \dot{H}$,

bind5 (5) for each $s \in S_\beta$ and $\{s_i : i < n\} \in S^{\to}$ with $o(s_0) = \beta$, s forces that $\{\dot{F} : (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}^{\dagger}_\beta\}$ is a maximal filter on the S-sequentially closed subsets of $Y^{\{s_i:i<n\}}$.

In case α is a limit, let $\mathcal{SB}_\alpha$ be the upward closure of the union of

$$\{(s\oplus\tilde s,\{s_i:i<n\},\dot F):\tilde s\in S_\alpha \text{ and } (s,\{s_i:i<n\},\dot F)\in\bigcup_{\beta<\alpha}\mathcal{SB}_\beta\}$$

and $\bigcup_{\gamma<\alpha}(\mathcal{SB}_\gamma\cup\mathcal{SB}_\gamma^{\uparrow\alpha})$. Induction hypotheses (1) and (2) are immediate. For (3), let $s,\tilde s\in S_\alpha$ and $(s,\{s_i:i<n\},\dot F)\in\mathcal{SB}^\dagger_\alpha$. It follows from the definition that $(s,\{s_i:i<n\},\dot F)$ must be in $\mathcal{SB}_\gamma^{\uparrow\alpha}$ for some $\gamma<\alpha$. That is, there is an S-name $\dot H$ such that $\dot F=\dot H^{\oplus s}$ and $(s{\restriction}\gamma,\{s_i{\restriction}\gamma: i<n\},\dot H)\in\mathcal{SB}^\dagger_\gamma$. Choose $\beta\ge\gamma$ so that $s{\restriction}[\beta,\alpha)=\tilde s{\restriction}[\beta,\alpha)$. By inductive assumptions (1) and (2), $(s{\restriction}\beta,\{s_i{\restriction}\beta:i<n\},\dot H^{\oplus s\restriction\beta})$ is in $\mathcal{SB}^\dagger_\beta$. By (3), $(\tilde s{\restriction}\beta,\{s_i{\restriction}\beta:i<n\},(\dot H^{\oplus s\restriction\beta})^{s\restriction\beta}_{\tilde s\restriction\beta})$ is in $\mathcal{SB}_\beta$. By the choice of β, $\{s_i:i<n\}$ is equal to $\{s_i{\restriction}\beta\oplus\tilde s:i<n\}$, so we have that $(\tilde s,\{s_i:i<n\},(\ (\dot H^{\oplus s\restriction\beta})^{s\restriction\beta}_{\tilde s\restriction\beta}\ )^{\oplus\tilde s})$ is in $\mathcal{SB}_\alpha$. Now it follows that $(\tilde s,\{s_i:i<n\},\dot F^s_{\tilde s})$ is in $\mathcal{SB}_\alpha$ simply because we took the upward closure and

$$(\ (\dot H^{\oplus s\restriction\beta})^{s\restriction\beta}_{\tilde s\restriction\beta}\ )^{\oplus\tilde s}=(\ (\dot H^{\oplus s\restriction\beta})^{s\restriction\beta}_{\tilde s\restriction\beta}\ )^{\oplus s}=(\ (\dot H)^{s\restriction\beta}_{\tilde s\restriction\beta}\ )^{\oplus s}\supset\dot F^s_{\tilde s}\ .$$

To verify induction hypothesis (5), fix any $s\in S_\alpha$ and $\{s_i:i<n\}\in S^\rightarrow$ with $o(s_0)=\alpha$. Choose $\beta<\alpha$ so that there is a $\{\bar s_i:i<n\}\in S^\rightarrow$ and $s_i=\bar s_i\oplus s$ for each $i<n$. By the induction hypotheses,

$$s\restriction\beta\Vdash\{\dot F:(s{\restriction}\beta,\{\bar s_i:i<n\},\dot F)\in\mathcal{SB}^\dagger_\beta\}$$

is a maximal filter on the sequentially closed subsets of $Y^{\{\bar s_i:i<n\}}$. Therefore, it is immediate that s forces that$\{\dot F^{\oplus s}:(s{\restriction}\beta,\{\bar s_i:i<n\},\dot F)\in\mathcal{SB}^\dagger_\beta\}$ is a maximal filter on the sequentially closed subsets of $Y^{\{s_i:i<n\}}$. To complete this limit step we just have to verify that for $s\in S_\alpha$, $\mathcal{SB}^\dagger_\alpha$ is finitely directed as in (4). It should be clear that it suffices to consider those elements that were not added by taking the upward closure. Let $\ell_0\in\omega$ and $\{(s,\{s^\ell_i:i<n_\ell\},\dot F_\ell):\ell<\ell_0\}$ be a subset of $\mathcal{SB}^\dagger_\alpha$. It follows from the definition of $\mathcal{SB}_\alpha$, and the fact that α is a limit, that there is a $\beta<\alpha$ such that $\{(s,\{s^\ell_i:i<n_\ell\},\dot F_\ell):\ell<\ell_0\}$ is a subset of $\bigcup_{\gamma<\beta}\mathcal{SB}_\gamma^{\uparrow\alpha}$. For each $\ell<\ell_0$, there is an $\dot H_\ell$ so that $\dot F_\ell=\dot H_\ell^{\oplus s}$ and $(s{\restriction}\beta,\{s^\ell_i{\restriction}\beta:i<n_\ell\},\dot H)\in\mathcal{SB}^\dagger_\beta$. Applying (1) and (4) to β, we may choose $\dot H$ so that

$$(s{\restriction}\beta,\bigcup\{\{s^\ell_i{\restriction}\beta:i<n_\ell\}:\ell<\ell_0\},\dot H)\in\mathcal{SB}^\dagger_\beta$$

so that s forces that $\vec y\restriction\{s^\ell_i{\restriction}\beta:i<n_\ell\}\in\dot H_\ell$ for each $\vec y\in\dot H$. Therefore $(s,\{s_i:i<n\},\dot H^{\oplus s})$ is the required member of $\mathcal{SB}^\dagger_\alpha$ to complete the verification of (4).

Now assume that $\alpha = \beta + 1$. In preparation for defining $\mathcal{SB}_\alpha$, let $\mathcal{G}_\alpha$ be the upward closure of the union of $\mathcal{SB}_\beta$, $\mathcal{SB}_\beta^{\uparrow\alpha}$, and the family

$$\{(s \oplus \tilde{s}, \{s_i : i < n\}, \dot{F}) : \tilde{s} \in S_\alpha \text{ and } (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_\beta\} \ .$$

If $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}_\alpha^\dagger$ and $s \in S_\alpha$, then $(s, \{s_i : i < n\}, \dot{F}) \in \left(\mathcal{SB}_\beta^\dagger\right)^{\uparrow\alpha}$. For each $s \in S_\alpha$, let $\mathcal{G}_\alpha(s)$ be the set of $(s, \{s_i : i < n\}, \dot{F})$ that are in $\mathcal{G}_\alpha^\dagger$. Since $\mathcal{SB}_\beta^\dagger$ is finitely directed, it is immediate that $\mathcal{G}_\alpha(s)$ is finitely directed for each $s \in S_\alpha$. We will define a special collection $\mathcal{H}_\alpha$ of S-names of S-sequentially closed subsets of Y^{S_α}.

For the remainder of the proof fix any $s \in S_\alpha$ and let $s^\dagger \in S_\alpha \setminus \{s\}$ be the other successor of $s{\restriction}\beta$. Let

$$\mathcal{H}_\alpha(s) = \{\dot{F}(S_\alpha) : (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}_\alpha(s)\}$$

and similarly

$$\mathcal{H}_\alpha(s^\dagger) = \{\dot{F}(S_\alpha) : (s^\dagger, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}_\alpha(s^\dagger)\} \ .$$

The fact that each of $\mathcal{G}_\alpha(s)$ and $\mathcal{G}_\alpha(s^\dagger)$ are finitely directed easily implies that each of $\mathcal{H}_\alpha(s)$ and $\mathcal{H}_\alpha(s^\dagger)$ are similarly finitely directed. We define a collection $(\mathcal{H}_\alpha(s^\dagger))_s$ by

$$(\mathcal{H}_\alpha(s^\dagger))_s = \{(\dot{F}_s^{s^\dagger})(S_\alpha) : (s^\dagger, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}_\alpha\} \ .$$

It should be clear that for any $t \in S$, $\vec{y} \in Y^{S_\alpha}$, and $(s^\dagger, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}_\alpha^\dagger$,

$$s \oplus t \Vdash \vec{y} \in (\dot{F}_s^{s^\dagger})(S_\alpha) \ \text{ iff } \ s^\dagger \oplus t \Vdash \vec{y} \in \dot{F}(S_\alpha) \tag{1}$$

We also note that the support of each member of $\mathcal{H}_\alpha(s)$ is disjoint from the support of each member of $\mathcal{H}_\alpha(s^\dagger)$. This is simply because for any $\tilde{s} \in S_\alpha$, $s'(\beta)$ will equal $\tilde{s}(\beta)$ for any s' in the support of any element of $\mathcal{H}_\alpha(\tilde{s})$. Now we set

$$\mathcal{H}_\alpha = \mathcal{H}_\alpha(s) \ \cup \ (\mathcal{H}_\alpha(s^\dagger))_s$$

and prove that s forces that $\mathcal{H}_\alpha$ is finitely directed. Pick any pair $(s, \{s_i^1 : i < n_1\}, \dot{F}_1), (s^\dagger, \{s_i^2 : i < n_2\}, \dot{F}_2)$ from $\mathcal{G}_\alpha^\dagger$. Let t be any extension of s in S. Choose $t \leq t^1$ and $\vec{y}_1 \in Y^{\{s_i^1 : i < n_1\}}$ so that $t^1 \Vdash \vec{y}_1 \in \dot{F}_1$. Then choose any $t^1 \leq t^2$ and $\vec{y}_2 \in Y^{\{s_i^2 : i < n_2\}}$ so that $s^\dagger \oplus t^2 \Vdash \vec{y}_2 \in \dot{F}_2$. Since the support of $\dot{F}_1$ is disjoint from the support of $\dot{F}_2$, we may choose a $\vec{y} \in Y^{S_\alpha}$ satisfying $\vec{y} \restriction \{s_i^1 : i < n_1\} = \vec{y}_1$ and $\vec{y} \restriction \{s_i^2 : i < n_2\} = \vec{y}_2$. We have shown that $t^2 \Vdash \vec{y} \in \dot{F}_1(S_\alpha)$ and $s^\dagger \oplus t^2 \Vdash \vec{y} \in \dot{F}_2(S_\alpha)$. It follows from equation (1) that $t^2 \Vdash \vec{y} \in ((\dot{F}_2)_s^{s^\dagger})(S_\alpha)$. Since t was arbitrary and each of $\mathcal{H}_\alpha(s)$ and $\mathcal{H}_\alpha(s^\dagger)$

are finitely directed, this completes the proof that s forces that $\mathcal{H}_\alpha$ is finitely directed.

Concerning induction hypotheses (3), we note that for all $\tilde{s} \in S_\alpha$ and $(\tilde{s}, \{s_i : i < n\}, \dot{F})$ from $\mathcal{G}_\alpha(\tilde{s})$, if $s(\beta) = \tilde{s}(\beta)$, then $(s, \{s_i : i < n\}, \dot{F}_s^{\tilde{s}}) \in \mathcal{G}_\alpha(s)$, and otherwise, $(s^\dagger, \{s_i : i < n\}, \dot{F}_{s^\dagger}^{\tilde{s}}) \in \mathcal{G}_\alpha(s^\dagger)$. It then follows that $\mathcal{H}_\alpha$ has the property that for each $\tilde{s} \in S_\alpha$ and $(\tilde{s}, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}_\alpha(\tilde{s})$, $(\dot{F}(S_\alpha))_s^{\tilde{s}}$ is in $\mathcal{H}_\alpha$. Let us abbreviate this statement by saying that inductive hypothesis (3) holds for $\mathcal{H}_\alpha$.

Now we can choose an S-name $\dot{\mathcal{F}}_\alpha$ of a maximal filter of S-sequentially closed subsets of Y^{S_α} that is forced by s to include the collection $\mathcal{H}_\alpha$. We define

$$\mathcal{SB}_\alpha(s) = \{(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{V}_\alpha^\dagger : s \Vdash \dot{F}(S_\alpha) \in \dot{\mathcal{F}}_\alpha\} \ .$$

It is useful to note that inductive hypothesis (3) holding for $\mathcal{H}_\alpha$ ensures that we have that $(s, \{s_i : i < n\}, (\dot{F})_s^{\tilde{s}})$ is in $\mathcal{SB}_\alpha(s)$ for each $(\tilde{s}, \{s_i : i < n\}, \dot{F}) \in \mathcal{G}_\alpha^\dagger$. It is clear, by the maximality of $\dot{\mathcal{F}}_\alpha$, that for each $\{s_i : i < n\} \subset S_\alpha$, s forces that, as in induction hypothesis (5), $\{\dot{F} : (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_\alpha(s)\}$ is a maximal filter on the S-sequentially closed subsets of $Y^{\{s_i : i<n\}}$. Similarly it follows that, for each $\tilde{s} \in S_\alpha$, $\tilde{s}$ forces that $\{\dot{F}_{\tilde{s}}^s : (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_\alpha(s)\}$ is a maximal filter on the S-sequentially closed subsets of $Y^{\{s_i : i<n\}}$.

We are ready to define $\mathcal{SB}_\alpha$ in this successor case:

$$\begin{aligned} &\mathcal{SB}_\alpha = \mathcal{G}_\alpha \ \cup \ \mathcal{SB}_\beta \cup \\ &\{(\tilde{s} \oplus t, \{s_i : i < n\}, \dot{F}) : t \in S_\alpha \text{ and } (\tilde{s}, \{s_i : i < n\}, \dot{F}) \in \bigcup_{\gamma<\alpha} \mathcal{SB}_\gamma\} \\ \cup &\{(\tilde{s}, \{s_i : i < n\}, (\dot{F})_{\tilde{s}}^s) : \tilde{s} \in S_\alpha, \text{ and } (s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_\alpha(s)\} \ . \end{aligned}$$

We check that the inductive hypotheses hold for $\{\mathcal{SB}_\beta : \beta < \alpha + 1\}$. Inductive hypotheses (1) and (2) are immediate, and, so long as (4) holds, (5) was proven above. Hypothesis (4) for members of $\mathcal{SB}_\alpha(s)$ was verified when we proved that $\mathcal{H}_\alpha$ was finitely directed. We complete the verification of (4) after verifying (3).

Towards verifying (3), let t play the role of s in the statement (3) and suppose that $(t, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_\alpha^\dagger \setminus \bigcup_{\beta<\alpha} \mathcal{SB}_\beta$ and let $\tilde{s} \in S_\alpha$. Since $o(s_0) = \alpha$, we have that either $(t, \{s_i : i < n\}, \dot{F})$ is in $\mathcal{G}_\alpha$ or that $\dot{F}$ is equal to $(\dot{H})_t^s$ for some $(s, \{s_i : i < n\}, \dot{H}) \in \mathcal{SB}_\alpha(s)$. If $(t, \{s_i : i < n\}, \dot{F})$ is in $\mathcal{G}_\alpha$ then the induction hypothesis (3) applied to some $\beta < \alpha$ ensures that $(\tilde{s}, \{s_i : i < n\}, (\dot{F})_{\tilde{s}}^t) \in \mathcal{G}_\alpha$. So now assume that there is some $\dot{H}$ as above. By the definition of $\mathcal{SB}_\alpha$,

$(\tilde{s}, \{s_i : i < n\}, (\dot{H})^s_{\tilde{s}})$ is also in $\mathcal{SB}_\alpha$, and this completes the verification since $(\dot{F})^t_{\tilde{s}} = ((\dot{H})^s_t\ )^t_{\tilde{s}} = (\dot{H})^s_{\tilde{s}}$.

We complete the verification of (4). Let $\{(\tilde{s}, \{s^\ell_i : i < n_\ell\}, \dot{F}_\ell) : \ell < \ell_0\} \subset \mathcal{SB}^\dagger_\alpha$. As proven above, the family $\{(s, \{s^\ell_i : i < n_\ell\}, (\dot{F}_\ell)^{\tilde{s}}_s) : \ell < \ell_0\}$ is a subset of $\mathcal{SB}_\alpha(s)$. Since (4) holds for this collection, we choose $\dot{H}$ such that $(s, \bigcup\{\{s^\ell_i : i < n_\ell\} : \ell < \ell_0\}, \dot{H}) \in \mathcal{SB}_\alpha(s)$, and s forces that $\vec{y} \restriction \{s^\ell_i : i < n_\ell\} \in (\dot{F}_\ell)^{\tilde{s}}_s$ for each $\vec{y} \in \dot{H}$. By the definition of $\mathcal{SB}_\alpha$, $(\tilde{s}, \bigcup\{\{s^\ell_i : i < n_\ell\} : \ell < \ell_0\}, (\dot{H})^s_{\tilde{s}})$ is in $\mathcal{SB}_\alpha$. Now suppose that $\tilde{s} < t$ and t forces that $\vec{y} \in (\dot{H})^s_{\tilde{s}}$. It follows that $s \oplus t \Vdash \vec{y} \in \dot{H}$ and so $s \oplus t \Vdash \vec{y} \restriction \{s^\ell_i : i < n_\ell\} \in (\dot{F}_\ell)^{\tilde{s}}_s$ (for each $\ell < \ell_0$). The final unraveling is that $\tilde{s} \oplus t = t \Vdash \vec{y} \restriction \{s^\ell_i : i < n_\ell\} \in (\ (\dot{F}_\ell)^{\tilde{s}}_s\ )^s_{\tilde{s}} = \dot{F}_\ell$.

Evidently, the induction hypotheses holding for $\mathcal{SB}_{max}$ implies that $\mathcal{SB}_{max}$ is a symmetric $\mathcal{F}_0$-filter base, and this completes the proof. ☐

needA **Definition 3.17.** *Let $\mathcal{A}$ denote the family of all $(s, \{s_i : i < n\}, \dot{A})$ such that*

(1) *$s \Vdash \dot{A} \subset Y^{\{s_i : i<n\}}$ and $(s, \{s_i : i < n\}, \dot{A}^{(\omega_1)}) \in \mathcal{V}$,*
(2) *for all $\dot{F}$ such that $(s, \{s_i \oplus s : i < n\}, \dot{F}) \in \mathcal{SB}_{max}$, $s \Vdash \dot{A}^{\oplus s} \cap \dot{F} \neq \emptyset$.*

The family $\mathcal{A}$ has properties analogous to those of symmetric $\mathcal{F}_0$-filter bases.

goupLem **Lemma 3.18.** *If $(s, \{s_i\colon i < n\}, \dot{A}) \in \mathcal{A}$, then $(\tilde{s}, \{s_i \oplus \tilde{s}\colon i < n\}, \dot{A}^{\oplus\tilde{s}}) \in \mathcal{A}$ for all $s \leq \tilde{s}$. If $o(s_0) = o(s) = o(t)$ and $(s, \{s_i\colon i < n\}, \dot{A}) \in \mathcal{A}$, then $(s, \{s_i : i < n\}, \dot{A}^{(\omega_1)})$ is in $\mathcal{SB}_{max}$ and $(t, \{s_i : i < n\}, \dot{A}^s_t) \in \mathcal{A}$.*

Proof. Let $(s, \{s_i\colon i < n\}, \dot{A}) \in \mathcal{A}$ and $s \leq \tilde{s}$. Let $\dot{F}$ be an S-name such that $(\tilde{s}, \{s_i \oplus \tilde{s} : i < n\}, \dot{F}) \in \mathcal{SB}_{max}$. To prove that $(\tilde{s}, \{s_i \oplus \tilde{s} : i < n\}, \dot{A}^{\oplus\tilde{s}})$ is in $\mathcal{A}$, we just have to prove that $\tilde{s} \Vdash \dot{A}^{\oplus\tilde{s}} \cap \dot{F}$ is not empty. Since $s \leq \tilde{s}$, this is equivalent to proving that $\tilde{s} \Vdash (\dot{A}^{\oplus s})^{\oplus\tilde{s}} \cap \dot{F}$ is not empty. By property (4) of $\mathcal{SB}_{max}$, we have that s forces $\mathcal{F}_{s,\{s_i \oplus s : i<n\}}$ is maximal. It therefore also follows that $\tilde{s}$ forces that $\{\dot{H}^{\oplus\tilde{s}} : (s, \{s_i \oplus s : i < n\}, \dot{H}) \in \mathcal{SB}_{max}\}$ is maximal. In addition, by properties (1) and (2) of $\mathcal{SB}_{max}$, $\tilde{s}$ forces that $\{\dot{H}^{\oplus\tilde{s}} : (s, \{s_i \oplus s : i < n\}, \dot{H}) \in \mathcal{SB}_{max}\}$ is equal to $\mathcal{F}_{\tilde{s},\{s_i \oplus \tilde{s} : i<n\}}$. Since s forces that $\dot{A}^{\oplus s} \cap \dot{H}$ is not empty for all $(s, \{s_i \oplus s : i < n\}, \dot{H}) \in \mathcal{SB}_{max}$, it follows that $\tilde{s} \Vdash \dot{A}^{\oplus\tilde{s}} \cap \dot{F}$ is not empty.

Now assume that $o(s_0) = o(s)$ and note that $(s, \{s_i : i < n\}, \dot{A}^{(\omega_1)})$ is in $\mathcal{V}^\dagger_{o(s)}$. With the notation from Lemma 3.16, we claim that $s \Vdash \dot{A}^{(\omega_1)} \in \mathcal{F}_{s,\{s_i : i<n\}}$. Since s forces that $\mathcal{F}_{s,\{s_i : i<n\}}$ is a maximal filter of S-sequentially closed sets, it suffices to show that $s \Vdash \dot{A}^{(\omega_1)} \cap \dot{F}$ is not

empty for all S-names $\dot{F}$ such that $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_{max}$. Of course this is true since $s \Vdash \dot{A} \cap \dot{F}$ is not empty for all $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_{max}$. Now let $t \in S_{o(s)}$ and since $s_i \oplus t = s_i$ for each $i < n$, we consider $(t, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_{max}$ and show that $t \Vdash \dot{A}^s_t \cap \dot{F} \neq \emptyset$. Choose any extension $\tilde{t}$ of t. By property (2) of $\mathcal{SB}_{max}$, we have that $(s, \{s_i : i < n\}, \dot{F}^t_s) \in \mathcal{SB}_{max}$ and so we may find a $\vec{y} \in Y^{\{s_i : i<n\}}$ and an extension $\tilde{s}$ of $s \oplus \tilde{t}$ forcing that $\vec{y} \in \dot{A} \cap \dot{F}^t_s$. Naturally it follows that $t \oplus \tilde{s}$ forces that $\vec{y} \in \dot{A}^s_t$ and $\vec{y} \in \dot{F}$. □

This next lemma is the main ingredient for constructing a suitable S-preserving proper poset for the application of PFA(S). It asserts the existence, for any countable elementary submodel M, of a very special member of Y^{S_δ} where $M \cap \omega_1 = \delta$.

Lemma 3.19. *Suppose that M is a countable elementary submodel of $(H(\kappa), \prec)$ including S, Y, and $\mathcal{SB}_{max}$. Let $M \cap \omega_1 = \delta$ and let s_δ be the $\prec$-least element of S_δ. There is a sequence $\langle y^M(s) : s \in S_\delta \rangle \in Y^{S_\delta}$ such that, for each $\tilde{s} < s_\delta$ and $(\tilde{s}, \{s_i : i < n\}, \dot{A}) \in \mathcal{A} \cap M$ and $(\tilde{s}, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_{max} \cap M$,* Melement

1. *$y^M(s)$ has domain T^M_s,*
2. *$y^M(s) \restriction (T^M_s)_t$ is in M for each $t \in T^M_s$ above the root,*
3. *s_δ forces that $y^M \restriction \{s_i \oplus s_\delta : i < n\} \in \dot{F}^{\oplus s_\delta}$,*
4. *if, for each $s \in S_\delta$, $O(s)$ is a Seq_ω-open neighborhood of* $\mathrm{root}(T^M_s)$ *in T^M_s, there is a sequence $\{t_i : i < n\} \subset \omega^{<\omega}$ such that each $t_i \in O(s_i \oplus s_\delta)$ and there is a $\vec{b} \in M$ such that $s_\delta \Vdash \vec{b} \in \dot{A}$ and, for each $i < n$, $\vec{b}_i$ is congruent to $y^M(s_i \oplus s_\delta) \restriction (T^M_{s_i \oplus s_\delta})_{t_i}$.*

Proof. Let $\{s_{\delta,j} : j \in \omega\}$ be an enumeration of S_δ with $s_{\delta,0} = s_\delta$, and for each $\xi < \delta$, let $s_\xi = s_\delta \restriction \xi$. Choose an increasing sequence $\{\delta_\ell : \ell \in \omega\} \subset \delta$ cofinal in δ and satisfying that $(s_{\delta,\ell} \restriction \delta_\ell) \oplus s_\delta = s_{\delta,\ell}$ for each $\ell < \omega$. Fix an enumeration $\{(s^m, \{s^m_i : i < n_m\}, \dot{F}_m) : m \in \omega\}$ of all those $(s, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_{max} \cap M$ satisfying that $s < s_\delta$ and $o(s_0) = o(s)$. By induction, we choose a subsequence $\{\beta_\ell : \ell \in \omega\} \subset \omega$ and an accompanying sequence $\{\vec{y}_\ell : \ell \in \omega\} \subset M$ so that for all $k \leq \ell$

(i) $\delta_\ell < o(s^{\beta_\ell})$,
(ii) $s_\delta \Vdash \vec{y}_\ell \in \dot{F}_{\beta_\ell}$,
(iii) $s_{\delta,k} \in \{s^{\beta_\ell}_i \oplus s_\delta : i < n_{\beta_\ell}\}$,
(iv) $\{s^k_i \oplus s^{\beta_\ell} : i < n_k\} \subset \{s^{\beta_\ell} : i < n_{\beta_\ell}\}$,
(v) s_δ forces that $\vec{y}_\ell \restriction \{s^k_k \oplus s^{\beta_\ell} : i < n_k\} \in \dot{F}_k^{\oplus s^{\beta_\ell}}$.

To start, let $\xi_0 = \max(\delta_0, o(s^0)) + 1$. By properties (1) and (2) of Definition 3.15, each of $(s_{\xi_0}, \{s_{\delta,0} \restriction \xi_0\}, \check{Y}^{\{s_{\delta,0} \restriction \xi_0\}})$ and $(s_{\xi_0}, \{s^0_i \oplus s_{\xi_0} : i <$

$n_0\}, \dot{F}_0^{\oplus s_{\xi_0}})$ are in $\mathcal{SB}_{max}$. By property (3), there is a value $\beta_0 \in \omega$ so that $\{s_i^{\beta_0} : i < n_{\beta_0}\} \supseteq \{s_{\delta,0}\restriction\xi_0\} \cup \{s_i^0 \oplus s_{\xi_0} : i < n_0\}$ and s_δ forces that $\vec{y} \restriction \{s_i^0 \oplus s_{\xi_0} : i < n_0\} \in \dot{F}_0^{\oplus s_{\xi_0}}$ for all $\vec{y} \in \dot{F}_{\beta_0}$. Choose any $\vec{y}_0 \in M$ such that $s_\delta \Vdash \vec{y}_0 \in \dot{F}_{\beta_0}$.

Now assume we have chosen $\{\beta_k, \vec{y}_k : k < \ell\}$ and let $m_\ell \in \omega \setminus \ell$ be chosen so that $\beta_k \leq m_\ell$ for all $k < \ell$. Let $\xi_\ell < \delta$ be greater than the maximum of the set $\{\delta_\ell\} \cup \{o(s^k) : k \leq m_\ell\}$. Again, we have that each of $(s_{\xi_\ell}, \{s_{\delta,\ell}\restriction\xi_\ell\}, \check{Y}^{\{s_{\delta,\ell}\restriction\xi_\ell\}})$ and $(s_{\xi_\ell}, \{s_i^k \oplus s_{\xi_\ell} : i < n_k\}, \dot{F}_k^{\oplus s_{\xi_\ell}})$, for all $k \leq m_\ell$, are in $\mathcal{SB}_{max}$. By property (3) of Definition 3.15, we may choose a $\beta_\ell \in \omega$ so that, for each $k \leq m_\ell$, $\{s_{\delta,\ell}\restriction\xi_\ell\} \cup \{s_i^k \oplus s_{\xi_\ell} : i < n_k\} \subset \{s_i^{\beta_\ell} : i < n_{\beta_\ell}\}$ and s_δ forces that $\vec{y} \restriction \{s_i^k \oplus s_{\xi_\ell} : i < n_k\} \in \dot{F}_k^{\oplus s_{\xi_\ell}}$ for each $\vec{y} \in \dot{F}_{\beta_\ell}$. Choose any $\vec{y}_\ell \in M$ such that $s_\delta \Vdash \vec{y}_\ell \in \dot{F}_{\beta_\ell}$. We have $s_{\delta,\ell} \oplus s_{\xi_\ell} \in \{s_i^{\beta_\ell} : i < n_{\beta_\ell}\}$, hence $s_{\delta,\ell} \in \{s_i^{\beta_\ell} \oplus s_\delta : i < n_{\beta_\ell}\}$. For each $k < \ell$, we have, by the induction hypotheses, that $s_{\delta,k} \in \{s_i^{\beta_k} \oplus s_\delta : i < n_{\beta_k}\}$ and so $s_{\delta,k}\restriction\xi_\ell \in \{s_i^{\beta_\ell} : i < n_{\beta_\ell}\}$. Therefore $s_{\delta,k} \in \{s_i^{\beta_\ell} \oplus s_\delta : i < n_{\beta_\ell}\}$ for each $k \leq \ell$. This completes the recursive construction of the sequence $\{\beta_\ell : \ell \in \omega\}$ and $\{\vec{y}_\ell : \ell \in \omega\}$.

For each $\ell \in \omega$ and $i < n_{\beta_\ell}$, $\vec{y}_\ell(s_i^{\beta_\ell})$ is in Y and so, we recall from Definition 3.7, that, for each $t \in T_{\vec{y}_\ell(s_i^{\beta_\ell})}$, $(\vec{y}_\ell(s_i^{\beta_\ell}))(t)$ can be interpreted as an S-name of an element of $\dot{X}$. In particular, we introduced the notation $e(\vec{y}_\ell(s_i^{\beta_\ell}))$ to denote the S-name for the element so named by $(\vec{y}_\ell(s_i^{\beta_\ell}))(\mathrm{root}(T_{\vec{y}_\ell(s_i^{\beta_\ell})}))$. By a simple length ω recursion using Lemma 3.6, choose a decreasing sequence $\{J_m : m \in \omega\}$ of infinite subsets of ω satisfying that, for all $m \in \omega$, $\{e(\vec{y}_\ell(s_{\delta,m}\restriction o(s^{\beta_\ell}))) : m < \ell \in J_m\}$ is an S-converging sequence in $\dot{X}$. Let J be an infinite set that is almost included in each J_m. Let $\{j_k : k \in \omega\}$ be an order-preserving indexing of J. For each integer $k \geq \ell$, $\ell \leq j_k$ and so $s_{\delta,\ell}\restriction o(s^{\beta_{j_k}})$ is in $\{s_i^{\beta_{j_k}} : i < n_{\beta_{j_k}}\}$. This means that, for each $m \in \omega$, the sequence $\langle e(\vec{y}_{j_k}(s_{\delta,m}\restriction o(s^{\beta_{j_k}}))) : m < k \in \omega\rangle$ is well-defined and is S-converging. We choose $y^M(s_{\delta,m}) \in Y$ to be the limit in Y where the root of $T^M_{y^M(s_{\delta,m})}$ is the empty sequence and so that

$$y^M(s_{\delta,m}) \restriction (T^M_{y^M(s_{\delta,m})})_{\langle k\rangle} \approx \vec{y}_{j_k}(s_{\delta,m}\restriction o(s^{\beta_{j_k}}))$$

for each $m < k \in \omega$. Using a small abuse of indexing notation, this is what we have accomplished: for each $\ell \in \omega$,

$$\{\vec{y}_{j_k} \restriction \{s_i^\ell \oplus s^{\beta_{j_k}} : i < n_\ell\} : \ell < k \in \omega\}\ Y\text{-converges to } y^M \restriction \{s_i^\ell \oplus s_\delta : i < n_\ell\}\,.$$

More formally, consider any $\ell \in \omega$. For each $\ell < k$, let $\vec{z}^{\,\ell}_k$ be the vector $\langle \vec{y}_{j_k}(s_i^\ell \oplus s^{\beta_{j_k}}) : i < n_\ell\rangle$, then by induction hypothesis (iv), we have that

$\vec{z}^{\,\ell}_k$ is an element of Y^{n_ℓ}. Also, let $\vec{z}^{\,\ell}_\omega$ be the vector $\langle y^M(s^\ell_i \oplus s_\delta) : i < n_\ell\rangle$. Then we do indeed have that $\{\vec{z}^{\,\ell}_k : \ell < k \in \omega\}$ Y-converges to $\vec{z}^{\,\ell}_\omega$. By inductive condition (v), and the fact that $s^{\beta_k} < s_\delta$ for all k, s_δ forces that $\vec{y}_{j_k} \restriction \{s^\ell_i \oplus s_\delta : i < n_\ell\}$ is in $\dot{F}^{\oplus s_\delta}_\ell$ for each $k > \ell$. This verifies that item (3) holds.

We finish by verifying the conclusion in (4) by considering any $\tilde{s} < s_\delta$ and $(\tilde{s}, \{s_i : i < n\}, \dot{A}) \in \mathcal{A} \cap M$. By Lemma 3.18 we may choose an integer ℓ so that $(s^\ell, \{s^\ell_i : i < n_\ell\}, \dot{F}_\ell)$ is equal to $(\tilde{s}, \{s_i \oplus \tilde{s} : i < n\}, (\dot{A}^{\oplus \tilde{s}})^{(\omega_1)})$. We are also given, for each $i < n$, a Seq_ω-open neighborhood, $O(s_i \oplus s_\delta)$, of $\mathrm{root}(T^M_{s_i \oplus s_\delta})$ in $T^M_{s_i\oplus s_\delta}$. That is, for each $i < n$, we have that $O_i = O(s_i \oplus s_\delta)$ is a Seq_ω-open subset of $T_{\vec{z}^{\,\ell}_\omega(i)}$ with $\mathrm{root}(T_{\vec{z}^{\,\ell}_\omega(i)}) \in O_i$. We are going to apply Lemma 3.12 to this sequence and the vector $\vec{z}^{\,\ell}_\omega$ on the set

$$B = \{\vec{b} \in Y^n : (\exists \vec{y} \in M)(\forall i < n)\ \vec{b}(i) = \vec{y}(s_i) \text{ and } \ s_\delta \Vdash \vec{y} \in \dot{A}\} \ .$$

For each $k \in \omega \setminus \ell$, since $\vec{y}_{j_k} \in M$, it follows that s_δ forces that $\vec{y}_{j_k} \restriction \{s^\ell_i \oplus s^{\beta_{j_k}} : i < n_\ell\}$ is in $((\dot{A})^{(\delta)})^{\oplus s^{\beta_{j_k}}}$. Therefore $\vec{z}^{\,\ell}_k \in B^{(\delta)}$. We now apply Lemma 3.12 and we obtain $\{t_i : i < n\}$ with each $t_i \in O(i)$ such that there is a $\vec{b} \in B$ where $\vec{b}(i)$ is congruent to $\vec{z}^{\,\ell}_\omega(i) \restriction (T_{\vec{z}^{\,\ell}_\omega})_{t_i}$. Choose $\vec{y} \in M$ so that $s_\delta \Vdash \vec{y} \in \dot{A}$ and $\vec{b}(i) = \vec{y}(s_i)$ for each $i < n$. We now have that $\vec{y}(s_i)$ is congruent to $y^M(s_i \oplus s_\delta) \restriction (T^M_{s_i\oplus s_\delta})_{t_i}$, completing the proof of the Lemma. □

Lemma 3.20. *If M and $\langle y^M(s) : s \in S_\delta\rangle \in Y^{S_\delta}$ are as described in Lemma 3.19, then for all $s \in S_\delta$ and $(\tilde{s}, \{s_i : i < n\}, \dot{F}) \in \mathcal{SB}_{max} \cap M$ with $\tilde{s} < s$, s forces that $y^M \restriction \{s_i \oplus s \colon i < n\}$ is an element of $\dot{F}^{\oplus s}$.* strongMelement

Proof. Let $(\tilde{s}, \{s_i : i < n\}, \dot{F})$ be in $\mathcal{SB}_{max} \cap M$ with $\tilde{s} < s$. Choose $\xi \in M$ large enough so that $s = (s{\restriction}\xi) \oplus s_\delta$ and $o(s_0) \leq \xi$. Let $\tilde{s}_\xi = s{\restriction}\xi$ and $s_\xi = s_\delta{\restriction}\xi$. With $\dot{H} = \dot{F}^{\oplus \tilde{s}_\xi}$ we have, by Definition 3.15, that $(\tilde{s}_\xi, \{s_i \oplus \tilde{s}_\xi : i < n\}, \dot{H})$ and, subsequently, $(s_\xi, \{s_i \oplus \tilde{s}_\xi : i < n\}, \dot{H}^{\tilde{s}_\xi}_{s_\xi})$, is in $\mathcal{SB}_{max} \cap M$. Note that, for $i < n$, $(s_i \oplus \tilde{s}_\xi) \oplus s_\delta$ is equal to $s_i \oplus s$. Apply Lemma 3.19 (3) to $(s_\xi, \{s_i \oplus \tilde{s}_\xi : i < n\}, \dot{H}^{\tilde{s}_\xi}_{s_\xi})$ to conclude that s_δ forces that $\langle y^M(s_i \oplus s) : i < n\rangle$ is in $((\dot{H})^{\tilde{s}_\xi}_{s_\xi})^{\oplus s_\delta}$. Since $\tilde{s}_\xi \oplus s_\delta = s$, it then follows that s forces that $\langle y^M(s_i \oplus s) : i < n\rangle$ is in $(\dot{H})^{\oplus s_\delta}$ which, in turn, is equal to $(\dot{F}^{\oplus \tilde{s}_\xi})^{\oplus s_\delta} = \dot{F}^{\oplus s}$. □

3.3. S-**preserving proper forcing.** Now we are ready to define our poset $\mathcal{P}$. Recall that we have a fixed assignment $\{\dot{U}(x, n) : x \in \mathfrak{c}, n \in \omega\}$ of S-names, which we will denote as $\mathcal{U}$, of neighborhoods (regular descending for each x) for each x in $\dot{X}$ (the sequential closure of ω_1).

We also have that, for each $x \in \mathfrak{c}$ and $n \in \omega$, 1 forces that the closure of $\dot{U}(x,n)$ meets ω_1 in a countable set. We have already chosen above a well-ordering $\prec$ of $H(\kappa)$, and for each countable elementary submodel M of $H(\kappa)$ as in Lemma 3.19, let $\langle y^M(s) : s \in S_{M\cap\omega_1}\rangle$ denote the $\prec$-least sequence with the properties in Lemma 3.19. We recall that for each $y \in Y$, we defined the associated S-name $\dot{y}$ in Definition 3.7.

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Lemma 3.21. *For each countable elementary submodel M of $(H(\kappa), \prec, Y)$, there is a $\gamma \in \omega_1$ such that $M \cap \omega_1 = \delta < \gamma$, and for all $s \in S$ with $\gamma \leq o(s)$, s decides the value of $e(y^M(s \restriction \delta))$ and of $M \cap \dot{U}(e(y^M(s \restriction \delta)), n)$ for all $n \in \omega$. Furthermore, for all $y \in \{y^M(s \restriction \delta)\} \cup (M \cap Y)$ and $n \in \omega$, there is a Seq_ω-open set $O(y, M, s, n)$ of T_y containing the root, such that*

1. *if $s \Vdash e(y) \notin \dot{U}(e(y^M(s \restriction \delta)), n))$, then $O(y, M, s, n)$ equals T_y,*
2. *if $s \Vdash e(y) \in \dot{U}(e(y^M(s \restriction \delta)), n))$, then $\mathrm{root}(T_y)^\frown t \in O(y, M, s, n)$ if and only if $s \Vdash \dot{y}(\mathrm{root}(T_y)^\frown t)) \in \dot{U}(e(y^M(s \restriction \delta)), n)$*

Proof. Lemma 2.1 implies that there is a γ satisfying that each $s \in S_\gamma$ decides the values of $y^M(s \restriction \delta)$ and of $M \cap \dot{U}(e(y^M(s \restriction \delta)), n)$ for each $n \in \omega$. Therefore each $s \in S$ with $\gamma \leq o(s)$ does the same. We also have, by elementarity, that $s \notin M$ implies that s decides the value (in $M \cap \mathfrak{c}$) of $\dot{y}(t)$ for all $y \in Y \cap M$ and all $t \in T_y$. Finally, since s forces that the function $\dot{y} : T_y \to \dot{X}$ is continuous, and since s decides the value of $M \cap \dot{U}(e(y^M(s \restriction \delta)), n)$, it follows s decides the value of the set $O(y, M, s, n)$ as defined in (1) and (2) and that it is a Seq_ω-open subset of T_y. □

We establish notation for the objects from Lemma 3.21.

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Definition 3.22. *For each countable elementary submodel M of $(H(\kappa), \prec, Y)$, let $\gamma(M)$ denote the minimal γ as described in Lemma 3.21. For each $s \in S$ with $\gamma(M) \leq o(s)$ and each $y \in \{y^M(s \restriction (M \cap \omega_1))\} \cup (M \cap Y)$ and $n \in \omega$, let $O(y, M, s, n)$ denote the Seq_ω-open subset of T_y as described in Lemma 3.21. We will also use $T^M_{\tilde{s}}$ to denote this T_y where $\tilde{s} = s \restriction (M \cap \omega_1)$ and $y = y^M(\tilde{s})$.*

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Definition 3.23. *A condition $p \in \mathcal{P}$ is simply a function into S with domain $\mathcal{M}_p$, a finite $\in$-chain of countable elementary submodels of $(H(\kappa), \prec)$, satisfying*

1. *for each $M \in \mathcal{M}_p$, $\{\mathcal{U}, Y\} \in M$ and $\gamma(M) \leq o(p(M))$,*
2. *for $M_1 \in M_2$ both in $\mathcal{M}_p$, $p(M_1) \in M_2$.*

Before defining the ordering on $\mathcal{P}$ we establish some notation. For each $p \in \mathcal{P}$, we let $C_p = \{M \cap \omega_1 : M \in \mathcal{M}_p\}$ and $S_p = \{p(M) : M \in \mathcal{M}_p\}$.

For convenience, we also let $\{M^p_\delta : \delta \in C_p\}$ *be the enumeration of* $\mathcal{M}_p$ *such that* $M^p_\delta \cap \omega_1 = \delta$. *Then we define* $S^\downarrow_p$ *to be the finite set* $\{s \restriction \delta : s \in S_p,\ \delta \in C_p \cap o(s)\}$. *Then for* $s \in S^\downarrow_p \cap S_\delta$ *and* $y = y^{M^p_\delta}(s)$, $W(p, s)$ *is the* Seq_ω*-open subset of* $T^{M_\delta}_s$ *equal to the intersection of the family* $\{O(y, M^p_\eta, s', n) \colon \delta \leq \eta \in C_p,\ n \leq |\mathcal{M}_p|,\ s < s' \in S_p \setminus M^p_\eta\ \}$.

The definition of $p < q$ *is that* $p \supseteq q$ *and for each* $\delta \in C_p \setminus C_q$, *if* $\delta < \max(C_q)$, *then for* $\gamma = \min(C_q \setminus \delta)$ *and for all* $s \in S^\downarrow_q \cap S_\gamma$, *the* Y*-value* $y^{M^p_\delta}(s \restriction \delta)$ *is congruent to* $y^{M^q_\gamma}(s) \restriction (T^{M^q_\gamma}_s)_t$ *for some* $t \in W(q, s)$.

We note that $W(p, s)$ defined in Definition 3.23 will contain the root of $T^{M^p_\delta}_s$. The ordering on $\mathcal{P}$ is more complicated than is usual in such posets, so we verify that it is transitive. Before doing so we record an immediate but important consequence (and purpose) of the ordering relation.

Lemma 3.24. *Let* $p < q \in \mathcal{P}$, $s \in S^\downarrow_q$, $\gamma = o(s)$ *and* $\beta = \max(\{0\} \cup (C_q \cap \gamma))$. *Then for each* $s < s' \in S_q$, s' *forces that* $e(y^{M^p_\gamma}(s \restriction \delta)) \in$ purpose
$\dot{U}(e(y^{M^q_\gamma}(s)), n)$ *for all* $n \leq |\mathcal{M}_q|$ *and* $\beta < \delta \in C_p \cap \gamma$.

Proof. Let $p < q$ and s, s', γ, δ be as in the Lemma. For convenience, let y denote $y^{M^q_\gamma}(s)$. Since $s' \in S_q$ and $\gamma \in C_q \cap o(s')$, we have that $\gamma(M^q_\gamma) < o(s')$. Fix any $n \in \omega$ and note that it is immediate from the definitions that s' forces that $e(y) \in \dot{U}(e(y), n)$. Therefore, by Lemma 3.21, s' forces that $\dot{y}(\mathrm{root}(T_y)^\frown t)) \in \dot{U}(e(y), n)$ for all $t \in O(y, M^q_\gamma, s', n)$. Now let $t \in W(q, s)$ be as postulated in the definition of $p < q$ so that $y^{M^p_\gamma}(s{\restriction}\delta) = y^{M^q_\gamma}(s{\restriction}\delta)$ is congruent to $y \restriction (T_y)_t$. By the definition of $W(q, s)$, we have that $t \in O(y, M^q_\gamma, s', n)$ and so s' forces that $e(y^{M^p_\gamma}(s{\restriction}\delta)) = e(\dot{y}(\mathrm{root}(T_y)^\frown t))$ is in $\dot{U}(e(y), n)$. □

Lemma 3.25. *If* q, r *are in* $\mathcal{P}$ *and* $q < r$, *then for all* $s \in S^\downarrow_q$, $W(q, s) \subset W(r, s)$. *Furthermore, the ordering on* $\mathcal{P}$ *is transitive.*

Proof. Assume that p, q, r are in $\mathcal{P}$ and that $p < q$ and $q < r$. Since $q \supset r$, it is immediate that $S_q \supset S_r$ and $C_q \supset C_r$, and $S^\downarrow_q \supset S^\downarrow_r$. We prove, for $s \in S^\downarrow_r$, that $W(q, s) \subset W(r, s)$. In fact, letting $y = y^{M^r_\gamma}(s)$ with $\gamma = o(s)$, the family $\{O(y, M^q_\eta, s', n) \colon \delta \leq \eta \in C_p,\ n \leq |\mathcal{M}_q|,\ s < s' \in S_q \setminus M^q_\eta\ \}$ is easily seen to include the family $\{O(y, M^r_\eta, s', n) \colon \delta \leq \eta \in C_r,\ n \leq |\mathcal{M}_r|,\ s < s' \in S_r \setminus M^r_\eta\ \}$.

Now it follows now that $W(p, s) \subset W(q, s) \subset W(r, s)$ for each $s \in S^\downarrow_r$. Now we assume that $\delta \in C_p \setminus C_r$ and that there is a $\gamma \in C_r$ with $\delta < \gamma = \min(C_r)$. We consider $s \in S^\downarrow_r \cap S_\gamma$, and we have to show that $y^{M^p_\delta}(s \restriction \delta)$ is congruent to $y^{M^r_\gamma}(s) \restriction (T^{M^r_\gamma}_s)_t$ for some $t \in W(r, s)$. If

$\delta \in C_q$, then this follows from the fact that $q < r$. If $\delta \notin C_q$, then let $\gamma' = \min(C_q \setminus \delta)$. Let us use simply y to equal $y^{M^r_\delta}(s)$. Using that $p < q$ and $q < r$ we can choose $t_1 \in W(M^q_{\gamma'}, s \restriction \gamma')$ such that $y^{M^p_\delta}(s \restriction \delta)$ is congruent to $y^{M^q_{\gamma'}}(s \restriction \gamma') \restriction (T^{M^q_{\gamma'}}_{s\restriction\gamma'})_{t_1}$ and choose $t_2 \in W(M^r_\gamma, s)$ so that $y^{M^q_{\gamma'}}(s \restriction \gamma')$ is congruent to $y \restriction (T_y)_{t_2}$.

Let T be the tree witnessing that $y^{M^q_{\gamma'}}(s \restriction \gamma')$ is congruent to $y \restriction (T_y)_{t_2}$. Choose $t \in T$ so that $t_1 = \mathrm{root}(T^{M^q_{\gamma'}}_{s\restriction\gamma'})^\frown t$. It follows that $t_2^\frown t$ is in T_y. It also follows that $y^{M^p_\delta}(s \restriction \delta)$ is congruent to $y \restriction (T_y)_{t_2^\frown t}$. We simply have to show that $t_2' = t_2^\frown t$ is in $W(r, s)$, which we do by showing that $t_2' \in O(y, M^r_\eta, s', n)$ for all $s < s' \in S_r$, $n \leq |\mathcal{M}_r|$, $\delta \leq \eta \in C_r$, and $s' \notin M^r_\eta$. Fix such a list s', n and η. Since $\gamma(M^r_\eta) \leq o(s')$, we may let x, x', z denote the values in $\mathfrak{c}$ such that $s' \Vdash x = e(y^{M^r_\gamma}(s))$, $s' \Vdash x' = e(y^{M^q_{\gamma'}}(s \restriction \gamma'))$, and $s' \Vdash z = e(y^{M^p_\delta}(s \restriction \delta))$. Let us note that s' also forces that $z = \dot{y}^{M^q_{\gamma'}}(t_1) = \dot{y}(t_2')$. Let $U(s', M^r_\eta, n)$ denote the value forced by s' to equal $M^r_\eta \cap \dot{U}(x, n)$. If $x \notin U(s', M^r_\eta, n)$, then $O(y, M^r_\eta, s', n)$ is all of T_y and so t_2' is in $O(y, M^r_\eta, s', n)$. If x is in $U(s', M^r_\eta, n)$, then we have that $x' \in U(s', M^r_\eta, n)$ since $t_2 \in O(y, M^r_\eta, s', n)$ and s' forces that $\dot{y}(t') \in U(s', M^r_\eta, n)$ for all $t' \in O(y, M^r_\eta, s', n)$. So now, from the definition of $O(y^{M^q_{\gamma'}}, M^r_\eta, s', n)$ and the fact that $t_1 \in W(q, s \restriction \gamma')$, we have that s' forces that $z = \dot{y}^{M^q_{\gamma'}}(t_1))$ is in $U(s', M^r_\eta, n)$. But this also means that s' forces that $z = \dot{y}(t_2')$ is in $U(s', M^r_\eta, n)$, and completes the proof that t_2' is in $O(y, M^r_\mu, s', n)$. ☐

extendM **Proposition 3.26.** *For each $q \in \mathcal{P}$ and countable $M \prec (H(\kappa), \prec)$ with $\{q, \mathcal{U}, Y\} \in M$, each function of the form $p = q \cup \{(M, s)\}$ with $s \in S_{\gamma(M)}$ satisfies that $p \in \mathcal{P}$ and $p < q$.*

Proof. It is trivial to check that $p \in \mathcal{P}$. It is clear that $p \subset q$ and since $(C_p \setminus C_q) \cap \max(C_q)$ is empty, we have that $p < q$. ☐

Before we show that $\mathcal{P} \times S$ is proper, let us verify that we will then have the ω_1-sequences that we need.

Proposition 3.27. *If $\mathcal{P}$ is proper and S-preserving, then PFA(S) im-*
3.9 *plies that S forces that $\dot{X}$*

(1) *includes a free ω_1-sequence, and*
(2) *if $\dot{X}$ is first countable, includes a copy of ω_1.*

Proof. It follows from Proposition 3.26 that, for each $m \in \omega$ and $s \in S$, the set $D_{s,m} = \{p \in \mathcal{P} : |\mathcal{M}_p| \geq m$ and $(\exists M \in \mathcal{M}_p)$ $(s \in M$ and $s < p(M))\}$ is a dense open subset of $\mathcal{P}$. For any condition $q \in \mathcal{P}$, let $\mathcal{E}(q)$

denote the collection of all M such that there exists a $p < q$ such that $M \in \mathcal{M}_p \backslash \mathcal{M}_q$, and for each $\delta \in \omega_1$, let $\mathcal{E}(q, \delta) = \{M \in \mathcal{E}(q) : M \cap \omega_1 < \delta\}$. Now we prove, similar to Claim (3) in Proposition 2.7, that

Claim 9. *For each $\delta \in \omega_1$,*

$$D_\delta = \{p \in \mathcal{P} : (\forall M' \in \mathcal{E}(p, \delta))(\exists M \in \mathcal{M}_p)(\delta \notin M \text{ and } M' \in M)\}$$

is a dense open subset of $\mathcal{P}$.

Proof of Claim 9: To prove that D_δ is dense it suffices to choose $s \in S_\delta$ and, using that $D_{s,1}$ is dense, to show that each $q \in D_{s,1}$ has an extension p in D_δ. So let $q \in D_{s,1}$ be arbitrary. If there is any $p < q$ with $\delta \in C_p$, then p is our desired extension in D_p, since $M' \in \mathcal{E}(p, \delta)$ implies $M' \in M^p_\delta$. So now we assume that $\delta \notin C_p$ for all $p < q$, and that $q \notin D_\delta$. Let $\xi = \min(C_q \setminus \delta)$, and, witnessing that $q \notin D_\delta$, there must be some $M' \in \mathcal{E}(q, \delta)$ such that $\max(C_q \cap \delta) \in M'$. Choose $r < q$ so that $M' \in \mathcal{M}_r$. Since $r < q$, we note that $\mathcal{M}_q \in M'$ and $\{M', \gamma(M')\} \in M^p_\xi$. Choose any $s' \in S \cap M^q_\xi$ so that $r(M') \leq s'$ and $\delta \leq o(s')$. Set $p = q \cup \{(M', s')\}$. The facts that $r \in \mathcal{P}$ and $r < q$, and $M' \in \mathcal{M}_r$, imply that $p \in \mathcal{P}$. In the definition of $p < q$ the only value $\tilde{\delta}$ in $C_p \setminus C_q$ is an element of $C_r \setminus C_q$ and $M^p_{\tilde{\delta}} = M^r_{\tilde{\delta}}$. Thus it follows that $p < q$. $\square$

Consider the family $\mathcal{D} = \{D_\delta \cap D_{s,m} : \delta \in \omega_1, m \in \omega,\ s \in S\}$ and let G be a $\mathcal{D}$-generic filter. Let $C = \bigcup\{C_p : p \in G\}$ and let $\{M_\delta : \delta \in C\}$ be an enumeration of $\bigcup\{\mathcal{M}_p : p \in G\}$ enumerated so that $M_\delta \cap \omega_1 = \delta$. We show that C is a closed subset of ω_1. Suppose that α is a limit ordinal and that $\alpha \notin C$. Choose any $s \in S_\alpha$ and let $p \in G \cap D_\alpha \cap D_{s,1}$. Since $\alpha \notin C_p$ and is a limit, we can choose $\beta < \alpha$ so that $C_p \cap \alpha \subset \beta$. We finish by proving that $C \cap \alpha \subset \beta$. Suppose that $\xi \in C \cap \alpha$. Since G is a filter, we may choose an $r < p$ in G such that $\xi \in C_r$. It follows then that $M^r_\xi \in \mathcal{E}(p, \alpha)$ and so there is an $M \in \mathcal{M}_p$ such that $M^r_\xi \in M$ and $M \cap \omega_1 < \alpha$. Therefore $M^r_\xi \cap \omega_1 < M \cap \omega_1 \in C_p \cap \alpha < \beta$.

Let $g \subset S$ be a generic filter. For each $\delta \in C$, let $s_\delta \in S_\delta \cap g$ and let $x_\delta = val_g(e(y^{M_\delta}(s_\delta)))$. We claim that the map sending x_γ to γ is a continuous map from $W = \{x_\gamma : \gamma \in C\}$ onto C endowed with the order topology. This will show that W includes an uncountable free sequence. To prove this claim, it suffices to show that for each pair $\beta < \delta \in C$, x_δ is not in the closure of $\{x_\alpha : \alpha \in C \cap (\beta + 1)\}$ and that x_η is not in $val_g(\dot{U}(x_\delta, 0))$ for $\eta > \delta$. For each $\alpha \in C$, let α^+_C denote the minimum element of C above α. For all $\alpha \in C$, the closure of $val_g(\dot{U}(x_\alpha, 1))$ is included in $val_g(\dot{U}(x_\alpha, 0))$ and, by elementarity, the ordinal $x_\alpha \in M_{\alpha^+_C}$ and the supremum of $val_g(\dot{U}(x_\alpha, 0)) \cap \omega_1$ is less than

α_C^+. For $\eta \in C \setminus \alpha_C^+$, x_η is in the closure of $\omega_1 \setminus \xi$ for all $\xi \in \alpha_C^+$, and therefore x_η is not in the closure of $val_g(\dot{U}(x_\alpha, 1))$. Now to show that x_δ is not in the closure of $\{x_\alpha : \alpha \in C \cap (\beta+1)\}$ we show that there is a finite subset $\{\beta_i : i < \ell\}$ of $C \cap (\beta+1)$ such that $\{x_\alpha : \alpha \in C \cap (\beta+1)\}$ is included in $\bigcup_{i \leq \ell} val_g(\dot{U}(x_{\beta_i}, 1))$. Consider any $\alpha \leq \beta$ and choose any $q \in G$ so that $\{M_\alpha, M_\delta\} \subset \mathcal{M}_q$ and so that $q \in D_\delta \cap D_{s_\delta, 2}$. Fix $\bar{s} \in S_q$ so that $s_\delta < \bar{s}$. It follows that $s_\delta \in S_q^{\downarrow}$ and that $\gamma(M_\delta) < o(\bar{s})$. Then, by Lemma 3.24, there is an $\beta_\alpha < \alpha$ such that $x_\xi \in val_g(\dot{U}(x_\alpha, 1))$ for all $\beta_\alpha < \xi \leq \alpha$. Now set $\gamma_0 = \beta$ and recursively set, for $\gamma_i > 0$, $\gamma_{i+1} = \beta_{\gamma_i} < \gamma_i$. Once we find ℓ so that $\gamma_\ell = 0$ we stop.

Now we prove (2) of the Proposition. That is, we assume that 1 forces that $\{\dot{U}(x, n) : n \in \omega\}$ is a neighborhood base at x for each $x \in \dot{X}$. We prove that W is a copy of ω_1 by proving that it is homeomorphic to C. Lemma 3.24 implies that if, for each $\delta \in C$, $\{val_g(\dot{U}(x_\delta, n)) \colon n \in \omega\}$ is a neighborhood base at x_δ, then the map sending $x_\delta \in W$ to $\delta \in C$ is a closed mapping, and thus a homeomorphism. This completes the proof. □

All we have to do now is to prove that

ppithm **Theorem 3.28.** *The poset $\mathcal{P}$ is proper and S-preserving.*

Proof. Following Lemma 2.6 we just have to prove that $S \times \mathcal{P}$ is proper. That is, by Definition 2.2, we assume that $S, \mathcal{P} \in M \prec H(\mu)$ (for some suitably large μ) is countable, and that $M_0 = M \cap H(\kappa) \in \mathcal{M}_p$ for some condition p. Let $M \cap \omega_1 = \delta$ and let s_δ be the $\prec$-least element of S_δ as in Lemma 3.19. Let $D \in M$ be a dense open subset of $S \times \mathcal{P}$ and assume that $(s^\dagger, r) \in D$ is below (s_δ, p). By applying Lemma 3.26 at most twice, and by ensuring that $r(\max(\mathcal{M}_r)) < s^\dagger$, we can assume that $s_\delta \in S_r^{\downarrow}$ and that $s^\dagger \restriction \gamma \in S_r^{\downarrow}$ for all $\gamma \in C_r$. Let $\{M_i^r : i < \bar{\ell}\}$ and $\{\delta_i^r : i < \bar{\ell}\}$ enumerate $\mathcal{M}_r \setminus M$ and $C_r \setminus M$ in increasing order. For each $i < \bar{\ell}$, let $s_i^r = p(M_i^r)$. Let $\{s_i : i < \ell\} \in S^{\rightarrow}$ enumerate $S_\delta \cap S_p^{\downarrow}$, and for each $j < \bar{\ell}$, let $S_j^r = \{s_i \oplus (s^\dagger \restriction \delta_j^r) \colon i < \ell\}$. We note that S_j^r is not necessarily included in $S_r^{\downarrow}$. The set S_j^r is also equal to $\{s_i \oplus (s_{\bar{\ell}-1}^r \restriction \delta_j^r) \colon i < \ell\}$. Choose $\bar{\alpha} \in M \cap \omega_1$ large enough so that $M' \cap \omega_1 < \bar{\alpha}$ for all $M' \in \mathcal{M}_r \cap M$, and so that $s \restriction [\bar{\alpha}, \delta) = s' \restriction [\bar{\alpha}, \delta)$ for all $s, s' \in S_r$, and let $\bar{s}_i = s_i \restriction \bar{\alpha}$ for each $i < \ell$.

The remainder of the proof is similar to the proof that the poset used in the proof of Theorem 3.4 was proper. Let us note that $r \cap M = r_M$ is an element of $\mathcal{P}$ and $r < r_M$. We will find a condition $q \in \mathcal{P} \cap M$ so that $q < r_M$ and so that $(s_\delta, q) \in D$. The hard part is to choose q so that $q \not\perp r$ which requires that, for each $i < \ell$ and $\eta \in C_q \cap (\bar{\alpha}, \delta)$, $y^{M_\gamma^q}(s_i \restriction \eta)$ is congruent to $y^{M_\delta^r}(s_i) \restriction (T_{s_i}^{M_\delta^r})_t$ for some $t \in T_{s_i}^{M_\delta^r}$.

Let us say that a condition q is **like** r (or $q \equiv r$) providing:

(1) $q < r_M$ and $C_q \cap \bar\alpha = C_r \cap \bar\alpha$,
(2) $\mathcal{M}_q \setminus \mathcal{M}_r$ listed in ϵ-increasing order is $\{M^q_i : i < \bar\ell\}$,
(3) let $C_q \setminus \bar\alpha = \{\delta^q_i\colon i < \bar\ell\}$ where $M^q_i \cap \omega_1 = \delta^1_i$,
(4) $S^q_0 = \{s \restriction \delta^q_0 : s \in S_q\}$ is equal to $\{\bar s_i \oplus (s^q_{\bar\ell-1} \restriction \delta^q_0)\colon i < \ell\}$.

When $q \equiv r$, let us also define for $0 < j < \bar\ell$, $S^q_j = \{\bar s_i \oplus (s^q_{\bar\ell-1} \restriction \delta^q_j) : i < \ell\}$, and for $j < \bar\ell$, let $\vec y^{\,q}_j$ denote the element in $Y^{S^q_j}$ equal to $y^{M^q_j} \restriction S^q_j$.

Recursively define a collection of sets and names. First we have the S-name:

$$\dot{\mathcal{Y}}_{\bar\ell} = \{(s, \langle \vec y^{\,q}_k : k < \bar\ell\rangle) : (s,q) \in D\ , s^q_{\bar\ell-1} < s\ ,\ C_q \subset o(s),\ \text{and } q \equiv r\}\,.$$

As usual, we have that $\dot{\mathcal{Y}}_{\bar\ell} \in M$ since $q \equiv r$ can be described within M, and $\dot{\mathcal{Y}}_{\bar\ell}$ is in M_0 because $\dot{\mathcal{Y}}_{\bar\ell} \in H(\kappa)$. Now define, for $k \in \{\bar\ell - 1, \bar\ell - 2, \ldots, 0\}$ (in that order) and $q \equiv r$,

$$\dot A(q,k) = \{(\bar s, \vec y)\colon (\exists (\bar s, \langle \vec y^{\,\bar q}_j : j \le k\rangle) \in \dot{\mathcal{Y}}_{k+1})\ \vec y \in Y^{\{\bar s_i\colon i<\ell\}} \text{ and } (\vec y)^{\oplus(\bar s \restriction \delta^{\bar q}_k)} = \vec y^{\,\bar q}_k\ \}$$

and let (for $k < \bar\ell$)

$$\dot{\mathcal{Y}}_k = \{(s, \langle \vec y^{\,q}_j : j < k\rangle) : (s, \{\bar s_i : i < \ell\}, \dot A(q,k)) \in \mathcal{A}\}\ .$$

We use the notation $\dot A(q,k)$ even though the definition depends only on the parameters $\langle \vec y^{\,q}_j : j < k\rangle$ but we can thus observe that $\dot A(q,k)$ and $\dot{\mathcal{Y}}_{k+1}$ are in M^q_k. Notice that $\dot A(q,k)$ is forced to be a subset of $Y^{\{\bar s_i : i < \ell\}}$. Clearly $(s^\dagger, \langle \vec y^{\,r}_k : k < \bar\ell\rangle) \in \dot{\mathcal{Y}}_{\bar\ell}$ and, for readability, temporarily let $k = \bar\ell - 1$. We then have that

$$(2)\qquad s^\dagger \Vdash \dot A(r,k) \subset Y^{\{\bar s_i : i<\ell\}} \ \text{ and }\ \vec y^{\,r}_k \in (\dot A(r,k))^{\oplus(s^\dagger \restriction \delta^r_k)}\ .$$

We show, by elementarity, that $(s^\dagger {\restriction} \xi, \{\bar s_i : i < \ell\}, \dot A(r,k)) \in \mathcal{A}$ for some $\xi \in M^r_k$. In M^r_k we have the set

$$S(\dot A(r,k)) = \{s \in S : (s, \{\bar s_i : i < \ell\}, \dot A(r,k)) \in \mathcal{A}\}$$

and we need to show that $s^\dagger \in S(\dot A(r,k))$. By Lemma 3.18, $S(\dot A(r,k))$ is an upwards closed subset of S, i.e if $s < \tilde s$ and $s \in S(\dot A(r,k))$, then $\tilde s \in S(\dot A(r,k))$. The set of minimal elements, $\min(S(\dot A(r,k)))$, is an antichain of S and so there is a $\xi \in M^r_k$ such that for each $s \in S_\xi$, either $s \in S(\dot A(r,k))$ or no element above s is in $S(\dot A(r,k))$. We assume that $s_\xi = s^\dagger \restriction \xi \notin S(\dot A(r,k))$ and obtain a contradiction. By Definition 3.17, there is an $(s_\xi, \{\bar s_i \oplus s_\xi : i < n\}, \dot F) \in \mathcal{SB}_{max}$ such that $s_\xi \Vdash (\dot A(r,k))^{\oplus s_\xi} \cap \dot F = \emptyset$. By elementarity, there is such an element in M^r_k. The contradiction is that $s_\xi < s^\dagger \restriction \delta^r_k < s^\dagger$ and $s^\dagger \Vdash \dot A(r,k)^{\oplus s_\xi} \cap \dot F \neq \emptyset$

because, by equation (2), $s^\dagger \Vdash \vec{y}^{\,r}_k \in (\dot{A}(r,k)^{\oplus s_\xi})^{\oplus (s^\dagger \restriction \delta^r_k)}$ and by Lemma 3.20, $s^\dagger \Vdash \vec{y}^{\,r}_k \in \dot{F}^{\oplus s^\dagger \restriction \delta^r_k}$.

Continuing this standard argument, walking down from $s^\dagger$, shows that, for each $k < \bar{\ell}$, there is a $\xi_k \in M^r_k$ such that $(s^\dagger \restriction \xi_k, \langle \vec{y}^{\,r}_m : m < k\rangle)$ is a member of $\dot{\mathcal{Y}}_{k+1}$. Now we have that there is some $\xi_0 \in M^r_0$ such that $(s^\dagger \restriction \xi_0, \emptyset)$ is a member of $\dot{\mathcal{Y}}_0$; and more importantly that $(s^\dagger \restriction \xi_0, \{\bar{s}_i \colon i < \ell\}, \dot{A}(r,0)) \in \mathcal{A} \cap M^r_0$. By Lemma 3.19, there is a $\vec{y}_0 \in M$ such that $s_\delta < s^\dagger$ forces that $\vec{y}_0 \in \dot{A}(r,0)$ and, for each $i < \ell$, there is a $t \in W(r, s_i)$ such that $\vec{y}_0(\bar{s}_i)$ is congruent to $y^{M^r_0}(s_i) \restriction (T^{M^r_0}_{s_i})_t$.

By elementarity and Lemma 2.1, there is a $\beta_0 \in \delta$ such that $(s_\delta \restriction \beta_0, \vec{y}_0) \in \dot{A}(r,0)$. Similarly, there is a $q_0 \in M \cap \mathcal{P}$ witnessing that $(s_\delta \restriction \beta_0, \vec{y}_0)$ is in $\dot{A}(r,0)$; namely that $(s_\delta \restriction \beta_0, \langle \vec{y}^{\,q_0}_0 \rangle) \in \dot{\mathcal{Y}}_1$ and $(\vec{y}_0)^{\oplus (s_\delta \restriction \delta^{q_0}_0)} = \vec{y}^{\,q_0}_0$. It follows that, for $i < \ell$, $y^{M^{q_0}_0}(s_i \restriction \delta^q_0)$ is congruent to $y^{M^r_0}(s_i) \restriction (T^{M^r_0}_{s_i})_t$ for some $t \in W(r, s_i)$. Now that we have that $(s_\delta \restriction \beta_0, \{\bar{s}_i : i < \ell\}, \dot{A}(q_0, 1))$ is in $\mathcal{A} \cap M$ and we repeat the argument above, beginning with the assertion that $(s^\dagger \restriction \xi_0, \{\bar{s}_i \colon i < \ell\}, \dot{A}(r,0))$ is in $\mathcal{A} \cap M^r_0$, replaced by $(s_\delta \restriction \beta_0, \{\bar{s}_i : i < \ell\}, \dot{A}(q_0,1))$ is in $\mathcal{A} \cap M$. In this way, we find a $q_1 \in M \cap \mathcal{P}$ and a $\beta_1 \in \delta$ satisfying that there is a $\vec{y}_1 \in Y^{\{\bar{s}_i \colon i < \ell\}}$ with $(s_\delta \restriction \beta_1, \vec{y}_1) \in \dot{A}(q_0, 1)$ and witnessed by $(s_\delta \restriction \beta_1, \langle \vec{y}^{\,q_1}_p : j \leq 1\rangle) \in \dot{\mathcal{Y}}_1$. It follows that $\vec{y}^{\,q_1}_0 = \vec{y}^{\,q_0}_0$, and for each $i < \ell$, there is a $t \in W(r, s_i)$ such that $y^{M^{q_1}_1}(s_i \restriction \delta^q_1)$ is congruent to $y^{M^r_0}(s_i) \restriction (T^{M^r_0}_{s_i})_t$. We repeat this argument for $\bar{\ell}$ steps until we find $q = q_{\bar{\ell}} \equiv r$ with the property that $(s_\delta, q) \in D$ and, for each $k < \bar{\ell}$, and for each $i < \ell$, $y^{M^q_{\delta_k}}(\bar{s}_i \oplus (s_\delta \restriction \delta^q_k))$ is congruent to $y^{M^r_0}(s_i) \restriction (T^{M^r_0}_{s_i})_t$ for some $t \in W(r, s_i)$. It now follows that q is compatible with r, and, by elementarity and Lemma 2.1, that there is an $s' < s_\delta$ such that $(s', q) \in D$ as required. □

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4. On the consistency of **GA**

The goal of this section is to prove that the consistency of ZFC is sufficient to prove that **GA** is also consistent with ZFC. The method is standard in that we assume that we have a ground model of ZFC in which CH and a diamond principle on ω_2 also hold. This assumption is well-known to follow from the consistency of ZFC. We will then construct a poset via a countable support iteration of proper posets (see [25, III 3.1]) designed to force that **GA** holds in the forcing extension. We will need that this poset satisfies that every antichain has cardinality less than $\aleph_2$ (the $\aleph_2$-cc). A proper poset with the $\aleph_2$-cc will ensure that sets in the extension that have cardinality at most $\aleph_2$ will have names that also have cardinality at most $\aleph_2$. To accomplish this, we will use the κ-p.i.c. (for "proper isomorphism condition") scheme (with $\kappa = \aleph_2$) introduced by Shelah. The diamond sequence on ω_2 will help us decide which proper $\aleph_2$-p.i.c. posets of size $\aleph_2$ to use in such an iteration, by predicting initial segments of the size $\aleph_2$ objects that we must finally consider to verify that, for example, $\mathbf{PPI}^+$ holds in the extension.

We use the method of Todorcevic [30] in which side conditions are finite sets (or matrices) of elementary submodels rather than the more common method in which side conditions are simple finite $\in$-chains of elementary submodels. It is this change which is the key in making the resulting posets satisfy the $\aleph_2$-p.i.c. and thereby removing the need for large cardinals to prove the results. This approach was also used in [5] in the PFA context. 3.2

Definition 4.1 ([25, Ch. VIII])**.** *A poset P satisfies the $\aleph_2$-p.i.c. provided the following holds for big enough λ (for example $[P]^{\aleph_1} \in H(\lambda)$):*

Suppose $\prec$ is a well-ordering of $H(\lambda)$, $i < j < \omega_2$, N_i and N_j are countable elementary submodels of $\langle H(\lambda), \prec, \in\rangle$ such that $\{\aleph_2, P\} \in N_i \cap N_j$, $i \in N_i$, $j \in N_j$, $N_i \cap \omega_2 \subset j$, $N_i \cap i = N_j \cap j$, $p \in P \cap N_i$, and $h : N_i \to N_j$ is an isomorphism such that $h(i) = j$ and h is the identity on $N_i \cap N_j$, then there is a $q \in P$ such that:

(1) *$q < p$, $q < h(p)$ and q is both N_i and N_j generic ,*
(2) *if $r \in N_i \cap P$ and $q' < q$ there is a $q'' < q'$ so that $q'' < r$ if and only if $q'' < h(r)$.*

We record the next three results from [25, Ch. VIII].

Proposition 4.2. *A countable support iteration of length less than ω_2 of $\aleph_2$-p.i.c. proper posets is again $\aleph_2$-p.i.c.. Furthermore if CH holds and the iteration has length at most ω_2 then the iteration satisfies the $\aleph_2$-cc .* 3.3

Proposition 4.3. *A proper poset of cardinality $\aleph_1$ satisfies the $\aleph_2$-pic.*

Proposition 4.4 (CH)**.** *If P is a proper $\aleph_2$-p.i.c. poset and G is P-generic over V then $V[G] \models \mathfrak{c} = \omega_1$.* chaincondition

This next proposition is taken from [25, VIII 2.2].

Proposition 4.5. *If CH and $2^{\aleph_1} = \aleph_2$ and $P \subset H(\aleph_2)$ is a poset that satisfies Definition 4.1 for P and $\lambda = \aleph_3$, then P satisfies the $\aleph_2$-p.i.c., and Definition 4.1 holds for all $\lambda > 2^{\aleph_3}$.* aleph3

Following [30], for a countable elementary submodel N of $H(\aleph_2)$, we let $\overline{N}$ be its transitive collapse, and we let $h_N : N \to \overline{N}$ be the collapsing map, i.e. $h_N(x) = \{h_N(y) : y \in x \cap N\}$.

homega1 **Lemma 4.6.** *Suppose that N_1, N_2 are countable elementary submodels of $H(\aleph_2)$ such that $\overline{N_1} = \overline{N_2}$, and let h_{N_1,N_2} denote the map $h_{N_2}^{-1} \circ h_{N_1}$. Then h_{N_1,N_2} is the identity on $H(\aleph_1) \cap N_1$ and for each $A \in N_1$ with $A \subset H(\aleph_1)$, $A \cap N_1 = h_{N_1,N_2}(A) \cap N_2$.*

Proof. It follows by $\in$-induction that each $x \in N_1 \cap H(\aleph_1)$, $x \subset N_1$ and so $x \in \overline{N_1}$. Therefore we also have, by $\in$-induction, that $h_{N_1}(x) = x = h_{N_2}^{-1}(x)$. □

A family, denoted $[\mathcal{N}]$, is an elementary matrix if, for some integer $n > 0$,

(1) $[\mathcal{N}] = \{\mathcal{N}_1, \ldots, \mathcal{N}_n\}$
(2) for each $1 \leq i \leq n$, $\mathcal{N}_i$ is a finite set of countable elementary submodels of $H(\aleph_2)$
(3) for each $1 \leq i \leq n$, $\overline{N_1} = \overline{N_2}$ for each pair $N_1, N_2 \in \mathcal{N}_i$,
(4) for each $1 \leq i < j \leq n$ and each $N_i \in \mathcal{N}_i$, there is an $N_j \in \mathcal{N}_j$ with $N_i \in N_j$.

It will be convenient to let $N \in [\mathcal{N}]$, for an elementary submodel N of $H(\aleph_2)$, be an abbreviation for $N \in \mathcal{N}_i$ for some $\mathcal{N}_i \in [\mathcal{N}]$.

p22pic **Lemma 4.7** (CH). *If $\dot{\mathcal{I}}$ is an S-name of a P-ideal on ω_1 such that 1 forces that $\dot{\mathcal{I}} \cap [E]^{\aleph_0}$ is not empty for all stationary sets $E \subset \omega_1$, then there is an S-preserving $\aleph_2$-p.i.c. proper poset $\mathcal{P} \subset H(\aleph_2)$ such that $\mathcal{P}$ forces that there is an S-name $\dot{E}$ of a stationary set with $1 \Vdash [\dot{E}]^{\aleph_0} \subset \dot{\mathcal{I}}$.*

ppipic **Lemma 4.8** (CH). *If $\dot{X}$ is an S-name of a sequentially compact non-compact space of cardinality $\aleph_1$, then there is an S-preserving $\aleph_2$-p.i.c. proper poset $\mathcal{Q} \subset H(\aleph_2)$ such that $\mathcal{Q}$ forces that there is an S-name $\{\dot{x}_\gamma : \gamma \in \omega_1\} \subset \dot{X}$ that is forced to include an uncountable free sequence, and, if $\dot{X}$ is first countable, to be a homeomorphic copy of ω_1.*

The proofs are very similar, with the same underlying idea in that we replace elementary chains from the original proofs with elementary matrices. The usage of elementary matrices is the device to make the poset satisfy the $\aleph_2$-p.i.c. The proof that the modified poset is proper and S-preserving relies on the fact that CH guarantees that the key combinatorics take place within $H(\aleph_1)$ and so, by Lemma 4.6 no new arguments or constructions are required. Since it is newer, we sketch the proof of Lemma 4.8 and leave the proof of Lemma 4.7 to the interested reader. In actual fact, this method is not really needed for the consistency of $\mathbf{P}_{22}$ because the needed poset can be chosen to have cardinality $\mathfrak{c}$. The reason this is not true for $\mathbf{PPI}^+$ is that we must utilize the construction of the maximal filter of S-sequentially closed sets which may have cardinality $2^{\aleph_1}$. We simply indicate the modifications needed to the proof of Lemma 3.28.

Proof. Let $\dot X$ be the S-name as formulated in Lemma 4.8. Since $\dot X$ is forced to be countably compact but not compact, there is an open cover of $\dot X$ of cardinality $\aleph_1$ that has no countable subcover. Let $\{\dot U_\alpha : \alpha \in \omega_1\}$ be the S-names for such a cover. For each α, let $\dot x_\alpha$ be the S-name of a point that is forced by 1 to not be an element of $\dot U_\beta$ for all $\beta < \alpha$. It is forced by 1 that the set $\{\dot x_\alpha : \alpha \in \omega_1\}$ has no complete accumulation point.

Now by simply renaming the elements of the original base set for $\dot X$, we can assume that $\omega_1 \times \omega_1$ is the base set and that $\omega_1 \times \{0\}$ is a subspace with no complete accumulation point. For each point $x = (\alpha, \beta)$ of $\dot X$, we fix a family $\{\dot U(x, n) : n \in \omega\}$ of S-names of open sets so that 1 forces that $\dot U(x, 0) \cap (\omega_1 \times \{0\})$ is countable, and that, for each n, $x \in \dot U(x, n+1)$ and the closure of $\dot U(x, n+1)$ is included in $\dot U(x, n)$. If $\dot X$ is assumed to be first countable, then also we assume that $\{\dot U(x, n) \colon n \in \omega\}$ is a local base at x.

This entire family of S-names in the topology for $\dot X$ can be coded as a single subset, τ, of $S \times \omega \times \omega_1^4$ where $(s, m, \alpha, \beta, \gamma, \delta) \in \tau$ codes the fact that s forces that (γ, δ) is in $\dot U((\alpha, \beta), m)$. The family $\bigcup\{Y_\alpha : \alpha \in \omega_1\}$ and **WF** as defined in §3.1 are already subsets of $H(\aleph_1)$. We also fix a well-order $\prec_{\omega_1}$ of $H(\aleph_1)$.

Finally, with no changes, the families $\mathcal{SB}$ defined in Lemma 3.16 and $\mathcal{A}$ as defined in Definition 3.17 are subsets of $H(\aleph_2)$. For each countable elementary submodel M of $H(\aleph_2)$ satisfying that $\{\tau, \prec_{\omega_1}, \mathcal{SB}, \mathcal{A}\} \in M$, the sequence $\{y^M(s) : s \in S_{M \cap \omega_1}\}$, is chosen as in Lemma 3.19 to be the $\prec_{\omega_1}$-minimal such sequence. Let $\mathcal{P}$ be the poset defined in Definition 3.23 using $\kappa = \aleph_2$. For each $p \in \mathcal{P}$ and $s \in S_p^{\downarrow}$, let $W(p, s)$ be defined as in Definition 3.23. We will use the construction and properties of $\mathcal{P}$ to simplify the construction of the poset $\mathcal{Q}$ in this proof.

Claim 10. *Consider any set $\mathcal{N}$ of pairwise isomorphic countable elementary submodels of $(H(\aleph_2), \tau, \prec_{\omega_1}, \mathcal{SB}, \mathcal{A})$; i.e. $\overline{N} = \overline{N'}$ for $N, N' \in \mathcal{N}$. Let $\delta = N \cap \omega_1$ for any $N \in \mathcal{N}$. Let N_1, N_2 be elements of $\mathcal{N}$. We* claimsame *then have that the two sequences $\langle y^{N_1}(s) : s \in S_\delta\rangle$ and $\langle y^{N_2}(s) : s \in S_\delta\rangle$ are the same.*

Proof of Claim 10: To prove the claim, let $(\bar s, \{s_i : i < n\}, \dot A)$ be any member of $\mathcal{A} \cap N_1$ and assume that $\bar s < s \in S_\delta$. Choose $B \subset Y^n \cap N_1$ such that $s \Vdash B \subset \dot A$ and $s \Vdash \langle y^{N_1}(s \oplus s_i) : i < n\rangle \in B^{(\delta+1)}$. By Lemma 4.6, $h_{N_1,N_2}((\bar s, \{s_i : i < n\}, \dot A))$ is in $\mathcal{A} \cap N_2$. Since $\dot A \subset H(\aleph_1)$ we also have, by Lemma 4.6, that $h_{N_1,N_2}(\dot A) \cap N_2$ is equal to $\dot A \cap N_1$. Therefore, we have that s also forces that B is a subset of $h_{N_1,N_2}(\dot{A})$. Well, this shows that $\langle y^{N_1}(s \oplus s_i) : i < n\rangle$ satisfies this particular requirement of $\langle y^{N_2}(s \oplus s_i) : i < n\rangle$ with respect to $h_{N_1,N_2}((\bar{s}, \{s_i : i < n\}, \dot{A}))$. Since h_{N_1,N_2} is an isomorphism, this shows that $\langle y^{N_1}(s) : s \in S_\delta\rangle$ works as a choice for $\langle y^{N_2}(s) : s \in S_\delta\rangle$, and so, indeed, they are the same. □

A condition $q \in \mathcal{Q}$ consists of a pair $([\mathcal{N}_q], S_q)$ where $[\mathcal{N}_q]$ is an elementary matrix of submodels of $(H(\aleph_2), \prec_{\omega_1}, \dot{\tau}, \mathcal{SB}, \mathcal{A})$ such that there is a $p_q \in \mathcal{P}$ satisfying that $\mathcal{M}_{p_q}$ is a subset of $[\mathcal{N}_q]$, $S_{p_q} = S_q$, and $C_p = C_q = \{N \cap \omega_1 : N \in [\mathcal{N}_q]\}$. The choice of p_q is not unique, so let $[q]_{\mathcal{P}}$ equal the set of $p \in \mathcal{P}$ such that $\mathcal{M}_p \subset [\mathcal{N}_q]$ and $S_p = S_q$. We have that $\mathcal{P} \subset \{p_q : q \in \mathcal{Q}\}$ because, for each $p \in \mathcal{P}$, the pair $(\{\{N\} : N \in \mathcal{M}_p\}, S_p)$ is a condition in $\mathcal{Q}$. The ordering on $\mathcal{Q}$ is that $q_2 < q_1$ providing that each $N \in [\mathcal{N}_{q_1}]$ is an element of $[\mathcal{N}_{q_2}]$, and there is a $p_2 \in [q_2]_{\mathcal{P}}$ such that $p_1 = p_2 \restriction \{N \in \mathcal{M}_{p_2} : N \in [\mathcal{N}_{q_1}]\}$ satisfies that $p_2 \leq_{\mathcal{P}} p_1$ and $p_1 \in [q_1]_{\mathcal{P}}$. Using Claim 10, it follows that if $q_2 < q_1$, then an element $p_2 \in [q_2]_{\mathcal{P}}$ is below an element of $[q_1]_{\mathcal{P}}$ providing each $N \in \mathcal{M}_{p_2}$ with $N \cap \omega_1 \in C_{q_1}$ is in $[\mathcal{N}_{q_1}]$. This implies that the ordering on $\mathcal{Q}$ is transitive.

Qproper **Claim 11.** *$S \times \mathcal{Q}$ is proper.*

Proof of Claim 11: Consider a countable elementary submodel M as in Definition 2.2 for $S \times \mathcal{Q}$. Since $\mathcal{P}$ is definable from $\mathcal{Q}$, we also have that $\mathcal{P}$ is in M. Choose any $(\bar{s}, \bar{q}) \in M \cap (S \times \mathcal{Q})$ and any dense open set $D \in M$ of $S \times \mathcal{Q}$. Choose any $\bar{p} \in [\bar{q}]_{\mathcal{P}} \cap M$ and $p^\dagger < \bar{p}$ so that $\mathcal{M}_{p^\dagger} = \mathcal{M}_{\bar{p}} \cup \{M \cap H(\aleph_2)\}$. Since $\bar{q} \in M$, $[\mathcal{N}_{\bar{q}}]$ is a subset of M and so it follows that $([\mathcal{N}_{\bar{q}}] \cup \{M \cap H(\aleph_2)\}, S_{\bar{p}} \cup \{p^\dagger(M \cap H(\aleph_2))\})$ is in $\mathcal{Q}$. Now we know we can choose a condition $(s^\dagger, q) \in D$ below $(\bar{s}, \bar{q})$ such that $M \cap H(\aleph_2) \in [\mathcal{N}_q]$. We show that

$$D_{\mathcal{P}} = \{(s,p) \in S \times \mathcal{P} : (\exists q \in \mathcal{Q})\ p \in [q]_{\mathcal{P}} \ \text{ and } \ (s,q) \in D\} \in M$$

is a dense subset of $S \times \mathcal{P}$. Fix any $(s', p') \in \mathcal{P}$. Choose any $r_1 \in \mathcal{Q}$ so that $p' \in [r_1]_{\mathcal{P}}$ and let $(s_2, r_2) \in D$ be below (s', r_1). Fix any $p_2 \in [r_2]_{\mathcal{P}}$. Let $\gamma = \max(C_{p'})$, and let $N_\gamma \in \mathcal{M}_{p_2}$ and $M_\gamma \in \mathcal{M}_{p'}$ so that $\gamma = N_\gamma \cap \omega_1 = M_\gamma \cap \omega_1$. We define a condition $\tilde{r}_2 \in \mathcal{Q}$ so that $(s_2, \tilde{r}_2) < (s_2, r_2)$ and so that there is a $p_3 \in [\tilde{r}_2]_{\mathcal{P}}$ satisfying that $p_3 < p'$. Since $(s_2, p_3) \in D_{\mathcal{P}}$, this will show that $D_{\mathcal{P}}$ is dense. For each $\alpha \in C_{r_2}$, let $\mathcal{N}^{r_2}_\alpha = \{N \in [\mathcal{N}_{r_2}] : N \cap \omega_1 = \alpha\}$. For each $\alpha \in C_{r_2} \cap \gamma$, let $\mathcal{N}^{\tilde{r}_2}_\alpha = \mathcal{N}^{r_2}_\alpha \cup \{h_{N_\gamma, M_\gamma}(N) : N \in \mathcal{M}_{p_2} \cap N_\gamma\}$. For $\alpha \in C_{r_2} \setminus \gamma$, set $\mathcal{N}^{\tilde{r}_2}_\alpha$ equal to $\mathcal{N}^{r_2}_\alpha$. Since $\{h_{N_\gamma, M_\gamma}(N) : N \in \mathcal{M}_{p_2} \cap N_\gamma\} \cup \{M_\gamma\}$ is an $\in$-chain, it follows easily that $[\mathcal{N}_{\tilde{r}_2}]$ is an elementary matrix. Also, there is an $\in$-chain, which we will call $\mathcal{M}_{p_3}$, of elements of $[\mathcal{N}_{\tilde{r}_2}]$ extending this chain so that $C_{r_2} = \{N \cap \omega_1 : N \in \mathcal{M}_{p_3}\}$. For each $N \in \mathcal{M}_{p_3}$, choose $N' \in \mathcal{M}_{p_2}$ so that $N \cap \omega_1 = N' \cap \omega_1$ and set $p_3(N) = p_2(N')$. It follows easily that $p_3 \in [\tilde{r}_2]_{\mathcal{P}}$, and since $r_2 < r_1$, it follows from Claim 10, that $p_3 < p'$ – completing the proof that $D_{\mathcal{P}}$ is dense.

Now, as in the proof of Lemma 3.28, there is a $\bar{\delta} < \xi < \delta$ and a pair $(s_\xi, p_\xi) \in D_{\mathcal{P}} \cap M$ such that

(1) $C_{p_\xi} \cap \bar{\delta} = C_q \cap \delta$ and $\bar{\delta} \in C_{p_\xi}$,
(2) for any $s_\delta \in S^{\downarrow}_{p_q} \cap S_\delta$, $S^{\downarrow}_{p_q} \cap S_\delta = \{s \oplus s_\delta : s \in S^{\downarrow}_{p_\xi} \cap S_{\bar{\delta}}\}$
(3) $s_\xi = s^\dagger \restriction \xi$ and (s_ξ, p_ξ) is compatible with $(s^\dagger, p_q)$.

The condition $\bar{r} \in \mathcal{P}$ with simply $\bar{r} = p_q \cup p_\xi$ satisfies that $\bar{r} < p_q$ and $\bar{r} < p_\xi$. Now choose $(s_\xi, q_\xi) \in D \cap M$ so that $p_\xi \in [q_\xi]_{\mathcal{P}}$. We are ready to define an $r \in \mathcal{Q}$ so that $(s^\dagger, r)$ is below each of $(s^\dagger, q)$ and (s_ξ, q_ξ) – completing the proof that $S \times \mathcal{Q}$ is proper. The definition of S_r is simply $S_q \cup S_{q_\xi}$. For $\alpha \in C_q \setminus \delta$, we let $\mathcal{N}^r_\alpha = \mathcal{N}^q_\alpha = \{N \in [\mathcal{N}_q] : N \cap \omega_1 = \alpha\}$. For $\alpha \in C_{q_\xi}$, let $\mathcal{N}^r_\alpha = \{N \in [\mathcal{N}^q] \cup [\mathcal{N}^{q_\xi}] : N \cap \omega_1 = \alpha\}$. Since $q_\xi \in M$, it follows that $[\mathcal{N}_{q_\xi}] \subset M \cap H(\aleph_2)$, and that condition (4) of the property of being an elementary matrix holds for $[\mathcal{N}^r] = \{\mathcal{N}^r_\alpha : \alpha \in C_q \cup C_{q_\xi}\}$. Since $\bar{r} \in [r]_{\mathcal{P}}$, we have that $r < q$ and $r < q_\xi$ holds. □

Now we have to prove that $\mathcal{Q}$ satisfies the $\aleph_2$-p.i.c.

Claim 12. *$\mathcal{Q}$ satisfies the $\aleph_2$-p.i.c.* itspic

Proof of Claim 12: Let $\lambda = \aleph_3$ and fix a well-ordering $\prec$ of $H(\lambda)$. Let $i < j < \omega_2$ be such that there are two countable elementary submodels N_i and N_j of $\langle H(\lambda), \prec, \in\rangle$ such that $\mathcal{Q}$ is in $N_i \cap N_j$, $i \in N_i$, $j \in N_j$, $N_i \cap i = N_j \cap j$, and suppose further that we are given $p \in \mathcal{Q} \cap N_i$ and an isomorphism $h : N_i \to N_j$ such that $h(i) = j$ and h is the identity on $N_i \cap N_j$. Since $S \subset 2^{<\omega_1}$, $h(s) = s$ for each $s \in 2^{<\omega_1} \cap N_i$. Also, since $S_p \subset N_i$, we have that $h(S_p) = S_p$.

We must show that there is a $q \in \mathcal{Q}$ such that :

(1) $q < p$, $q < h(p)$ and q is both N_i and N_j generic ,
(2) if $r \in N_i \cap P$ and $q' < q$ there is a $q'' < q'$ so that
$q'' < r$ if and only if $q'' < h(r)$.

Since $\mathcal{Q} \in N_i \cap N_j$, we have that $\{\prec_{\omega_1}, \tau, \mathcal{SB}, \mathcal{A}\} \in N_i \cap N_j$. The reason is that the collection $\{N : (\exists p \in \mathcal{Q}) N \in [\mathcal{N}_p]\}$ is in $N_i \cap N_j$. It follows that $N'_i = N_i \cap H(\aleph_2)$ is an elementary submodel of $(H(\aleph_2), \prec_{\omega_1}, \tau, \mathcal{SB}, \mathcal{A})$. N'_j, defined similarly, is as well. The definition of the $[\mathcal{N}_q]$ for q is canonical. Given that $[\mathcal{N}_p] = \{\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_n\}$, we set $[\mathcal{N}_q] = \{\mathcal{N}_1 \cup h(\mathcal{N}_1), \ldots, \mathcal{N}_n \cup h(\mathcal{N}_n), \{N'_i, N'_j\}\}$. The existence of h ensures that $\overline{N'_i} = \overline{N'_j}$. Since $[\mathcal{N}_p] \in N'_i$ and $h([\mathcal{N}_p]) = [\mathcal{N}_{h(p)}] \in N'_j$, we have that $[\mathcal{N}_q]$ is an elementary matrix. Now we have to choose S_q.

It follows from Claim 10 and Lemma 3.21, that the value of $\gamma(N_i')$ (as in Definition 3.22) is equal to $\gamma(N_j')$. Let s_p be the element in S_p with $o(s_p)$ a maximum. Choose any $s \in S_{\gamma(N_i')}$ such that $s_p < s$. Now set $S_q = S_p \cup \{s\}$. To show that $q = ([\mathcal{N}_q], S_q) \in \mathcal{Q}$, we just note that if $p_1 \in [p]_{\mathcal{P}}$, then the function p_2 equalling $p_1 \cup \{(N_i', s)\}$ is in $[q]_{\mathcal{P}}$.

We already know that q is both N_i-generic and N_j-generic from the discussion above explaining that $S \times \mathcal{Q}$ is proper. Finally, as in (2) above, let $r \in N_i \cap \mathcal{Q}$ and $q' < q$ with $q' \in \mathcal{Q}$. Towards verifying (2), we may assume, by symmetry and by possibly extending q', that q' is also below r. Let $[\mathcal{N}_{q'}]$ be listed as $\{\mathcal{N}_1^{q'}, \ldots, \mathcal{N}_k^{q'}\}$ and let $1 < \ell \leq k$ be chosen so that $N_i' \in \mathcal{N}_\ell^{q'}$. For $1 \leq m < \ell$, let $\mathcal{N}_m^i = \mathcal{N}_m^{q'} \cap N_i$. Of course we have that $N \in N_j'$ for each $1 \leq m < \ell$ and each $N \in h(\mathcal{N}_m^i)$. It is then easily verified that

$$\{h(\mathcal{N}_1^i) \cup \mathcal{N}_1^{q'}, \ldots, h(\mathcal{N}_{\ell-1}^i) \cup \mathcal{N}_{\ell-1}^{q'}, \mathcal{N}_\ell^{q''}, \ldots, \mathcal{N}_k^{q'}\}$$

is an elementary matrix, and so $q'' \in \mathcal{Q}$ where $S_{q''} = S_{q'}$ and

$$[\mathcal{N}_{q''}] = \{h(\mathcal{N}_1^i) \cup \mathcal{N}_1^{q'}, \ldots, h(\mathcal{N}_{\ell-1}^i) \cup \mathcal{N}_{\ell-1}^{q'}, \mathcal{N}_\ell^{q'}, \ldots, \mathcal{N}_k^{q'}\} \ .$$

It is immediate that $q'' < q'$, and so $q'' < r$. We just have to show that q'' is also below $h(r)$.

Since $q'' < r$, we have that $[\mathcal{N}_r] \in N_i$ is a submatrix of $\{N_i \cap \mathcal{N}_1, \ldots, N_i \cap \mathcal{N}_{\ell-1}\}$, and so $[\mathcal{N}_{h(r)}] \in N_j$ is a submatrix of $[\mathcal{N}_{q''}]$. Choose $p_1 \in [h(r)]_{\mathcal{P}}$ so that $h^{-1}(p_1) \in [r]_{\mathcal{P}}$ has an extension in $\mathcal{P}$ to some $p_2 \in [q'']_{\mathcal{P}}$. We know that $|\mathcal{M}_{p_2}| = k$ and now let $\mathcal{M}_{p_2}$ be listed in increasing order as $\{M_m^{p_2} : 1 \leq m \leq k\}$. As above, $\ell \leq k$ is the value so that $N_j' \in \mathcal{N}_\ell^{q''}$, and we noted above that $\mathcal{M}_{p_1} \subset N_j'$. Let $\bar{N}_\ell = N_j'$, and for $\ell < m \leq k$, recursively choose $\bar{N}_m \in \mathcal{N}_m^{q''}$ so that $\bar{N}_{m-1} \in \bar{N}_m$. For $1 \leq m < \ell$, let $\bar{N}_m = h(M_m^{p_2})$; note that $\bar{N}_m \in \mathcal{M}_{p_1}$ if $\bar{N}_m \cap \omega_1 \in C_{p_1}$. Define a function p_3 so that the domain of p_3 is the set $\mathcal{M}_{p_3} = \{\bar{N}_m : 1 \leq m \leq k\}$ and, for each $1 \leq m \leq k$, $p_3(\bar{N}_m) = p_2(M_m^{p_2})$. It follows that $p_3 \in [q'']_{\mathcal{P}}$ and that $p_3 < p_1$. This proves that $q'' < h(r)$. □

This completes the proof of Lemma 4.8. □

Definition 4.9. *The stationary set of ordinals $\lambda \in \omega_2$ with uncountable cofinality is denoted as S_1^2. The principle $\diamondsuit(S_1^2)$ is the statement: There is a family $\{X_\lambda : \lambda \in S_1^2\}$ such that*

1. *for each $\lambda \in S_1^2$, $X_\lambda \subset \lambda$,*
2. *for each $X \subset \omega_2$, the set $E_X = \{\lambda \in S_1^2 : X \cap \lambda = X_\lambda\}$ is stationary.*

zfc **Theorem 4.10.** *Assume CH and $\diamondsuit(S^2_1)$. There is a proper poset $\mathbb{P}$ so that in the forcing extension by $\mathbb{P}$ there is a coherent Souslin tree S such that, in the full forcing extension by $\mathbb{P} * S$, the statement* **GA** *holds.*

Proof. We construct a countable support iteration sequence $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$. By induction, we assume that $\mathbb{P}_\alpha$ is proper, has cardinality at most $\aleph_2$, and that

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha \text{ satisfies the } \aleph_2\text{-p.i.c} \ .$$

Note that by Propositions 4.2 and 4.4 we will have that, for each $\alpha < \omega_2$, CH holds in the forcing extension by $\mathbb{P}_\alpha$. We may assume that $\dot{\mathbb{Q}}_0$, and therefore $\mathbb{P}_1$ is constructed so that there is a $\mathbb{P}_1$-name, $\dot{S}$ of a coherent Souslin tree (henceforth we suppress the dot on the S). We further demand of our induction that, for $\alpha \geq 1$

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha \text{ is } S\text{-preserving} \ .$$

For each ordinal $0 < \alpha \in \omega_2 \setminus S^2_1$, we let $\dot{\mathbb{Q}}_\alpha$ denote the $\mathbb{P}_\alpha$-name of the standard Hechler poset for adding a dominating real. This ensures that $\mathfrak{b} = \omega_2$ in the forcing extension by $\mathbb{P}_{\omega_2}$. For the rest of the construction, fix any function h from ω_2 onto $H(\aleph_2)$. Also let $\{X_\lambda : \lambda \in S^2_1\}$ be a $\diamondsuit(S^2_1)$-sequence.

Now consider $\lambda \in S^2_1$ and let $x_\lambda = h[X_\lambda]$. We define $\dot{\mathbb{Q}}_\lambda$ according to cases:

(1) if x_λ is the $\mathbb{P}_\lambda * S$-name of a P-ideal on ω_1 such that

$$1 \Vdash [E]^{\aleph_0} \cap x_\lambda \text{ is not empty for all stationary sets } E \subset \omega_1,$$

then $\dot{\mathbb{Q}}_\lambda$ is the $\mathbb{P}_\lambda$-name of the poset from Theorem 4.7,

(2) if x_λ is the $\mathbb{P}_\lambda * S$-name of a subset of $\lambda \times \lambda \times \lambda$ so that if we define, for $\xi, \eta \in \lambda$, $\dot{U}(\xi, \eta)$ to be the $\mathbb{P}_\lambda * S$-name of the subset of λ such that

$$\{(\xi,\eta)\} \times \dot{U}(\xi,\eta) = x_\lambda \cap (\{(\xi,\eta)\} \times \lambda) \ ,$$

i.e. for $((p,s),(\xi,\eta,\gamma))$ in the set x_λ, $((p,s),\gamma)$ is in the name $\dot{U}(\xi,\eta)$, and $\mathbb{P}_\lambda * S$ forces that the family $\{\dot{U}(\xi,\eta) : \eta \in \lambda\}$ is a local base for ξ in a sequentially compact regular topology on λ, and that no finite subset of $\{\dot{U}(\xi,\eta) : \xi, \eta \in \lambda\}$ covers λ, then $\dot{\mathbb{Q}}_\lambda$ is the $\mathbb{P}_\lambda$-name of the poset from Theorem 4.8.

(3) in all other cases, $\dot{\mathbb{Q}}_\lambda$ is the $\mathbb{P}_\lambda$-name of the Cohen poset $2^{<\omega}$.

Assume that $\dot{\mathcal{I}}$ is a $\mathbb{P}_{\omega_2} * S$-name of a P-ideal on ω_1 satisfying that there is some $(p_0, s_0) \in \mathbb{P}_{\omega_2} * S$ forcing that $[E]^{\aleph_0} \cap \dot{\mathcal{I}}$ is not empty for all stationary sets $E \subset \omega_1$. In order to apply case (1) of the definition of

$\mathbb{P}_{\omega_2}$ we construct another $\mathbb{P}_{\omega_2} * S$-name $\dot{\mathcal{J}}$ so that $(p,s) \Vdash \dot{\mathcal{J}} = \dot{\mathcal{I}}$, and 1 forces that $[E]^{\aleph_0} \cap \dot{\mathcal{I}}$ is not empty for all stationary sets $E \subset \omega_1$. More precisely, we choose $\dot{\mathcal{J}}$ so that $(p,s) \Vdash \dot{\mathcal{J}} = \dot{\mathcal{I}}$ and $(p',s') \Vdash \dot{\mathcal{J}} = [\omega_1]^{\leq\aleph_0}$ for all $(p',s') \in \mathbb{P}_{\omega_2} * S$ that are incompatible with (p,s). Let $X \subset \omega_2$ be chosen to be the set of all $\xi \in \omega_2$ with the property that there is a $\mu < \omega_2$ such that $h(\xi)$ is a $\mathbb{P}_\mu * S$-name with $1 \Vdash h(\xi) \in \dot{\mathcal{J}}$. There is a cub $C \subset \omega_2$ such that for each $\mu < \mu' \in C$:

(1) the collection $\{h(\xi) : \xi \in X \cap \mu\}$ is a collection of $\mathbb{P}_{\mu'} * S$-names,
(2) for each countable subset $\{\xi_n : n \in \omega\}$ of $X \cap \mu$ there is a $\xi < \mu'$ such that 1 forces that $h(\xi_n)$ is almost included in $h(\xi)$ for each n,
(3) every $\mathbb{P}_\mu * S$-name that is forced by 1 to be a member of $\dot{\mathcal{J}}$ is congruent to a name in $\{h(\xi) : \xi \in X \cap \mu'\}$.

Therefore, there is a $\lambda \in E_X \cap C$ such that $X_\lambda = X \cap \lambda$. We may of course assume that $p_0 \in \mathbb{P}_\lambda$. Routine checking now shows that x_λ satisfies clause (1) in the definition of $\dot{\mathbb{Q}}_\lambda$. It follows that $\mathbb{P}_{\omega_2} * S$ is a model of $\mathbf{P}_{22}$.

Now suppose that we have a $\mathbb{P}_{\omega_2} * S$-name of a sequentially compact non-compact space. We note that $\mathbb{P}_{\omega_2} * S$ forces that $2^{\aleph_0} = \aleph_2$. By Proposition 3.3, we can assume that either the space is separable, or equal to the sequential closure of some subspace of cardinality at most $\aleph_1$ that has no complete accumulation point. In either case, every point of the space has a neighborhood that is separable. To see the latter case, consider the (sequential) closure of a set A of size $\aleph_1$ with no complete accumulation point. Let x be in that (sequential) closure. Take an open set U about x which has countable intersection with A. That intersection is dense in U. Now we see that this local separability ensures that every point has a neighborhood base of cardinality at most $\aleph_2$. Then, by possibly taking the free union with the Cantor set, we can assume the space has cardinality exactly ω_2, and that the base set for the space is the ordinal ω_2. Let $\dot{Z}$ denote the $\mathbb{P}_{\omega_2} * S$-name of this space. Let $\{\dot{U}(\xi,\eta) : \xi,\eta \in \omega_2\}$ be the list of $\mathbb{P}_{\omega_2} * S$-names of the neighborhood bases of the points, and chosen so that no finite subcollection covers. We define X to be the set of all those $\alpha \in \omega_2$ such that $h(\alpha)$ is a tuple of the form $((p,s),(\xi,\eta,\gamma))$, i.e. a $\mathbb{P}_{\omega_2} * S$-name of a member of $\omega_2 \times \omega_2 \times \omega_2$, where $(p,s) \Vdash \gamma \in \dot{U}(\xi,\eta)$.

We again want to choose a λ in $E_X \cap C$ for some special cub set C and in this case it is much simpler to make use of uncountable elementary submodels. Let κ be any regular cardinal greater than 2^{ω_2}, and let $\{M_\alpha : \alpha \in \omega_2\}$ be chosen so that, for each $\alpha \in \omega_2$:

(1) X, h and $\mathbb{P}_{\omega_2} * S$ are in M_α

(2) $\omega_1 \subset M_\alpha$ and M_α has cardinality $\aleph_1$,
(3) for each $\beta < \alpha$, every countable subset of M_β is an element of M_α,
(4) M_α is an elementary submodel of $H(\kappa)$,
(5) if α is a limit ordinal, then $M_\alpha = \bigcup\{M_\beta : \beta < \alpha\}$.

Items (2) and (4) guarantee that $M_\alpha \cap \omega_2$ is an initial segment of ω_2. The chain $\{M_\alpha : \alpha \in \omega_2\}$ is a continuous chain because of item (5), and so $C = \{\alpha \in \omega_2 \colon M_\alpha \cap \omega_2 = \alpha\}$ is a closed and unbounded subset of ω_2. Now we choose $\lambda \in E_X \cap C$. Using items (1), (3) and (4) and the fact that $\lambda \in S^2_1$, it is now easy to show that x_λ will satisfy the requirement (2) in the construction of $\dot{\mathbb{Q}}_\lambda$. It then follows, as in the proof of Lemma 3.27, that $\mathbb{P}_{\lambda+1} * S$ will force the existence of the necessary ω_1-sequence showing that $\dot{Z}$ is not a counterexample to $\mathbf{PPI}^+$. $\square$

5. Acknowledgement

The authors are very grateful for the efforts of the excellent referee, which led us to greatly improve the presentation of these results.

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