

ON THE CARDINALITY OF SEPARABLE PSEUDORADIAL SPACES

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ABSTRACT. The aim of this paper is to consider questions concerning the possible maximum cardinality of various *separable pseudo-radial* (in short: SP) spaces. The most intriguing question here is if there is in ZFC a regular (or just Hausdorff) SP space of cardinality $> \mathfrak{c}$. While this question is left open, we establish a number of non-trivial results that we list below.

- It is consistent with $MA + \mathfrak{c} = \aleph_2$ that there is a *countably tight* and compact SP space of cardinality $2^{\mathfrak{c}}$.
- If κ is a measurable cardinal then in the forcing extension obtained by adding κ many Cohen reals, every *countably tight* regular SP space has cardinality at most \mathfrak{c} .
- If $\kappa > \aleph_1$ Cohen reals are added to a model of GCH then in the extension every pseudocompact SP space with a countable dense set of isolated points has cardinality at most \mathfrak{c} .
- If $\mathfrak{c} \leq \aleph_2$ then there is a 0-dimensional SP space with a countable dense set of isolated points that has cardinality greater than \mathfrak{c} .

1. INTRODUCTION

The class of pseudoradial spaces is a natural and well-studied class that is a generalization of the radial property in the same way that the class of sequential spaces is a generalization of the class of Frechet-Urysohn spaces. For a cardinal κ , a sequence $\{x_\alpha : \alpha < \kappa\}$ is said to converge to x in a space X , if for every neighborhood U of x there is a $\beta < \kappa$ such that $\{x_\alpha : \beta < \alpha < \kappa\}$ is a subset of U . A set A is radially closed in X if, for every cardinal $\kappa \leq |X|$, no κ -sequence of points of A converges to a point not in A . A space X is pseudoradial if every radially closed set is closed. A set A is sequentially closed if no ω -sequence of points from A converges to a point not in A and a space is sequential if every sequentially closed set is closed.

Date: December 4, 2020.

2020 Mathematics Subject Classification. 54A35 .

Key words and phrases. tightness, pseudoradial.

First author supported by NSF grant DMS-1501506 and

Second author supported by NKFIH grant number K 129211.

As stated in the abstract, we are exploring the question of whether separable pseudoradial spaces of cardinality greater than \mathfrak{c} exist. We may say that a space is large if it has cardinality greater than \mathfrak{c} . We thank A. Bella for informing us that it was shown in [2] that it is consistent that the usual product space 2^{ω_1} is SP.

One of the most interesting results about compact pseudoradial spaces arose when Sapirovsii [21] proved that the continuum hypothesis implied that a compact space is pseudoradial so long as it is sequentially compact. A space is sequentially compact if every infinite sequence has a limit point. This was improved in [16] where it was shown that it follows from $\mathfrak{c} \leq \aleph_2$ that compact sequentially compact spaces are pseudoradial and that this bound can not be improved in ZFC. In this paper we are able to show that this same assumption of $\mathfrak{c} \leq \aleph_2$ is sufficient to produce examples of regular Hausdorff separable pseudoradial spaces of cardinality equal to $2^{\mathfrak{c}}$. In this paper we will restrict our investigation to regular Hausdorff spaces and note that $2^{\mathfrak{c}}$ is the upper bound on the cardinality of any separable space.

Returning to the class of compact separable spaces, we recall that every pseudoradial compact space is sequentially compact. More generally in a pseudoradial space, a countable discrete set is closed if it contains no converging sequence, and so clearly, a compact pseudoradial space is sequentially compact. It is well-known that the sequential closure of a countable set has cardinality at most \mathfrak{c} and therefore every sequential separable space has cardinality at most \mathfrak{c} . Balogh [3] proved that the proper forcing axiom implies that every compact space of countable tightness is sequential, and therefore compact separable spaces of countable tightness have cardinality at most \mathfrak{c} . Of course this paper of Balogh's was in answer to the celebrated Moore-Mrowka problem. Recall that the proper forcing axiom also implies that $\mathfrak{c} = \aleph_2$ [18, 23]. This background motivates one to ask more about separable pseudoradial spaces of countable tightness both with and without the extra assumption of compactness.

2. MARTIN'S AXIOM

This project began when we were made aware of a question in connection to the Moore-Mrowka problem posed by S. Spadaro in MathOverflow. We thank K.P. Hart for bringing the question to our attention. We refer the reader to [17] for details about Martin's Axiom and to [3] for the statement of the proper forcing axiom.

Assume MA_{\aleph_1} . Is it true that every compact pseudoradial space of countable tightness is sequential?

We answer this question in the negative but will rely on quoting two related results from the literature. It would be too ambitious to reproduce the proofs from these publications. However we can connect the investigations there to the current one.

Definition 2.1. *A space X is initially \aleph_1 -compact if every open cover of cardinality at most \aleph_1 has a finite subcover.*

An initially \aleph_1 -compact space is countably compact and a first-countable initially \aleph_1 -compact space is sequentially compact. It is shown in [10, 7.1] that every compactification of an initially \aleph_1 -compact space of countable tightness has countable tightness. It was proven in [15] that there is a model of $\mathfrak{c} = \aleph_2$ in which there exists a separable first-countable locally compact initially \aleph_1 -compact space that is not compact. The one-point compactification of this space is compact, separable, sequentially compact, countably tight and not sequential. Since $\mathfrak{c} \leq \aleph_2$ holds in this model, this space is also pseudoradial. Finally, it was shown explicitly in [11, 5.11] (and implicitly in [10, 6.3]) that one can perform a further ccc forcing to make MA_{\aleph_1} and $\mathfrak{c} = \aleph_2$ hold while ensuring that the original space generates a space in the extension with all the same properties. This answers Spadaro's question in the negative.

Now we consider our generalization of Spadaro's question and ask if Martin's Axiom is sufficient to ensure that compact separable pseudoradial spaces of countable tightness have cardinality at most \mathfrak{c} . For this we use the example constructed in [10, 5.5,6.3] that had, in answer to a question of Arhangel'skii, an example of an initially \aleph_1 -compact first-countable space of cardinality greater than \mathfrak{c} . We cite the exact statement that we will need. This was also shown to hold in a model of Martin's Axiom (MA) and $\mathfrak{c} = \aleph_2$. Clearly this space was not compact and it also contained a separable closed subset that had a compactification of cardinality $2^{\mathfrak{c}}$.

Proposition 2.2. [10, 5.5,6.3] *There is a model of $MA + \mathfrak{c} = \aleph_2$ in which there is a separable first-countable initially \aleph_1 -compact space X with the property that $\beta X \setminus X$ has cardinality 2^{\aleph_2} and contains no infinite compact subsets.*

Corollary 2.3. *It is consistent with MA_{\aleph_1} that there is a compact separable pseudoradial space of countable tightness and cardinality greater than \mathfrak{c} .*

Proof. Let X be the space and in the model as stated in Proposition 2.2. We show that βX is the desired example. The only property that needs to be established is that βX is pseudoradial and has countable tightness. We remarked above that every compactification of a initially \aleph_1 -compact space has countable tightness. We are working in a model of $\mathfrak{c} = \aleph_2$, so to prove that βX is pseudoradial, it is enough to prove that it is sequentially compact. The space X is sequentially compact so we consider an infinite countable subset D of $\beta X \setminus X$. The closure of D is not contained in $\beta X \setminus X$ by the statement in Proposition 2.2, hence there is a point $x \in X$ that is a limit point of D . It is easily checked that if $\{U_n : n \in \omega\}$ is a local base in X for x consisting of cozero subsets of X , then the family $\{\beta X \setminus \text{cl}_{\beta X}(X \setminus U_n) : n \in \omega\}$ is a local base for x in βX . It follows then that there is a sequence from D converging to x , and this completes the proof that βX is sequentially compact and pseudoradial. \square

3. CARDINALITY OF SEPARABLE PSEUDORADIAL SPACES

In this section we begin our investigation of separable pseudoradial spaces in the absence of the assumption of compactness. We first consider the effect of the proper forcing axiom in the context of a strengthening of countable tightness and then, using large cardinals, we establish the consistency of there being no large separable pseudoradial spaces with countable tightness.

We introduce a natural generalization of the property of a set being sequentially closed in a space X . In particular, in this next definition, we would say that a sequentially closed set is $<\aleph_1$ -closed.

Definition 3.1. *For a cardinal κ and a set A in a space X , say that A is $<\kappa$ -radially closed if for every cardinal $\lambda < \kappa$, no λ -sequence of points of A converges to a point of X outside of A .*

For a set A in a space X , let $A^{(<\kappa)}$ denote the smallest $<\kappa$ -radially closed set containing A .

Needless to say, $A^{(<\kappa)}$ exists in every space since the intersection of $<\kappa$ -radially closed sets is $<\kappa$ -radially closed. Alternatively, $A^{(<\kappa)}$ can be recursively constructed in a manner analogous to the usual constructions of the sequential closure.

Proposition 3.2. *Let A be a subset of a space X and let κ be a cardinal. By induction on $\alpha \leq \kappa^+$, define $A_\alpha \subset X$ as follows:*

- (1) $A_0 = A$,
- (2) for limit $\alpha \leq \kappa^+$, $A_\alpha = \bigcup\{A_\beta : \beta < \alpha\}$,

(3) $A_{\alpha+1}$ is the set of all points of X such that there is a cardinal $\lambda < \kappa$ and a λ -sequence of points of A_α that converges to x .

Then $A_{\kappa^+} = A^{(<\kappa)}$.

Proof. It follows easily, by induction on $\alpha \leq \kappa^+$ that A_α is a subset of $A^{(<\kappa)}$. So it suffices to note that A_{κ^+} is itself $<\kappa$ -radially closed. \square

A κ -sequence $\{x_\alpha : \alpha < \kappa\}$ is a free sequence [1] in a space X if, for all $\delta < \kappa$, the initial segment $\{x_\alpha : \alpha < \delta\}$ and the final segment $\{x_\alpha : \delta \leq \alpha < \kappa\}$ have disjoint closures. The tightness degree, $t(X)$, of a space X is of course a well-known cardinal invariant. Similarly, the invariant, $F(X)$, is the supremum of the lengths of free sequences of X . Of course Arhangel'skii showed that $t(X) \leq F(X)$ holds for any compact space X , and it was shown by Bella [5] that this inequality also holds for pseudoradial spaces. We record this for future reference.

Proposition 3.3 ([5]). *A pseudoradial space of uncountable tightness contains uncountable free sequences.*

3.1. PFA and countable tightness.

Proposition 3.4. *PFA implies that every separable regular pseudoradial space of cardinality greater than \mathfrak{c} contains uncountable free sequences.*

Proof. Let X be a separable regular pseudoradial space of cardinality greater than \mathfrak{c} . Since we simply wish to prove that X contains an uncountable free sequence, we may assume, by Corollary 3.3, that X has countable tightness. Let κ be a large enough regular cardinal so that X is an element of $H(\kappa)$ (see [17, IV §6]). Let X be an element of an elementary submodel M of $H(\kappa)$ where M is chosen so that $|M| = 2^{\aleph_1} = \mathfrak{c}$ and so that every subset of M of cardinality at most \aleph_1 is an element of M . We note that $Y = X \cap M$ is $<\aleph_2$ -radially closed. We may assume that ω is a dense subset of X . Let $\mathcal{W}_z = \{W \subset \omega : z \in \text{int}_X \text{cl}_X(W)\}$. Define the family \mathcal{H}_z to be all countable subsets of Y that have z in their closure. Let \mathcal{F}_z denote the family of all closed subsets of Y that contain a member of the family $\{\text{cl}_Y(H) : H \in \mathcal{H}_z\}$.

Claim 1. \mathcal{F}_z is a countably complete maximal filter of closed subsets of Y .

Proof of Claim: Let $\{H_n : n \in \omega\}$ be a subset of \mathcal{H}_z . Let $F_\omega = \bigcap \{\text{cl}_Y(H_n) : n \in \omega\}$ and assume there is $W \in \mathcal{W}_z$ such that $\text{cl}_X(W)$ is disjoint from F_ω . Since $W \in M$, we can replace each H_n by $\text{int}_X \text{cl}_X(W) \cap H_n$ since the latter is also in \mathcal{H}_z . However, this is impossible since $z \in \text{cl}_X(H_n)$ for all $n \in \omega$, by elementarity, there are points of Y in

F_ω . Now it follows that z is in the closure of F_ω , and by countable tightness, there is an $H_\omega \in \mathcal{H}_z$ with $H_\omega \subset F_\omega$. \square

It is evident from its definition that \mathcal{F}_z has a base of separable sets. Only minor modifications of any number of published proofs in connection to the Moore-Mrowka problem (e.g. [8, Theorem 3.3]) that these hypotheses on the filter \mathcal{F}_z are sufficient to conclude that there is a proper poset P that will force an uncountable free sequence in the space X . We provide a sketch. Fix a neighborhood assignment $\mathcal{U}_Y = \{U_y : y \in Y\}$ so that for each $y \in Y$, z is not in the closure of U_y . A member p of the poset P is a function into Y with a finite domain \mathcal{M}_p . The domain \mathcal{M}_p is a finite \in -chain of countable elementary submodels of $H(\kappa)$ satisfying that $\{X, Y, \mathcal{F}_z, \mathcal{U}_Y\}$ are members of each. For each $\{M_1, M_2\} \subset \mathcal{M}_p$ with $M_1 \in M_2$, the value of $p(M_1)$ is an element of $M_2 \cap F$ for all $F \in \mathcal{F}_z \cap M_1$. The poset is ordered by the conditions that $p < q$ providing $p \supset q$ and for all $M_1 \in \mathcal{M}_p \setminus \mathcal{M}_q$, if there is a (minimal) $M_2 \in \mathcal{M}_q$ such that $M_1 \in M_2$, then $p(M_1) \in U_{p(M_3)}$ for all $M_3 \in \mathcal{M}_q$ satisfying that $p(M_2) \in U_{p(M_3)}$. We omit the proof that P is proper (see any of []). It is easily shown that for each $\delta \in \omega_1$, the set $D_\delta = \{p \in P : (\exists M_p \in \mathcal{M}_p) \delta \in M_p\}$ is a dense subset of P . By PFA, there is a filter $G \subset P$ such that $G \cap D_\delta$ is not empty for all $\delta < \omega_1$. This implies there is an uncountable set $C_G \subset \omega_1$ such that $\gamma \in C_G$ if and only if there is an $M_\gamma \in \bigcup \{\mathcal{M}_p : p \in G\}$ with $M_\gamma \cap \omega_1 = \gamma$. For each $\gamma \in C_G$, choose any $p_\gamma \in G$ such that $M_\gamma \in \mathcal{M}_{p_\gamma}$. Since G is a filter, we may assume that for all $\gamma \in C_G$, $M_{\min(C_G)} \in \mathcal{M}_{p_\gamma}$.

For each $\gamma \in C_G$, also let $F_\gamma = \bigcap \mathcal{F}_z \cap M_\gamma$. The sequence $\{x_\gamma = p_\gamma(M_\gamma) : \gamma \in C_G\}$ satisfies that for each $\delta \in C_G$, F_δ contains the set $\{x_\gamma : \delta \leq \gamma\}$. It also follows that, for $\gamma < \delta$ both in C_G , the closure of U_{x_γ} is disjoint from F_δ because of the condition that z is not in the closure of U_{x_γ} . Finally, for each $\min(C_G) < \delta \in C_G$, there is a $\beta \in C_G \cap \delta$ such that $\{x_\xi : \beta < \xi \in C_G \cap \delta\}$ is contained U_{x_δ} . Indeed, then there is a finite $S_\delta \subset \delta+1$ such that $\{x_\xi : \xi \in C_G \cap \delta\}$ is contained in $\bigcup \{U_{x_\xi} : \xi \in S_\delta\}$. This completes the proof that the closure of $\{x_\xi : \xi \in C_G \cap \delta\}$ is disjoint from the closure of $\{x_\xi : \delta < \xi \in C_G\}$. \square

In compact spaces, the property of being countably tight is equivalent to the property that every subset has countable π -character [19]. A family of non-empty open sets \mathcal{U} is a local π -base in X at x if every neighborhood of x contains a member of \mathcal{U} . A point has countable π -character if it has a countable local π -base. It was discovered by Eisworth [13] that in the absence of compactness, the property of hereditary countable π -character was an important strengthening of countable tightness. In particular, it was shown in [12] and [9] that

it was consistent that countably compact subsets of such spaces are closed. Using this result we note the following application.

Corollary 3.5. *PFA implies that every separable countably compact pseudoradial regular space has cardinality at most \mathfrak{c} if the space also has hereditary countable π -character.*

Proof. Assume that X is a separable countably compact pseudoradial regular space. The sequential closure of the countable dense set is countably compact and has cardinality \mathfrak{c} . It was proven in [9] that PFA implies that countably compact subsets are closed in such spaces. \square

Question 1. Let X be a countably compact regular pseudoradial space of countable tightness. Does PFA imply any of the following?

- (1) X has cardinality at most \mathfrak{c} .
- (2) Countably compact subsets of X are closed.
- (3) Every subset of X has countable π -character.

3.2. Consistency of no large countably tight examples. In this section we prove that it is consistent, modulo large cardinals, that every separable regular pseudoradial space of countable tightness has cardinality at most \mathfrak{c} . We give the result using a measurable cardinal for simplicity, but the same proof can be modified to work for a weakly compact cardinal. A cardinal κ is measurable if there is a κ -complete free ultrafilter on κ .

Before proving the theorem we make note of a simple property shared by pseudoradial spaces.

Proposition 3.6. *Assume that a point z of a pseudoradial regular space X is a limit point of a countable subset A of X . Let $A^{(1)}$ denote the set of points of X that are limit points of a converging sequence from A . Then z is in the closure of $A^{(1)} \setminus A$.*

Proof. Since z is a limit point of A we may assume that $z \notin A$ and, to be clear, we note that $A \subset A^{(1)}$. If z is an element of $A^{(1)}$ then there is nothing to prove. Let W be a neighborhood of z and assume that $A_W = \overline{W} \cap A^{(1)}$ is contained in A . Certainly z is in the closure of A_W but this contradicts that X is pseudoradial since A_W is clearly radially closed. \square

Theorem 3.7. *If κ is a measurable cardinal then in the forcing extension obtained by adding κ many Cohen reals, every separable regular pseudoradial space of countable tightness has cardinality at most \mathfrak{c} .*

Proof. Let κ be a measurable cardinal. By [14, (8.7),10.20] , there is a normal κ -complete ultrafilter \mathcal{U} on κ . This means that for every $U \in \mathcal{U}$

and regressive function f on U (i.e. $f(\alpha) < \alpha$ for all $\alpha \in U$) there is a $U_1 \in \mathcal{U}$ satisfying that $f(\lambda) = f(\mu)$ for all $\lambda, \mu \in U_1$. Note also that it follows that each $U \in \mathcal{U}$ is a stationary subset of κ [14, 10.19]. More surprisingly, there is an element U of \mathcal{U} consisting only of regular limit cardinals. To review this, first we note that the function sending μ^+ to μ is regressive on the set of successor cardinals and so this set can not be an element of \mathcal{U} . Now we prove that there is a member of \mathcal{U} consisting only of regular cardinals. This is because if the function $f(\lambda) = \text{cf}(\lambda)$ was regressive on a set U of \mathcal{U} , i.e. every member of U would have the same cofinality $\delta < \kappa$, then we would have δ -many regressive functions (to elements of cofinal sequences) and one of these could not be constant on a member of \mathcal{U} . Needless to say, it is also true that κ is a strongly inaccessible cardinal.

Claim 2. If $\{S_\xi : \xi \in \kappa\}$ is any sequence of countable subsets of κ , then there is a set \bar{S} and a set $U \in \mathcal{U}$ such that for all $\lambda, \mu \in U$ with $\lambda < \mu$, $\bar{S} = S_\lambda \cap \lambda$ and $S_\lambda \subset \mu$.

Proof of Claim: For each $\xi \in \kappa$, let $\delta_\xi < \omega_1$ be the order-type of $S_\xi \cap \xi$ and let $\{\beta_\eta^\xi : \eta < \delta_\xi\}$ be an enumeration of $S_\xi \cap \xi$. Choose $U_0 \in \mathcal{U}$ and $\delta < \omega_1$ so that $\delta_\lambda = \delta$ for all $\lambda \in U_0$. Similarly, for each $\eta < \delta$, choose $\beta_\eta < \kappa$ and $U_{\eta+1} \in \mathcal{U}$ so that $\beta_\eta^\lambda = \beta_\eta$ for all $\lambda \in U_{\eta+1}$. Let $\bar{U} \in \mathcal{U}$ equal $U_0 \cap \bigcap \{U_{\eta+1} : \eta < \delta\}$. Choose any cub $C \subset \kappa$ so that for all $\lambda \in C$ and $\gamma < \lambda$, $S_\gamma \subset \lambda$. Now it follows that $C \cap \bar{U} \in \mathcal{U}$ has the required properties. \square

We say that an indexed subset $\{p_\xi : \xi < \delta\}$ of the poset $\text{Fn}(\kappa, 2)$ is a Δ -system if there is a $\bar{p} \in \text{Fn}(\kappa, 2)$ such that for all $\xi \neq \eta < \delta$, $p_\xi \upharpoonright \text{dom}(\bar{p}) = \bar{p}$ and the members of the family $\{\text{dom}(p_\xi) \setminus \text{dom}(\bar{p}) : \xi < \delta\}$ are pairwise disjoint. We say that \bar{p} is the root of the Δ -system. We record two well-know facts about Cohen forcing.

Claim 3. If $\{p_\xi : \xi < \delta\}$ is a Δ -system in $\text{Fn}(\kappa, 2)$ with root \bar{p} , then for every generic filter G with $\bar{p} \in G$ and every infinite subset $J \subset \delta$ the set $\{\xi \in J : p_\xi \in G\}$ has cardinality $|J|$.

Claim 4. For any uncountable regular cardinal $\mu \leq \kappa$ and sequence $\{p_\xi : \xi < \mu\}$, there is a set J cofinal in μ so that $\{p_\xi : \xi \in J\}$ is a Δ -system.

Now suppose that $\theta > \kappa$ is a cardinal and that in the extension by the poset $\text{Fn}(\kappa, 2)$ there is a regular pseudoradial topology on θ in which ω is dense and which has countable tightness.

If G is a $\text{Fn}(\kappa, 2)$ -generic filter, then in $V[G]$, by [17, VII]), $\mathfrak{c} = \kappa$ and, for all infinite cardinals $\lambda < \kappa$, $2^\lambda = \kappa$. Similarly, the cardinals of

V and the cardinals of $V[G]$ coincide. It therefore follows that, in $V[G]$, the cardinality of $\omega^{(<\kappa)}$ is at most κ . We will use the convention that a nice name of a subset of θ is a set \dot{A} of the form $\bigcup\{\{\alpha\} \times A_\alpha : \alpha \in I\}$ where I is any subset of θ and for each $\alpha \in I$, A_α is a countable subset of $\text{Fn}(\kappa, 2)$. We will say that \dot{A} is a nice name of a subset of I to mean that $\{\alpha \in \theta : A_\alpha \neq \emptyset\} \subset I$. When we say that \dot{A} is an I -name we will mean in this sense. Usually each A_α is taken to be an antichain but it will be convenient to use our generalization. If G is a $\text{Fn}(\kappa, 2)$ -generic filter, then $\text{val}_G(\dot{A})$ is defined as $\{\alpha : G \cap A_\alpha \neq \emptyset\}$. It is shown in [17] that, in $V[G]$, every subset of θ is equal to $\text{val}_G(\dot{A})$ for some nice name \dot{A} (in the ground model). Moreover, for every infinite subset J of θ in $V[G]$ there is a ground model set I and a nice name of a subset of I for J , where I and J have the same cardinality in $V[G]$. Similarly, it can be shown that for every subset \mathcal{A} of the power set of θ in $V[G]$, there is a family \mathcal{N} of nice names of subsets of θ such that \mathcal{A} is equal to $\{\text{val}_G(\dot{A}) : \dot{A} \in \mathcal{N}\}$. Therefore, we may suppose that we have families $\{\mathcal{N}_\alpha : \alpha \in \theta\}$ where, for each $\alpha < \theta$, \mathcal{N}_α is the family of nice names of the open subsets of θ that contain α .

Another subset of θ that is of interest is the set that will evaluate to $\omega^{(<\kappa)}$ in $V[G]$. There is a set $I \subset \theta$ of cardinality at most κ so that there is a nice name \dot{Y} of a subset of I satisfying that, for any generic filter $G \subset \text{Fn}(\kappa, 2)$, $\text{val}_G(\dot{Y})$ contains $\omega^{(<\kappa)}$. Next, we may choose a condition $\bar{p} \in G$ and an ordinal $z \in \theta \setminus I$ such that \bar{p} forces that there is a κ -sequence from $\omega^{(<\kappa)}$ that converges to z . Again we may suppose that there is a sequence $\{\dot{y}_\xi : \xi < \kappa\}$ where for each $\xi < \kappa$, \dot{y}_ξ is a nice name of a single ordinal that is forced by \bar{p} to be in \dot{Y} and so that the sequence is forced to converge to z . It follows that each \dot{y}_ξ is an I_ξ -name for some countable (or finite) set I_ξ .

For each $\xi < \kappa$, we may choose a pair (α_ξ, p_ξ) that is in the name \dot{y}_ξ and such that \bar{p} is compatible with p_ξ . That is, if $\bar{p} \cup p_\xi \in G$, then $\text{val}_G(\dot{y}_\xi)$ is equal to α_ξ and is in $\omega^{(<\kappa)}$. A minor modification of Claim 2 show that there is a $\bar{q} \leq \bar{p}$ and a set $U_0 \in \mathcal{U}$ such that, for all $\lambda < \mu$ with both from U_0 , $\bar{q} = (\bar{p} \cup p_\lambda) \upharpoonright \lambda$ and $\text{dom}(p_\lambda) \subset \mu$.

We define a new nice name $\dot{Z} = \{(\alpha_\lambda, \bar{p} \cup p_\lambda) : \lambda \in U_0\}$ and observe that, by Claim 3, if $\bar{p} \in G$ (for a $\text{Fn}(\kappa, 2)$ -generic filter G) the sequence $\text{val}_G(\dot{Z})$ is a κ -sequence that converges to z . Let C_0 be a cub in κ satisfying that, for each $\mu \in C_0$, $U_0 \cap \mu$ is cofinal in μ . Since z is forced to not be an element of $\omega^{(<\kappa)}$, we may choose, for each $\mu \in C_0$, a nice name $\dot{A}_\mu \in \mathcal{N}_z$, so that there is a $I_\mu \subset U_0 \cap \mu$ and a nice name \dot{J}_μ for a

subset of I_μ satisfying that \dot{J}_μ is forced to be cofinal in μ and α_ξ is not \dot{A}_μ for all $\xi \in \dot{J}_\mu$. We emphasize that, in $V[G]$, z is not in the closure of $\{\alpha_\lambda : \lambda \in \dot{J}_\mu\}$. Recall that a condition $q \in \text{Fn}(\kappa, 2)$ forces that an ordinal $\xi \in \dot{J}_\mu$ if there is some pair $(\xi, p) \in \dot{J}_\mu$ with $q \leq p$. Similarly, q forces that $\xi \in \dot{Z}$ if $\xi \in U_0$ and $q \leq \bar{p} \cup p_\xi$.

For each $\lambda \in U_0$ and $\lambda < \mu \in C_0$, choose a pair $(\xi(\lambda, \mu), q(\lambda, \mu))$ so that

- (1) $\lambda \leq \xi(\lambda, \mu) \in U_0 \cap \mu$,
- (2) $q(\lambda, \mu) \in \text{Fn}(\kappa, 2)$ and $q(\lambda, \mu)$ forces $\xi(\lambda, \mu) \in \dot{Z}$,
- (3) $q(\lambda, \mu)$ forces that $\xi(\lambda, \mu) \in \dot{J}_\mu$.

There is a cub C_1 satisfying that, for all $\zeta \in C_1$, $\{q(\lambda, \mu) : \mu < \zeta \text{ and } \lambda \in U_0 \cap \mu\}$ is a subset of $\text{Fn}(\zeta, 2)$. Choose any $U_1 \in \mathcal{U}$ so that $U_1 \subset C_1$ and every element of U_1 is a regular cardinal.

Applying Claim 4, we may choose, for each $\mu \in U_1$, a strictly increasing function $h_\mu : \mu \mapsto U_0 \cap \mu$ so that the sequence $\{q(h_\mu(\alpha), \mu) : \alpha \in \mu\}$ is a Δ -system. For each $\mu \in U_1$, let \tilde{q}_μ denote the root of this Δ -system, and, by possibly shrinking U_1 , we can also assume that the sequence $\{\tilde{q}_\mu : \mu \in U_1\}$ is a Δ -system.

Now we come to a stronger version of Claim 3.

Claim 5. There is a strictly increasing function h from κ into U_0 and a sequence $\{(\bar{\xi}_\alpha, \bar{q}_\alpha) : \alpha < \kappa\}$ such that for all $\omega < \delta < \kappa$, there is a set $U_\delta \in \mathcal{U}$ so that, for all $\mu \in U_\delta$,

$$\{(h(\alpha), \bar{\xi}_\alpha, \bar{q}_\alpha) : \alpha < \delta\} = \{(h_\mu(\alpha), \xi(h_\mu(\alpha), \mu), q(h_\mu(\alpha), \mu) \upharpoonright \mu) : \alpha < \delta\}.$$

Proof of Claim: Choose any enumeration $e : \kappa \mapsto \kappa \times \kappa \times \text{Fn}(\kappa, 2)$. Choose a cub $C_e \subset \kappa$ so that for all $\gamma < \delta \in C_e$, $e(\gamma) \in \delta \times \delta \times \text{Fn}(\delta, 2)$. For each $\alpha \in \kappa$, the function sending μ to the ordinal γ_μ satisfying that $(h_\mu(\alpha), \xi(h_\mu(\alpha)), q(h_\mu(\alpha), \mu) \upharpoonright \mu) = e(\gamma_\mu)$ is regressive on a set in \mathcal{U} . Therefore there is a $W_\alpha \in \mathcal{U}$ and a value $g(\alpha) \in \kappa$ so that

$$(h_\mu(\alpha), \xi(h_\mu(\alpha), \mu), q(h_\mu(\alpha), \mu) \upharpoonright \mu) = e(g(\alpha)) \quad \text{for all } \mu \in W_\alpha.$$

Let $(h(\alpha), \bar{\xi}_\alpha, \bar{q}_\alpha)$ be defined so as to equal $e(g(\alpha))$. Now, for each $\delta < \kappa$, the set $U_\delta = \bigcap_{\alpha < \delta} W_\alpha$ is an element of \mathcal{U} and, after some straightforward unravelling, we have that for all $\mu \in U_\delta$,

$$\{(h(\alpha), \bar{\xi}_\alpha, \bar{q}_\alpha) : \alpha < \delta\} = \{(h_\mu(\alpha), \xi(h_\mu(\alpha), \mu), q(h_\mu(\alpha), \mu) \upharpoonright \mu) : \alpha < \delta\}.$$

□

The family $\{\bar{q}_\alpha : \alpha \in \kappa\}$ is a Δ -system of $\text{Fn}(\kappa, 2)$ since for each $\delta < \kappa$, there is a $\mu \in U_1$ such that the initial segment $\{\bar{q}_\alpha : \alpha < \delta\}$ is a

subset of the Δ -system $\{q(h_\mu(\alpha), \mu) \upharpoonright \mu : \alpha < \delta\}$. In addition, each \bar{q}_α forces that $\bar{\xi}_\alpha$ is an element of the set $\{\dot{y}_\zeta : \alpha \leq \zeta < \kappa\}$.

We briefly pass to the extension $V[G]$ and note that, by Claim 3, the set $J = \{\alpha : \bar{q}_\alpha \in G\}$ is cofinal in κ . Since the sequence $\{\text{val}_G(\dot{y}_\zeta) : \zeta < \kappa\}$ converges to z , and by countable tightness, there is a countable subset J_1 of J such that $\{\bar{\xi}_\alpha : \alpha \in J_1\}$ has z in its closure. Now we apply Proposition 3.6 to the point z and the countable set $A = \{\bar{\xi}_\alpha : \alpha \in J_1\}$ and we choose a countable set J_2 of θ (i.e. the space X) so that $J_2 \subset A^{(1)} \setminus A$ and z is in the closure of J_2 .

Now we return to V and we choose countable $I_1 \subset \kappa$ and countable $I_2 \subset \theta$ so that, in $V[G]$, $J_1 \subset I_1$ and $J_2 \subset I_2$. Fix a nice name, \dot{J}_2 , for the subset J_2 of I_2 and notice that the name \dot{J}_1 for J_1 is simply the set $\{(\alpha, \bar{q}_\alpha) : \alpha \in I_1\}$. Next we want to choose sufficiently many nice names for sequences contained in $\{\bar{\xi}_\alpha : \alpha \in J_1\}$ that converge to points of J_2 . We can do this by a simple examination of the name \dot{J}_2 . For each ordinal $x \in I_2$ and condition $p \in \text{Fn}(\kappa, 2)$ such that (x, p) is an element of the name \dot{J}_2 (thus $p \Vdash x \in \dot{J}_2$) choose a nice name $\dot{S}(x, p)$ of a subset of I_1 such that p forces that $\dot{S}(x, p)$ is a subset of J_1 and that $\{\bar{\xi}_\alpha : \alpha \in \dot{S}(x, p)\}$ converges to x . Certainly we have only chosen countably many nice names of the form $\dot{S}(x, p)$. Fix any $\delta < \kappa$ so that

- (1) $I_1 \cup I_2 \subset \delta$,
- (2) \dot{J}_2 is a nice $\text{Fn}(\delta, 2)$ -name,
- (3) for all $x \in I_2$ and $(x, p) \in \dot{J}_2$, $\dot{S}(x, p)$ is a nice $\text{Fn}(\delta, 2)$ -name,

and choose any $\mu \in U_\delta$ such that $\tilde{q}_\mu \in G$.

We are ready for our final contradiction (to the assumption that X has cardinality greater than κ). The contradiction will be that z is not in the closure of the set $A_\mu = \{\xi(h_\mu(\alpha), \mu) : \alpha \in I_1 \text{ and } h_\mu(\alpha) \in \text{val}_G(\dot{J}_\mu)\}$, but z is in the closure of J_2 . We will show that J_2 is contained in the closure of A_μ . To show this, it suffices to prove that for each $x \in I_2$ and $p \in G$ such that $(x, p) \in \dot{J}_2$, the sequence $\{\bar{\xi}_\alpha : \alpha \in \text{val}_G(\dot{S}(x, p))\}$ meets A_μ in an infinite set. Fix any $x \in I_2$ and p so that $(x, p) \in \dot{J}_2$. Let H be any finite subset of I_1 and we will show that there is an $\alpha \in \text{val}_G(\dot{S}(x, p)) \setminus H$ such that $\bar{\xi}_\alpha \in A_\mu$. To do so, we will work in V and have to use the genericity of G . Following [17, VII], it suffices to show that if r is any element of G , there is an $\alpha \in I_1 \setminus H$ and an extension $r_\alpha \in \text{Fn}(\kappa, 2)$ so that, with $\bar{\xi}_\alpha$ r_α forces that $\alpha \in \dot{S}(x, p)$ and r_α forces that $h_\mu(\alpha) \in \dot{J}_\mu$. This is the same as proving that the set of conditions of the form r_α is dense below the

condition p . This ensures that G will include one such r_α and that $\bar{\xi}_\alpha$ will be the desired member of A_μ with $\alpha \in \dot{S}(x, p)$.

Now we use that $\{q(h_\mu(\alpha), \mu) : \alpha \in I_1 \setminus H\}$ is a Δ -system. By possibly increasing the size of H we can assume that $\text{dom}(r) \cup \text{dom}(\tilde{q}_\mu) \cup \text{dom}(p)$ is disjoint from $\text{dom}(q(h_\mu(\alpha), \mu)) \setminus \text{dom}(\tilde{q}_\mu)$ for all $\alpha \in I_1 \setminus H$. Next we simply note that, since $\{\bar{\xi}_\alpha : \alpha \in \text{val}_G(\dot{S}(x, p))\}$ converges to x , there an $\alpha \in I_1 \setminus H$ and a condition $\bar{r}_\alpha \in G$ so that (α, \bar{r}_α) is an element of the name $\dot{S}(x, p)$. Recalling that p forces that $\dot{S}(x, p)$ is a subset of J_1 , it follows also that $\bar{r}_\alpha \leq \bar{q}_\alpha$. By the choice of δ , $\bar{r}_\alpha \in \text{Fn}(\delta, 2)$ and $\bar{r}_\alpha \leq q(h_\mu(\alpha), \mu) \upharpoonright \mu$. Finally, it follows that with $r_\alpha = \bar{r}_\alpha \cup q(h_\mu(\alpha), \mu)$ we have completed the proof. \square

4. PSEUDORADIAL SPACES IN MODELS OF $\mathfrak{c} \leq \aleph_2$

In this section we prove that if \mathfrak{c} is most \aleph_2 , then there are separable regular pseudoradial spaces of cardinality $2^\mathfrak{c}$. These examples are also 0-dimensional with a countable dense set of isolated points. We first prove that CH implies the stronger result that there are such space that are compact.

Theorem 4.1 (CH). *There is a compactification of ω that is pseudoradial and has cardinality 2^{\aleph_1} .*

Proof. Let X be an η_1 -set of cardinality \aleph_1 and let \prec denote the corresponding (strict) linear ordering on X . Recall that an η_1 -set has the property that if A and B are countable subsets of X and that $a \prec b$ for all $a \in A$ and $b \in B$, then there is an $x \in X$ with $a \prec x \prec b$ for all $a \in A$ and $b \in B$. Next, let \mathbb{D} denote the standard Dedekind completion of X . Since X is dense in \mathbb{D} it follows that \mathbb{D} has weight \aleph_1 and no points of countable character. It also follows then that \mathbb{D} has cardinality 2^{\aleph_1} . Next, let K be the space obtained by doubling every point of \mathbb{D} that has countable one-sided character. Again, using that X is dense, it follows that K has weight \aleph_1 . Since K is a linearly ordered space, it follows that K is radial. We observe that K has the property that every countably infinite set has subsequence converging to a point of countable character.

Finally, it follows from CH that there is a compactification $\gamma\omega$ that has remainder K . This is the requisite example. It remains only to prove that $\gamma\omega$ is sequentially compact. Since K is sequentially compact, it suffices to prove that every infinite subset A of ω has a converging subsequence. If A has only finitely many accumulation points then this is clear. On the other hand, if A has infinitely many accumulation points, one of them will be a point of countable character. \square

Now we turn to the proof in the case that $\mathfrak{c} = \aleph_2$. Before constructing the example we establish an instructive constraint on the nature of possible examples. This is based on the paper [4] concerning partitioner algebras. A similar application in [7] established that it was consistent that compact separable spaces of cardinality greater than \mathfrak{c} were not sequentially compact. The proof is similar to the proof of Theorem 3.7.

Theorem 4.2. *It is consistent with $\mathfrak{c} = \aleph_2$ that every pseudocompact pseudoradial regular space that contains a countable dense discrete space has cardinality at most \mathfrak{c} . Moreover, if V is a model of GCH and $\kappa > \aleph_1$ is a regular cardinal, then in the forcing extension by $\text{Fn}(\kappa, 2)$ every pseudocompact pseudoradial regular space with a countable dense discrete subset has cardinality at most κ .*

Proof. Clearly it suffices to prove the second statement of the Theorem. Suppose that GCH holds and that $\kappa > \aleph_1$ is a regular cardinal. Let G be a $\text{Fn}(\kappa, 2)$ -generic filter. We assume that $\theta > \kappa$ is a cardinal and that we have a $\text{Fn}(\kappa, 2)$ -name for a topology on $X = \theta$ such that, in $V[G]$, ω is open, dense, and discrete. Similar to Theorem 3.7, let, for each $\alpha < \lambda$ and $p \in \text{Fn}(\kappa, 2)$, $\mathcal{N}_\alpha(p)$ be the set of all nice names of subsets of ω that are forced by p to have α in the interior of their closure. Also let $\mathcal{S}_\alpha(p)$ be the set of all nice names of subsets of ω that are forced by p to simply have α in their closure. To avoid confusion, we will let 1_P denote the maximal element \emptyset of $\text{Fn}(\kappa, 2)$.

Since κ is a regular cardinal, and we are assuming GCH, we may fix an elementary submodel M of $H(\theta^+)$ of cardinality κ such that $\{\{\mathcal{N}_\alpha(1_P), \mathcal{S}_\alpha(1_P)\} : \alpha \in \theta\}$ is an element of M , and every subset of M of cardinality less than κ is an element of M . Let z be any element of $\theta \setminus M$. In this proof we will choose a κ -sequence $\{y_\xi : \xi < \kappa\} \subset M \cap \theta$ satisfying that, in $V[G]$, the sequence $\{y_\xi : \xi < \kappa\}$ converges to z . Fix enumerations $\{\dot{S}_\beta : \beta < \kappa\}$ of $\mathcal{S}_z(1_P)$ and $\{\dot{W}_\beta : \beta < \kappa\}$ of $\mathcal{N}_z(1_P)$. Fix any $\lambda < \kappa$. By assumption, the sequence $\{\dot{S}_\beta : \beta < \lambda\}$ and $\{\dot{W}_\beta : \beta < \lambda\}$ are elements of M . Since

$$H(\theta^+) \models (\exists z \in \theta)(\{\dot{S}_\beta : \beta < \lambda\} \subset \mathcal{S}_z(1_P) \text{ and } \{\dot{W}_\beta : \beta < \lambda\} \subset \mathcal{N}_z(1_P))$$

there is a $y_\lambda \in M \cap \theta$ satisfying that $\{\dot{S}_\beta : \beta < \lambda\} \subset \mathcal{S}_{y_\lambda}(1_P)$ and $\{\dot{W}_\beta : \beta < \lambda\} \subset \mathcal{N}_{y_\lambda}(1_P)$.

It is easily verified that $\{y_\lambda : \lambda < \kappa\}$ converges to z in $V[G]$ but this is not actually needed in the remainder of the proof. Rather, we will prove that ω is forced to have an infinite subset S that contains no converging sequence. If X is pseudoradial, S would be closed and, since ω is open in X , this would imply that X is not pseudocompact.

For limit $\lambda < \kappa$ of uncountable cofinality, no cofinal subsequence of $\{y_\alpha : \alpha < \lambda\}$ converges to z since z is not an element of M . We translate this into the forcing language.

Claim 6. For each limit $\lambda < \kappa$ of uncountable cofinality, there is a value $\zeta_\lambda < \kappa$ and $\dot{W}_{\zeta_\lambda} \in \mathcal{N}_z(1_P)$ satisfying that, for all $\beta < \lambda$ and $p \in \text{Fn}(\kappa, 2)$, there are $\beta < \alpha < \lambda$ and $p_\alpha < p$ such that the name $\omega \setminus \dot{W}_{\zeta_\lambda}$ is an element of $\mathcal{N}_{y_\alpha}(p_\alpha)$.

The last statement in Claim 6 is simply asserting that p_α forces that y_α is not in the closure of \dot{W}_{ζ_λ} . We also note that the statement of Claim 6 asserts that, for each $\beta < \lambda$, the set of p_α as in the statement is a dense subset of $\text{Fn}(\kappa, 2)$. For each λ , choose another ordinal ρ_λ so that 1_P forces that the closure of $\dot{W}_{\rho_\lambda} \in \mathcal{N}_z(1_P)$ is contained in the interior of the closure of \dot{W}_{ζ_λ} .

Let Λ denote the set of $\lambda < \kappa$ of uncountable cofinality. For all $\lambda \in \Lambda$, let \dot{W}_{ζ_λ} be the name $\{(n, p(\lambda, n, k)) : n \in B_\lambda, k \in \omega\}$ and let \dot{W}_{ρ_λ} be the name $\{(n, q(\lambda, n, k)) : n \in D_\lambda, k \in \omega\}$. Note that $D_\lambda \subset B_\lambda$ and we may assume that, for all $n \in D_\lambda$ and $k \in \omega$, there is a $j \in \omega$ so that $q(\lambda, n, k) < p(\lambda, n, j)$ (since $q(\lambda, n, k) \Vdash n \in \dot{W}_{\zeta_\lambda}$). Let I_λ denote the union of the set $\bigcup\{\text{dom}(q(\lambda, n, k)) : n \in B_\lambda, k \in \omega\}$. For any $p \in \text{Fn}(\kappa, 2)$ and partial injection $\sigma : \kappa \mapsto \kappa$ such that $\text{dom}(p) \subset \text{dom}(\sigma)$, we let $\widehat{\sigma}(p)$ be the condition with domain $\sigma(\text{dom}(p))$ and satisfying that $\widehat{\sigma}(p)(\sigma(\alpha)) = p(\alpha)$ for all $\alpha \in \text{dom}(p)$.

Fix a cub $C \subset \kappa$ satisfying that for all $\mu \in C$, $\{\zeta_\lambda\} \cup I_\lambda \subset \mu$ for all $\lambda \in \Lambda \cap \mu$. By the usual pressing down lemma, and some straightforward enumeration techniques as used in the proof of Theorem 3.7, there is a stationary set $E \subset \Lambda \cap C$ satisfying that, for all $\lambda, \mu \in E$ with $\lambda < \mu$, there is an order preserving isomorphism $\sigma_{\lambda, \mu}$ from I_λ to I_μ so that:

- (1) $B_\lambda = B_\mu$ and $I_\lambda \cap \lambda = I_\mu \cap \mu$,
- (2) $\sigma_{\lambda, \mu}(I_\lambda \setminus \lambda) = I_\mu \setminus \mu$,
- (3) for all $n \in B_\lambda$ and $k \in \omega$, $p(\mu, n, k) = \widehat{\sigma}_{\lambda, \mu}(p(\lambda, n, k))$,
- (4) for all $n \in D_\lambda$ and $k \in \omega$, $q(\mu, n, k) = \widehat{\sigma}_{\lambda, \mu}(q(\lambda, n, k))$.

Let μ_0 be any element of E . Next, using that E is stationary it follows that we can find $\mu_0 < \mu_1 \in E$ so that every $\text{Fn}(\mu_1, 2)$ -name in $\mathcal{S}_z(1_P)$ is in the list $\{\dot{S}_\beta : \beta < \mu_1\}$. Additionally we can ensure that, for every nice $\text{Fn}(\mu_0, 2)$ -name, \dot{S} , for a subset of ω , and every $p \in \text{Fn}(\kappa, 2)$ such that $\dot{S} \in \mathcal{S}_z(p)$, then $\dot{S} \in \mathcal{S}_z(p \upharpoonright \mu_1)$. This latter condition is a simple consequence of the fact that $\text{Fn}(\kappa, 2)$ is ccc.

Next, extend $\{\mu_0, \mu_1\}$ to any strictly increasing sequence $\{\mu_\ell : \ell \in \omega\} \subset E_1$. Our next step is to prove that the family $\{\dot{W}_{\zeta_{\mu_\ell}} : \ell \in \omega\}$ is forced to be an independent family, or rather that $\{\{\dot{W}_{\rho_{\mu_\ell}}, \omega \setminus \dot{W}_{\zeta_{\mu_\ell}}\} :$

$\ell \in \omega\}$ is forced to be an independent family of pairs. We will use the fact $\text{Fn}(\kappa, 2)$ is isomorphic to $\text{Fn}(\mu_0, 2) \times \text{Fn}(\kappa \setminus \mu_0, 2)$ and so when we pass to the extension $V[G_{\mu_0}]$ (where $G_{\mu_0} = G \cap \text{Fn}(\mu_0, 2)$), we can use the fact that $V[G]$ is equal to $V[G_{\mu_0}][G \cap \text{Fn}(\kappa \setminus \mu_0, 2)]$. We are trying to minimize our appeal to advanced forcing techniques but we need a minor one here.

Claim 7. For each $S \in [\omega]^{\aleph_0} \cap V[G_{\mu_0}]$, if $V[G] \models z \in \text{cl}_X(S)$, then there is an $\beta < \mu_1$ such that $\dot{S}_\beta \in \mathcal{S}_z(1_P)$ is a $\text{Fn}(\mu_1, 2)$ -name and $\text{val}_G(\dot{S}_\beta) = S$.

Proof of Claim: By definition, if $S \in V[G_{\mu_0}] \cap [\omega]^{\aleph_0}$ there is a $\text{Fn}(\mu_0, 2)$ -name \dot{S} such that $S = \text{val}_{G_{\mu_0}}(\dot{S})$. Now assume that, in $V[G]$, that z is in the closure of \dot{S} . By the forcing theorem, there is a condition $p \in G$ that forces z is in the closure of \dot{S} . Of course this means that $\dot{S} \in \mathcal{S}_z(p)$. By the assumption on μ_1 , we can assume that $p \in \text{Fn}(\mu_1, 2)$. It is a routine exercise to prove that there is a nice $\text{Fn}(\mu_1, 2)$ -name \dot{T} such that $p \Vdash \dot{T} = \dot{S}$ and, all $q \in \text{Fn}(\mu_1, 2)$ that are incomparable with p force that $\dot{T} = \omega$. It follows now, from the assumption on μ_1 , that \dot{T} is in the list $\{\dot{S}_\alpha : \alpha < \mu_1\}$. \square

Let $B = \{n \in B_{\mu_0} : (\exists k)p(\mu_0, n, k) \upharpoonright \mu_0 \in G_{\mu_0}\}$ and $D = \{n \in D_{\mu_0} : (\exists k)q(\mu_0, n, k) \upharpoonright \mu_0 \in G_{\mu_0}\}$. Let I denote the countable set $I_{\mu_0} \setminus \mu_0$ and let, for each $\ell < \omega$, \dot{U}_ℓ denote the canonical $\text{Fn}(I_{\mu_\ell} \setminus \mu_\ell, 2)$ -name in $V[G_{\mu_0}]$ for the set $\dot{W}_{\zeta_{\mu_\ell}}$ and let \dot{V}_ℓ denote the canonical $\text{Fn}(I_{\mu_\ell} \setminus \mu_\ell, 2)$ -name in $V[G_{\mu_0}]$ for the set $\dot{W}_{\rho_{\mu_\ell}}$. More precisely, for each $n \in B$ and $k \in \omega$,

$$(n, p(\mu_\ell, n, k) \upharpoonright I) \in \dot{U}_0 \text{ if and only if } p(\mu_\ell, n, k) \upharpoonright \mu_0 \in G_{\mu_0}.$$

For all $\ell > 0$, $\dot{U}_\ell = \{(n, \hat{\sigma}_{\mu_0, \mu_\ell}(p)) : (n, p) \in \dot{U}_0\}$. For convenience, let σ_{μ_0, μ_0} denote the identity function on I and recall that the sets $\{I_{\mu_\ell} \setminus \mu_\ell = \sigma_{\mu_0, \mu_\ell}(I) : \ell \in \omega\}$ are pairwise disjoint. The definition of \dot{V}_ℓ is defined similarly.

For each $p \in \text{Fn}(I, 2)$, let $T_p = \{n \in B : p \Vdash n \in \dot{U}_0\}$ and let $V_p = \{n \in D : (\exists q < p) q \Vdash n \in \dot{V}_0\}$. We note that T_p and V_p are elements of $V[G_{\mu_0}]$.

Claim 8. For each $p \in \text{Fn}(I, 2)$, z is not in the closure, in $V[G]$, of T_p . Similarly, z is in the interior of the closure of V_p .

Proof of Claim: Assume that the first statement fails for some $p \in \text{Fn}(I, 2)$. By Claim 7, there is an $\beta < \mu_1$ such that $T_p = \text{val}_G(\dot{S}_\beta)$. The family $\{\sigma_{\mu_0, \mu_\ell}(p) : 1 \leq \ell \in \omega\}$ is a Δ -system (with empty root) and so

there is an $\ell > 0$ such that $\sigma_{\mu_0, \mu_\ell}(p) \in G$. By Claim 6, there is an α and a condition $p_\alpha < \sigma_{\mu_0, \mu_\ell}(p)$ such that $p_\alpha \upharpoonright \mu_0 \in G_{\mu_0}$, $\beta < \alpha < \mu_\ell$ and $p_\alpha \Vdash \omega \setminus \dot{W}_\ell \in \mathcal{N}_{y_\alpha}(p_\alpha)$. This is equivalent to the assertion that p_α forces that y_α is not in the closure of \dot{W}_ℓ . On the other hand, we have that $\dot{S}_\beta \in \mathcal{S}_{y_\alpha}(1_p)$. This implies that 1_p forces that y_β is in the closure of T_p and yet, in $V[G_{\mu_0}]$, p_α forces $T_p \subset \dot{U}_\ell$ is disjoint from a neighborhood of y_α .

The second statement is proven similarly by simply noting that $\sigma_{\mu_0, \mu_\ell}(p)$ forces that \dot{V}_ℓ is a subset of V_p . \square

Now we prove that $\{\{\dot{V}_\ell, \omega \setminus \dot{U}_\ell\} : \ell \in \omega\}$ is forced to be an independent family. Fix any pair of disjoint finite sets L_0, L_1 of ω . Let $q \in \text{Fn}(\kappa \setminus \mu_0, 2)$ be any condition. We produce a condition $\tilde{q} < q$ and an integer n so that \tilde{q} forces that $n \in \dot{V}_\ell$ for all $\ell \in L_0$ and \tilde{q} forces that $n \notin \dot{U}_\ell$ for all $\ell \in L_1$. For each $\ell \in L_0 \cup L_1$, let $p_\ell \in \text{Fn}(I, 2)$ be chosen so that $\sigma_{\mu_0, \mu_\ell}(p_\ell) = q \upharpoonright I_{\mu_\ell} \setminus \mu_\ell$. Let $T = \bigcup\{T_{p_\ell} : \ell \in L_1\}$ and $U = \bigcap\{V_{p_\ell} : \ell \in L_0\}$. Choose any integer $n \in U \setminus T$. For each $\ell \in L_0$, $n \in U_{p_0}$ and so we may choose $q_\ell < p_\ell$ in $\text{Fn}(I, 2)$, such that $q_\ell \Vdash n \in \dot{V}_0$. For each $\ell \in L_1$, since $n \in T_{p_\ell}$, $p_\ell \nVdash n \in \dot{U}_0$, and so we may choose $q_\ell < p_\ell$ in $\text{Fn}(I, 2)$ such that $q_\ell \Vdash n \notin \dot{U}_0$. It now follows that $\tilde{q} = q \cup \bigcup\{\sigma_{\mu_0, \mu_\ell}(q_\ell) : \ell \in L_0 \cup L_1\}$ is an element of $\text{Fn}(\kappa \setminus \mu_0, 2)$ and that $\tilde{q} \Vdash n \in \dot{V}_\ell$ for all $\ell \in L_0$ and $\tilde{q} \Vdash n \notin \dot{U}_\ell$ for all $\ell \in L_1$ as required.

We reassure the reader that we are nearly at the end of the proof. Fix any $\lambda < \kappa$ such that $\mu_\ell < \lambda$ for all $\ell \in \omega$ and let \dot{L}_λ be the canonical subset of ω given by $n \in \dot{L}_\lambda$ if and only if $\bigcup G(\lambda + n) = 1$. That is \dot{L}_λ is the name $\{(n, \{(\lambda + n, 1)\}) : n \in \omega\}$. We have proven that the family $\{\dot{V}_\ell : \ell \in \dot{L}_\lambda\} \cup \{\omega \setminus \dot{U}_\ell : \ell \in \omega \setminus \dot{L}_\lambda\}$ has the finite intersection property. We can define \dot{S} to be the sequence $\{\dot{n}_\ell : \ell \in \omega\}$ where \dot{n}_ℓ is the minimum element above $\dot{n}_{\ell-1}$ (for $\ell > 0$) of the set

$$\bigcap\{\dot{V}_j : j \leq \ell \text{ and } j \in \dot{L}_\lambda\} \setminus \bigcup\{\dot{U}_j : j \leq \ell \text{ and } j \notin \dot{L}_\lambda\}.$$

Each \dot{n}_ℓ exists was proven in the previous paragraph. Now suppose that \dot{J} is a nice $\text{Fn}(\kappa \setminus \mu_0, 2)$ -name of an infinite subset of ω . Choose any $\mu \in E$ satisfying that $\lambda + \omega < \mu$ and \dot{J} is a $\text{Fn}(\mu \setminus \mu_0, 2)$ -name. We complete the proof by showing that each of \dot{W}_{ρ_μ} and $\omega \setminus \dot{W}_{\zeta_\mu}$ hit the sequence $\{\dot{n}_\ell : \ell \in \dot{J}\}$ in an infinite set. Since \dot{W}_{ρ_μ} and $\omega \setminus \dot{W}_{\zeta_\mu}$ have disjoint closures, and \dot{J} was arbitrary, this shows that \dot{S} does not have any limit points. The proof is a density argument. Analogous to the definitions of $\dot{V}_\ell, \dot{W}_\ell$ from $\dot{W}_{\rho_{\mu_\ell}}, \dot{W}_{\zeta_{\mu_\ell}}$, let us define $\dot{V}_\mu = \{(n, \widehat{\sigma}_{\mu_0, \mu}(p)) :$

$(n, p) \in \dot{V}_0\}$ and $\dot{U}_\mu = \{(n, \widehat{\sigma}_{\mu_0, \mu}(p)) : (n, p) \in \dot{U}_0\}$. It follows that \dot{V}_μ is the $\text{Fn}(\kappa \setminus \mu_0, 2)$ -name for $\text{val}_G(\dot{W}_{\rho_\mu})$ and \dot{U}_μ is the $\text{Fn}(\kappa \setminus \mu_0, 2)$ -name for $\text{val}_G(\dot{W}_{\zeta_\mu})$. Let p be an arbitrary element of $\text{Fn}(\kappa \setminus \mu_0, 2)$ and let m be any integer. We will produce $q < p$ and a pair $m < \ell_1 < \ell_2$ so that q forces that $\{\ell_1, \ell_2\} \subset \dot{J}$, $\dot{n}_{\ell_1} \in \dot{V}_\mu$ and $\dot{n}_{\ell_2} \notin \dot{U}_\mu$. By possibly increasing m , we can assume that $\text{dom}(p) \cap [\lambda, \lambda + \omega)$ is contained in $[\lambda, \lambda + m]$. Similarly, we can assume that $\text{dom}(p) \cap I_{\mu_\ell} \setminus \mu_\ell$ is empty for all $\ell > m$. Choose $\bar{p} \in \text{Fn}(I, 2)$ so that $\sigma_{\mu_0, \mu}(\bar{p})$ is equal to $p \upharpoonright I_\mu \setminus \mu$. Let $q_1 \in \text{Fn}(\mu \setminus \mu_0, 2)$ be any extension of $(p \upharpoonright \mu) \cup \sigma_{\mu_0, \mu_{m+1}}(\bar{p})$ such that $q_1(\lambda + m + 1) = 1$, and there is an $\ell_1 > m$ and an $n_{\ell_1} \in \omega$ such that $q_1 \Vdash \ell_1 \in \dot{J}$ and $q_1 \Vdash \dot{n}_{\ell_1} = n_{\ell_1}$. Notice that $q_1 \Vdash n_{\ell_1} \in \dot{V}_{m+1}$. Now let $\bar{p}_1 \in \text{Fn}(I, 2)$ be chosen so that $\sigma_{\mu_0, \mu_{m+1}}(\bar{p}_1) = q_1 \upharpoonright I_{\mu_{m+1}} \setminus \mu_{m+1}$ and note that $\bar{p}_1 < \bar{p}$. Let us note that $\sigma_{\mu_0, \mu}(\bar{p}_1)$ forces that $n_{\ell_1} \in \dot{V}_\mu$. Choose next a sufficiently large $m_2 > \ell_1$ so that $\text{dom}(q_1) \cap [\lambda, \lambda + \omega) \subset [\lambda, \lambda + m_2)$ and $\text{dom}(q_1) \cap I_{\mu_\ell} \setminus \mu_\ell$ is empty for all $\ell > m_2$. Choose $q_2 \in \text{Fn}(\mu \setminus \mu_0, 2)$ so that $q_2 < \sigma_{\mu_0, \mu_{m_2}}(\bar{p}_1)$, $q_2(\lambda + m_2) = 0$, and, such that there are an $\ell_2 > m_2$ and $n_{\ell_2} \in \omega$ such that $q_2 \Vdash \ell_2 \in \dot{J}$ and $q_2 \Vdash \dot{n}_{\ell_2} = n_{\ell_2}$. Let $\bar{p}_2 \in \text{Fn}(I, 2)$ such that $\sigma_{\mu_0, \mu_{m_2}}(\bar{p}_2) = q_2 \upharpoonright I_{\mu_{m_2}} \setminus \mu_{m_2}$. Since $q_2 \Vdash m_2 \notin \dot{I}_\lambda$, we have that $q_2 \Vdash n_{\ell_2} \in \omega \setminus \dot{U}_{m_2}$. Therefore $\bar{p}_2 \Vdash n_{\ell_2} \notin \dot{U}_0$, which implies that $\sigma_{\mu_0, \mu}(\bar{p}_2) < \sigma_{\mu_0, \mu}(\bar{p}_1)$ forces that $n_{\ell_2} \notin \dot{U}_\mu$ and $n_{\ell_1} \in \dot{V}_\mu$. The required condition q is $q_2 \cup \sigma_{\mu_0, \mu}(\bar{p}_2)$. We have that $\sigma_{\mu_0, \mu}(\bar{p}_2)$ forces that $n_{\ell_1} \in \dot{V}_\mu$ and that $n_{\ell_2} \notin \dot{U}_\mu$. \square

We do not know if we had to assume that ω was a discrete subset in Theorem 4.2 so was raise this question.

Question 2. Is it consistent that every separable pseudocompact pseudoradial regular space has cardinality at most \mathfrak{c} ?

Now we prove what may be the main result of the paper.

Theorem 4.3. *If $\mathfrak{c} \leq \aleph_2$, then there is a separable 0-dimensional pseudoradial space of cardinality greater than \mathfrak{c} .*

Proof. As we learned in our previous results the space will necessarily contain a large independent family of clopen sets, and in fact the family must have special properties that will support the existence of many converging sequences. We will use the ideas introduced by P. Simon [22] to facilitate the construction of such families by using a countable set with special structure. For each $n \in \omega$, let $D_n = (2^n)^{2^n}$. Our countable dense set will be $D = \bigcup \{D_n : n \in \omega\}$.

Our construction will be to define a Boolean subalgebra \mathcal{B} of $\mathcal{P}(D)$ and to then choose a subspace of the space of ultrafilters, i.e. Stone

space, of \mathcal{B} . As usual, the finite subsets of D will be members of \mathcal{B} and we identify the fixed ultrafilters of \mathcal{B} with the points of D .

We will define a subtree of $2^{<\omega_2}$ that has cardinality \mathfrak{c} and has more than \mathfrak{c} many cofinal branches. If $2^{\aleph_1} = \aleph_2$ then this tree will necessarily have height ω_2 . We first present the proof in the case that $2^{\aleph_1} = \aleph_2$ and indicate at the end of the proof the minor modification needed for the case that $2^{\aleph_1} > \aleph_2$.

We will let T denote the chosen subtree of $2^{<\omega_2}$ (including its cofinal branches) and use the elements of T to enumerate the ultrafilters of \mathcal{B} that are chosen to be members of our space X . This approach ensures that our space is zero-dimensional and Hausdorff. Thus X will, technically have as base set $D \cup \{\mathcal{F}_t : t \in T\}$ but when proving that X is pseudoradial it will be convenient to identify $t \in T$ with the point \mathcal{F}_t in $S(\mathcal{B})$. The structure of T will be very helpful in understanding the convergence structure of X . In particular, for many, but not all, elements t of T that lie on limit levels of uncountable cofinality, the set of predecessors of t will be a well-ordered sequence that converges to t . This gives some of the insight into the proof that X is pseudoradial. As mentioned in the first paragraph, the structure of D is chosen to allow us to more strategically define the sequences from D that converge to points of X .

For $t \in 2^{<\omega_2}$, let $o(t)$ denote the domain or level of t . Let S_0^2 and S_1^2 denote the stationary subsets of ω_2 consisting of the cofinality ω and, respectively, ω_1 limits. Let $(S_1^2)'$ denote the set of all limits of S_1^2 and let $S_0 = S_0^2 \cap (S_1^2)'$ and $S_1 = S_1^2 \cap (S_1^2)'$. Let T_0 be the subtree of $2^{<\omega_2}$ where $t \in T_0$ if and only if $t^{-1}(1)$ is a subset of S_1 (i.e. we are only branching at levels in S_1). It should be clear that $T_0 \cap 2^{\omega_2}$ has cardinality 2^{\aleph_2} . We will need to add some more nodes to our tree because we want to have a large set of auxillary successors to associate with each $t \in T_0$ where $o(t) \in S_0$. Therefore T will consist of T_0 together with all $t \frown \sigma$ (concatentation) where $t \in T_0$, $\sigma \in 2^{<\omega}$, and $o(t) \in S_0$. For each $t \in T \setminus T_0$, let $\delta_t \in S_0$ and $\sigma_t \in 2^{<\omega}$ denote the values such that $t = t \upharpoonright \delta_t \frown \sigma_t$. For $s, t \in T$, let $s \wedge t$ be the maximum element of T satisfying that $s \wedge t \leq s$ and $s \wedge t \leq t$.

Now we choose our special independent subfamily of $\mathcal{P}(D)$ and a special family of converging sequences. It will make the proof more readable to use the set $T \cap 2^{<\omega_2}$ to enumerate the family. Let $\{r_t : t \in T_0 \cap 2^{<\omega_2}\}$ be any one-to-one enumeration of $2^\omega \setminus \{\vec{0}\}$ (where $\vec{0}$ is the constant zero function).

$$\text{For each } t \in T_0 \cap 2^{<\omega_2}, \quad A_t = \bigcup_n \{d \in D_n : d(r_t \upharpoonright n) = r_t \upharpoonright n\}.$$

Let \mathcal{L} denote the family of finite non-empty subsets of $T_0 \cap 2^{<\omega_2}$. For each $L \in \mathcal{L}$, let B_L be the set of $d \in \bigcap\{A_t : t \in L\}$ satisfying that $d(\sigma) = \vec{0} \upharpoonright n$ providing $d \in D_n$ and $\sigma \in 2^n \setminus \{r_t \upharpoonright n : t \in L\}$ (i.e. $d \in B_L \cap D_n$ if d is the identity on $\{r_t \upharpoonright n : t \in L\}$ and otherwise takes on value $\vec{0} \upharpoonright n$).

Claim 9. For each $L \in \mathcal{L}$, B_L is an infinite subset of $\bigcap\{A_t : t \in L\}$ and for $L \neq L' \in \mathcal{L}$, $B_L \cap B_{L'}$ is finite.

Proof of Claim: By symmetry we may assume that $L \setminus L' \neq \emptyset$ and let $t_L \in L \setminus L'$. Choose n_0 large enough so that the elements of $\{r_t \upharpoonright n_0 : t \in L \cup L'\}$ are pairwise distinct and not equal to $\vec{0} \upharpoonright n_0$. For all $n > n_0$ and $d \in B_{L'}$, $d(r_{t_L} \upharpoonright n) = \vec{0} \upharpoonright n$ and, for all $d \in B_L$, $d(r_{t_L} \upharpoonright n) = r_{t_L} \upharpoonright n \neq \vec{0} \upharpoonright n$. \square

It follows from Claim 9 that the family $\{A_t : t \in T_0 \cap 2^{<\omega_2}\}$ is an independent family. Let \mathcal{I} be the ideal of subsets of D that are almost disjoint from every such B_L , i.e. \mathcal{I} is often denoted as $(\{B_L : L \in \mathcal{L}\})^\perp$.

Let \mathcal{B} denote the Boolean subalgebra of $\mathcal{P}(D)$ generated by the family

$$[D]^{<\aleph_0} \cup \{A_t : t \in T_0 \cap 2^{<\omega_2}\} \cup \{B_L : L \in \mathcal{L}\} \cup \mathcal{I}$$

and we will choose, for each $t \in T$, a (free) ultrafilter \mathcal{F}_t on \mathcal{B} .

For each $t \in T$, we will first make an assignment H_t consisting of a non-empty set of nodes in T that are less or equal to t , and this assignment will determine the ultrafilter \mathcal{F}_t as described below.

We begin with the assignment $\langle H_t : t \in T \rangle$.

- (1) For $t \in T_0$ with $o(t) \in S_1 \cup \{\omega_2\}$, $H_t = \{t \upharpoonright \alpha : \alpha \in o(t)\}$.
- (2) For each $t \in T_0$ such that $o(t) = \mu + \omega_1$ for some $\mu \in S_1$, ensure that $\{H_{t \upharpoonright \xi} : \mu < \xi < \mu + \omega_1\}$ is a one-to-one listing of all the finite subsets of $\{t \upharpoonright \alpha : \alpha < \mu + \omega_1\}$ that have at least one element strictly extending $t \upharpoonright \mu$ (and satisfy the requirement that $t \upharpoonright \xi \in H_{t \upharpoonright \alpha}$ implies $\xi \leq \alpha$).
- (3) For each $t \in T_0$ such that $o(t) = \mu + \omega_1$ for some $\mu \in \{0\} \cup \overline{S_1^2} \setminus S_1$, ensure that $\{H_{t \upharpoonright \xi} : \mu \leq \xi < \mu + \omega_1\}$ is a one-to-one listing of all the finite subsets of $\{t \upharpoonright \alpha : \alpha < \mu + \omega_1\}$ that have at least one element above $t \upharpoonright \mu$ (and satisfy the requirement that $t \upharpoonright \xi \in H_{t \upharpoonright \alpha}$ implies $\xi \leq \alpha$).
- (4) For each $t \in T_0$ with $o(t) \in S_0$, ensure that $\{H_{t \frown \sigma} : \sigma \in 2^{\leq \omega} \text{ and } t \frown \sigma \notin T_0\}$ is a one-to-one listing of all countable cofinal subsets of $\{t \upharpoonright \alpha : \alpha < o(t)\}$. Note that, in

the previous item, $H_{t \frown \sigma}$ has already been defined as a finite set if $t \frown \sigma \in T_0$.

Claim 10. If $t \neq s$ and $t, s \in T$, then $H_t \neq H_s$.

Proof of Claim: It is clear that $H_t \neq H_s$ if either of them is uncountable. Similarly, if they are countably infinite, then it follows from clause (4) that $\delta_t = \delta_s$, and thus that $H_t \neq H_s$ since the assignment in (4) was chosen to be one-to-one. Finally we suppose that H_t and H_s are finite. We may choose minimal μ_t so that $\mu_t \leq o(t) \leq \mu_t + \omega_1$. Similarly choose minimal μ_s so that $\mu_s \leq o(s) \leq \mu_s + \omega_1$. It follows by the minimality, that μ_t and μ_s are limit points of S_1^2 . It follows easily from clauses (2) and (3) that if $H_t = H_s$, then $\mu_t = \mu_s$.

If $\mu_t \in S_1$ clause (2) ensures that if $t \upharpoonright \mu_t + 1 = s \upharpoonright \mu_t + 1$ then H_s, H_t are distinct elements of $\{H_{\bar{t} \upharpoonright \xi} : \mu_t < \xi < \mu_t + \omega_1\}$ for some $\bar{t} \in 2^{\mu_t + \omega_1}$. Otherwise $t \upharpoonright \mu_t + 1 \neq s \upharpoonright \mu_t + 1$ and clause (2) ensures that H_t and H_s contain incomparable elements. Finally, if $\mu_t \notin S_1$, then s and t are distinct but comparable and again we have that H_t and H_s are distinct elements of the list $\{H_{\bar{t} \upharpoonright \xi} : \mu_t \leq \xi < \mu_t + \omega_1\}$ for some $\bar{t} \in 2^{\mu_t + \omega_1}$. \square

Now we make the simple assignment of the family $\langle \mathcal{F}_t : t \in T \rangle$:

- (5) If H_t is finite, then define \mathcal{F}_t to be the unique free ultrafilter of \mathcal{B} with $B_{H_t} \in \mathcal{F}_t$.
- (6) If H_t is infinite, then \mathcal{F}_t is the ultrafilter satisfying
 - (a) $A_s \in \mathcal{F}_t$ if and only if $s \in H_t$,
 - (b) $B_L \notin \mathcal{F}_t$ for all $L \in \mathcal{L}$,
 - (c) $I \notin \mathcal{F}_t$ for all $I \in \mathcal{I}$,
 - (d) $[D]^{<\aleph_0} \cap \mathcal{F}_t$ is empty.

It should be clear that the subspace $D \cup X_1$ is simply equal to the traditional Mrowka-Isbell type space constructed from the almost disjoint family $\{B_L : L \in \mathcal{L}\}$. Let us also note that for each $I \in \mathcal{I}$, I is closed in X , and this implies this next claim.

Claim 11. If $Y \subset D$ and $\bar{Y} \cap X_1$ is empty, then Y is closed.

Proof of Claim: If no point of X_1 is in the closure of Y , then $Y \cap B_L$ is finite for all $L \in \mathcal{L}$. Thus $Y \in \mathcal{I}$. \square

It will be convenient to let $K_t = \{A_s : s \in (T_0 \cap 2^{<\omega_2}) \setminus H_t\}$, i.e. $K_t = \{A_s : A_s \notin \mathcal{F}_t\}$. For disjoint finite subsets H, K of $T_0 \cap 2^{<\omega_2}$, let $[H; K]$ denote the clopen subset of X corresponding to the closure of this element of \mathcal{B} :

$$[H; K] = \bigcap \{A_t : t \in H\} \setminus \left(\bigcup \{A_s : s \in K\} \cup \bigcup \{B_L : \emptyset \neq L \subset H\} \right) .$$

For any $t \in X_2 \cup X_3$, the family $\{[H; K] : H \in [H_t]^{<\aleph_0}, K \in [K_t]^{<\aleph_0}\}$ is easily seen to be a local filter base in the subspace $X \setminus D$. Let $X_0 = D$, $X_1 = \{t \in T_0 : |H_t| < \aleph_0\}$, $X_2 = \{t \in T : |H_t| = \aleph_0\}$ and let $X_3 = \{t \in T : |H_t| > \aleph_0\}$.

Claim 12. X_3 is a closed radial subspace of X .

Proof of Claim: We first show that X_3 is closed. Clearly $X_0 \cup X_1$ is open so consider any $t \in X_2$. Since H_t is a countable cofinal subset of the uncountable ordinal δ_t , there are $\beta < \alpha < \delta_t$ such that $t \upharpoonright \beta \in K_t$ and $t \upharpoonright \alpha \in H_t$. Note that $\{t \upharpoonright \alpha\}; \{t \upharpoonright \beta\}$ is disjoint from X_3 since H_s is a downward closed subset of T_0 .

Now consider any $Y \subset X_3$ and assume that $t \in X_3$ is a limit point of Y . We will assume that $o(t) < \omega_2$ and leave the case when $o(t) = \omega_2$ to the reader. Choose any strictly increasing cofinal sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset o(t)$. Note that $t \in K_t$. For each $\xi < \omega_1$, choose $y_\xi \in Y \cap [\{t \upharpoonright \alpha_\xi\}; \{t\}]$. Consider any finite $H \subset H_t$ and finite $K \subset K_t$. Let $\tilde{K} = \{s \in K_t : t \not\leq s\}$ and choose $\xi < \omega_1$ so that $t \upharpoonright \alpha_\xi \not\leq s$ for all $s \in \tilde{K}$. It follows easily that $y_\eta \in [H; K]$ for all $\xi < \eta < \omega_1$ and thereby proving that $\langle y_\xi : \xi \in \omega_1 \rangle$ converges to t . \square

We continue the proof that X is pseudoradial. Let us note that it follows from Claim 11 that it suffices to proof that $X \setminus D$ is pseudoradial. For the remainder of the proof we will say that an elementary submodel M is suitable to mean that $M \prec H(\aleph_3)$ and that $\{T_0, \mathcal{L}, \{A_t : t \in T_0 \cap 2^{<\omega_2}\}, \{H_t : t \in T\}, \{B_L : L \in \mathcal{L}\}\}$ is an element of M . Here is one of the key properties of the space.

Claim 13. If $Y \subset X \setminus D$ and $t \in X_2 \cup X_3$ is a limit point of Y , then for any countable suitable elementary submodel M such that $Y, t \in M$, there is a converging sequence $\{t_n : n \in \omega\} \subset Y \cap M$ such that t is the limit if $t \in X_2$ and, if $t \in X_3$, the limit is $t_M \in X_2$ where

- (1) $\delta_{t_M} = \sup(M \cap o(t))$ and $t_M = t \upharpoonright \delta_{t_M} \hat{\ } \sigma_{t_M}$,
- (2) $H_{t_M} = \{t \upharpoonright \alpha : \alpha \in M \cap o(t)\}$.

Proof of Claim: Let \bar{t} equal to t if $t \in X_2$ and let $\bar{t} = t_M$ if $t \in X_3$. In either case, we have that $H_{\bar{t}} \subset M$. If $t \in X_3$, then observe that $t \in K_t$ and $K_t \cap \{t \upharpoonright \alpha : \alpha \in M \cap o(t)\}$ is empty. If $t \in X_2$, then, by elementarity, $K_t \cap \{t \upharpoonright \alpha : \alpha \in M \cap o(t)\}$ is non-empty. Define $K_0 \in M$ to be $\{t\}$ if $t \in X_3$ and to be any finite subset of $K_t \cap \{t \upharpoonright \alpha : \alpha \in M \cap o(t)\}$ if $t \in X_2$. Choose H_0 to be any finite subset of $H_{\bar{t}}$ so that, if $t \in X_2$, then there are $\alpha < \beta$ so that $t \upharpoonright \alpha \in K_0$ and $t \upharpoonright \beta \in H_0$. Since $[H_0; K_0] \in M$, there is a $y_0 \in M \cap Y \cap [H_0; K_0]$.

Choose any sequence $\{[H_n; K_n] : n \in \omega\} \subset M$ so that:

- (1) $\{H_n : n \in \omega\}$ is a strictly increasing sequence of finite sets whose union is $H_{\bar{t}}$,
- (2) $\{K_n : n \in \omega\}$ is a strictly increasing sequence of finite sets whose union equals $(K_{\bar{t}} \setminus H_t) \cap M$,

We show that we can choose $y_n \in Y \cap M \cap [H_n; K_n]$ for all n . If $t = \bar{t}$, then $t \in [H_n; K_n]$ and so $Y \cap M \cap [H_n; K_n]$ is not empty by elementarity and the assumption that t is a limit point of Y . If $t \neq \bar{t}$, then $H_n \subset H_{\bar{t}} \subset H_t$ and $K_n \subset K_{\bar{t}} \setminus H_t$. Therefore, we again have that $t \in [H_n; K_n]$ and so, by elementarity, there is a $y_n \in Y \cap M \cap [H_n; K_n]$.

Now we show that the sequence $\{y_n : n \in \omega\}$ converges to \bar{t} . Consider any finite $H \subset H_{\bar{t}}$ and finite $K \subset K_{\bar{t}}$. Choose $\beta < \delta_{\bar{t}}$ large enough so that $o(s \wedge (t \upharpoonright \delta_{\bar{t}})) < \bar{\beta}$ for all $s \in K$ such that $t \upharpoonright \delta_{\bar{t}} \not\leq s$. We may assume that $t \upharpoonright \beta \in H_{\bar{t}}$. Choose n so that $H \cup \{t \upharpoonright \beta\} \subset H_n$, and $(K \setminus H_t) \cap M$ is a subset of K_n . We show that $y_m \in [H; K]$ for each $m > n$. Since $H \subset H_m$ and $y_m \in [H_m; K_m]$ it is clear that $H \subset H_{y_m}$ and $K_m \subset K_{y_m}$. Also, $H_{y_m} \setminus H_m$ is not empty (in the case that H_{y_m} is finite). To show that $y_m \in [H; K]$, we must simply show that $H_{y_m} \cap K$ is empty.

In the case that $t = \bar{t}$, then $K_{\bar{t}} \setminus H_t = K_{\bar{t}}$, and so $H_{y_m} \cap K \subset H_{y_m} \cap K_m = \emptyset$. So now we assume that $\bar{t} \neq t \in X_3$ and recall that $t \in K_0$. In this case, $K_{\bar{t}} \cap M$ is also disjoint from H_t , hence $K \cap M \subset K_n$. If $y_m \notin X_3$, then $H_{y_m} \subset M$ and so $H_{y_m} \cap K \subset H_{y_m} \cap K \cap M \subset H_{y_m} \cap K_m = \emptyset$. Now suppose that $y_m \in X_3$ and assume that $s \in H_{y_m} \cap K$. Note that we have that $t \upharpoonright \beta \in H_{y_m}$ and so $t \upharpoonright \delta_{\bar{t}} \leq s \leq y_m$. Also, $\{t \upharpoonright \alpha : \alpha < \delta_{\bar{t}}\}$ is a subset of H_{y_m} . Since $t \in K_0$ implies that $o(t \wedge y_m) \in M \cap o(t) = \delta_{\bar{t}}$, this contradicts that $t \upharpoonright \delta_{\bar{t}} \leq s$. \square

Claim 14. If $Y \subset X \setminus D$ and $t \in X_3$ is a limit point of Y , then there is a sequence $\{y_\alpha : \alpha \in \omega_1\}$ converging to a point in X_3 where each y_α is in the sequential closure of Y .

Before we prove this Claim, we observe that this completes the proof that X is pseudoradial. If Y is a subset of $X \setminus D$ then, by Claim 13 every limit point in X_2 is in the sequential closure, and by Claim 14, the points of X_3 in the radial closure of Y are dense in $\bar{Y} \cap X_3$. Since X_3 is radial (Claim 12), this implies that the radial closure of Y is closed.

Proof of Claim: Fix any increasing sequence $\{M_\alpha : \alpha \in \omega_1\}$ of suitable countable elementary submodels satisfying that Y, t are elements of M_0 . There is an ordinal $\theta \in S_1$ such that $\theta = \bigcup \{M_\alpha \cap \omega_2 : \alpha \in \omega_1\}$ (see [17]). If $o(t) < \omega_2$, let $\lambda = o(t) \in S_1$. If $o(t) = \omega_2$, let $\lambda = \theta$.

For each $\alpha \in \omega_1$, let $\delta_\alpha = \sup(M_\alpha \cap \lambda)$. Following Claim 13, let $y_\alpha = t \upharpoonright \delta_\alpha \cap \sigma_\alpha$ where σ_α is chosen so that $H_{y_\alpha} = \{t \upharpoonright \gamma : \gamma \in M_\alpha \cap \lambda\}$. It follows from Claim 13, that y_α is in the sequential closure of Y .

Now we simply check that $\{y_\alpha : \alpha \in \omega_1\}$ converges to $t \upharpoonright \lambda$. Fix any basic open set $[H; K]$ of $t \upharpoonright \lambda$. Choose any $\beta < \omega_1$ so that $H \in M_\beta$. Observe that $H \subset H_{y_\alpha}$ for all $\beta < \alpha \in \omega_1$. Similarly, $H_{y_\alpha} \subset H_{t \upharpoonright \delta_\alpha} \subset H_{t \upharpoonright \lambda}$ for all $\alpha \in \omega_1$. Therefore $K \cap H_{y_\alpha} = \emptyset$, and so $y_\alpha \in [H; K]$, for all $\beta < \alpha \in \omega_1$. This completes the proof of the Claim. \square

This completes the proof of the Theorem in the case that $2^{\aleph_1} = \aleph_2$. The only modification that is needed for the case $2^{\aleph_1} > \aleph_2$ is to pass to the subtree $T_0 \cap 2^{\leq \theta}$ where $\theta \in S_1$ is the minimum level of T_0 which has cardinality greater than \aleph_2 . This ensures that $T_0 \cap 2^{< \theta}$ has cardinality \aleph_2 . Establish the new enumeration $\{r_t : t \in T_0 \cap 2^{< \theta}\}$ of $2^\omega \setminus \{0\}$ and the proof proceeds exactly as above. \square

Question 3. Is it consistent that there is no separable pseudoradial space of cardinality $2^{\mathfrak{c}}$?

Question 4. Do separable regular pseudoradial spaces of cardinality greater than \mathfrak{c} exist?

We also do not know if large separable Hausdorff pseudoradial spaces of cardinality greater than \mathfrak{c} exist.

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