

# NOT OCA AND PRODUCTS OF FRÉCHET SPACES

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ABSTRACT. We continue the investigation of the question of whether the product of two countable Fréchet spaces must be M-separable. We are especially interested in this question in the presence of Martin's Axiom. The question has been shown to be independent of Martin's Axiom but only in models in which  $\mathfrak{c} \leq \omega_2$ . In fact, OCA implies an affirmative answer.

## 1. INTRODUCTION

A space is Fréchet (or Fréchet-Urysohn) if it satisfies that if a point  $x$  is in the closure of a set  $A$ , there is a standard  $\omega$ -sequence of points from  $A$  converging to  $x$ . A space  $X$  is M-separable, also known as selectively separable, if for every countable family  $\{D_n : n \in \omega\}$  of dense subsets, there is selection  $\{H_n \in [D_n]^{<\aleph_0} : n \in \omega\}$  satisfying that  $H = \bigcup_n H_n$  is dense. Every separable Fréchet space is M-separable [4]. A product of two separable Fréchet spaces need not be Fréchet, but might the product still be M-separable?

There are two main results of this paper. The first is that in models of Martin's Axiom in which there are special  $(\mathfrak{c}, \mathfrak{c})$ -gaps, there will be pairs of countable Fréchet spaces with a product that is not M-separable. The second is that in standard models of MA( $\sigma$ -linked), it will hold that the product of any two countable Fréchet spaces will be M-separable. In both results there is no (new) restriction on the size of the continuum. MA( $\sigma$ -linked) is the statement (see [2]) that Martin's Axiom holds for ccc posets that can be expressed as a countable union of linked subsets.

Any countable space with  $\pi$ -weight less than  $\mathfrak{d}$  is M-separable and, in the Cohen model every countable Fréchet space has  $\pi$ -weight at most  $\aleph_1$  [5]. Therefore we are more interested in the question in models in which there are countable Fréchet spaces with  $\pi$ -weight at least  $\mathfrak{d}$ . In fact in this paper we focus on models in which  $\mathfrak{b} = \mathfrak{c}$ . It is interesting that it was shown in [17] that  $\mathfrak{b} = \mathfrak{d}$  implies there are countable M-separable spaces whose product is not M-separable, but the status of this statement in ZFC is very much open. The cardinals  $\mathfrak{p}$ ,  $\mathfrak{b}$ , and  $\mathfrak{d}$  are the usual cardinal invariants corresponding to mod finite orderings on subsets of  $\omega$  known as the pseudointersection number, the bounding number, and the dominating number.

Back to the product of countable Fréchet spaces in models of  $\mathfrak{b} = \mathfrak{c}$ . The known results seem to point to a close connection to the open coloring axiom and gaps. It was shown in [3] that in a model of Martin's Axiom plus  $\mathfrak{c} = \omega_2$  in which there was a strong failure of OCA, this strong failure of OCA was crucial to the construction of two countable Fréchet spaces whose product was not M-separable. Improving on the PFA result in [5], it was shown in [8] that the version of OCA from [21], which was shown to imply  $\mathfrak{b} = \omega_2$ , implies that the product of two countable Fréchet

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2020 *Mathematics Subject Classification.* 54A35 (Primary), 03E50, 03E35, 54D55.  
*Key words and phrases.* gaps, Martin's Axiom, Fréchet-Urysohn.

spaces is necessarily M-separable. The key step in this proof was that this version of OCA implies that most unseparated pairs of orthogonal ideals on  $\omega$  will contain a Luzin gap (see Definition 2.3). Two families of subsets of a countable set are said to be orthogonal if each member of the first family has finite intersection with each member of the second. The other well-known version of OCA, denoted  $\text{OCA}_{[\text{ARS}]}$  in [15], is from [1]. On the other hand, adding to the seeming OCA connection, it was also shown in [8] that Martin's Axiom plus not CH implies there are three countable Fréchet spaces whose product is not M-separable **because**, by Avilés-Todorčević [2], in such models there are no three dimensional analogues of the above mentioned Luzin gaps.

We note that a space  $X$  is M-separable if every countable filter base  $\mathcal{D}$  consisting of dense subsets of a space  $X$  has a pseudointersection that is dense. Since M stands for Menger, one could define the Menger degree of a space  $X$ ,  $\mathfrak{p}\text{-M}(X)$ , to be the minimum cardinality of a filter base of dense subsets of  $X$  that has no dense pseudointersection. A natural family to consider is those countable Fréchet spaces with Menger degree equal to the pseudointersection number  $\mathfrak{p}$ . We did begin work on this paper by considering whether this smaller family of spaces may have better behavior in products but could find no results. We leave this remark here as a simple suggestion for further research.

Motivated by the paper [10], we had hoped to completely solve this question in this paper, but it remains open.

*Question 1.* Does  $\text{MA} + \mathfrak{c} > \omega_2$  imply there are two countable Fréchet spaces whose product is not M-separable?

## 2. A FEW COMBINATORIAL TOOLS

The following observation, a strengthening of Arhangel'skii's  $\alpha_1$ -property for first countable spaces, has proven useful in a number of papers. This statement and proof is taken right from [8] and is included for completeness.

**Proposition 2.1.** *Let  $\mathcal{I}$  be a family of sequences in a countable space  $X$  all converging to a single point  $x$  that has countable character. If  $\mathcal{I}$  has cardinality less than  $\mathfrak{b}$ , then there is a single sequence  $S$  converging to  $x$  that mod finite contains every member of  $\mathcal{I}$ .*

*Proof.* Fix a descending neighborhood basis,  $\{U_n : n \in \omega\}$ , for  $x$  with  $U_0 = X$ . For each  $n \in \omega$ , let  $X_n = U_n \setminus U_{n+1}$ . There is nothing to prove if  $x$  has a neighborhood that is simply a converging sequence, so we may assume that each  $X_n$  is infinite. For each  $n \in \omega$ , choose an enumeration,  $\{x(n, m) : m \in \omega\}$  of  $X_n$ . For each  $I \in \mathcal{I}$ , there is a function  $f_I \in \omega^\omega$  satisfying that  $I \subset \bigcup_n \{x(n, m) : m < f_I(n)\}$ . Therefore, if  $|\mathcal{I}| < \mathfrak{b}$ , we may choose a function  $f \in \omega^\omega$  so that  $f$  is eventually larger than each  $f_I$ . It is easy to check that  $S = \bigcup_n \{x(n, m) : m < f(n)\}$  is a sequence that converges to  $x$  and which satisfies that  $I \setminus S$  is finite for all  $I \in \mathcal{I}$ .  $\square$

For a set  $A$  in a space  $X$ , we let  $A^{(1)}$  be the set of points  $x$  of  $X$  for which there is a countable, possibly constant, sequence from  $A$  converging to  $x$ .

**Proposition 2.2** ([11]). *If a space  $X$  has character less than  $\mathfrak{b}$ , then for every set  $A \subset X$ , the set  $(A^{(1)})^{(1)} = A^{(1)}$ .*

The next two items are taken from [21, §8] and [12, Theorem 2.2.1]

**Definition 2.3.** A family  $\{(I_\alpha, J_\alpha) : \alpha < \omega_1\}$  is a Luzin gap if  $\bigcup\{I_\alpha \cup J_\alpha : \alpha \in \omega_1\}$  is countable, , for each  $\alpha \neq \beta$ ,  $I_\alpha \cap J_\alpha$  is empty,  $I_\alpha \cap J_\beta$  is finite, and  $(I_\alpha \cap J_\beta) \cup (I_\beta \cap J_\alpha)$  is not empty.

**Proposition 2.4.** *If  $\{(I_\alpha, J_\alpha) : \alpha < \omega_1\}$  is a Luzin gap, then the family  $\{I_\alpha : \alpha \in \omega_1\}$  can not be mod finite separated, or split, from the family  $\{J_\alpha : \alpha \in \omega_1\}$ .*

For completeness, we include the easy proof.

*Proof.* Fix an enumeration  $e : \omega \rightarrow \bigcup\{I_\alpha \cup J_\alpha : \alpha \in \omega_1\}$ . Suppose that  $A$  is a set satisfying that each of  $I_\alpha \setminus A$  and  $J_\alpha \cap A$  are finite for all  $\alpha \in \omega_1$ . Choose a finite subset  $F$  of  $\omega$  so that there is an uncountable subset  $\Gamma$  of  $\omega_1$  satisfying that, for all  $\alpha \in \Gamma$ ,  $I_\alpha \setminus A \subset e(F)$  and  $J_\alpha \cap A \subset e(F)$ . If necessary, shrink  $\Gamma$  further so that, for all  $\alpha, \beta \in \Gamma$ ,  $I_\alpha \cap e(F) = I_\beta \cap e(F)$  and  $J_\alpha \cap e(F) = J_\beta \cap e(F)$ . Note that for  $\alpha \neq \beta \in \Gamma$ ,  $I_\alpha \setminus e(F) \subset A$  and  $J_\beta \setminus e(F)$  is disjoint from  $A$ . Since, in addition,  $I_\alpha \cap e(F)$  is disjoint from  $J_\alpha \cap e(F) = J_\beta \cap e(F)$ , this contradicts that the family was Luzin.  $\square$

### 3. MA, A FAILURE OF OCA AND TWO FRÉCHET SPACES

**Definition 3.1.** Say that two ideals  $\mathcal{I}_1, \mathcal{I}_2$  form a tight  $\omega^\omega$ -gap if

- (1) every member of  $\mathcal{I}_1 \cup \mathcal{I}_2$  is a subset of  $\omega \times \omega$ ,
- (2) for each  $I \in \mathcal{I}_1 \cup \mathcal{I}_2$ ,  $I$  is a subset of  $f^\downarrow = \{(n, m) : m < f(n)\}$  for some  $f \in \omega^\omega$ ,
- (3)  $\mathcal{I}_1$  and  $\mathcal{I}_2$  are orthogonal,
- (4) for each  $f \in \omega^\omega$ , there are  $a_f \in \mathcal{I}_1$  and  $b_f \in \mathcal{I}_2$  such that  $f^\downarrow = a_f \cup b_f$ ,
- (5) for any  $X \subset \omega \times \omega$  such that  $\{n \in \omega : X \cap (\{n\} \times \omega) \text{ is infinite}\}$  is infinite, there is an  $f \in \omega^\omega$  such that each of  $X \cap a_f$  and  $X \cap b_f$  are infinite.

*Remark 3.1.* CH implies there are tight  $\omega^\omega$ -gaps. Todorčević [21] showed that OCA implies there is no tight  $\omega^\omega$ -gap. It is proven in [10] that it is consistent with Martin's Axiom and  $\mathfrak{c}$  arbitrarily large that there is no tight  $\omega^\omega$ -gap.

The proof of the following Lemma is technical and is simply following the methods of Laver [14], see also Rabus [16, Theorem 1] and Scheepers [19], showing that  $(\omega_1, \omega_1)$ -gaps that are added generically, can be split by a ccc poset. We postpone the proof to the last section.

**Lemma 3.2.** *For any cardinal  $\kappa > \omega_1$  such that  $\kappa^{<\kappa} = \kappa$ , there is a ccc poset  $P$  such that in the forcing extension by  $P$ , Martin's Axiom holds,  $\mathfrak{c} = \kappa$ , and there is a tight  $\omega^\omega$ -gap.*

**Theorem 3.3.** *If there is a tight  $\omega^\omega$ -gap and  $\mathfrak{b} = \mathfrak{c}$ , then there are two countable Fréchet spaces whose product is not  $M$ -separable.*

*Proof.* Assume that  $\mathfrak{b} = \mathfrak{c}$  and that  $\mathcal{I}_1, \mathcal{I}_2$  form a tight  $\omega^\omega$ -gap. Let  $\{f_\alpha : \alpha \in \mathfrak{c}\} \subset \omega^\omega$  be a standard scale in the sense that, each  $f_\alpha$  is a strictly increasing function,  $f_\alpha <^* f_\beta$  for any  $\alpha < \beta < \mathfrak{c}$ , and for all  $f \in \omega^\omega$  there is an  $\alpha < \mathfrak{c}$  such that  $f \leq f_\alpha$ .

For each  $\alpha < \mathfrak{c}$ , let  $a_\alpha \in \mathcal{I}_1$  and  $b_\alpha \in \mathcal{I}_2$  be disjoint sets such that  $a_\alpha \cup b_\alpha = f_\alpha^\downarrow$ .

We start our construction as in [3], and similar to [8]. Let  $\tau_0 = \sigma_0$  be any countable clopen base for a topology on  $\omega$  that is homeomorphic to the rationals. Fix a partition  $\{E_n : n \in \omega\}$  of  $\omega$  so that each  $E_n$  is  $\tau_0$  dense. Now choose any bijection  $\rho$  on  $\omega$  satisfying that, for each  $n \in \omega$ , the graph of  $\rho \upharpoonright E_n$ , namely

$D_n = \{(k, \rho(k)) : k \in E_n\}$  is a dense subset of  $\omega \times \omega$  with respect to the product topology  $\tau_0 \times \sigma_0$ . Observe that  $\rho[E_n]$  is dense with respect to  $\sigma_0$ . Let  $D = \bigcup_n E_n = \{(k, \rho(k)) : k \in \omega\}$ , i.e.  $D$  is the graph of  $\rho$ . We will let  $\pi_1$  denote the first coordinate projection on  $\omega \times \omega$  and  $\pi_2$  the second coordinate projection.

Choose any countable elementary submodel  $M_0$  of  $H(\mathfrak{c})$  such that each of  $\{\{E_n : n \in \omega\}, \rho, \tau_0\} \in M_0$ . This is simply a convenient way of choosing a good starting family of converging sequences from each of  $\tau_0$  and  $\sigma_0$ . Let  $\mathcal{I}_0$  be all sets  $I \in M_0$  such that, for some  $n \in \omega$ ,  $I$  is a subset of  $E_n$  and  $\tau_0$ -converges. Similarly let  $\mathcal{J}_0$  be all sets  $J \in M_0$  that, for some  $n \in \omega$ ,  $J$  is a subset of  $\rho[E_n]$  and  $\sigma_0$ -converges.

Let us note here that if  $\tau' \supset \tau_0$  and  $\sigma' \supset \sigma_0$  are larger bases for topologies that preserve that  $\mathcal{I}_0$  and  $\mathcal{J}_0$  respectively remain converging, then, for each  $n \in \omega$ ,  $D_n = \rho \upharpoonright E_n$  is dense in the product topology. To see this let  $U \in \tau'$  and  $W \in \sigma'$ . Choose any  $m \in U$  and  $k \in W$ . In  $M_0$  choose an infinite sequence  $S \subset D_n$  such that  $S$  converges to  $(m, k)$ . Therefore  $S = \{(i, \rho(i)) : i \in I\}$  for some  $I \in M_0$ . Note that  $I \in \mathcal{I}_0$  and converges to  $m$ . Similarly,  $\rho[I] = J$  is an element of  $\mathcal{J}_0$  and  $J$  converges to  $k$ . By assumption  $I$  is almost contained in  $U$  and  $J$  is almost contained in  $W$ . Of course this implies that  $U \times W$  almost contains  $S \subset D_n$ .

Choose a bijection  $\psi : \omega \times \omega \rightarrow \omega$  such that  $\psi(\{n\} \times \omega) = E_n$ . For each  $\alpha < \mathfrak{c}$ , let  $A_\alpha = \rho(\psi(a_\alpha))$  and  $B_\alpha = \rho(\psi(b_\alpha))$ . Now we have that the ideals generated by  $\{\rho[E_n] : n \in \omega\}$ ,  $\{A_\alpha : \alpha < \mathfrak{c}\}$ , and  $\{B_\alpha : \alpha < \mathfrak{c}\}$  are orthogonal ideals on  $D$  whose union is dense in  $D$ . If  $X$  is a subset of  $D$  that meets infinitely many of the elements of  $\{\rho[E_n] : n \in \omega\}$  in an infinite set, then there is an  $\alpha < \mathfrak{c}$  such that  $X \cap A_\alpha$  and  $X \cap B_\alpha$  are both infinite.

We will recursively construct increasing chains,  $\{\tau_\alpha : \alpha < \mathfrak{c}\}$  and  $\{\sigma_\alpha : \alpha < \mathfrak{c}\}$  of clopen bases of cardinality less than  $\mathfrak{c}$  for topologies on  $\omega$ . We will also, simultaneously choose increasing chains,  $\{\mathcal{I}_\alpha : \alpha < \mathfrak{c}\}$  and  $\{\mathcal{J}_\alpha : \alpha < \mathfrak{c}\}$ , of sequences that must converge in  $\tau_\beta$ , respectively  $\sigma_\beta$ , for all  $\beta < \mathfrak{c}$ . Naturally the purpose of choosing these chains of sets of converging sequences is to ensure that each of  $(\omega, \tau_\mathfrak{c})$  and  $(\omega, \sigma_\mathfrak{c})$  are Fréchet as witnessed by  $\mathcal{I}_\mathfrak{c}$  and  $\mathcal{J}_\mathfrak{c}$  respectively.

As mentioned above, with these inductive assumptions, we will have ensured that each member of the sequence  $\{D_n : n \in \omega\}$  remains  $\tau_\alpha \times \sigma_\alpha$ -dense for every  $\alpha$ . The next goal is to ensure that if  $H \subset D$  satisfies that  $H \cap D_n$  is finite for all  $n$ , then  $H$  is closed and discrete. This of course ensures that the product is not M-separable. Here is the plan for ensuring that such an  $H$  is closed and discrete. Notice that there will be an  $\alpha < \mathfrak{c}$  such that  $H \subset^* A_\alpha \cup B_\alpha$ . We will ensure that the first coordinate projection,  $\pi_1(H \cap B_\alpha)$ , is closed and discrete in  $(\omega, \tau_{\alpha+1})$  and that the second coordinate projection,  $\pi_2[H \cap A_\alpha]$ , is closed and discrete in  $(\omega, \sigma_{\alpha+1})$ . To ensure this is possible, we will necessarily also have the inductive hypotheses that if  $I \in \mathcal{I}_\alpha$  is almost disjoint from each  $E_n$ , then  $\rho \upharpoonright I$  is contained in some  $A_\gamma$ . Similarly, if  $J \in \mathcal{J}_\alpha$  and is almost disjoint from each  $\rho[E_n]$ , then  $\rho^{-1} \upharpoonright J = \{(\rho^{-1}(j), j) : j \in J\}$  is almost contained in some  $B_\gamma$ . These conditions are vacuously true for  $\mathcal{I}_0$  and  $\mathcal{J}_0$ .

Fix an enumeration  $\{X_\xi : \xi < \mathfrak{c}, \xi \text{ a limit}\}$  of the infinite subsets of  $\omega$ . Let  $0 < \lambda < \mathfrak{c}$  and assume that we have constructed the following increasing sets  $\{\tau_\alpha : \alpha < \lambda\}$ ,  $\{\sigma_\alpha : \alpha < \lambda\}$ ,  $\{\mathcal{I}_\alpha : \alpha < \lambda\}$ , and  $\{\mathcal{J}_\alpha : \alpha < \lambda\}$  satisfying the following inductive assumptions for all  $\beta < \alpha < \lambda$ :

- (1) every  $I \in \mathcal{I}_\alpha$  is a  $\tau_\alpha$ -converging,
- (2) every  $J \in \mathcal{J}_\alpha$  is a  $\sigma_\alpha$ -converging,

- (3) for each  $I \in \mathcal{I}_\alpha$ , the graph  $\rho \upharpoonright I$  is mod finite contained in  $D_n \cup A_\gamma$  for some  $n \in \omega$  and  $\gamma < \mathfrak{c}$ ,
- (4) for each  $J \in \mathcal{J}_\alpha$ , the set  $\rho^{-1} \upharpoonright J = \{(\rho^{-1}(j), j) : j \in J\}$  is mod finite contained in  $D_n \cup B_\gamma$  for some  $n \in \omega$  and  $\gamma < \mathfrak{c}$ ,
- (5) the set  $\pi_1[B_\beta]$  is closed and discrete with respect to  $\tau_\alpha$ ,
- (6) the set  $\pi_2[A_\beta]$  is closed and discrete with respect to  $\sigma_\alpha$ ,
- (7) if  $\beta$  is a limit and  $\beta + m < \alpha$ , then if  $m$  is a  $\tau_\alpha$ -limit point of  $X_\beta$ , there is an  $I \in \mathcal{I}_\alpha$  converging to  $m$  such that  $I \subset X_\beta$ ,
- (8) if  $\beta$  is a limit and  $\beta + k \leq \alpha$ , then if  $k$  a  $\sigma_\alpha$ -limit point of  $X_\beta$ , there is a  $J \in \mathcal{J}_\alpha$  converging to  $k$  such that  $J \subset X_\beta$ .

If  $\lambda$  is a limit ordinal, then the inductive hypotheses are satisfied by simply taking unions:  $\tau_\lambda = \bigcup_{\alpha < \lambda} \tau_\alpha$ ,  $\sigma_\lambda = \bigcup_{\alpha < \lambda} \sigma_\alpha$ ,  $\mathcal{I}_\lambda = \bigcup \{\mathcal{I}_\alpha : \alpha < \lambda\}$ , and  $\mathcal{J}_\lambda = \bigcup \{\mathcal{J}_\alpha : \alpha < \lambda\}$ .

Now suppose that  $\lambda = \alpha + 1$  and, if  $\omega \leq \alpha$ , let  $\beta$  be the largest limit below  $\lambda$  and let  $\beta + \bar{m} + 1 = \lambda$ . There are two tasks for each of  $\tau_\lambda$  and  $\sigma_\lambda$  to deal with  $X_\beta$ ,  $A_\alpha$  and  $B_\alpha$  as in items (5)-(8). These are done independently, but symmetrically, for  $\tau_\lambda, \mathcal{I}_\lambda$  and  $\sigma_\lambda, \mathcal{J}_\lambda$ , so we just provide the construction for  $\tau_\lambda$  and  $\mathcal{I}_\lambda$ .

Let us first consider the closure of  $X_\beta$  with respect to  $\tau_\alpha$ . For each  $m \in \omega$  for which there is a sequence  $I \subset X_\beta$  that converges to  $m$  and satisfies that  $\rho \upharpoonright I \subset E_n \cup A_\gamma$  for some  $n \in \omega$  and  $\gamma < \mathfrak{c}$ , ensure there is such an  $I \in \mathcal{I}_\lambda$ . Let  $X_\beta^{(1)\lambda}$  denote the set  $X_\beta$  together with all the points that are Fréchet limits with respect to  $\mathcal{I}_\lambda$ . Naturally this step is only required at stage  $\lambda = \beta + 1$ . Note that it follows from Proposition 2.2 that  $X_\beta^{(1)\lambda}$  is almost disjoint from every  $I \in \mathcal{I}_\lambda$  such that  $I$  converges to a point not in  $X_\beta^{(1)\lambda}$ .

Apply Lemma 2.1 to choose, for each  $m \in \omega$  a sequence  $S_m$  that  $\tau_0$  converges to  $m$  and satisfies that  $I \subset^* S_m$  for all  $I \in \mathcal{I}_\lambda$  that converge to  $m$ . Clearly the family  $\{S_m : m \in \omega\}$  is almost disjoint, and so by removing a finite set from each, we will assume they are pairwise disjoint. A second reduction is that we can replace each  $S_m$  by  $S_m \setminus \pi_1[B_\lambda]$  since, by our inductive assumptions, each  $I \in \mathcal{I}_\lambda$  is almost disjoint from  $\pi_1[B_\lambda]$ . Our third, and final reduction, is that for each  $m \notin X_\beta^{(1)\lambda}$ , we can assume that  $S_m \cap X_\beta^{(1)\lambda}$  is empty, but this needs a proof since we can not apply Proposition 2.2 directly because of the new restriction that we must respect the  $\omega^\omega$ -gap.

Assume that there is a sequence  $\{s_n : n \in \omega\} \subset X_\beta^{(1)\lambda}$  that converges to  $m$ . We prove that  $m$  is also in  $X_\beta^{(1)\lambda}$ . For each  $n \in \omega$ , fix a sequence  $I_n$  that converges to  $n$  and such that  $I_n \subset X_\beta$  and, by definition of  $X_\beta^{(1)\lambda}$ , either  $I_n \subset E_{k_n}$  (case 1) or  $I_n \subset \pi_1[A_{\gamma_n}]$  (case 2). By passing to a subsequence of  $\{s_n : n \in \omega\}$  we may assume that either, for all  $n$ ,  $I_n \subset E_{k_n}$  (case 1) or for all  $n$ ,  $I_n \subset \pi_1[B_{\gamma_n}]$  for some  $\gamma_n < \mathfrak{c}$  (case 2).

Since it is easier, we complete the proof for case 2 first. Choose any  $\gamma < \mathfrak{c}$  so that  $\gamma_n < \gamma$  for all  $n$ . By removing finite subset from each  $I_n$  we may assume that  $\bigcup_n I_n \subset \pi_1[A_\gamma]$ . Now the character of  $m$  in the subspace  $\{m\} \cup \{s_n : n \in \omega\} \cup \{I_n : n \in \omega\}$  is less than  $\mathfrak{b}$  and if we set  $A = \bigcup_n I_n$ , we can apply Proposition 2.2 to conclude there is a sequence  $I \subset \bigcup_n I_n$  that converges to  $m$ . This implies that  $m$  is in  $X_\beta^{(1)\lambda}$ .

Now we deal with case 1. If there is a  $k$  so that  $k_n = k$  for infinitely many  $n$ , then clearly, by Proposition 2.2, there is a sequence contained in  $X_\beta \cap E_k$  that converges to  $m$ , showing that  $m \in X_\beta^{(1)\lambda}$ . Finally, again by passing to a subsequence, we may assume that  $\{k_n : n \in \omega\}$  is strictly increasing. Let  $\{U_\xi : \xi < \lambda\}$  enumerate the neighborhood base at  $m$  with respect to the topology  $\tau_\alpha$ . For each  $\xi < \lambda$ , there is a function  $h_\xi \in \omega^\omega$  so that  $I_n \setminus h_\xi(n) \subset U_\xi$  for all but finitely many  $n \in \omega$ . Since  $\lambda < \mathfrak{b}$ , there is a  $\gamma < \mathfrak{c}$  such that, for all  $\xi < \lambda$ ,  $I_n \setminus f_\gamma(k_n)$  is a subset of  $U_\xi$  for all but finitely many  $n \in \omega$ . Let  $X = \bigcup_n \rho[I_n \setminus f_\gamma(k_n)]$  and note  $X$  is a subset that meets infinitely many of the elements of  $\{\rho[E_k] : k \in \omega\}$  in an infinite set. By the assumption on the  $\omega^\omega$ -gap, there is a  $\delta < \mathfrak{c}$  such that  $X \cap A_\delta$  is infinite. Since  $A_\delta \cap \rho[I_n]$  is finite for every  $n$ , it follows that  $X \cap A_\delta$  is mod finite contained in  $\rho[U_\xi]$  for every  $\xi < \lambda$ . Equivalently,  $\pi_1[X \cap A_\delta] \subset X_\beta$  is a sequence that converges to  $m$  with respect to  $\tau_\alpha$ , showing that  $m \in X_\beta^{(1)\lambda}$ .

Now we construct countably many new clopen sets to add to  $\tau_\lambda \supset \tau_\alpha$  by defining a function  $g : \omega \rightarrow \omega$  and adding  $g^{-1}(k)$  to  $\tau_\lambda$  for each  $k \in \omega$ . Then let  $\tau_\lambda$  be closed under finite intersections.

Let  $g_0$  be any 1-to-1 function from  $\bar{m} \cup \pi_1[B_\alpha]$  into  $\omega$ . For convenience choose  $g_0(\bar{m}) = 0$ . We define  $g$  as  $\bigcup g_n$  where for each  $n \in \omega$ ,  $\text{dom}(g_n)$  equals  $\{\bar{m}\} \cup \pi_1[B_\alpha] \cup n \cup \bigcup \{S_m : m < n\}$ . Note that with this assumption,  $\text{dom}(g_n)$  is almost disjoint from  $S_\ell$  for all  $\ell \geq n$ . Two additional inductive assumption are that for each  $m, j \in \text{dom}(g_n)$ ,

- (1) if  $m < n$ , then  $g(s) = g(m)$  for all but finitely many  $i \in S_m$ ,
- (2)  $j \in X_\beta^{(1)\lambda}$  then  $g_n(j) \neq 0 = g_n(\bar{m})$ .

The first inductive assumption on  $g$  ensures that every member of  $\mathcal{I}_\lambda$  will be  $\tau_\lambda$ -converging, and the second ensures that  $g^{-1}(0)$  is a  $\tau_\lambda$ -neighborhood of  $\bar{m}$  that is disjoint from  $X_\beta$ .

Our definition of  $g_0$  ensures that  $\pi_1[B_\alpha]$  is closed and discrete. Assume then that  $g_n$  has been defined and note that the inductive assumptions ensure that the range,  $R_n$ , of  $g_n \upharpoonright (\text{dom}(g_n) \setminus \pi_1[B_\alpha])$  is finite. If  $n$  is not in  $\text{dom}(g_n)$ , then define  $g_{n+1}(n)$  to be any value not in  $R_n$ . So long as  $g_{n+1}(n) \neq 0$ , then simply define  $g_{n+1}(i) = g_{n+1}(n)$  for all  $i \in S_n \setminus \text{dom}(g_n)$ . This choice of  $g_{n+1} \upharpoonright S_n$  preserves both the inductive hypotheses. If  $g_{n+1}(n) = 0$ , then it is because  $g_n(n) = 0$  and by the induction hypothesis,  $n \notin X_\beta^{(1)\lambda}$  and  $S_n$  is also disjoint from  $X_\beta^{(1)\lambda}$ . For these reasons, we may again define  $g_{n+1}(i) = 0$  for all  $i \in S_n \setminus \text{dom}(g_n)$  and preserve the induction hypotheses.

Let us verify that  $(\omega, \tau_\mathfrak{c})$  is Fréchet, where  $\tau_\mathfrak{c} = \bigcup_{\alpha < \mathfrak{c}} \tau_\alpha$ . Consider any  $\bar{m} \in \omega$  and subset  $X_\beta \subset \omega$  for some limit  $\beta \in \mathfrak{c}$ . If  $\bar{m}$  is in the set  $X_\beta^{(1)\beta+1}$ , then there is a sequence  $I \in \mathcal{I}_{\beta+1}$  that is contained in  $X_\beta$  and converges to  $\bar{m}$ . Otherwise, by induction hypothesis (7),  $\bar{m}$  is not in the  $\tau_{\beta+\omega}$ -closure of  $X_\beta$ .  $\square$

#### 4. PRODUCTS WHICH ARE M-SEPARABLE

In this section we prove that in standard models of weak forms of Martin's Axiom products of two countable Fréchet spaces are M-separable. It is known that this holds in models of OCA but we are interested in models in which  $\mathfrak{c}$  is larger than  $\omega_2$ . It is shown in [5] that this also holds in all standard Cohen real forcing extensions. We do not know if this can hold in models of Martin's Axiom with  $\mathfrak{c} > \omega_2$ , so we

make do with Martin's Axiom for  $\sigma$ -linked posets ([2]) as was used for products of three Fréchet spaces in [8].

**Theorem 4.1.** *Let  $\kappa > \omega_1$  satisfy that  $\kappa^{<\kappa} = \kappa$ . Then if  $P_\kappa$  is the standard finite support iteration of length  $\kappa$  consisting of factors that are names of  $\sigma$ -linked posets, then in the forcing extension any product of two countable Fréchet spaces is  $M$ -separable.*

The following is a direct consequence of [21, Theorem 4.4], see also [15, Lemma 1]. It is the key result behind the proof in [21, Theorem 8.0] that PFA implies OCA.

**Lemma 4.2 (CH).** *If  $X$  is a separable metric space and  $G \subset X^2 \setminus \Delta_X$  is a symmetric open relation on  $X$ , then either there is a countable cover,  $\mathcal{Y}$ , of  $X$  by sets  $Y \in \mathcal{Y}$  satisfying that  $Y^2 \cap G$  is empty, or the poset  $Q = \{F \in [X]^{<\aleph_0} : F^2 \setminus \Delta_F \subset G\}$  is ccc when ordered by reverse inclusion.*

Therefore, using the standard countably closed collapsing trick (see again [21, Theorem 8.0]) and the fact that the iteration of a countably closed poset and a ccc poset is proper we have the same result in a more convenient form.

**Corollary 4.3.** *If  $X$  is a separable metric space and  $G \subset X^2 \setminus \Delta_X$  is a symmetric open relation on  $X$  for which there is no countable cover,  $\mathcal{Y}$ , of  $X$  by sets  $Y \in \mathcal{Y}$  satisfying that  $Y^2 \cap G$  is empty, then there is a proper poset  $P$  that forces there to be an uncountable set  $Z \subset X$  satisfying that  $Z^2 \setminus \Delta_Z$  is a subset of  $G$ .*

Using this result and the method from [8] we have this next technical Lemma concerning products of Fréchet spaces. We can loosely view it as forcing the product to be  $M$ -separable.

**Theorem 4.4.** *Let  $(\omega, \tau)$  and  $(\omega, \sigma)$  be Fréchet spaces. Assume that  $\{D_n : n \in \omega\}$  are dense subsets of  $\omega \times \omega$  with respect to the product topology. Let  $(x, y) \in \omega \times \omega$  be arbitrary and let  $\mathcal{I}, \mathcal{J}$  be the family of all sequences that  $\tau$ -converge, respectively  $\sigma$ -converge, to  $x$  and  $y$  respectively.*

*If  $Q$  is any  $\sigma$ -linked poset that adds a dominating real  $f$ , then in the forcing extension by  $Q$ , if  $\hat{\tau}$  and  $\hat{\sigma}$  are topologies extending  $\tau$  and  $\sigma$  respectively satisfying that every member of  $\mathcal{I}$  and  $\mathcal{J}$  respectively remain as converging sequences, then  $(x, y)$  is in the closure of  $H_f = \bigcup \{D_n \cap ([0, f(n)] \times [0, f(n)]) : n \in \omega\}$  with respect to the product topology given by  $\hat{\tau}$  and  $\hat{\sigma}$ .*

*Proof.* Let  $x, y, \mathcal{I}$  and  $\mathcal{J}$  be as described in the statement of the Lemma. If  $x$  is isolated, then  $\{x\} \times (\omega, \sigma)$  is  $M$ -separable, and, in fact, there is a sequence  $S \subset \{x\} \times \omega$  converging to  $(x, y)$  satisfying that  $S$  is mod finite contained in  $\bigcup \{D_n : n > m\}$  for all  $m \in \omega$ . Therefore we assume that neither  $x$  nor  $y$  are isolated and we choose infinite sequences  $\langle x_n : n \in \omega \rangle$  converging to  $x$  and  $\langle y_n : n \in \omega \rangle$  converging to  $y$ . Fix pairwise disjoint families  $\{U_n : n \in \omega\}$  and  $\{W_n : n \in \omega\}$  of clopen sets in  $\tau$  respectively  $\sigma$  so that, for all  $n$ ,  $x_n \in U_n$  and  $y_n \in W_n$ .

Define the set  $D = \bigcup \{D_n \cap (U_n \times W_n) : n \in \omega\}$ . Clearly  $(x_n, y_n)$  is in the closure of  $D$  for all  $n \in \omega$ . Assume there is some  $I \in \mathcal{I}$  and  $J \in \mathcal{J}$  satisfying that  $(x, y)$  is in the closure of  $D \cap (I \times J)$ . Since  $\{(x, y)\} \cup (D \cap (I \times J))$  is a metric space, it is  $M$ -separable, so again, there is sequence  $S \subset D$  that converges to  $(x, y)$ . By the choice of  $D$ ,  $S \setminus \bigcup \{D_m : m \geq n\}$  is finite for all  $m$ .

So it remains to prove the Theorem in the case where  $(x, y)$  is not in the closure of  $D \cap (I \times J)$  for all  $I \in \mathcal{I}$  and  $J \in \mathcal{J}$ . Notice that this is equivalent to the case that for each such  $I, J$  pair, we can remove a finite set from each and have that  $D \cap (I \times J)$  is empty.

For each  $f \in \omega^\omega$ , let  $H_f = \bigcup \{D_n \cap (f(n) \times f(n)) : n \in \omega\}$  as in the statement of the Theorem. Let  $X$  be the following set

$$X = \{(I, J, f) : I \in \mathcal{I}, J \in \mathcal{J}, f \in \omega^\omega, D \cap (I \times J) = \emptyset\} .$$

We will identify  $(I, J, f)$  from  $X$  with the pair  $(I(f), J(f))$  where  $I(f) = H_f \cap (I \times \omega)$  and  $J(f) = H_f \cap (\omega \times J)$ . For  $(I, J, f) \in X$ ,  $I(f) \cap J(f) \subset D \cap (I \times J)$  and so is empty. We equip  $X$  with the standard topology on  $(\mathcal{P}(\omega \times \omega))^2$  through this identification. The standard subbasic clopen subsets of  $\mathcal{P}(\omega \times \omega)$  are sets of the form  $\{a \subset \omega \times \omega : (j, k) \in a\}$ .

Define the set  $G \subset X^2 \setminus \Delta_X$  by the relation that  $((I_1, J_1, f_1), (I_2, J_2, f_2))$  is in  $G$  providing

$$I_1(f_1) \cap J_2(f_2) \neq \emptyset \text{ or } I_2(f_2) \cap J_1(f_1) \neq \emptyset .$$

It is trivial that  $G$  is an open relation.

Suppose that  $\{Y_k : k \in \omega\}$  is a family of subsets of  $X$  satisfying that  $Y_k^2$  is disjoint from  $G$  for each  $k \in \omega$ . Assume towards a contradiction that  $\bigcup_k Y_k = X$ . Let  $X_0$  be the set of pairs  $(I, J) \in \mathcal{I} \times \mathcal{J}$  that satisfy that  $D \cap (I \times J)$  is empty. For each  $(I, J) \in X_0$ , since  $\{(I, J)\} \times \omega^\omega$  is a subset of  $\bigcup_k Y_k$ , there is a  $k(I, J) \in \omega$  (minimal to be definite) so that  $\{f \in \omega^\omega : (I, J, f) \in Y_k\}$  is  $<^*$ -cofinal in  $\omega^\omega$ . Let  $X_0[k] = \{(I, J) : k(I, J) = k\}$  and observe that, since  $Y_k^2 \setminus \Delta_X$  is disjoint from  $G$ , it follows that for  $(I_1, J_1), (I_2, J_2) \in X_0[k]$ ,  $(I_1 \cup I_2) \times (J_1 \cup J_2)$  is disjoint from  $D$ . For each  $k \in \omega$ , set

$$A_k = \bigcup \{I \in \mathcal{I} : (\exists J \in \mathcal{J}) (I, J) \in X_0[k]\}$$

and

$$B_k = \bigcup \{J \in \mathcal{J} : (\exists I \in \mathcal{I}) (I, J) \in X_0[k]\} .$$

Notice that  $A_k \times B_k$  is disjoint from  $D$  for all  $k \in \omega$ .

Fix any  $n \in \omega$  and set  $U_n^0 = U_n$  and  $W_n^0 = W_n$ . Clearly  $(x_n, y_n)$  is a limit point of the interior of  $\overline{D \cap (U_n^0 \times W_n^0)}$ . By recursion on  $k < n$ , we define  $U_n^{k+1}$  and  $W_n^{k+1}$  so that

- (1)  $(x_n, y_n)$  is a limit point of the interior of  $\overline{D \cap (U_n^{k+1} \times W_n^{k+1})}$ , and
- (2) either  $U_n^{k+1} = U_n^k \setminus A_k$ , or
- (3)  $W_n^{k+1} = W_n^k \setminus B_k$ .

Suppose we have chosen  $U_n^k$  and  $W_n^k$  and let  $S$  be the interior of  $\overline{D \cap (U_n^k \times W_n^k)}$ . By assumption  $(x_n, y_n)$  is a limit point of  $S$ . Work briefly in the subspace  $S \cup \{(x_n, y_n)\}$ . The sets  $S \cap (D \cap ((U_n^k \cap A_k) \times W_n^k))$  and  $S \cap (D \cap (U_n^k \times (W_n^k \cap B_k)))$  are disjoint and so one of their interiors will not be dense in a neighborhood of  $(x_n, y_n)$  (in the subspace  $S \cup \{(x_n, y_n)\}$ ). By symmetry assume this is so for  $S \cap (D \cap ((U_n^k \cap A_k) \times W_n^k))$ . Choose an open subset  $S_0$  of  $S$  so that  $S_0 \cup \{(x_n, y_n)\}$  is open in  $S \cup \{(x_n, y_n)\}$  and so that  $S_0 \cap (D \cap ((U_n^k \cap A_k) \times W_n^k))$  has empty interior. In this case we set  $U_n^{k+1} = U_n^k \setminus A_k$  and  $W_n^{k+1} = W_n^k$ , and notice that  $D \cap ((U_n^k \setminus A_k) \times W_n^k)$  is dense in  $S_0$ . It follows that  $S_0$  is contained in  $\overline{D \cap (U_n^{k+1} \times W_n^{k+1})}$ . Let  $F_n$  be the set of  $k < n$  such that  $U_n^k$  is disjoint from  $A_k$ . Notice that for  $k < n$  and  $k \notin F_n$ ,  $W_n^k$  is disjoint from  $B_k$ . Let  $s_n \in 2^n$  denote the characteristic function of  $F_n$ .

Let  $T \subset 2^{<\omega}$  denote the tree consisting of all  $\{s_n \upharpoonright k : k \leq n \in \omega\}$  and let  $s \in 2^\omega$  be a branch of  $T$ . For each  $m \in \omega$ , choose  $n_m \in \omega$  so that  $s_{n_m} \upharpoonright m = s \upharpoonright m$ . Without loss of generality we can assume that the sequence  $S = \{n_m : m \in \omega\}$  is strictly increasing. For each  $m \in \omega$ , choose a sequence  $I_{n_m} \subset U_{n_m}^m$  that converges to  $x_{n_m}$ . Since  $x$  is in the closure of  $\bigcup\{I_{n_m} : m \in \omega\}$ , we can choose a sequence  $I \subset \bigcup\{I_{n_m} : m \in \omega\}$  that converges to  $x$ . Let  $L = \{m \in \omega : I \cap I_{n_m} \neq \emptyset\}$ . For each  $m \in L$ , choose  $J_{n_m} \subset W_{n_m}^m$  that converges to  $y_{n_m}$  and choose a sequence  $J \subset \bigcup\{J_{n_m} : m \in L\}$  that converges to  $y$ . Let  $L_2 \subset L$  be the set of  $m$  such that  $J \cap J_{n_m}$  is not empty.

Let us check that, for each  $k \in \omega$ ,  $(I, J) \notin X_0[k]$ . Indeed, consider any  $k \in \omega$  and choose  $m \in L_2$  so that  $k < n_m$ . If  $k \in F_{n_m}$ , then  $(I, J) \notin X_0[k]$  because  $I \cap A_k$  is empty. If  $k \notin F_{n_m}$ , then  $(I, J) \notin X_0[k]$  because  $J \cap B_k$  is empty.

At this stage of the proof, we have established, by Lemma 4.3, that there is a proper poset  $P$  that forces there is a subset, which we will denote  $\{(I_\alpha, J_\alpha, f_\alpha) : \alpha \in \omega_1\} \subset X$ , satisfying that  $((I_\alpha, J_\alpha, f_\alpha), (I_\beta, J_\beta, f_\beta))$  is in  $G$  for all  $\alpha < \beta < \omega_1$ . Note that the family  $\{(I_\alpha(f_\alpha), J_\alpha(f_\alpha)) : \alpha < \omega_1\}$  is a Luzin gap (see Definition 2.3) and recall that a Luzin gap remains a Luzin gap in any forcing extension that preserves  $\omega_1$ .

Now consider a  $\sigma$ -linked poset  $Q$  that adds a dominating function  $f_Q$  over the ground model. Suppose that  $\mathcal{F}$  is a filter on  $Q$  that is generic over the ground model. Assume, towards a contradiction, that there is  $Q$ -name  $\dot{A}$  for a subset of  $D$  that satisfies, for any  $Q$ -generic filter  $\mathcal{F}$ ,  $I(f) \subset^* \text{val}_{\mathcal{F}}(\dot{A})$  and  $J(f) \cap \text{val}_{\mathcal{F}}(\dot{A})$  is finite for all  $(I, J, f) \in X$ . This property of  $\dot{A}$  will continue to hold in the forcing extension by  $P$  since all the relevant dense open subsets of  $Q$  will remain dense. In addition, forcing by  $P$  will preserve that  $Q$  is  $\sigma$ -linked, and so the iteration  $P * Q$  will also be proper, and therefore preserve  $\omega_1$ . However this is a contradiction, since the Luzin gap  $\{(I_\alpha(f_\alpha), J_\alpha(f_\alpha)) : \alpha \in \omega_1\}$  added by  $P$  can not be split by the valuation of the  $Q$ -name  $\dot{A}$ .

Now, with  $f_Q$  being the dominating real added by  $Q$ , consider the set  $H_Q = \bigcup\{D_n \cap (f_Q(n) \times f_Q(n)) : n \in \omega\}$  and note that, for all  $f \in \omega^\omega$  in the ground model,  $H_f \subset^* H_Q$ . Therefore, for all  $(I, J, f) \in X$ ,  $I(f) \cup J(f)$  is mod finite contained in  $H_Q$ .

Assume that  $\hat{\tau}$  and  $\hat{\sigma}$  are topologies as in the statement of the Theorem. Assume that  $x \in U \in \hat{\tau}$  and  $y \in W \in \hat{\sigma}$ . Let  $(I, J, f)$  be any element of  $X$ . Clearly  $A = H_Q \cap (U \times \omega)$  mod finite contains  $I(f)$  and  $B = H_Q \cap (\omega \times W)$  mod finite contains  $J(f)$ . Since  $A$  mod finite contains  $I(f)$  for all  $(I, J, f) \in X$ , it must meet  $J(f)$  in an infinite set for some  $(I, J, f) \in X$ , so fix such an element  $(I, J, f)$  of  $X$ . Since  $J(f)$  is mod finite contained in  $B$  and  $A \cap B = H_Q \cap (U \times W)$ , it follows that  $H_Q \cap (U \times W)$  is infinite.

This completes the proof.  $\square$

## 5. PROOF OF LEMMA 3.2

The goal of this section is to prove that if  $\kappa^{<\kappa} = \kappa$ , then there is a ccc poset  $P_\kappa$  of cardinality  $\kappa$  that produce a model of Martin's Axiom in which there is a tight  $\omega^\omega$ -gap. We regret that we have to prove this, but we are unable to find a suitable reference. For a partial or total function  $s$  from  $\omega$  to  $\omega$ , let  $s^\downarrow = \{(m, j) : m \in \text{dom}(s) \text{ and } j < s(m)\}$ . For functions  $f, g \in \omega^\omega$ , let  $f \vee g$  denote the function  $(f \vee g)(n) = \max(f(n), g(n))$  for all  $n \in \omega$ .

Suppose that  $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{b}\} \subset \omega^\omega$  is a mod finite increasing chain that is also dominating. Suppose that, for each  $\alpha < \mathfrak{b}$ ,  $h_\alpha$  is a 2-valued function with domain  $f_\alpha^\perp$ . Say that  $\mathcal{H}_\lambda = \{h_\alpha : \alpha < \lambda\}$  is coherent if, for all  $\beta < \alpha < \lambda$ , the set of  $(j, k) \in f_\alpha^\perp \cap f_\beta^\perp$  such that  $h_\alpha(j, k) \neq h_\beta(j, k)$  is finite. Clearly it is necessary to prove there is such a coherent family  $\mathcal{H}_\mathfrak{b}$  in the final model.

**Definition 5.1.** Say that  $\mathcal{H}_\lambda = \{h_\alpha : \alpha < \lambda\}$  is a linear coherent family if it is a coherent family of 2-valued functions, such that for each  $\alpha$ ,  $\text{dom}(h_\alpha) = f_\alpha^\perp$  for some  $f_\alpha \in \omega^\omega$ , and  $\{f_\alpha : \alpha < \lambda\}$  is  $\leq^*$ -increasing.

**Definition 5.2.** If  $\mathcal{H}_\lambda$  is a linear coherent family, then  $Q(\mathcal{H}_\lambda)$ , also denoted  $Q(\{h_\alpha : \alpha < \lambda\})$ , is the following poset. A condition  $q \in Q(\mathcal{H}_\lambda)$ , is a tuple  $(s_q, h_q, F_q, f_q)$  satisfying

- (1)  $s_q \in \omega^{<\omega}$  with domain  $n_q$  and  $f_q \in \omega^\omega$ ,
- (2)  $h_q$  is a 2-valued function with domain  $s_q^\perp$ ,
- (3)  $F_q$  is a finite subset of  $\lambda$ ,
- (4) for all  $\alpha \in F_q$  and  $\delta_q = \max(F_q)$ , and for all  $n_q \leq m$ ,  $f_\alpha(m) \leq f_{\delta_q}(m) \leq f_q(m)$ , and for all  $(m, j) \in \text{dom}(h_\alpha) \cap \text{dom}(h_{\delta_q})$ ,  $h_\alpha(m, j) = h_{\delta_q}(m, j)$ .

The ordering on  $Q(\mathcal{H}_\lambda)$  is that  $q \leq r$  providing  $s_q \supset r_q$ ,  $h_q \supset h_r$ ,  $F_q \supset F_r$ ,  $f_q \geq f_r$ , and for all  $(m, j) \in \text{dom}(h_q) \cap \text{dom}(h_{\delta_r})$  with  $n_r \leq m$ ,  $s_q(m, j) = h_{\delta_r}(m, j)$ .

We let  $\dot{f}$  and  $\dot{h}$  be the two canonical  $Q(\mathcal{H}_\lambda)$ -names, (and in context we would denote them as  $\dot{f}_\lambda$  and  $\dot{h}_\lambda$ ) where, if  $G$  is a  $Q(\mathcal{H}_\lambda)$ -generic filter,  $\text{val}_G(\dot{f}) = \bigcup\{s_q : q \in G\}$  and  $\text{val}_G(\dot{h}) = \bigcup\{h_q : q \in G\}$ .

It should be clear that  $\dot{f}$  is a dominating real added by  $Q(\mathcal{H}_\lambda)$  (even if  $\lambda = 0$ ) and that, for each  $\alpha \in \lambda$ , the set of  $n \in \omega$  such that there are  $f_\alpha(n) < j < k < f_\lambda(n)$  with  $h_\lambda(n, j) \neq h_\lambda(n, k)$  is cofinite. Also, by the next proposition,  $\{h_\alpha : \alpha < \lambda + 1\}$  is a linear coherent family extending  $\mathcal{H}_\lambda$ .

**Proposition 5.3.** For each  $\beta \in \lambda$ , the set  $D_\beta = \{q \in Q(\{h_\alpha : \alpha < \lambda\}) : \beta \in F_q\}$  is dense. Furthermore, if  $q \in Q(\{h_\alpha : \alpha < \lambda\})$  and  $h_{\delta_q} \cup h_\beta \upharpoonright ([n_q, \omega) \times \omega)$  is a function, then  $(s_q, h_q, F_q \cup \{\beta\}, f_q \vee f_\beta)$  is an extension of  $q$ .

**Lemma 5.4.** For each  $\delta \in \lambda$ , the subset  $S_\delta = \{q \in Q(\{h_\alpha : \alpha \in \lambda\}) : \delta_q = \delta\}$  is  $\sigma$ -centered.

*Proof.* If  $q, r \in S_\delta$  and  $h_q = h_r$ , then  $(s_q, h_q, F_q \cup F_r, f_q \vee f_r)$  is an extension of both  $q$  and  $r$  and is in  $S_\delta$ .  $\square$

**Corollary 5.5.** If  $\lambda$  has countable cofinality, then  $Q(\{h_\alpha : \alpha \in \lambda\})$  is ccc for every linear coherent sequence of length  $\lambda$ .

This next result is also a standard fact about gaps, but in a new setting.

**Corollary 5.6.** If  $\{h_\alpha : \alpha \leq \lambda\}$  is a linear coherent gap, then  $Q(\{h_\alpha : \alpha < \lambda\})$  is  $\sigma$ -centered.

*Proof.* By Lemmas 5.4 and 5.3,  $S_\lambda = \{q \in Q(\{h_\alpha : \alpha \leq \lambda\}) : \lambda \in F_q\}$  is dense and  $\sigma$ -centered in  $Q(\{h_\alpha : \alpha \leq \lambda\})$ . Also, the poset  $Q(\{h_\alpha : \alpha < \lambda\})$  is subset of  $Q(\{h_\alpha : \alpha \leq \lambda\})$  and therefore is also  $\sigma$ -centered.  $\square$

However, if  $\lambda$  has cofinality  $\omega_1$ ,  $Q(\mathcal{H}_\lambda)$  may not be ccc. This next well-known result is due to Kunen (see [19]) when applied to standard gaps.

**Lemma 5.7.** *For an uncountable linear coherent gap,  $\{h_\alpha : \alpha < \lambda\}$ , the poset  $Q(\{h_\alpha : \alpha < \lambda\})$  is ccc if and only if for every uncountable  $X \subset \lambda$ , there are  $\alpha < \beta$  in  $X$  such that  $h_\alpha \cup h_\beta$  is a function.*

*Proof.* Assume first that  $Q(\{h_\alpha : \alpha < \lambda\})$  is ccc and consider any uncountable  $X \subset \lambda$ . By passing to a subset we may assume that  $X$  has order-type  $\omega_1$ . Suppose first that  $X$  has an upper bound  $\mu < \lambda$ . Then for each  $\xi \in X$ , there is an  $n_\xi \in \omega$  such that  $h_\xi \upharpoonright ((n_\xi, \omega) \times \omega) \subset h_\mu$ . Choose  $\xi < \alpha$  both in  $X$  so that  $n = n_\xi = n_\alpha$  and  $h_\xi \upharpoonright n \times \omega$  and  $h_\alpha \upharpoonright n \times \omega$  (which are both finite) are equal. Then  $h_\xi \cup h_\alpha$  is a function.

Now suppose that  $\lambda$  has cofinality  $\omega_1$ . In this case we can force with  $Q(\{h_\alpha : \alpha < \lambda\})$ , thus preserving  $\omega_1$ , and repeat the argument in the previous paragraph using  $\lambda = \mu$ , i.e. the new function  $h_\lambda$  added by  $Q(\{h_\alpha : \alpha < \lambda\})$ .

Now we prove the other direction and assume that for every uncountable  $X \subset \lambda$ , there are distinct  $\alpha, \beta \in X$  satisfying that  $h_\alpha \cup h_\beta$  is a function. Let  $\{q_\xi : \xi \in \omega_1\}$  be any subset of  $Q(\{h_\alpha : \alpha \in \omega_1\})$ . By passing to a subcollection, we can assume that there is a pair  $s, h$  such that  $(s, h) = (s_{q_\xi}, h_{q_\xi})$  for all  $\xi \in \omega_1$ . Let  $X = \{\delta_{q_\xi} : \xi \in \omega_1\}$  and choose distinct  $\xi, \eta \in \omega_1$  so that  $h_{\delta_{q_\xi}} \cup h_{\delta_{q_\eta}}$  is a function. It is easy to check that  $(s, h, F_{q_\xi} \cup F_{q_\eta}, f_{q_\xi} \vee f_{q_\eta})$  is a common extension of  $q_\xi$  and  $q_\eta$ .  $\square$

The dominating real aspect of the linear coherent sequence poset introduces some complications when utilized in an iteration which we deal with by introducing an alternate, but equivalent, formulation of the poset. We will separate each of the components  $s$  and  $h$  into two pieces where one piece is not allowed to be a name. This is just to emphasize which portion has been forced to have a specific value (or *determined* as it is often called). We will abuse the standard notation  $\check{a}$  to mean that  $\check{a}$  is a finite ground model set.

**Definition 5.8.** For a poset  $P$  and  $P$ -names,  $\{\dot{f}_\alpha, \dot{h}_\alpha : \alpha < \lambda\}$ , that is forced to be a linear coherent sequence, we define the  $P$ -name  $\dot{Q}'(\{\dot{h}_\alpha : \alpha < \lambda\})$  as follows. A condition  $q \in \dot{Q}'(\{\dot{h}_\alpha : \alpha < \lambda\})$  is a tuple  $(\check{n}_q, \check{s}_q, \tau_q, \check{h}_q, \pi_q, \check{F}_q, \check{f}_q)$  where the following are forced by  $1_P$ :

- (1)  $n_q \in \omega$ ,  $s_q \in \omega^{n_q}$ ,  $s_q \leq \tau_q \in \omega^{n_q}$ ,
- (2)  $h_q$  is a 2-valued function with domain  $s_q^\downarrow$ ,
- (3)  $h_q \subset \pi_q$  is a 2-valued function with domain  $\tau_q^\downarrow$ ,
- (4)  $F_q$  is a finite subset of  $\lambda$ ,
- (5) for each  $\alpha \in F_q$ ,  $\dot{f}_\alpha \leq \dot{f}$ .

Say that a condition  $q \in \dot{Q}'(\{\dot{h}_\alpha : \alpha < \lambda\})$  is pure if  $\tau_q = \check{s}_q$  and  $\pi_q = \check{h}_q$ .

For each  $q \in \dot{Q}'(\{\dot{h}_\alpha : \alpha < \lambda\})$ , let  $\hat{q} = (\tau_q, \pi_q, F_q, \check{f}_q)$ . We note that  $\hat{q}$  is forced to be an element of  $Q(\{\dot{h}_\alpha : \alpha < \lambda\})$  and we define the ordering on  $\dot{Q}'(\{\dot{h}_\alpha : \alpha < \lambda\})$  by  $q_1 \leq q_2$  if  $\hat{q}_1 \leq \hat{q}_2$ .

Suppose that  $s \in \omega^n$  and  $\tau$  is a  $P$ -name for an element of  $\omega^n$  such that  $1 \Vdash s \leq \tau$ . For any 2-valued function  $h$  with domain  $s^\downarrow$ , let  $h \oplus 0_\tau$  denote the name of the function with domain  $\tau^\downarrow$  that extends  $h$  and has value 0 at all  $(m, j) \in \tau^\downarrow \setminus s^\downarrow$ .

**Lemma 5.9.** *Suppose that  $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$  is a finitely supported iteration, and  $\{\dot{f}_\alpha, \dot{h}_\alpha : \alpha < \lambda\}$  are  $P_\lambda$ -names satisfying*

- (1) for each  $\alpha < \lambda$ , each of  $\dot{f}_\alpha$  and  $\dot{h}_\alpha$  are  $P_{\alpha+1}$ -names satisfying that  $\dot{f}_\alpha$  is forced to be in  $\omega^\omega$  and  $\dot{h}_\alpha$  is a 2-valued function with domain  $\dot{f}_\alpha^\downarrow$ ,
- (2) for each  $\beta < \lambda$ ,  $\dot{Q}_\beta$  is a  $P_\beta$ -name of a ccc poset,
- (3) for each even ordinal  $\beta < \lambda$ , if  $P_\beta$  forces that  $\{\dot{h}_\alpha : \alpha < \beta\}$  is a linear coherent family, then  $\dot{Q}_\beta$  is the  $P_\beta$ -name  $\dot{Q}'(\{\dot{h}_\alpha : \alpha < \beta\})$ , and  $\dot{f}_\beta, \dot{h}_\beta$  are the canonical  $P_{\beta+1}$ -names associated with  $\dot{Q}_\beta$ , otherwise  $\dot{Q}_\beta = 2^{<\omega}$  and  $\dot{f}_\beta = \dot{f}_0$ , and  $\dot{h}_\beta = \dot{h}_0$ ,
- (4) if  $\beta = \alpha + 1 < \lambda$  is an odd ordinal, then  $\dot{f}_\beta = \dot{f}_\alpha$  and  $\dot{h}_\beta = \dot{h}_\alpha$ .

Then for each  $\beta \leq \lambda$ ,  $P_\beta$  forces that  $\{\dot{h}_\alpha : \alpha < \beta\}$  is a linear coherent family. Furthermore, if  $\beta$  has uncountable cofinality, then  $P_\beta$  forces that for every uncountable  $X \subset \beta$ , there are distinct  $\xi, \eta \in X$  such that  $\dot{h}_\xi \cup \dot{h}_\eta$  is a function.

*Proof.* It is immediate from the remarks immediately after Definition 5.2 that, for each  $\beta < \lambda$ ,  $P_\beta$  forces that  $\{\dot{h}_\alpha : \alpha < \beta\}$  is a linear coherent family. It, however, is not immediate that  $P_\beta$  is ccc. We prove the second stated conclusion of the Lemma by induction on  $\beta \leq \lambda$ . Since this conclusion is vacuous for  $\beta < \omega_1$ , we might as well simply assume that it holds for all  $\beta < \lambda$  and prove it for  $\lambda$ . It follows from Corollaries 5.6 and 5.7 that for uncountable  $X \subset \beta < \lambda$ , it remains true, i.e. not destroyed by further forcing, that there are  $\xi < \eta \in X$  such that  $h_\xi \cup h_\eta$  is a function. Needless to say, there is nothing to prove unless  $\lambda$  has cofinality  $\omega_1$ .

In preparation we make some observations, stated as Facts, about  $P_\lambda$ .

*Fact 1.* Let  $\tilde{P}_\lambda$  be the set of conditions that satisfy, for each even  $\beta \in \text{dom}(p)$  and each odd  $\alpha + 1 \in F_{p(\beta)}$ , we also have that  $\alpha$  is in  $F_{p(\beta)}$ . Then  $\tilde{P}_\lambda$  is a dense subset of  $P_\lambda$ .

*Fact 2.* Let  $p \in \tilde{P}_\lambda$  and suppose that  $\alpha < \beta$  are even ordinals in  $\text{dom}(p)$ . Then  $p$  will force that  $\dot{h}_\alpha \cup \dot{h}_\beta$  is a function if

- (1)  $\alpha \in F_{p(\beta)}$ ,
- (2)  $n_{p(\alpha)} = n_{p(\beta)}$ ,  $s_{p(\alpha)} = s_{p(\beta)}$ ,  $h_{p(\alpha)} = h_{p(\beta)}$ ,
- (3)  $\pi_{p(\alpha)} = h_{p(\alpha)} \oplus 0_{\tau_{p(\alpha)}}$  (as in Definition 5.8), and
- (4)  $\pi_{p(\beta)} = h_{p(\beta)} \oplus 0_{\tau_{p(\beta)}}$ .

*Fact 3.* Let  $p \in \tilde{P}_\lambda$  and suppose that  $\alpha < \beta$  are even ordinals in  $\text{dom}(p)$  and that  $\alpha \in F_{p(\beta)}$ . Then  $\bar{p}$  is an extension of  $p$  where  $\bar{p}(\gamma) = p(\gamma)$  for all  $\beta \neq \gamma \in \text{dom}(p)$  and  $\bar{p}(\beta)$  is equal to  $(n_{p(\beta)}, s_{p(\beta)}, \tau_{p(\beta)}, h_{p(\beta)}, \pi_{p(\beta)}, F_{p(\beta)} \cup F_{p(\alpha)}, \dot{f}_{p(\beta)})$ .

*Fact 4.* Suppose that  $p \in \tilde{P}_\lambda$  and that for even  $\alpha < \beta$  both in  $\text{dom}(p)$  we have the conditions (2)-(4) of Fact 2 holding and that  $\delta_{p(\alpha)}$  is an element of  $F_{p_\xi(\beta)}$ . Then the condition  $\bar{p} \leq p$  forces that  $\dot{h}_\alpha \cup \dot{h}_\beta$  is a function where  $\bar{p}$  is defined as follows:  $\bar{p}(\gamma) = p(\gamma)$  for all  $\beta \neq \gamma \in \text{dom}(p)$ , and

$$\bar{p}(\beta) = (n_{p(\beta)}, s_{p(\beta)}, \tau_{p(\beta)}, h_{p(\beta)}, h_{p(\beta)} \oplus 0_{\tau_{p(\beta)}}, F_{p(\beta)} \cup \{\alpha\}, \dot{f}_{p(\beta)} \vee \dot{f}_\alpha) .$$

*Fact 5.* Let  $p \in \tilde{P}_\lambda$  and let  $\beta$  be an even ordinal in  $\text{dom}(p)$  such that  $p(\beta)$  is a pure condition and let  $\delta = \delta_{p(\beta)}$  be the maximum even ordinal in  $\max(F_{p(\beta)})$ . Choose any  $m \in \omega$  such that  $n = n_{p(\beta)} < m$  and any  $\bar{p} \in P_{\delta+1}$  that forces values  $\bar{s}, \bar{h}$  on  $\dot{f}_\delta \upharpoonright [n, m)$  and  $\dot{h}_\delta \upharpoonright \bar{s}^\downarrow$  respectively. Recall that  $1_{P_\beta}$  forces that  $\dot{f}_\delta \leq \dot{f}_{p(\beta)}$ . Then  $\tilde{p}$  is an extension of  $\bar{p}$  and  $p$  where  $\tilde{p} \upharpoonright \delta = \bar{p}$ ,  $\tilde{p}(\gamma) = p(\gamma)$  for  $\beta \neq \gamma \in$

$\text{dom}(p) \setminus \delta + 1$ , and  $\bar{p}(\beta) = (m, s_{p(\beta)} \cup \bar{s}, \bar{\tau}, h_{p(\beta)} \cup \bar{h}, (h_{p(\beta)} \cup \bar{h}) \oplus 0_{\bar{\tau}}, F_{p(\beta)}, \dot{f}_{p(\beta)})$  where  $\bar{\tau} = s_{p(\beta)} \cup \dot{f}_{p(\beta)} \upharpoonright [n, m)$ .

Moreover, if for some even  $\alpha < \beta$ ,  $\bar{p}$  forces that  $\dot{h}_\delta \cup (\dot{h}_\alpha \upharpoonright ([m, \omega) \times \omega))$  is a function, then we could instead define  $\bar{p}(\beta)$  to equal  $(m, s_{p(\beta)} \cup \bar{s}, \bar{\tau}, h_{p(\beta)} \cup \bar{h}, (h_{p(\beta)} \cup \bar{h}) \oplus 0_{\bar{\tau}}, F_{p(\beta)} \cup \{\alpha\}, \dot{f}_{p(\beta)} \vee \dot{f}_\alpha)$

With the benefit of the above Facts we are ready to prove Theorem 5.9. Let  $\dot{X}$  be a  $P_\lambda$ -name of an uncountable subset of  $\lambda$ . By the definition of the family  $\{\dot{h}_\alpha : \alpha < \lambda\}$ , we may assume that  $\dot{X}$  is forced to consist of even ordinals. Since we are proceeding by induction, we may assume that  $\dot{X}$  is forced to be cofinal in  $\lambda$ . Let  $e$  be a strictly increasing function from  $\omega_1$  to a cofinal subset of  $\lambda$ . For each  $\xi < \omega_1$ , choose a condition  $p_\xi \in \dot{P}_\lambda$  that forces some  $\beta_\xi \in \lambda \setminus e(\xi)$  is an element of  $\dot{X}$ . For each  $\xi \in \omega_1$ , let  $\beta_\xi \in H_\xi$  denote the finite support of  $p_\xi$ .

For each  $\xi \in \omega_1$ , we make some additional assumptions about  $p_\xi$ . Let  $\beta$  be the maximum even ordinal in  $H_\xi$ . By possibly strengthening  $p_\xi \upharpoonright \beta$  we can ensure that  $p_\xi \upharpoonright \beta$  forces that  $p_\xi(\beta)$  is pure. We can also ensure that  $F_{p_\xi(\beta)} \cap \alpha$  is a subset of  $\text{dom}(p_\xi)$ , and for each even  $\alpha \in \text{dom}(p_\xi) \cap \beta$ ,  $F_{p_\xi(\beta)} \cap \alpha$  is a subset of  $F_{p_\xi(\alpha)}$ . We can also ensure that  $n_{p_\xi(\alpha)} \geq n_{p_\xi(\beta)}$  for all even  $\alpha \in \text{dom}(p_\xi)$ . This is step 1 of a finite recursion (since every descending sequence of ordinal is finite). In this way we can assume that, for each even ordinal  $\beta$  in  $\text{dom}(p_\xi)$ ,  $p_\xi \upharpoonright \beta$  forces that  $p_\xi(\beta)$  is pure and for even  $\alpha \in H_\xi \cap \beta$ ,  $F_{p_\xi(\beta)} \cap \alpha \subset F_{p_\xi(\alpha)}$ .

By passing to an uncountable subset we can assume that each  $H_\xi$  has cardinality  $\ell$  and fix an increasing enumeration,  $\{\alpha(\xi, i) : i < \ell\}$  of  $H_\xi$ . For each  $i < \ell$  and  $\xi, \eta < \omega_1$ , we may assume that  $\alpha(\xi, i)$  is even if and only if  $\alpha(\eta, i)$  is even, and that there is a fixed  $\bar{i} < \ell$  so that  $\beta_\xi = \alpha(\xi, \bar{i})$  for all  $\xi$ . Let  $E$  denote the set of  $i < \ell$  such that (each)  $\alpha(\xi, i)$  is even. We can also assume that for  $\xi < \eta$  and  $i \in E$ ,  $(n_i, s_i, h_i) = (n_{p_\xi(\alpha(\xi, i))}, s_{p_\xi(\alpha(\xi, i))}, h_{p_\xi(\alpha(\xi, i))}) = (n_{p_\xi(\alpha(\xi, i))}, s_{p_\eta(\alpha(\eta, i))}, h_{p_\eta(\alpha(\eta, i))})$ .

Notice that  $\{\alpha(\xi, \bar{i}) : \xi \in \omega_1\}$  is unbounded in  $\lambda$ . By a recursion of length at most  $\bar{i}$ , we can repeatedly pass to an uncountable subset of  $\xi \in \omega_1$  so as to ensure, for each  $i < \bar{i}$ , either  $\{\alpha(\xi, i) : \xi \in \omega_1\}$  is unbounded in  $\lambda$  or has an upper bound  $\mu_i < \lambda$ . Let  $i_0 \leq \bar{i}$  be minimal so that  $\{\alpha(\xi, i_0) : \xi \in \omega_1\}$  is unbounded. Choose any even  $\mu < \lambda$  so that  $\alpha(\xi, i) < \mu$  for all  $\xi \in \omega_1$  and  $i < i_0$ . For simple convenience assume that, for all  $\xi \in \omega_1$ ,  $\alpha(\xi, i_0)$  is an even ordinal. Let  $\{i_k : k < \bar{\ell}\}$  be an enumeration of  $E \setminus i_0$ .

By the inductive assumption,  $P_{\mu+1}$  is ccc and so we may choose a generic filter  $G_{\mu+1}$  for  $P_{\mu+1}$  such that  $\Gamma = \{\xi \in \omega_1 : p_\xi \upharpoonright \mu \in G_\mu\}$  is uncountable. By re-indexing we may assume that  $\mu + 1 < \alpha(\xi, i_0)$  for all  $\xi \in \omega_1$ , and by again choosing an uncountable subsequence we can assume that, for  $\xi < \eta$  both in  $\Gamma$ ,  $H_\xi \subset \alpha(\eta, i_0)$ .

For each  $\xi \in \Gamma$ , let  $\delta_\xi$  denote the maximum element of  $F_{p_\xi(\alpha(\xi, i_0))}$ . Since  $\delta_\xi < \mu$  we may let  $f_{\delta_\xi}$  and  $h_{\delta_\xi}$  be the valuations of  $\dot{f}_{\delta_\xi}$  and  $\dot{h}_{\delta_\xi}$  respectively by the filter  $G_{\mu+1}$ . Let  $f_\mu$  and  $h_\mu$  be defined analogously. Now choose a value  $\bar{m} \in \omega$  and an uncountable  $\Gamma_1 \subset \Gamma$  satisfying that  $f_{\delta_\xi} \upharpoonright [\bar{m}, \omega) \leq f_\mu$ , and  $h_{\delta_\xi} \upharpoonright [\bar{m}, \omega) \times \omega$  is a subset of  $h_\mu$  for all  $\xi \in \Gamma_1$ .

We work in the extension  $V[G_{\mu+1}]$  and fix any  $\xi \in \Gamma_1$ . We can ignore  $p_\xi \upharpoonright \mu$  since we have that  $p_\xi \upharpoonright \mu \in G_{\mu+1}$ . For each even  $\beta \in H_\xi \setminus \mu$ , we have that  $\delta_\xi$  is an element of  $F_{p_\xi(\beta)}$  and is the maximum of  $F_{p_\xi(\beta)} \cap \mu + 1$ . Let  $\bar{s} = f_{\delta_\xi} \upharpoonright [n_{i_0}, \bar{m})$  and  $\bar{h} = h_{\delta_\xi} \upharpoonright \bar{s}^\perp$ . Applying the ‘‘moreover’’ clause of Fact 5, we have an extension  $\bar{p}_\xi$

of  $p_\xi$  such that  $\bar{p}_\xi \upharpoonright \mu+1 \in G_{\mu+1}$  forces that  $\bar{s} = \dot{f}_{\delta_\xi} \upharpoonright [n_{i_0}, \bar{m})$  and  $\bar{h} = \dot{h}_{\delta_\xi} \upharpoonright \bar{s}^\perp$ ,  $\bar{p}_\xi(\gamma) = p_\xi(\gamma)$  for  $\alpha(\xi, i_0) < \gamma \in H_\xi$  and  $\bar{p}_\xi(\alpha(\xi, i_0))$  is as indicated in Fact 5 with  $\mu$  added to  $F_{\bar{p}_\xi(\alpha(\xi, i_0))}$ . Next, by a finite recursion, we keep applying the first clause of Fact 5, so as to arrange that  $\alpha(\xi, i_k) \in F_{\bar{p}_\xi(\alpha(\xi, i_{k+1}))}$  and  $n_{\bar{p}_\xi(\alpha(\xi, i_k))} = n_{\bar{p}_\xi(\alpha(\xi, i_{k+1}))}$  for all  $k < \bar{\ell} - 1$ . Actually we can stop at  $\alpha(\xi, \bar{\nu})$  but nothing is saved. Additionally, by Fact 3, we can assume that  $F_{\bar{p}_\xi(\alpha(\xi, i_k))} \subset F_{\bar{p}_\xi(\alpha(\xi, i_{k+1}))}$  for all  $k < \bar{\ell}$ .

Choose  $\xi < \eta$  both in  $\Gamma_1$  so that, for all  $k < \bar{\ell} - 1$ ,

$$(n_{\bar{p}_\xi(\alpha(\xi, i_k))}, s_{\bar{p}_\xi(\alpha(\xi, i_k))}, h_{\bar{p}_\xi(\alpha(\xi, i_k))}) = (n_{\bar{p}_\eta(\alpha(\xi, i_k))}, s_{\bar{p}_\eta(\alpha(\xi, i_k))}, h_{\bar{p}_\eta(\alpha(\xi, i_k))}).$$

Since  $\bar{p}_\xi \upharpoonright \mu + 1$  and  $\bar{p}_\eta \upharpoonright \mu + 1$  are both in  $G_{\mu+1}$ , and  $H_\xi \cap H_\eta \subset \mu$ , there is a condition  $\bar{p} \in P_\lambda$  satisfying that  $\bar{p} \upharpoonright \mu + 1 = (\bar{p}_\xi \upharpoonright H_\xi \setminus (\mu + 1)) \cup (\bar{p}_\eta \upharpoonright H_\eta \setminus (\mu + 1))$ .

Recall that  $\beta_\xi = \alpha(\xi, \bar{\nu})$ . Note that  $\mu$  is the largest element of  $F_{\bar{p}_\eta(\alpha(\eta, i_0))}$  and that  $\bar{p}_\xi$  forces that  $h_\mu \upharpoonright [n_{\bar{p}_\xi(\beta_\xi)}, \omega) \times \omega$  is a subset of  $\dot{h}_{\beta_\xi}$ . Therefore, by the *moreover* clause of Fact 5, we can extend  $\bar{p}$  to a condition  $p'$  (but only in the coordinate  $\alpha(\eta, i_0)$ ) as in Fact 5 so that  $F_{p'(\alpha(\eta, i_0))}$  is obtained by adding  $\beta_\xi$  to  $F_{\bar{p}_\eta(\alpha(\eta, i_0))}$ , and, by Fact 3, we can also arrange that  $\beta_\xi$  is an element of  $F_{p'(\beta_\eta)}$ . The proof is finished by verifying that  $\alpha = \beta_\xi$  and  $\beta = \beta_\eta$  satisfy the conditions of Fact 2 for the condition  $p'$ .  $\square$

Now we can complete the proof of Lemma 3.2

*Proof of Lemma 3.2.* Let  $\kappa$  be an uncountable regular cardinal satisfying  $\kappa^{<\kappa} = \kappa$ . Fix an enumeration  $\{R_\alpha : \alpha < \kappa\}$  of  $H(\kappa)$  (the set of sets with transitive closure having cardinality less than  $\kappa$ , see [6]).

Define the system  $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$  as in Lemma 5.9 as well as the names  $\{\dot{f}_\alpha, \dot{h}_\alpha : \alpha < \lambda\}$ , where, for each odd ordinal  $\alpha < \kappa$ ,  $\dot{Q}_\alpha$  is chosen so that if  $R_{\gamma_\alpha} = \dot{Q}_\alpha$ , then  $\gamma_\alpha$  is the least ordinal  $\gamma < \kappa$  satisfying that  $R_\gamma$  is a  $P_\alpha$ -name of a ccc poset that is not an element of  $\{\dot{Q}_\eta : \eta < \alpha, \eta \text{ an odd ordinal}\}$ . By Lemma 5.9, it follows that  $\{\dot{f}_\alpha, \dot{h}_\alpha : \alpha < \kappa\}$  is forced to be a linear coherent family. By the definition of  $\dot{Q}_\beta$  for even ordinals  $\beta$ , it is clear that  $\{\dot{f}_\alpha : \alpha < \kappa\}$  is a dominating family. We now prove that the  $\omega^\omega$ -gap

$$\{(\dot{a}_{f_\alpha} = \dot{h}_\alpha^{-1}(0), \dot{b}_{f_\alpha} = \dot{h}_\alpha^{-1}(1)) : \alpha < \kappa\}$$

is a tight gap. Let  $\dot{X}$  be any  $P_\kappa$ -name of a subset of  $\omega \times \omega$  such that it is forced that there is an infinite set of  $n$  such that  $\dot{X} \cap (\{n\} \times \omega)$  is infinite. Since, by Lemma 5.9,  $P_\kappa$  is ccc, there is an even ordinal  $\lambda < \kappa$  such that  $\dot{A}$  and  $\dot{X}$  are equivalent to  $P_\lambda$ -names. Let  $G_\mu$  be a  $P_\mu$ -generic filter and let  $X$  be the valuation of  $\dot{X}$  by  $G_\mu$ . We prove that it will be forced that  $\dot{h}_\lambda \upharpoonright X$  takes on values 0 and 1 infinitely often. In  $V[G_\mu]$ , the valuation of the poset  $\dot{Q}_\lambda$  is equivalent to the poset  $Q = Q(\{\dot{h}_\alpha : \alpha < \lambda\})$ . To prove this, consider any condition  $r \in Q$  and simply note the trivial claim that there is an extension  $q \in Q$  satisfying that there is an  $m > n_r$  and values  $(m, i), (m, j) \in X$  such that  $h_q((m, i)) = 0$  and  $h_q((m, j)) = 1$ .

Finally we explain how our enumeration scheme ensured that Martin's Axiom holds in the forcing extension by  $P_\kappa$ . It suffices to prove that if  $\dot{Q} \in H(\kappa)$  is a  $P_\kappa$ -name of a ccc poset and if  $\{\dot{D}_\xi : \xi < \mu\}$ , for some  $\mu < \kappa$ , is a set of  $P_\kappa$ -names for dense subsets of  $\dot{Q}$ , then there is a  $P_\kappa$ -name  $\tilde{G}$  for a filter on  $\dot{Q}$  that meets every  $\dot{D}_\xi$ . Again, using that  $P_\kappa$  is ccc and that  $\mu$  and  $|\dot{Q}|$  are less than  $\kappa$ , there is a  $\beta < \kappa$  such that  $\dot{Q}$  and every  $\dot{D}_\xi$  is equivalent to  $P_\beta$ -names. Since we were lazy with our

enumeration method we play a little trick. Choose any  $\nu < \kappa$  large enough so that the  $P_\beta$ -name for the iteration  $\dot{Q} * \text{Fn}(\nu, 2)$  is not in the list  $\{\dot{Q}_\alpha : \alpha < \beta\}$ . Let  $\gamma < \kappa$  be such that  $R_\gamma = \dot{Q} * \text{Fn}(\nu, 2)$ . Since  $\dot{Q}$  is ccc is the forcing extension by  $P_\kappa$ , it is, for every  $\beta \leq \alpha < \kappa$ , a  $P_\alpha$ -name of a ccc poset. By the definition of the iteration sequence, there is an odd ordinal  $\alpha \geq \beta$  satisfying that  $\gamma_\alpha = \gamma$ . It is a standard exercise that  $P_{\alpha+1} = P_\alpha * \dot{Q} * \text{Fn}(\mu, 2)$  will add a filter on  $\dot{Q}$  that meets every  $\dot{D}_\xi$  (since these are all  $P_\alpha$ -names).  $\square$

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