PFA(S) AND AUTOMORPHISMS OF $\mathcal{P}(\mathbb{N})/\operatorname{fin}$

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Abstract. Todorcevic introduced the forcing axiom PFA(S) and established many consequences. We contribute to this project. In particular, we show that forcing with the Souslin tree S, as postulated by PFA(S), will preserve that all automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ are trivial.

1. Introduction

There are two models to consider. One is the (ground) model in which PFA(S) holds and the second is the forcing extension by the Souslin tree S of such a model, denote PFA(S)[S]. Farah used this method in to show that, what is now known as, the Open Graph Axiom (OGA) is not sufficient to prove to prove the well-known consequence of PFA concerning $\aleph_1$-dense sets of reals. It is not known if OGA is sufficient to establish that there are no non-trivial automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. We prove, though, that PFA(S)[S] does imply this. The literature on the question of the existence of non-trivial automorphisms on $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ is well-known and quite extensive (see [3, 4, 6–9, 13]).

The method of applying PFA(S) to prove results about either PFA(S), or the extension PFA(S)[S], is to produce a proper poset $\mathbb{P}$ and prove that it preserves that the Souslin tree S remains Souslin.

Lemma 1.1. *For a ccc poset $\mathbb{P}$ the following are equivalent*

(1) $\mathbb{P}$ preserves that S is Souslin,
(2) $\mathbb{P} \times S$ is ccc,
(3) S preserves that $\mathbb{P}$ is ccc.

A poset $\mathbb{P}$ is said to have property K if every uncountable subset of $\mathbb{P}$ has an uncountable linked subset. Kunen and Tall [5] showed that if $\mathbb{P}$ has property K then $\mathbb{P} \times T$ is ccc for each Souslin tree T. Therefore PFA(S) is a model of $MA_K(\omega_1)$.

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Lemma 1.2. *For a proper poset $\mathbb{P}$ the following are equivalent*

(1) *$\mathbb{P}$ preserves that S is Souslin,*
(2) *$\mathbb{P} \times S$ is proper.*

Definition 1.3. *A Souslin tree $S \subset \omega^{<\omega_1}$ is coherent if $s\Delta t = \{\xi \in \operatorname{dom}(s) \cap \operatorname{dom}(t) : s(\xi) \neq t(\xi)\}$ is finite for all $s, t \in S$. The axiom PFA(S) is the statement that there is a coherent Souslin tree and for all proper posets $\mathbb{P}$ such that forcing with $\mathbb{P}$ preserves that S is Souslin, for each family $\mathfrak{D}$ of at most ω_1 dense subsets of $\mathbb{P}$ there is a $\mathfrak{D}$-generic filter on $\mathbb{P}$.*

The homogeneous closure of a tree $S \subset \omega^{<\omega_1}$ will consist of all elements t of $\omega^{<\omega_1}$ that satisfy $s\Delta t$ is finite for each $s \in S_{\operatorname{dom}(t)}$. If S is a coherent Souslin tree then its homogeneous closure is as well. Henceforth we assume that S is equal to its homogeneous closure. For $s, t \in S$ we let $s \oplus t$ denote the element of S that is equal to $s \cup (t \restriction [\operatorname{dom}(s), \operatorname{dom}(t)))$. If g is any generic filter for S and $s \in S$, then $s \oplus g = \{s \oplus t : t \in g\}$ is also an S-generic filter.

2. PFA(S)[S] implies all automorphisms are trivial

We will need that the Ramsey axiom OGA holds in the PFA(S) model. It also holds in the PFA(S)[S] model. This was proven by Todorcevic in [12, 5.1], but also, one can deduce this fact from the earlier results in [2].

Definition 2.1. *OGA is the statement that every open graph on a separable metric space is countably chromatic unless it contains an uncountable complete subgraph.*

Definition 2.2. *If Φ is an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$, then an injection h induces Φ on $a \subset \mathbb{N}$ providing $a \setminus \operatorname{dom}(h)$ is finite, and for each $c \subset a$, $h(c)/\operatorname{fin}$ is equal to $\Phi(c/\operatorname{fin})$. We let $\mathsf{Triv}(\Phi)$ denote the ideal of sets $a \subset \mathbb{N}$ for which there is an injection h_a inducing Φ on a. As usual Φ is said to be non-trivial if Φ is a proper ideal.*

Lemma 2.3. *If an injection h does not induce Φ on some infinite $a \subset \operatorname{dom}(h)$, then there is an infinite $c \subset a$ such that $h(c)/\operatorname{fin} \wedge \Phi(c) = 0$ (i.e. $h(c)$ is almost disjoint from any d in the equivalence class $\Phi(c)$).*

Proof. Assuming that h does not induce Φ on a, there is some infinite $b \subset a$ such that $h(b)/\operatorname{fin}$ is not equal to $\Phi(b/\operatorname{fin})$. Let $b_\Phi \subset \mathbb{N}$ be any representative of $\Phi(b/\operatorname{fin})$. If $h(b) \setminus b_\Phi$ is infinite, then set $c = \{k \in b : h(k) \notin b_\Phi\}$. It follows that $h(c) \cap b_\Phi$ is empty, and since $\Phi(c/\operatorname{fin}) \leq \Phi(b/\operatorname{fin})$, we have that $(h(c)/\operatorname{fin}) \wedge \Phi(c/\operatorname{fin}) = 0$ as required. Otherwise we have that $b_\Phi \setminus h(b)$ is infinite. Choose any $c \subset b$ such that $\Phi(c/\operatorname{fin}) = (b_\Phi \setminus h(b))/\operatorname{fin}$. Then again, since $h(c) \subset h(b)$, we have that $(h(c)/\operatorname{fin}) \wedge \Phi(c/\operatorname{fin}) = 0$. □

Borrowing from [11] and [13] we define two separation conditions on an almost disjoint family.

Definition 2.4. *A family $\mathcal{A}$ of subsets of $\mathbb{N}$ is σ-separated if there is a countable family $\mathcal{B}$ of subsets of $\mathbb{N}$ such that for all distinct $a, a' \in \mathcal{A}$, there is a $b \in \mathcal{B}$ such that $a \subset^* b$ and $a' \subset^* \mathbb{N} \setminus b$.*

Say that $\mathcal{A}$ is $(\sigma, 2)$-separated if there are σ-separated families $\mathcal{A}_1, \mathcal{A}_2$ such that $\mathcal{A}$ is contained in the ideal generated by $\mathcal{A}_1 \cup \mathcal{A}_2 \cup [\mathbb{N}]^{<\aleph_0}$.

The next result is due to Velickovic [13] but we need to verify that the poset used has property K.

Proposition 2.5. *If $\mathcal{A}$ is an almost disjoint family of subsets of $\mathbb{N}$ then there is a property K poset $\mathbb{P} = \mathbb{P}(\mathcal{A})$ that forces $\mathcal{A}$ to be $(\sigma, 2)$-separated.*

Proof. Let $p \in \mathbb{P}$ if $p = (\langle k_i^p : i \le n_p \rangle, \langle \varphi_a^p : a \in \mathcal{A}_p \rangle)$ where

(1) $\langle k_i^p : i \le n_p \rangle$ is a strictly increasing sequence of integers,
(2) $\mathcal{A}_p$ is a finite subset of $\mathcal{A}$,
(3) $a \cap a' \subset k_n^p$ for distinct $a, a' \in \mathcal{A}_p$,
(4) φ_a^p is a function from n_p into $\{0, 1, 2\}$ such that $(\varphi_a^p)^{-1}(0)$ is an initial segment,
(5) if $0 < \varphi_a^p(i) = \varphi_{a'}^p(i)$ and if $a \cap [k_i^p, k_{i+1}^p) \ne a' \cap [k_i^p, k_{i+1}^p)$, then $a \cap a' \subset k_i^p$.

The ordering on $\mathbb{P}$ is that $p < q$ providing $n = n_p \ge n_q$, $k_i^p = k_i^q$ for $i \le n_q$, $\mathcal{A}_p \supset \mathcal{A}_q$ and $\varphi_a^q \subset \varphi_a^p$ for all $a \in \mathcal{A}_q$.

If $q \in \mathbb{P}$, a' is any member of $\mathcal{A} \setminus \mathcal{A}_q$, $n = n_q + 1$, and $k_n > k_{n_q}^q$ is any value such that $a \cap a' \subset k_n$ for all $a \in \mathcal{A}_q$, then each of

$$(\langle k_i^q : i < n \rangle^\frown k_n, \{\varphi_a^q \cup \{(n_q, 1)\} : a \in \mathcal{A}_q\} \cup \{\varphi_{a',2}\}) \text{ and}$$
$$(\langle k_i^q : i < n \rangle^\frown k_n, \{\varphi_a^q \cup \{(n_q, 2)\} : a \in \mathcal{A}_q\} \cup \{\varphi_{a',1}\})$$

are extensions of q, where $\varphi_{a',1}(i) = \varphi_{a',2}(i) = 0$ for $i < n_q$, $\varphi_{a',1}(n_q) = 1$ and $\varphi_{a',2}(n_q) = 2$.

Suppose that $\{p_\xi : \xi \in \omega_1\}$ is a subset of $\mathbb{P}$. By thinning we may assume that the collection $\{\mathcal{A}_{p_\xi} : \xi \in \omega_1\}$ is a Δ-system with root $\mathcal{C}$. Furthermore, we can assume that there is an enumeration $\{a_\ell^\xi : \ell < m\}$ of $\mathcal{A}_{p_\xi}$ such that, for each ξ, η and $\ell < m$,

(1) $n = n_{p_\xi} = n_{p_\eta}$, and $\langle k_i^{p_\xi} : i \le n \rangle = \langle k_i^{p_\xi} : i \le n \rangle$,
(2) $a_\ell^\xi \cap k_n = a_\ell^\eta \cap k_n$,

(3) $\varphi_{a_\ell^{p_\xi}} = \varphi_{a_\ell^{p_\eta}}$,

(4) $a_\ell^\xi \in \mathcal{C}$ iff $a_\ell^\eta \in \mathcal{C}$.

We now prove that this (uncountable subcollection) of elements are pairwise compatible. For any ξ, η we define a condition $p \in \mathbb{P}$ below each of p_ξ and p_η. We let $n_p = n_{p_\xi} + 1$ and choose $\langle k_i^p : i \leq n_p \rangle$ so that $k_i^p = k_i^{p_\xi}$ for $i < n_p$ and $k_{n_p}^p$ is large enough so that $\Delta(a, a') < k_{n_p}^p$ for all distinct $a, a' \in \mathcal{A}_p = \mathcal{A}_{p_\xi} \cup \mathcal{A}_{p_\eta}$. We define $\varphi_{n_p}^p(a) = 1$ for all $a \in \mathcal{A}_{p_\xi}$ and $\varphi_{n_p}^p(a') = 2$ for all $a' \in \mathcal{A}_{p_\eta} \setminus \mathcal{A}_{p_\xi}$.

Let G be a filter on $\mathbb{P}$ satisfying that for all $a \in \mathcal{A}$ and $n \in \omega$, there is a $p \in G$ such that $n_p > n$ and $a \in \mathcal{A}_p$. Let $\{k_i : i \in \omega\} = \bigcup\{\{k_i^p : i \leq n_p\} : p \in G\}$ (listed in increasing order). For each $a \in \mathcal{A}$, let $\varphi_a = \bigcup\{\varphi_a^p : p \in G \text{ and } a \in \mathcal{A}_p\}$. For each $a \in \mathcal{A}_p$, the set $\bigcup\{a \cap [k_i, k_{i+1}) : \varphi_a(i) = 1\}$ is in the family $\mathcal{A}_1$. Similarly the set $\bigcup\{a \cap [k_i, k_{i+1}) : \varphi_a(i) = 2\}$ is in the family $\mathcal{A}_2$.

Suppose that a_1, a_2 are distinct members of $\mathcal{A}$ and that there is a i such that $0 < e = \varphi_{a_1}(i) = \varphi_{a_2}(i)$ and $\Delta(a_1, a_2) < k_i$. Let $c = a_1 \cap [k_i, k_{i+1})$ and define

$$b(e,c) = \bigcup\{a \cap [k_j, k_{j+1}) : i \leq j \ , \varphi_a(j) = e = \varphi_a(i) \text{ and } a \cap [k_i, k_{i+1}) = a_1 \cap [k_i, k_{i+1})\}.$$

Clearly $\bigcup\{a_1 \cap [k_j, k_{j+1}) : i \leq j \text{ and } \varphi_{a_1}(j) = e\}$ is contained in $b(e,c)$. We prove that $\bigcup\{a_2 \cap [k_j, k_{j+1}) : i \leq j \text{ and } \varphi_{a_2}(j) = e\}$ is disjoint from $b(c,e)$. To see this choose $j > i$ so that $\varphi_{a_2}(j) = e$ and any a_3 so that $\varphi_{a_3}(i) = \varphi_{a_3}(j) = e$ and $c = a_3 \cap [k_i, k_{i+1})$. Choose a condition $p \in G$ so that $n_p > j$ and $\{a_1, a_2, a_3\} \subset \mathcal{A}_p$. Since $a_3 \cap [k_i, k_{i+1}) \neq a_2 \cap [k_i, k_{i+1})$, condition (5) of the definition of $\mathbb{P}$ ensures that $\Delta(a_2, a_3) < k_i$. Therefore it follows that $a_2 \setminus k_i$ is disjoint from a_3, and therefore from $b(c,e)$.

This completes the proof. □

Corollary 2.6. *PFA(S) implies that every almost disjoint family of cardinality $\aleph_1$ is $(\sigma, 2)$-separated.*

The following well-known result is due to Velickovic [13].

Proposition 2.7 (OGA). *If $\mathcal{A}$ is a $(\sigma, 2)$-separated family of subsets of $\mathbb{N}$ and Φ is an automorphism of $\mathcal{P}(\mathbb{N})/\text{fin}$, then $\mathcal{A} \setminus \mathsf{Triv}(\Phi)$ is countable.*

We will also need the following result of Todorcevic that appeared in [1, 3.13] as well as in [13].

Proposition 2.8. *OGA implies that if $\{h_f : f \in \omega^\omega\}$ is a family of integer-valued functions satisfying that $h_f \subset^* h_g$ whenever $f \leq^* g$, then there is a single function h satisfying that $h_f \subset^* h$ for all $f \in \omega^\omega$.*

An ideal is a P-ideal if it is countably upwards directed mod finite.

Lemma 2.9. *PFA(S)[S] implies that* $\mathsf{Triv}(\Phi)$ *is a dense P-ideal for each automorphism* Φ *of* $\mathcal{P}(\mathbb{N})/\operatorname{fin}$.

Proof. Of course $\mathsf{Triv}(\Phi)$ is a dense ideal by Proposition 2.7 and the fact that OGA holds. We show that it is a P-ideal. Let $\{a_n : n \in \omega\}$ be an increasing sequence of elements of $\mathsf{Triv}(\Phi)$. For each n, let h_n induce Φ on a_n. For each n there is a k_n so that for each $j \leq n$ and $m \in a_j \setminus k_n$, $h_j(m) = h_n(m)$. Evidently $h' = \bigcup\{h_n \restriction a_n \setminus k_n : n \in \omega\}$ is a 1-to-1 function that induces Φ on each a_n.

By OGA, $\mathfrak{b} = \mathfrak{d} = \omega_2$ holds in the PFA(S) model and so we may choose there a family $\{f_\gamma : \gamma \in \omega_2\} \subset \omega^\omega$ that is mod finite increasing and cofinal in the mod finite ordering on ω^ω. Since forcing with S adds no new subsets of ω and preserves cardinals, the family $\{f_\gamma : \gamma \in \omega_2\}$ remains a dominating family in the PFA(S)[S] model.

For each $\gamma \in \omega_2$, let $f_\gamma^\uparrow$ denote the set $\bigcup\{a_n \setminus f_\gamma(n) : n \in \omega\}$. Similarly we can let $f_\gamma^\downarrow$ denote $\mathbb{N} \setminus f_\gamma^\uparrow$. We must show that there is a $\alpha \in \omega_2$ so that $f_\alpha^\uparrow$ is in $\mathsf{Triv}(\Phi)$.

First we show that it suffices to find $\alpha \in \omega_2$ so that there is a single h that induces Φ on $f_\alpha^\uparrow \cap f_\delta^\downarrow$ for all $\delta \in \omega_2$; so assume that h is such a function. First note that if $c \subset f_\alpha^\uparrow$ and $(h(c)/\operatorname{fin}) \wedge \Phi(c) = 0$, then there is an $n \in \omega$ such that $c \subset a_n$ and $\{k \in c : h'(k) = h(k)\}$ is finite. Now let $a = \{k \in f_\alpha^\uparrow : h'(k) \neq h(k)\}$ and assume that $a \setminus a_n$ is infinite for each n. If there is an n such that h induces Φ on $a \setminus a_n$ then, by the previous sentence and Lemma 2.3, h induces Φ on $f_\alpha^\uparrow \setminus a_n$. Therefore, if $a \cap a_{j+1} \setminus a_j$ is finite for all $j > n$, then h induces Φ on $f_\alpha^\uparrow \setminus a_n$. Otherwise, let J be an infinite subset of ω such that $a \cap a_{j+1} \setminus a_j$ is infinite for each $j \in J$. By a standard Hausdorff disjoint refinement argument, there is $c \subset a$ such that $h(c) \cap h'(c) = \emptyset$ and $c \cap a_{j+1} \setminus a_j$ is infinite for all $j \in J$. Let $d \subset \mathbb{N}$ be a representative of $\Phi(c)$ in that $(d/\operatorname{fin}) = \Phi(c)$. Then $d \cap h'(a_{m+1} \setminus a_m)$ is almost equal $h'(c \cap (a_{m+1} \setminus a_m))$ for all $m \in \omega$. For each $j \in J$, $d_j = h(c \cap (a_{j+1} \setminus a_j))$ is almost disjoint from $h(f_\delta^\downarrow \cap f_\alpha^\uparrow)$ for each $\delta \in \omega_2$. Therefore, since we are assuming that h induces Φ on $f_\delta^\downarrow \cap f_\alpha^\uparrow$, we have that $(d_j/\operatorname{fin}) \wedge \Phi(f_\delta^\downarrow \cap f_\alpha^\uparrow) = 0$ for all $\delta \in \omega_2$. It then follows that there is an m such that $h(c \cap (a_{j+1} \setminus a_j))$ is mod finite contained in $h'(a_m)$. This all put together means that we can choose a value $v_j \in c \cap (a_{j+1} \setminus a_j)$ such that $h(v_j) \notin d$. We finally have our contradiction since $h(\{v_j : j \in J\})$ is disjoint from d while there is a $\delta \in \omega_2$ such that $\{v_j : j \in J\}$ is almost contained in $f_\delta^\downarrow \cap f_\alpha^\uparrow$, implying that h induces Φ on $\{v_j : j \in J\}$.

Now we show that there is an $\alpha \in \omega_2$ as above. There is a cub $C \subset \omega_2$ such that for each $\alpha < \delta \in C$, if Φ is trivial on $f^{\uparrow}_{\alpha} \cap f^{\downarrow}_{\delta}$, then Φ is trivial on $f^{\uparrow}_{\alpha} \cap f^{\downarrow}_{\delta}$ for all $\delta \in \omega_2$. Since S is ccc, there is a ground model (PFA(S)) cub contained in C. Therefore we may assume that C is in the PFA(S) model. For each $\delta \in C$ let δ^+ be the minimal member of C above δ. Since C is in the ground model, each $\aleph_1$-sized subset of $\{f^{\uparrow}_{\delta} \cap f^{\downarrow}_{\delta^+} : \delta \in C\}$ is $(\sigma, 2)$-separated. Therefore, by Lemma 2.7, there is an $\alpha \in C$ such that $f^{\uparrow}_{\alpha} \cap f^{\downarrow}_{\alpha^+}$ is in $\mathsf{Triv}(\Phi)$. By the definition of C it follows that $f^{\uparrow}_{\alpha} \cap f^{\downarrow}_{\delta}$ is in $\mathsf{Triv}(\Phi)$ for all $\delta > \alpha$. For each $\delta \in C \setminus \alpha$, choose h_δ that induces Φ on $f^{\uparrow}_{\alpha} \cap f^{\downarrow}_{\delta}$. For any $f \in \omega^\omega$, set $h_f = h_{f_\delta}$ where $\delta \in C \setminus \alpha$ is minimal such that $f \leq^* f_\delta$. By Proposition 2.8 we have our desired h that induces Φ on $f^{\uparrow}_{\alpha} \cap f^{\downarrow}_{\delta}$ for all $\delta \in \omega_2$. □

We are ready to finish the proof of the main theorem.

Theorem 2.10. *Each of PFA(S) and PFA(S)[S] imply that all automorphisms of* $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ *are trivial.*

Proof. It suffices to prove that PFA(S)[S] implies that all automorphisms are trivial since a non-trivial automorphism from the ground model would remain a non-trivial automorphism after forcing with S. We work in the PFA(S) model. Let $\dot\Phi$ be an S-name of an automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$. We assume, for a contradiction, that some condition forces that $\dot\Phi$ is not trivial. Since S is coherent, and therefore homogeneous, we may assume that condition is the root of S.

Let $\mathcal{I}$ denote the ideal of subsets of $\mathbb{N}$ where $a \in \mathcal{I}$ providing the root of S forces that $a \in \mathsf{Triv}(\dot\Phi)$. If every element of some level forces that $a \in \mathsf{Triv}(\dot\Phi)$, then a is in $\mathcal{I}$.

Claim 1. $\mathcal{I}$ is a dense P-ideal.

Proof of Claim: It is immediate from Lemma 2.7 that $\mathcal{I}$ is a dense ideal. Let $\{a_n : n \in \omega\} \subset \mathcal{I}$. There is a $\delta \in \omega_1$ such that, for each $s \in S_\delta$, there is an $a_s \subset \mathbb{N}$ such that $s \Vdash a_s \in \mathsf{Triv}(\dot\Phi)$ and, for each $n \in \omega$, $a_n \subset^* a_s$. Since S_δ is countable, there is an $a \subset \mathbb{N}$ such that for all $n \in \omega$ and all $s \in S_\delta$, $a_n \subset^* a \subset^* a_s$. It follows that each $s \in S_\delta$ forces that $a \in \mathsf{Triv}(\dot\Phi)$, and so $a \in \mathcal{I}$.

Let $\prec$ be any well-ordering of $H(\omega_1)$ in order-type ω_2. For each $a \in \mathcal{I}$, $\dot h_a$ is the $\prec$-minimal S-name such that the root of S forces that $\dot h_a$ evaluates to the $\prec$-minimal 1-to-1 function (in V) that induces $\dot\Phi$ on a. Also, for each countable $M \prec H(\omega_3)$ such that $\{S, \dot\Phi, \mathcal{I}, \prec\}$ is in M, let a_M denote the $\prec$-least element of $\mathcal{I}$ satisfying that every element of $M \cap \mathcal{I}$ is mod finite contained in a_M.

Now we define our poset $\mathbb{P}$ for applying PFA(S). A condition $p \in \mathbb{P}$ will be a tuple $(\mathcal{M}_p, C_p, \{s^p_\delta, c^p_\delta : \delta \in C_p\})$ where

(1) $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $(H(\omega_3), \in, S, \dot\Phi, \mathcal{I}, \prec)$,
(2) $C_p = \{M \cap \omega_1 : M \in \mathcal{M}_p\}$,
(3) we use $\{M^p_\delta : \delta \in C_p\}$ to enumerate $\mathcal{M}_p$ in increasing order,
(4) we use a^p_δ to denote $a_{M^p_\delta}$,
(5) $s^p_\delta \in S$ is not in M^p_δ and forces a value h^p_δ on $\dot h_{a^p_\delta}$,
(6) if $\beta < \delta$ are in C_p, then $s^p_\beta \in M^p_\delta$,
(7) c^p_δ is a finite subset of a^p_δ,
(8) $c^p_\delta \cap a^p_\beta = c^p_\beta \cap a^p_\delta$ for $\beta, \delta \in C_p$,
(9) we let L_p denote the maximum element of $\bigcup\{c^p_\delta : \delta \in C_p\}$,
(10) for $\beta < \delta$ are both in C_p such that $s^p_\beta < s^p_\delta$, there are $m_\beta \in a_\beta \cap L_p$, and $m_\delta \in a_\delta \cap L_p$ such that $h^p_\beta(m_\beta) = h^p_\delta(m_\delta)$ and $(c^p_\beta \cup c^p_\delta) \cap \{m_\beta, m_\delta\}$ is a singleton.

We define $p < q$ providing $\mathcal{M}_p \supset \mathcal{M}_q$, $s^p_\delta = s^q_\delta$ and $c^p_\delta \cap L_q = c^q_\delta$ for $\delta \in C_q$.

Suppose that G is a filter of conditions of $\mathbb{P}$ satisfying that $C_G = \bigcup\{C_p : p \in G\}$ is uncountable. For each $\delta \in C_p$, let $c_\delta = \bigcup\{c^p_\delta : \delta \in C_p\}$. Similarly, for each $\delta \in C$, let a_δ, h_δ be the unique pair such that $a_\delta = a^p_\delta$ and $h_\delta = h^p_\delta$ for some $p \in G$. The family $\{s^p_\delta : p \in G, \delta \in C_p\}$ is an uncountable subset of S, so there is a generic branch g such that $E = \{\delta \in C : s^p_\delta \in g\}$ is uncountable. Let $Y = \bigcup\{c_\delta : \delta \in E\}$ and notice that $Y \cap a_\delta = c_\delta$ for all $\delta \in E$. The contradiction is that there is no possible value for $\dot\Phi(Y)$ because condition (10) of the definition of $\mathbb{P}$ ensures that the collection $\{(a_\delta \setminus h_\delta(c_\delta), h_\delta(c_\delta)) : \delta \in E\}$ is an unsplittable gap.

Now we prove a general fact to assist with the proof that $\mathbb{P} \times S$ is proper.

Fact 1. Suppose that Φ is a non-trivial automorphism of $\mathcal{P}(\mathbb{N})/\operatorname{fin}$ and $\mathcal{I}$ is a dense P-ideal contained in $\mathsf{Triv}(\Phi)$. Suppose also that $\mathcal{H} = \{h_a : a \in \mathcal{I}\}$ is a fixed assignment of 1-to-1 functions where h_a induces Φ on a for each $a \in \mathcal{I}$. If $\mathcal{H} \in M$ for an elementary submodel M of $H(\theta)$ for a sufficiently large θ and $E \in M$ is a cofinal subset of $\mathcal{I}$ and $a \in \mathcal{I}$ contains, mod finite, every member of $E \cap M$, then for any integer L, there is an $e \in E \cap M$, and a distict pair $m_1 \in a$, $m_2 \in e$ such that $h_a(m_1) = h_e(m_2) > L$.

Proof of Fact 1. Set $R = \bigcup\{h_e : e \in E\}$ and note that $R \in M$. Let $J = \{j \in \mathbb{N} : |R \cap (\mathbb{N} \times \{j\})| = 1\}$, which is also in M. Let

$h_R = R \cap (\mathbb{N} \times J)$ and note that h_R is a 1-to-1 function in M. Since Φ is not trivial, h_R does not induce Φ. By Lemma 2.3, there is an infinite $c \subset \mathbb{N}$ in M such that $(h_R(c)/\text{fin}) \wedge \Phi(c) = 0$ (it may be that $c \cap \text{dom}(h_R) = \emptyset$). Since $\mathcal{I}$ is dense and E is cofinal in $\mathcal{I}$, there is an $e \in E \cap M$ such that $c \subset^* e$. Note that $h_e(c) \cap h_R(c)$ is finite. Since a mod finite contains e, h_a mod finite contains h_e. Choose $m_1 \in a \cap c$ such that $L < n = h_a(m_1) = h_e(m_1) \notin h_R(c)$. Since $n \neq h_R(c)$ and $(m_1, n) \in R$, we have that $n \notin J$. Choose $m_2 \neq m_1$ so that $(m_2, n) \in R$. Also choose $e_2 \in E \cap M$ so that $h_{e_2}(m_2) = n$. This completes the proof of the Fact.

Now we prove that $\mathbb{P} \times S$ is proper. Let θ be a sufficiently large regular cardinal and let $\mathbb{P} \times S$ be an element of a countable elementary submodel M of $H(\theta)$. We may assume also that $\dot{\Phi}$ and the well-ordering $\prec$ are elements of M. It suffices to prove that any condition $(p, s) \in \mathbb{P} \times S$ satisfying that $M \cap H(\omega_3) = M_0 \in \mathcal{M}_p$ is an $(M, \mathbb{P} \times S)$-generic condition. Choose any dense open set $D \in M$ (of $\mathbb{P} \times S$) and suppose that some (p, s) is D and that $M_0 \in \mathcal{M}_p$. There is no loss to assume that s satisfies that there is some elementary submodel M' of $H(\theta)$ such that $p \in M'$ and $s \notin M'$.

Let $\delta_0 = M \cap \omega_1$, and let $C_p \setminus M = \{\delta_0, \delta_1, \ldots, \delta_{m-1}\}$ be listed in increasing order. Next let $\{s_0, \ldots, s_{n-1}\}$ be a listing of $\{s^p_i \restriction \delta_0 : i < m-1\}$ so that $s_0 = s \restriction \delta_0$. Let σ be the map from m to n such that $s_{\sigma(\ell)} < s_\ell$ for $\ell < m$.

Choose an $\alpha \in M$ large enough so that $C_p \cap \delta_0 \subset \alpha$ and, since S is coherent, $s^p_i(\xi) = s^p_0(\xi)$ for all $i < n-1$ and all $\alpha < \xi < \delta_0$. Now let, for $i < n$, $\bar{s}_i = s_i \restriction \alpha$. Let $I = \{\ell < m-1 : \bar{s}_0 \oplus s^p_{\delta_\ell} = s_0 \oplus s^p_{\delta_\ell} \subset s\}$.

Now we can select a promising subset D_1 of D. Let $(r, s') \in D_1$ providing the following properties of (p, s) are shared by (r, s'):

(1) $(r, s') \in D$ and $C_p \cap \alpha$ is an initial segment of C_r,
(2) $M^r_\beta = M^p_\beta$ for $\beta \in C_p \cap \alpha$,
(3) $C_r \setminus \alpha$ equals $\{\beta^r_\ell : \ell < m\}$ listed in increasing order,
(4) $s' \restriction \alpha = \bar{s}_0$, and for each $\ell < m$, $\bar{s}_{\sigma(\ell)} \oplus s^r_{\beta_0} \subset s^r_{\beta_\ell}$,
(5) $I = \{\ell < m-1 : \bar{s}_0 \oplus s^r_{\beta_\ell} \subset s'\}$,
(6) for each $\beta \in C_r \cap \alpha$, $c^r_\beta = c^p_\beta$,
(7) for each $\ell < m$, $c^r_{\beta_\ell} = c^p_{\delta_\ell}$.

Assume we are able to find $(r, s') \in D_1$ such that $s' < s$ and such that there is a sequence of pairs $\{\{m^\ell_1, m^\ell_2\} : \ell \in I\}$ such that for each $\ell \in I$

(1) $L_p = L_r < \min\{m^\ell_1, m^\ell_2\}$,
(2) $\max\{m^\ell_1, m^\ell_2\} < \min\{m^{\ell'}_1, m^{\ell'}_2\}$ for $\ell < \ell' \in I$

(3) $m_1^\ell \in a^r_{\beta_\ell}$ and $m_2^\ell \in a^p_{\delta_\ell}$, and $h^r_{\beta_\ell}(m_1^\ell) = h^p_{\delta_\ell}(m_2^\ell)$,

then we have that (r, s') is compatible with (p, s). Define the (potential) condition q where $\mathcal{M}_q = \mathcal{M}_r \cup \mathcal{M}_p$, $\{s^q_\beta : \beta \in C_q\} = \{s^p_\delta : \delta \in C_p\} \cup \{s^r_\beta : \beta \in C_r\}$, $\{c^q_\gamma : \gamma \in C_p \cap \alpha\} = \{c^p_\gamma : \gamma \in C_p \cap \alpha\}$, and, for $\beta \in C_q \setminus \alpha$ we define c^q_β to be $c^r_\beta \cup (a^r_\beta \cap \{m_1^\ell : \ell \in I\})$ if $\beta \in C_r$ and similarly, to be $c^p_\delta \cup (a^p_\delta \cap \{m_1^\ell : \ell \in I\})$ if $\beta \in C_p$. If we show that $q \in \mathbb{P}$, then it is immediate that (q, s) is below each of (r, s') and (p, s). The only non-trivial detail of q being in $\mathbb{P}$ are showing that conditions (8) and (10) of the definition hold. Condition (8) is follows easily from the facts that (r, s') being in D_1 ensures that r is isomorphic to p and from the uniform definition of c^q_β for each $\beta \in C_q$. Similarly, for $\beta < \delta$ as in condition (10), the only case that needs checking is when there is an $\ell < m$ such that $\beta = \beta^r_\ell$ and $\delta = \delta_\ell$. So assume that $\ell < m$ and that $s^r_{\beta_\ell} < s^p_{\delta_\ell}$. It suffices to show that $\ell \in I$ because then the pair $\{m_1^\ell, m_2^\ell\}$ serves as the required pair in (10). We have that $s' < s$ and that $\bar{s}_{\sigma(\ell)} \oplus s^r_{\beta_\ell} = s^r_{\beta_\ell} < s^p_{\delta_\ell}$ and so $s^r_{\beta_\ell} < s_{\sigma(\ell)}$. But now, $\bar{s}_0 \oplus s^r_{\beta_\ell} < s_0$ and so $\bar{s}_0 \oplus s^r_{\beta_\ell} < s'$. By the isomorphism condition between r and p, this implies that $\bar{s}_0 \oplus s^p_{\beta_\ell} < s$, which is the condition that $\ell \in I$.

Now we prove that we can find such an $(r, s') \in D_1$ and required sequence $\{\{m_1^\ell, m_2^\ell\} : \ell \in I\}$. We start by noting that D_1 is an element of M. This means that the set

$$E = \{(\{a^r_{\beta^r_\ell} : \ell \in I\}, s') : (r, s') \in D_1\}$$

is also in $M \cap H(\omega_3) = M^p_{\delta_0}$. We will treat E as an S-name a family of I-tuples from $\mathcal{I}$ (in this proof we have the advantage that membership in $\mathcal{I}$ does not depend on the generic). Let g be any generic branch of S such that $s \in g$. We will use that $M[g]$ is an elementary submodel of the $H(\theta)$ in the forcing extension, and similarly, for $\ell \in I$, $M^p_{\delta_\ell}[g]$ is an elementary submodel of $H(\omega_3)$ in the forcing extension.

Let

$$E[g] = \{\ \{e_\ell : \ell \in I\} :\ (\{e_\ell : \ell \in I\}, s') \in E \text{ and } s' \in g\}$$

and we now go through the standard argument that $E[g]$ has a "large branching" subset. By default, members of $E[g]$ will be ordered by mod finite inclusion. We recursively define a sequence $\{E_\ell : \ell \in I\}$ (proceeding in descending order on I) so that $E_\ell \subset E_{\ell'} \subset E[g]$ for $\ell < \ell'$ in I. For $\ell = \max(I)$, let $E^+_\ell = E[g]$, and having defined $E_{\ell'}$ for $\ell < \ell' \in I$, let $E^+_\ell = E_{\ell'}$ where ℓ' is the minimal element of I that is larger than ℓ. For any ℓ and $\{e_k : k \in I\} \in E[g]$, we let $E^+_\ell\langle\{e_k : k \in I \cap \ell\}\rangle$ denote the set of e such that there is a sequence

$\{e'_k : k \in I\} \in E_\ell^+$ extending $\{e_k : k \in I \cap \ell\}$ such that $e'_k = e$. The definition of E_ℓ is simply that $\{e_k : k \in I\} \in E_\ell$ providing $\{e_k : k \in I\}$ is in E_ℓ^+ and $E_\ell^+\langle\{e_k : k \in I \cap \ell\}\rangle$ is a cofinal subset of $\mathcal{I}$.

Now $E[g]$ is in $M^p_{\delta_0}[g]$, and so, if $\ell = \max(I)$, $E_\ell^+(\langle\{a^p_{\delta_k} : k \in I \cap \ell\}\rangle)$ is an element of $M^p_{\delta_\ell}$. Since $a^p_{\delta_\ell}$ is in $E_\ell^+(\langle\{a^p_{\delta_k} : k \in I \cap \ell\}\rangle)$ and contains, mod finite, every member of $\mathcal{I} \cap M^p_{\delta_\ell}[g] = \mathcal{I} \cap M^p_{\delta_\ell}$, it follows that $\{a^p_{\delta_k} : k \in I\})$ is in E_ℓ. By the same reasoning, we have that $\{a^p_{\delta_k} : k \in I\})$ is in E_ℓ where $\ell = \min(I)$. Certainly each E_ℓ is in $M[g]$, and so $E_{\min(I)} \cap M[g]$ is not empty.

We are now ready to recursively choose a sequence $\{\{e^\ell_k : k \in I\} : \ell \in I\}\} \subset E_{\min(I)} \cap M$ so that for each $\ell < \ell'$ from I, $e^{\ell'}_\ell = e^\ell_\ell$. We let $h_{e^\ell_\ell}$ denote the $\prec$-minimal 1-to-1 function that is forced by $\bar{s}_{\sigma(\ell)} \oplus s$ to induce $\dot{\Phi}$ on e^ℓ_ℓ. Let us note that if $(r, s') \in D_1$ and $a^r_{\beta^r_\ell} = e^\ell_\ell$, then $h^r_{\beta^r_\ell}$ will be equal to $h_{e^\ell_\ell}$. When we choose e^ℓ_ℓ we must ensure that there is a pair $\{m^\ell_1, m^\ell_2\}$ so that $m^\ell_1 \in e^\ell_\ell$, $m^\ell_2 \in a^p_{\delta_\ell}$, the maximum of $L_p \cup \{m^{\ell'}_1, m^{\ell'}_2 : \ell' \in I \cap \ell\}$ is less than each of $\{m^\ell_1, m^\ell_2\}$ and so that $h_{e^\ell_\ell}(m^\ell_1) = h_{a^p_{\delta_\ell}}(m^\ell_2)$.

Recall that, since S is coherent, the forcing extension $V[s \oplus g]$ is equal to $V[g]$ for all $s \in S$. Similarly, for each $\ell \in I$, $M[\bar{s}_{\sigma(\ell)} \oplus g]$ is equal to $M[g]$ and so is an elementary submodel of $H(\theta)$ in $V[\bar{s}_{\sigma(\ell)}] = V[g]$. Suppose now we have chosen $\{e^{\ell'}_k : k \in I\} \in E_{\min(L)} \cap M[g]$ for $\ell' < \ell$ in I. Let L be the maximum of $L_p \cup \{m^{\ell'}_1, m^{\ell'}_2 : \ell' \in I \cap \ell\}$. We will apply Fact 1 to find the required $\{e^\ell_k : k \in \ell\} \in E_{\min(I)} \cap M$ and pair $\{m^\ell_1, m^\ell_2\}$. We have that $E_{\min(L)}(\langle\{e^{\ell'}_k : k \in I \cap \ell\}\rangle)$ is cofinal in $\mathcal{I}$. We also have that $\bar{s}_{\sigma(\ell)} \oplus s^p_{\delta_\ell}$ is in the generic $\bar{s}_{\sigma(\ell)} \oplus g$ because $\ell \in I$. Therefore applying Fact 1 with $a = a^p_{\delta_\ell}$ and similarly h_a, we can choose $e^\ell_\ell \in E_{\min(L)}(\langle\{e^{\ell'}_k : k \in I \cap \ell\}\rangle) \cap M$ (and any witness $\{e^\ell_k : k \in I\} \in M[g] \cap E_{\min(I)})$ so that there is a required pair $\{m^\ell_1, m^\ell_2\}$.

This completes the proof. □

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