CLOSURES OF DISCRETE SETS IN COMPACT SPACES

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Abstract. We consider the question of whether a compact space will always have a discrete subset whose closure has the same cardinality as the whole space. We obtain many positive results for compact spaces of countable tightness and a consistent negative result for a space of tightness and density ω_1.

Several authors have recently been investigating a notion called *discretely reflexive* where one seeks a discrete subset whose closure will reflect a particular property of the entire space. For example, in the paper [Tka87], if a space X is not compact then it possesses a discrete subset whose closure is not compact. A very interesting question remaining open is whether or not the same thing holds for the Lindelöf property ([ATW00], see also [DTTW02]). In this paper we are interested in a question raised by Arhangelski [Arh67] as to whether, in a compactum, the supremum of cardinalities of the closures of discrete subsets will always equal the cardinality of the space. This was shown to hold by Efimov [Efi69] for the class of dyadic compacta. We establish, at least consistently, this need not be the case. For compact spaces of countable tightness, we present some partial results but leave open the interesting general question. Our main (consistent) example is somewhat interesting in other ways: it is a compact $\aleph_2$-Luzin space, that is, every subset of size $\aleph_2$ is somewhere dense. The author wishes to acknowledge the contributions of the participants of the *Colloquium on General and Set-Theoretic Topology dedicated to the 60th birthday of Istvan Juhasz* held in Budapest, Hungary during August 2003. In particular, the author especially benefitted from conversations on this topic with I. Juhasz, K. Kunen, J. van Mill, and B. Weiss.

The spread of a space X, $s(X)$, is defined as the supremum of the cardinalities of discrete subsets of X. The depth of a space X, denoted $g(X)$ [Arh67], is defined as the supremum of cardinalities of closures of discrete subspaces. We will say that cardinality discretely reflects in X if X has a discrete subset D such that $|\overline{D}| = |X|$. Arhangelskii asked if for each compact space X, $g(X) = |X|$.

Lemma 1. *If X is compact then X contains a discrete set whose closure has size at least the minimum of $\{|X|, \mathfrak{c}\}$.*

Proof. Let D be the set of isolated points of X. We may as well assume that $\overline{D}$ has cardinality less than that of X, and therefore that $X \setminus \overline{D}$ has cardinality equal to $|X|$. It follows that if we let K denote the closure of $X \setminus \overline{D}$, then K is a compact space with no isolated points. Therefore K will map continuously onto the unit interval. Now the interval has a discrete set with closure having cardinality $\mathfrak{c}$ and X has a discrete set whose closure maps onto it. Therefore X has a discrete set whose closure has cardinality at least $\mathfrak{c}$. □

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Corollary 2. *Each compact space of cardinality at most $\mathfrak{c}$ is discretely reflexive.*

Definition 3. A subset x of a tree T is a *path* of T if x is a chain and for each $s < t \in T$, if $t \in x$, then $s \in x$. If T is a tree, then pT consists of the set of all paths of T and pT will be regarded as a subspace of the product space 2^T. For a finite subsets F, H of T, the basic clopen set $[F; H]$ will consist of those paths x such that $F \subset x$ and $x \cap H$ is empty. Note that $\emptyset$ is an element of pT, but if T has a root then $\emptyset$ is an isolated point of pT.

Proposition 4. *If T is a tree, then a point $x \in pT$ is isolated in pT if and only if there is a $t \in T$ such that $x = \{s \in T : s \leq t\}$ and t has only finitely many immediate successors.*

For the rest of the paper we shall make a mild restriction on what sort of trees we will consider. Specifically, we shall assume that every node in the tree has at least two immediate successors. In particular then, no hypothesized tree will have a maximal node.

Proposition 5. *If I is the set of isolated points of pT for a tree T, then $pT \setminus I$ has no isolated points.*

Proof. Suppose that x is a member of pT which is not isolated. By Proposition 4, either there is no $t \in T$ such that $x = \{s \in T : x \leq t\}$, or there is such a $t \in T$ but this t has infinitely many successors. We consider a typical basic open neighborhood of x, $[F; H]$, where F is a finite subset of x (recall that x is a path in T) and a H is a finite subset of $T \setminus x$. For each $h \in H$ such that there is some $f \in x$ which is not compatible with h, we add such an f to F. We must show there is some $y \in bT \setminus \{x\}$ which is also in $[F; H]$.

First consider the case that the path x has no maximal element. In this case there is some $t' \in x$ such that $F \subset \{s \in x : s < t'\}$. One of the immediate succssors of t', say t_0, will not be a member x. Note that no member of H is above t_0 since t' is above every member of F. By our assumption on T, there is a strictly increasing sequence $\{t_n : n \in \omega\}$ of members of T. There is a path $y \in bT$ such that $\{t_n : n \in \omega\}$ is cofinal in y. It follows by Proposition 4, that $y \in bT \cap [F; H]$. Clearly $y \neq x$.

Now assume that t is the maximal element of x, in which case t has infinitely many immediate successor. Clearly there is some immediate successor t_0 of t which is not below any member of H. Again finding a strictly increasing sequence $\{t_n : n \in \omega\}$ yields a path $y \in bT \cap [F; H]$ distinct from x. □

Definition 6. For a tree T and I denoting the isolated points of pT, we will let bT denote the compact subspace $pT \setminus I$.

Proposition 7 (Kunen)**.** *If κ is a strongly inaccessible cardinal and if T is a κ-Souslin tree, then bT is not discretely reflexive.*

Proof. Assume that $D = \{x_\alpha : \alpha < \kappa\} \subset bT$ is discrete. Since κ is strongly inaccessible, we may assume that for each $\alpha < \kappa$, the chain x_α has order type greater than α. Let S denote the subset of T consisting of all elements t such that the basis open set $[\{t\}; \emptyset]$ contains no elements of D. Let $A \subset S$ be any maximal antichain. Since T is κ-Souslin, it follows that A has cardinality less than κ. For each $\alpha \in \kappa$, let $[F_\alpha; H_\alpha]$ denote a basic clopen subset of pT which meets D in precisely the point x_α and, for each $\alpha \in \kappa$ let t_α be any member of x_α such that the maximal element of F_α is less than or equal to t_α. We may assume that $t_\alpha \in T \setminus 2^{<\alpha}$. Therefore, there are fewer than κ many α such that t_α is below some member of A, so fix any $\alpha < \kappa$ such that t_α is not below any member of A. Since $t_\alpha \in x_\alpha$ it also follows that t_α is not above any member of S. Since x_α is not isolated in pT, there is some $t' \in T$ such that $t_\alpha < t'$ and t' is incompatible with each member of H_α and $x_\alpha \notin [\{t'\}; \emptyset]$. Again, t' is not below any member of A nor above any member of S, hence there is some $\beta \in \kappa$ such that $t' \in x_\beta$. It follows immediately that $x_\beta \in [F_\alpha; H_\alpha]$, a contradiction. □

Kunen also noted that certain kinds of ω_2-Souslin trees also provide examples. This had been previously noted by Nyikos as well.

Proposition 8. *It is consistent with GCH that there is an ω_2-Souslin tree T with the property that every maximal branch has a countable cofinal sequence. In addition, this tree T has the property that $g(bT) = \aleph_1$ while $|bT| = \aleph_2$.*

Proof. We are not aware of an explicit reference to establish that there is such a tree. However it is not difficult to verify that the tree, attributed to Jensen, constructed in the Handbook of Logic [Dev77, Page 475, B5.11.3] yields such a tree when applied using the results B5.11.1 and B5.11.2 from the same reference. The reader can readily verify, using the notation from [Dev77, Page 475, B5.11.3], that if ν is a limit of C_α for some α, then T_ν, the ν-th level of the constructed tree T, will equal $\{p^x_\nu : x \in T \restriction \nu\}$. In addition, if for some ν, and $x, x' \in T \restriction \nu$, $p^x_\nu = p^{x'}_\nu$, then $p^x_\mu = p^{x'}_\mu$ for all $\nu \leq \mu \in C_\alpha$. Assume then, that $b \subset T$ is a maximal branch and that $b \subset T \restriction \alpha$ and that α has uncountable cofinality. Fix any limit $\nu_0 \in C_\alpha$, and, if it exists, let $x_0 \in b \cap T_{\nu_0}$. Since b is maximal, there is a $\nu_1 \in C_\alpha$ so that $\nu_0 < \nu_1$ and $p^{x_0}_{\nu_1} \notin b \cap T_{\nu_1}$. Again, if it exists, set $x_1 \in b \cap T_{\nu_1}$ and continue to choose an increasing sequence $\{\nu_k : k \in \omega\}$ and $x_k \in b \cap T_{\nu_k}$ in a similar manner. It follows that $b \cap T_\gamma$ is empty where γ is the supremum of $\{\nu_k : k \in \omega\}$ since $b \cap \{p^x_\nu : x \in T \restriction \nu\}$ will be empty.

We can proceed as in Proposition 7 to show that each discrete set of branches will be a subset of a bounded initial segment of T. Recall that each member of bT will either be of the form $\{s \in T : s < t\}$ for some $t \in T$ or will be a branch with a countable cofinal set. The fact then that each discrete set has size at most $\aleph_1$ follows from noting that for each $\alpha \in \omega_2$ with countable cofinality, there will be at most $\aleph_1$ members of bT which have a cofinal subset consisting of nodes of T whose levels are cofinal in α. □

We make three conjectures and raise a question as focus for the remainder of the paper.

Conjecture 1. Every compact separable space is cardinality discretely reflexive.

Conjecture 2. Every compact space of countable tightness is cardinality discretely reflexive.

Conjecture 3. It is not decidable in ZFC if every compact space of density $\aleph_1$ is cardinality discretely reflexive.

Question 4. Can each compact space of countable tightness be written as a union of at most $\mathfrak{c}$ many sets each of which has a dense discrete subset?

The following lemma is well-known for the spread and is proven the same way for the depth.

Proposition 9. *If $s(X)$ has countable cofinality, then there is a discrete set D with cardinality $s(X)$. Similarly, if $g(X)$ has countable cofinality, then there is a discrete set D with $|\overline{D}| = g(X)$.*

Proof. Suppose the $\{\lambda_n : n \in \omega\}$ is strictly increasing and cofinal in $\lambda = g(X)$ for a Hausdorff space X. For each $n \in \omega$, fix a discrete subset D_n of X so that $\overline{D_n}$ has cardinality at least λ_n. Since $\overline{D_n}$ is compact, we may choose a point x_n of $\overline{D_n}$ such that for each open set U_n with $x_n \in U_n$, the cardinality of $U_n \cap \overline{D_n}$ is at least λ_n. Let I be any infinite subset of ω so that the sequence $\{x_n : n \in I\}$ is discrete and let $\{W_n : n \in I\}$ be pairwise disjoint open sets so that $x_n \in W_n$ for each $n \in I$. It follows immediately that the set $D = \bigcup\{W_n \cap D_n : n \in I\}$ is a discrete set and that $\overline{D}$ has cardinality at least λ_n for each n. □

Theorem 10. *If X is compact with countable tightness then $|X| \leq g(X)^\omega$.*

Proof. Let X be compact with countable tightness and assume that M is an elementary submodel of some sufficiently large fragment of the universe such that X and its topology are members of M, the cardinal $g(X)$ is a subset of M, M has cardinality $g(X)^\omega$ and every countable subset of M is a member of M (e.g. see [Dow88]). We will proceed by showing that $Y = X \cap M$ is in fact all of X by first showing that each point of x is in the closure of a countable discrete subset of Y.

Fix any $x \in X$ and assume that x is not in the closure of any countable discrete subset of Y. Let $y \in Y$ and let $\mathcal{U}_y \in M$ denote the collection of open sets $U \subset X$ such that $y \notin \overline{U}$. According to Sapirovskii's lemma ([Juh80, 4.12]), there is a discrete set $D_y \in M$ and a subcollection $\mathcal{W}_y \in M$ with cardinality equal to that of D_y such that $X \setminus \{y\}$ is covered by $\overline{D_y}$ and $\bigcup \mathcal{W}_y$. Since $|\mathcal{W}_y| = |D_y| \leq g(X)$ and $\mathcal{W}_y \in M$, we have that $\mathcal{W}_y \subset M$. In addition, since D_y is discrete and by countable tightness, x is not in the closure of D_y, hence there is some $W_y \in \mathcal{W}_y$ such that $x \in W_y$. Since $y \notin \overline{W_y}$, we have that there is some open $U_y = X \setminus \overline{W_y} \in M$ such that $y \in U_y$ and $x \notin U_y$. Let $\mathcal{U}$ denote the collection of all open $U \in M$ such that x is not in the closure of U. We have shown that $\mathcal{U}$ is an open cover of $X \cap M$. We begin choosing points $\{y_\alpha : \alpha \in \omega_1\} \subset Y$ and countable subcollections $\mathcal{U}_\alpha$ of $\mathcal{U}$ as follows. Let $\mathcal{U}_0$ be any countable subset of $\mathcal{U}$. Since $x \notin \bigcup \mathcal{U}_0$, and $\mathcal{U}_0 \in M$, we may, by elementarity, select some $y_0 \in Y \setminus \bigcup \mathcal{U}_0$. At stage $0 < \alpha < \omega_1$, assume that x is not in the closure of the countable discrete set $\{y_\beta : \beta < \alpha\}$. By our assumptions on M, $\{y_\beta : \beta < \alpha\}$ is a member of M, and so is $\overline{\{y_\beta : \beta < \alpha\}}$. In addition, M will contain the cardinal $|\overline{\{y_\beta : \beta < \alpha\}}|$, hence M will contain the entire compact set $\overline{\{y_\beta : \beta < \alpha\}}$, which implies there is a finite collection $\mathcal{U}_\alpha \subset \mathcal{U}$ which covers it. Again, we may choose $y_\alpha \in Y \setminus \bigcup_{\beta \leq \alpha} \bigcup \mathcal{U}_\beta$. The sequence $\{y_\alpha : \alpha \in \omega_1\}$ we are constructing would be a free sequence in X which would contradict the countable tightness of X, therefore there is an $\alpha \in \omega_1$ such that x is in the closure of the discrete set $\{y_\beta : \beta < \alpha\}$.

Note that it now follows that $x \in M$, since if $D \subset Y$ is any countable discrete subset of Y with $x \in \overline{D}$, we have that $\overline{D} \subset M$ since $\overline{D} \in M$ and $|\overline{D}| \leq g(X)$. □

Corollary 11. *If X is compact with countable tightness and $|X| \leq \aleph_\omega$, then X is cardinality discretely reflexive.*

Finally we show that it is consistent with CH that there is a compact space of density and weight $\aleph_1$ which is not discretely reflexive. Recall that a tree T is a

Kurepa tree if T is a tree with ω_1 many levels, each level is countable and T has at least $\aleph_2$ many ω_1-branches. Notice that since bT has no isolated points, the closure of each discrete subset is nowhere dense.

Theorem 12. *It is consistent with GCH that there is a Kurepa tree T such that bT is a compact space of cardinality $\aleph_2$ which has the property that every subset of cardinality $\aleph_2$ is somewhere dense. In particular, bT is not cardinality discretely reflexive.*

Proof. Let V be any model of GCH and let P denote the standard poset for introducing a Kurepa tree (see [Jec78, p247]). A condition p of P consists of a triple (T_p, A_p, f_p) where T_p is a countable subtree of $2^{<\omega_1}$, A_p is a countable subset of ω_2 and f_p is a function from A_p into T_p. Furthermore, if we let δ_p denote the minimum $\delta \in \omega_1$ such that $T_p \cap 2^{\delta+1}$ is empty then for each $t \in T_p \cap 2^{<\delta}$ there must be at least two $t' \in T_p \cap 2^\delta$ such that $t < t'$. We will also demand that $f_p : A_p \to (T_p \cap 2^{\delta_p})$.

We define $p \leq q$ if

(1) $T_q \subseteq T_p$
(2) for each $\alpha \leq \delta_q$, $T_p \cap 2^\alpha = T_q \cap 2^\alpha$
(3) $A_q \subset A_p$
(4) for each $\beta \in A_q$, $f_q(\beta) \leq f_p(\beta)$.

The poset P is $\aleph_1$-directed closed for if $\{p_n : n \in \omega\} \subset P$ are such that $p_{n+1} \leq p_n$ for each n, then we can define $p \in P$ with $p \leq p_n$ for all n as follows. Set $A_p = \bigcup A_{p_n}$ and $\delta_p = \bigcup_n \delta_{p_n}$. For each $\alpha \in A_p$, let $J_\alpha = \{n \in \omega : \alpha \in A_n\}$. Notice that $\{f_{p_n}(\alpha) : n \in J_\alpha\}$ is a chain in $2^{<\omega_1}$ and we will set $f_p(\alpha) = \bigcup\{f_{p_n}(\alpha) : n \in J_\alpha\}$ for each $\alpha \in A_p$. Finally, T_p is any subtree of $2^{\leq \delta_p}$ which contains $\{f_p(\alpha) : \alpha \in A_p\} \cup \bigcup_n T_{p_n}$, and if necessary countably more branches of $\bigcup_n T_{p_n}$ are added so as to satisfy the condition that each $t \in T_p \cap 2^{<\delta_p}$ has at least two successors in $T_p \cap 2^{\delta_p}$.

If $p, q \in P$ are such that $T_p = T_q$ and $f_p \restriction (A_p \cap A_q) = f_q \restriction (A_p \cap A_q)$, then $(T_p, A_p \cup A_q, f_p \cup f_q)$ is a member of P which is below each of p and q. Therefore it follows from CH and a standard Δ-system argument, that P satisfies the $\aleph_2$-cc and therefore forcing with P preserves all cardinals.

Let G be P-generic and, in $V[G]$, let T denote the set $\bigcup\{T_p : p \in G\}$. It follows, as is well-known, from the definition of the ordering on P that T will be tree with countable levels and height ω_1. In fact T is a Kurepa tree.

For each $\alpha \in \omega_2$, let $\dot{x}_\alpha$ denote the following name of a subset of $2^{<\omega_1}$:

$$\dot{x}_\alpha = \{(p, \check{t}) : p \in P, t \in T_p, \alpha \in A_p, \text{ and } t \leq f_p(\alpha)\} .$$

A routine density argument establishes that if G is P-generic, the valuation of $\dot{x}_\alpha$ by G, namely x_α, will be an ω_1-branch of $2^{<\omega_1}$, and will also, of course, be a member of bT

We next show that the collection $\{x_\alpha : \alpha \in \omega_2\}$ will in fact be all the uncountable branches of T. Working in V again, let $\dot{x}$ be a P-name and assume that $p \in P$ forces that $\dot{x}$ is an ω_1-branch of T. Furthermore, towards a contradiction, assume that for each $\alpha \in \omega_2$, $p \Vdash \dot{x}_\alpha \neq \dot{x}$. We will show that there is a $q \leq p$ which forces $\dot{x}$ to be countable. Since P is ω_1-closed and A_p is countable, there is a condition $p_1 \leq p = p_0$ such that for each $\alpha \in A_p$, there is a $\delta_{p_0} \leq \gamma < \delta_{p_1}$ and a $t \in T_{p_1} \cap 2^\gamma$ such that

$$p_1 \Vdash \check{t} \in \dot{x} \text{ and } \check{t} \not\leq f_{p_1}(\alpha)$$

(i.e. p_1 forces that $\dot{x}$ and $\dot{x}_\alpha$ differ on the δ_{p_1}-level). Recursively choose a descending sequence p_n of conditions so that for each $n \in \omega$ and for each $\alpha \in A_{p_n}$, there is a $\delta_{p_n} \leq \gamma \leq \delta_{p_{n+1}}$ and a $t \in T_{p_{n+1}} \cap 2^\gamma$ such that

$$p_{n+1} \Vdash \check{t} \in \dot{x} \text{ and } \check{t} \not\leq f_{p_{n+1}}(\alpha).$$

For each $n \in \omega$, let t_n be the unique element of 2^{δ_n} such that $p_{n+1} \Vdash \check{t} \in \dot{x}$. Now let q be any lower bound of the chain $\{p_n : n \in \omega\}$ which is chosen so the chain $\{t_n : n \in \omega\}$ does not have an upper bound in T_q. The contradiction is that q forces that no member of $T \cap 2^{\delta_q}$ is in $\dot{x}$, hence $\dot{x}$ is not an uncountable branch.

We are now ready to prove that every nowhere dense subset of bT has cardinality less than $\aleph_2$. Assume that $\dot{Y}$ is a P-name of a subset of bT and that $p \Vdash |\dot{Y}| > \aleph_1$. Since bT is forced to have $\aleph_1$ many countable branches and $\aleph_2$ many uncountable branches, we may assume that $\dot{Y}$ is forced to be a subset of $\{x_\alpha : \alpha \in \omega_2\}$. Let I denote the set of all α such that there is a $p_\alpha \leq p$ such that $p_\alpha \Vdash \dot{x}_\alpha \in \dot{Y}$; we may assume that $\alpha \in A_{p_\alpha}$.

Since p forces $\dot{Y}$ has cardinality $\aleph_2$, it follows that I has cardinality $\aleph_2$. There is some $\delta \in \omega_1$ and $T' \subseteq 2^{\leq\delta}$ such that there is a $J \subset I$ with $|J| = \aleph_2$ such that $T_{p_\alpha} = T'$ for all $\alpha \in J$. In addition, we may assume that the family $\{A_{p_\alpha} : \alpha \in J\}$ forms a Δ-system with root A (i.e. $A_{p_\alpha} \cap A_{p_\beta} = A$ for all $\alpha < \beta \in J$). Finally, we can also assume that there is a function $f : A \to T'$ and a $t \in T'$ such that $f_{p_\alpha} \restriction A = f$ and $f_{p_\alpha}(\alpha) = t$ for all $\alpha \in J$. We will prove that the condition $p' = (T', A, f)$ forces that $\dot{Y}$ is dense in the clopen subset $[\{t\}; \emptyset]$ of bT. Since $p' \leq p$ this will complete the proof. Let $q \leq p'$ be any member of P and assume that $q \Vdash [\check{F}; \check{H}] \cap bT$ is a basic clopen set and that $[\check{F}; \check{H}] \cap bT \subset [\{t\}; \emptyset]$. It suffices to find some extension of q which forces that $\dot{Y}$ meets $[\check{F}; \check{H}]$.

We may assume that there is some $s \in F$ with $t \leq s$ and s is the maximum member of F. We may also assume that $F \cup H \subset T_q$. Since H is finite and $q \Vdash [F; H] \cap bT$ is not empty, it is not difficult to show that we can find some $s' \in T_q$, an extension of s such that no member of H is above s'; we leave this to the reader.

Now choose any $\alpha \in J$ such that $A_{p_\alpha} \setminus A$ is disjoint from A_q. Since $T' \subset T_q$, we can, for each $\beta \in A_{p_\alpha}$ find some element $g(\beta) \in T_q \cap 2^{\delta_q}$ such that $f_{p_\alpha}(\beta) \leq g(\beta)$. Since $t = f_{p_\alpha}(\alpha) \leq s'$, we can demand that $s' \leq g(\alpha)$. Set $q' = (T_q, A_q \cup A_{p_\alpha}, f_q \cup g)$. It is routine to check that $q' \in P$, $q' \leq q$, and $q' \leq p_\alpha$. In addition, q' forces that $\dot{x}_\alpha \in \dot{Y} \cap [\{s'\}; \emptyset] \subset [F; H]$ as required. □

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