# COMPACT SPACES AND THE PSEUDORADIAL PROPERTY, I

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ABSTRACT. We investigate two properties and their connection to the property of pseudoradiality in the context of compact spaces. The first is the WAP property introduced by P. Simon and the second is the $\aleph_0$-pseudoradial property introduced by B. Šapirovskii. We show that $\diamondsuit$ implies there is a compact space which is pseudoradial but not WAP. We show that there is a model in which CH fails and in which all compact spaces of weight at most $\aleph_2$ are $\aleph_0$-pseudoradial.

## 1. INTRODUCTION

A space $X$ is said to have the property of weak approximation by points, or WAP, if for every non-closed set $A$, there exists a point $x \in \overline{A} \setminus A$ such that, for some subset $B$ of $A$, $\overline{B} \setminus A = \{x\}$. A space $X$ is AP if it is hereditarily WAP, equivalently, if for every $x \in \overline{A} \setminus A$, there is a $B \subset A$ with $\overline{B} \setminus A = \{x\}$. Note that each Fréchet space is AP and each sequential space is WAP. A compact AP space is Fréchet

Much of this work is motivated by Sapirovskiĭ's CH result [4] (CH weakened to $\mathfrak{c} \leq \omega_2$ in an improvement by Juhasz and Szentmiklossy [2]) that a compact sequentially compact space is pseudoradial. Several similar results were shown to follow from the assumption that $2^{\omega_2}$ is not pseudoradial. Sapirovskiĭ asked if $2^{\omega_2}$ fails to be $\aleph_0$-pseudoradial and it is asked in [1] if it fails to be pseudoradial. We show that it is consistent to have $\mathfrak{c}$ be arbitrarily large and to have that $2^{\omega_2}$ is $\aleph_0$-pseudoradial.

P. Simon showed that a compact WAP space is pseudoradial and to our knowledge it was not known if the simpler WAP condition could characterize the pseudoradial spaces in the class of compact spaces. We do not know if there is a ZFC example of a compact pseudoradial space which is not WAP but we produce an example from $\diamondsuit$.

**Definition 1.1.** A set $A$ is $\omega$-closed in $X$ if $A$ contains the closure of each of its countable subsets. A space $X$ is $\aleph_0$-pseudoradial provided that every $\omega$-closed non-closed set $A \subseteq X$ contains a sequence converging to a point outside $A$.

We introduce a pair of properties that will serve to generalize the Fréchet and sequential properties. Given a point $x$ in a space $X$, let $\mathcal{U}_x$ denote the family of open subsets of $X$ containing the point $x$ ($X$ should be clear from the context).

**Definition 1.2.** $X$ is $(\aleph_1, \aleph_0)$-Fréchet if for each $\{x_\alpha : \alpha < \omega_1\}$ in $X$ and each complete accumulation point $x$ there is an uncountable subset $C$ of $\omega_1$ and a function $\varphi : \mathcal{U}_x \to \omega_1$ such that for each $\gamma \in C$, the family

$$\mathcal{U}_x^\gamma = \{U \cap \{x_\alpha : \beta < \alpha < \gamma\} : \beta \in \gamma, U \in \mathcal{U}_x \text{ and } \varphi(U) < \gamma\}$$

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has the finite intersection property.

A space $X$ is $(\aleph_0, \aleph_1)$-sequential if there is some complete accumulation point satisfying the above.

The idea behind the definition is that, if for each $\gamma \in C$, there is an adherent point $z_\gamma$ of the filter generated by $\mathcal{U}_x^\gamma$, then the sequence $\{z_\gamma : \gamma \in C\}$ would converge to $x$.

It is proven in [1, 2.9] (using slightly different notation) that each compact pseudoradial space is $(\aleph_0, \aleph_1)$-sequential.

**Lemma 1.3.** *If $X$ is a compact space of character at most $\omega_2$, then $X$ is $\aleph_0$-pseudoradial if $X$ is $(\aleph_1, \aleph_0)$-sequential.*

*Proof.* Assume that $A$ is an $\omega$-closed subset of $X$ which is not closed. Fix a neighborhood basis $\mathcal{B}$ of any limit point, $x$, which is not in $A$ so that $|\mathcal{B}| \leq \omega_2$. Notice that whenever $\mathcal{D}$ is a countable subset of $\mathcal{B}$, $A$ will meet $\bigcap \mathcal{D}$. This follows directly from the compactness of $X$ and the fact that $A$ is $\omega$-closed. For $\mathcal{D} \subset \mathcal{B}$ of size $\omega_1$, two cases are possible. In the first case, assume again that $A$ meets $\bigcap \mathcal{D}$ for each $\mathcal{D} \subset \mathcal{B}$ of size $\omega_1$. Fix a well-ordering $\{B_\alpha : \alpha \in \omega_2\}$ of $\mathcal{B}$ and pick a point $a_\alpha \in A \cap \bigcap \{B_\beta : \beta \in \alpha\}$ for each $\alpha \in \omega_2$. The sequence $\{a_\alpha : \alpha \in \omega_2\}$ converges to $x$.

In the other case, there is some $\mathcal{D} \subset \mathcal{B}$ of cardinality $\omega_1$ such that $A$ is disjoint from $\bigcap \mathcal{D}$. Choose any $\mathcal{E} \subset \mathcal{B}$ of cardinality $\omega_1$ that contains $\mathcal{D}$ and that has the property that for each $B \in \mathcal{E}$, there is a $B' \in \mathcal{E}$ whose closure is contained in $B$. Enumerate $\mathcal{E}$ as $\{B_\alpha : \alpha \in \omega_1\}$ and again pick $a_\alpha \in A \cap \bigcap \{B_\beta : \beta \in \alpha\}$ for $\alpha \in \omega_1$. Each complete accumulation point of $\{a_\alpha : \alpha \in \omega_1\}$ belongs to $\bigcap \mathcal{E}$ and so is not in $A$. Apply the definition of $(\aleph_1, \aleph_0)$-sequential to the above sequence to obtain a point $y$, an unbounded set $C \subset \omega_1$ and a function $\varphi : \mathcal{U}_y \to \omega_1$ as in Definition 1.2. Since, $X$ is compact and $A$ is $\omega$-closed, there is a point $y_\gamma \in A$ such that $y_\gamma$ is a limit of each member of $\mathcal{U}_y^\gamma$. It should be clear by the properties of Definition 1.2 that $\{y_\gamma : \gamma \in C\}$ does indeed converge to $y$. $\square$

**Proposition 1.4.** *The space $[0,1]^{\omega_2}$ is $\aleph_0$-pseudoradial iff it is $(\aleph_1, \aleph_0)$-sequential.*

## 2. WAP spaces

For completeness we include the proof (see [5]) that a compact WAP space is pseudoradial.

**Proposition 2.1.** *If $X$ is compact and WAP, then $X$ is pseudoradial.*

*Proof.* Assume that $X$ is compact and WAP and that $A$ is subset of $X$ which is not closed. Fix any $B \subset A$ such that there is an $x \in \overline{A} \setminus A$ such that $\overline{B} \setminus A = \{x\}$. Let $\kappa$ denote the minimum cardinality of a local basis for $x$ in $\overline{B}$ (i.e. the character of $x$ in $\overline{B}$). Let $\{U_\alpha : \alpha \in \kappa\}$ enumerate a local basis of open sets for $x$ in $\overline{B}$ where $\overline{U_{\alpha+1}} \subset U_\alpha$. Again, choose for each $\alpha \in \kappa$ any point $a_\alpha \in B \cap \bigcap \{U_\beta : \beta < \alpha\}$. Clearly if we are able to choose $a_\alpha$ for each $\alpha \in \kappa$, then $\{a_\alpha : \alpha \in \kappa\}$ converges to $x$ and shows that $X$ is pseudoradial. If there is no such $a_\alpha$, then the family of finite intersections from $\{U_\beta : \beta < \alpha\}$ would form a local basis at $x$ contradicting the minimality of $\kappa$ (i.e. character and pseudocharacter coincide in compact spaces). $\square$

**Definition 2.2.** [3, II.7.1] A sequence $\{E_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit$-sequence if for each subset $E$ of $\omega_1$, there is a stationary set $S$ in $\omega_1$ such that $E_\alpha = E \cap \alpha$ for each $\alpha \in \omega_1$.

It will be useful to record a prepratory Lemma whose simple proof is left to the reader before proving the main result of this section.

**Lemma 2.3.** *Let $t$ be a point in the Cantor set, $\mathbb{C}$, and let $\mathcal{K}$ be any countable collection of subsets of $\mathbb{C}$ such that $t$ is an accumulation point of each of them. Then $\mathbb{C} \setminus \{t\}$ can be partitioned into open sets $U_0, U_1, U_2$ so that so that $t$ is an accumulation point of $K \cap U_i$ for each $K \in \mathcal{K}$ and $i \in \{0, 1, 2\}$.*

**Theorem 2.4.** *Assume $\diamondsuit$, there is a compact space $X$ which is pseudoradial but is not WAP. The space $X$ also contains a dense first-countable sequentially compact subspace.*

*Proof.* Let $f$ be any bijection from $\omega_1$ onto $\omega_1 \times 3 \times \omega_1$ and let $\{E_\alpha : \alpha \in \omega_1\}$ be a $\diamondsuit$-sequence on $\omega_1$. We verify that there is a $\diamondsuit$-sequence for $\omega_1 \times 3 \times \omega_1$, i.e. a sequence $\{A_\alpha : \alpha \in \omega_1\}$ such that for each $A \subset \omega_1 \times 3 \times \omega_1$, there is a stationary set of $\alpha$ such that $A_\alpha = A \cap (\alpha \times 3 \times \alpha)$. In fact, we simply set $A_\alpha = f[E_\alpha]$ for each $\alpha$ such that $f[E_\alpha] \subset \alpha \times 3 \times \alpha$. For other values of $\alpha$, let $A_\alpha$ be empty (or any subset of $\alpha \times 3 \times \alpha$). We first show that there is a closed and unbounded set $C$ consisting of $\gamma$ such that $f[\gamma] = \gamma \times 3 \times \gamma$. For each $\alpha$, let $g(\alpha) \in \omega_1 \setminus \alpha$ be minimal such that $f[g(\alpha)] \supset \alpha \times 3 \times \alpha$ and $f[\alpha] \subset g(\alpha) \times 3 \times g(\alpha)$. Since $g$ is a monotone increasing continuous unbounded function from $\omega_1$ into itself, we can show that the set $C = \{\gamma : g(\gamma) = \gamma\}$ is closed and unbounded. The continuity of $g$ implies that $C$ is closed. To see that $C$ is unbounded, one checks that for each $\alpha \in \omega_1$, $\gamma = \sup\{g^n(\alpha) : n \in \omega\}$ is in $C$.

Now suppose that $A \subset \omega_1 \times 3 \times \omega_1$ and set $E = f^{-1}[A]$. Since the set $S = \{\gamma : E_\gamma = E \cap \gamma\}$ is stationary, $S$ meets each cub (this is the definition of stationary). It easily follows that $A_\gamma = f[E_\gamma] = f[E \cap \gamma] = A \cap f^{-1}[\gamma] = A \cap \gamma \times 3 \times \gamma$ for each $\gamma$ in the stationary set $S \cap C$.

Therefore, if we have $\{a_\alpha : \alpha \in \omega_1\} \subset 3^{\omega_1}$, there is a stationary set of $\lambda$ such that $A_\lambda = \bigcup\{\{\beta\} \times a_\beta \restriction \lambda : \beta < \lambda\}$, i.e we can consider $A_\lambda$ as a sequence $\{a(\lambda, \beta) : \beta < \lambda\}$ of points in $3^\lambda$. We will specifically set the value for $A_\omega$ below.

We will construct a sequence $\{x_n : n \in \omega\} \subset 3^{\omega_1}$. In order to do so, we will define by induction on $\alpha \in \omega_1$ the values $\{x_n \restriction \alpha : n \in \omega\}$. For each $\alpha \geq \omega$, let $X_\alpha$ denote the closure in $3^\alpha$ of the sequence $\{x_n \restriction \alpha : n \in \omega\}$. We will also inductively construct other elements of $X_\alpha$, $\{x_\beta \restriction \alpha, y_{\beta,0} \restriction \alpha, y_{\beta,1} \restriction \alpha : \beta < \alpha < \omega_1\}$ and we will define sets $T_\alpha \subset \alpha$ in order to ensure that $X = X_{\omega_1}$ has the desired properties. It will follow by induction that for each $\alpha < \omega_1$, $X_\alpha$ is homeomorphic to the Cantor set.

We define $\{x_n \restriction \omega : n \in \omega\}$ to be any dense subset of $3^\omega$. Set $A_\omega = \{a(\omega, n) : n \in \omega\}$ to be any subset of $X_\omega$ which does not have any limit in $\{x_n \restriction \omega : n \in \omega\}$.

The role of the $x_\alpha$'s is to form the sequential closure of the $x_n$'s and, additionally, to generally serve as sequential limits to ensure the space is sequentially compact. The role of the $y_{\beta,0} \restriction \alpha$ and $y_{\beta,1} \restriction \alpha$ is to be points eventually not in the sequential closure of $\{x_n : n \in \omega\}$ and to witness that certain sets do witness the failure of WAP. At each stage $\alpha \geq \omega$, we choose some $t_\alpha \in X_\alpha \setminus \{x_\xi \restriction \alpha : \xi < \alpha\}$ and apply Lemma 2.3 to obtain open subsets, $U_0^\alpha, U_1^\alpha$ and $U_2^\alpha$, of $X_\alpha \setminus \{t_\alpha\}$. For each $n \in \omega$, $x_n(\alpha)$ is defined to be the unique $i$ such that $x_n \restriction \alpha \in U_i^\alpha$, hence $X_{\alpha+1}$ is clearly defined. For each $\beta < \alpha$, $x_\beta \restriction (\alpha+1)$ is implicitly defined, and for $e \in 2$ such that $y_{\beta,e} \restriction \alpha \neq t_\alpha$, $y_{\beta,e} \restriction (\alpha+1)$ is also implicitly defined. For any $\beta < \alpha$ and $e \in 2$ such that $y_{\beta,e} \restriction \alpha = t_\alpha$ we will set $y_{\beta,e}(\alpha)$ to be $e$. Finally, we will set $x_\alpha \restriction (\alpha+1)$ to be the extension of $t_\alpha$ which has value 2 at $\alpha$, and similarly, $y_{\alpha,e} \restriction (\alpha+1)$ will be the extension of $t_\alpha$ which has value $e$ at $\alpha$. Recall also that we will define a set $T_\alpha \subset \alpha$.

There is nothing to do at limit stages, $\alpha$, of the induction but to realize that each of the elements $x_\beta \restriction \alpha$, $y_{\beta,e} \restriction \alpha$ ($e \in 2$) have been defined. Of course, $X_\alpha$ is the closure in $3^\alpha$ of the set $\{x_n \restriction \alpha : n \in \omega\}$ (and is also equal to the inverse limit of the previous $X_\beta$ under the obvious projection maps). For successor stages, we must define, for arbitrary $\alpha$, the space $X_{\alpha+1}$ by selecting $t_\alpha$ as well as the sets $T_\alpha$ and $U_i^\alpha$ for $i = 0, 1, 2$ in order to preserve the following inductive hypotheses.

(1) $t_\alpha \in X_\alpha \setminus \{x_\xi \restriction \alpha : \xi < \alpha\}$,
(2) $\{U_0^\alpha, U_1^\alpha, U_2^\alpha\}$ is a partition of $X_\alpha \setminus \{t_\alpha\}$ into open sets,
(3) $t_\alpha$ is a limit point of $U_i^\alpha \cap \{x_\xi \restriction \alpha : \xi \in T_\alpha\}$ for each $i \in 3$,
(4) for each $\beta < \alpha$, $e \in 2$ and $i \in 3$, if $t_\alpha = y_{\beta,e} \restriction \alpha$, then $U_i^\alpha \cap \{x_\xi \restriction \alpha : \xi \in T_\beta\}$ is infinite,
(5) if $\{a(\alpha, n) : n \in \omega\}$ is an infinite subset of $X_\alpha$ which has no limits in $\{x_\xi \restriction \alpha : \xi < \alpha\}$, then $t_\alpha$ is a limit of $U_2^\alpha \cap \{a(\alpha, n) : n \in \omega\}$,
(6) if $A_\alpha = \{a(\alpha, \xi) : \xi < \alpha\}$ is a subset of $\{x_\xi \restriction \alpha : \xi < \alpha\}$ which has a limit not in $\{x_\xi \restriction \alpha : \xi < \alpha\}$, then $t_\alpha$ will be such a limit and $T_\alpha$ will be contained in $\{\xi < \alpha : x_\xi \restriction \alpha \in A_\alpha\}$.

Assume first that the condition in 5 holds, then let $t_\alpha$ be any limit in $X_\alpha$ of $\{a(\alpha, n) : n \in \omega\}$. Set $T_\alpha = \omega$ and define $K_0 = \{a(\alpha, n) : n \in \omega\}$.

If both conditions 5 and 6 fail, then let $t_\alpha \in X_\alpha \setminus \{x_\xi \restriction \alpha : \xi < \alpha\}$ be arbitrary. Set $T_\alpha = \omega$ and $K_0 = X_\alpha$.

Now suppose that condition 6 holds. If 5 also holds, then there is no change. If 5 fails, then let $t_\alpha$ be any limit of $A_\alpha$ which is not in $\{x_\xi \restriction \alpha : \xi < \alpha\}$. Set $T_\alpha = \{\xi \in \alpha : x_\xi \restriction \alpha \in A_\alpha\}$ and $K_0 = \{x_\xi \restriction \alpha : \xi \in T_\alpha\}$.

We next let $\{K_n : n \in \omega \setminus \{0\}\}$ be any enumeration of those sets $\{x_\xi \restriction \alpha : \xi \in T_\beta\}$ such that $\beta < \alpha$, and and there is $e \in 2$ such that $y_{\beta,e} = t_\alpha$. If there are no such sets, then simply set $K_n = K_0$ for all $n \in \omega$.

The sets $U_0^\alpha, U_1^\alpha, U_2^\alpha$ are then obtained by simply applying Lemma 2.3 to the family $\{K_n : n \in \omega\}$.

This completes the induction, and we set $X$ to be the closure in $3^{\omega_1}$ of $\{x_n : n \in \omega\}$. We will show that $X_0 = \{x_\xi : \xi \in \omega_1\}$ is the sequential closure of $\{x_n : n \in \omega\}$ and though $X_0$ is not closed it contains no subset $B$ with a unique accumulation point not in $X_0$.

Clearly, $X$ is compact, and note that by the construction, especially conditions (1) and (2), it follows that $X$ is first countable at each $x_\xi$. For our purposes it suffices to note that a sequence $\{z_n : n \in \omega\} \subset X$ converges to $x_\alpha$ if and only if $\{z_n \restriction (\alpha+1) : n \in \omega\}$ converges to $x_\alpha \restriction (\alpha+1)$.

To see that $X_0$ is not closed, note that for each $\beta$ and $e \in 2$, we have the point $y_{\beta,e}$ which is clearly a limit point of $\{x_n : n \in \omega\}$. In the construction, $y_{\beta,e}$ extends $t_\beta$, hence $y_{\beta,e} \neq x_\xi$ for $\xi < \beta$. For each $\alpha \geq \beta$, whenever $y_{\beta,e} = t_\alpha = x_\alpha \restriction \alpha$, we ensured that $x_\alpha(\alpha) = 2 > y_{\beta,e}(\alpha) = e$. Therefore $y_{\beta,e}$ is not in $X_0$.

We next prove that every infinite sequence, $\{z_n : n \in \omega\} \subset X$, has a subsequence which converges to $x_\xi$ for some $\xi \in \omega_1$. Indeed, set $A = \{a_\alpha : \alpha \in \omega_1\}$ where

$a_n = z_n$ for $n \in \omega$ and $a_\alpha = x_0$ for $\alpha \in [\omega, \omega_1)$. Apply $\diamondsuit$ to find some $\alpha$ so that $A_\alpha = \{a_\beta \restriction \alpha : \beta \in \alpha\}$ and so that $\alpha$ is large enough such that $z_n \restriction \alpha \neq z_m \restriction \alpha$ for $n \neq m$. At stage $\alpha$ in the construction, we first considered if inductive condition 5 held. If it failed, it is because there is a $\xi < \alpha$ such that some converging subsequence of $\{z_n \restriction (\xi{+}1) : n \in \omega\}$ has $x_\xi \restriction (\xi{+}1)$ as a limit, which implies that $x_\xi$ is the limit of the corresponding converging subsequence of $\{z_n : n \in \omega\}$. If 5 held, it follows similarly from the conclusion of 5 that a subsequence of $\{z_n \restriction (\alpha{+}1) : n \in \omega\}$ will converge to $x_\alpha \restriction (\alpha{+}1)$. Then again, we have that a subsequence of $\{z_n : n \in \omega\}$ will converge to $x_\alpha$.

It certainly follows then that each of $X$ and $X_0$ is sequentially compact. Now suppose that some point $y$ is not in $X_0$ and that $Y = \{a_\beta : \omega \leq \beta < \omega_1\}$ is a subset of $X_0$ which has $y$ as a limit point. To show that $X$ is not WAP, we prove that $Y$ has another limit point not in $X_0$. For each $n \in \omega$, let $a_n$ also equal $a_\omega$ and set $A = \{a_\beta : \beta < \omega_1\}$. By a standard enumeration argument, there is a closed and unbounded $C \subset \omega_1$ such that for each $\alpha \in C$, $y \restriction \alpha$ is a limit of $\{a_\beta \restriction \alpha : \beta < \alpha\}$ and for each $\beta < \alpha$, $a_\beta \in \{x_\xi : \xi \in \alpha\}$. Note that since $y \notin X_0$, it follows that $y \restriction \alpha \notin \{x_\xi \restriction \alpha : \xi \in \alpha\}$ for all $\alpha$. Therefore, at stage $\alpha$, condition 6 applied, while 5 fails since $\{a_n : n \in \omega\}$ is not infinite. The conclusion of conditions 6 and 3 gives us that each of $y_{\alpha,0} \restriction (\alpha + 1)$ and $y_{\alpha,1} \restriction (\alpha + 1)$ are limit points of $\{x_\xi \restriction (\alpha + 1) : \xi \in T_\alpha\}$, while $\{x_\xi : \xi \in T_\alpha\}$ is a subset of $Y$. Conditions 4 and 2 together imply that the condition $y_{\alpha,e} \restriction \beta$ is a limit of $\{x_\xi \restriction \beta : \xi \in T_\alpha\}$ is preserved for all $\beta > \alpha$. Therefore $Y$ has at least two limit points not in $X_0$.

Finally, we observe that $X$ is pseudoradial since it is compact, sequentially compact and CH holds. □

## 3. Cohen Reals and $\aleph_0$-pseudoradial

In [1], it was shown that $2^{\omega_2}$ is $\aleph_0$-pseudoradial if CH and Kunen's principle $P_1$ hold. In this section we verify the conjecture from [1] that $2^{\omega_2}$ remains $\aleph_0$-pseudoradial if any number of Cohen reals are added. However, since $\mathfrak{s} = \omega_1$ in this model, even $2^{\omega_1}$ is not pseudoradial but it is interesting that we can get $\mathfrak{c}$ to be arbitrarily large and retain $\aleph_0$-pseudoradiality. Recall that $\mathfrak{c} \leq \omega_2$ implies that $2^{\omega_2}$ is not pseudoradial.

We first recall Kunen's principle $P_1$ and then introduce two weakenings that are closely related to the $(\aleph_1, \aleph_0)$-Fréchet and $(\aleph_1, \aleph_0)$-sequential properties.

**Definition 3.1.** [3, VIII.7.11] $P_1$ is the statement that whenever $\mathcal{A} \subset \wp(\omega_1)$, $|\mathcal{A}| < 2^{\omega_1}$, and

$$\forall F \subset \mathcal{A}(|F| < \omega_1 \to |\omega_1 \setminus \bigcup F| = \omega_1),$$

then there is an uncountable $d \subset \omega_1$ such that $d \cap A$ is countable for each $A \in \mathcal{A}$.

For our applications, we take $\omega_1$ as representing a subset of a space and $\mathcal{A}$ to be the complements of a neighborhood of a point. Note the hypothesis however, that no countable union of members of $\mathcal{A}$ can cover the set $\omega_1$. The set $d$ represents an $\omega_1$ subsequence that converges to the point in question. We introduce two set-theoretic principles which remove the restriction on the family $\mathcal{A}$. However we have adopted a filter approach, $sP_1$ for strong version and $wP_1$ for weak version.

**Definition 3.2.** A filter base on a cardinal $\kappa$ is said to be *uniform* if each member has cardinality $\kappa$.

(1) $sP_1$ is the statement that whenever $\mathcal{U}$ is a uniform filter base on $\omega_1$ with $|\mathcal{U}| < 2^{\omega_1}$, then there is an uncountable set $C \subset \omega_1$ and a function $\varphi : \mathcal{U} \to \omega_1$ such that for each $\gamma \in C$, the family
$$\{U \cap (\beta, \gamma) : \beta \in \gamma, U \in \mathcal{U}, \text{ and } \varphi(U) < \gamma\}$$
has the finite intersection property.

(2) $wP_1$ is the statement that whenever $\mathcal{X} \subset \wp(\omega_1)$ and $|\mathcal{X}| < 2^{\omega_1}$, then there are a uniform filter base $\mathcal{U}$ on $\omega_1$ so that $|\mathcal{U} \cap \{X, \omega_1 \setminus X\}| = 1$ for each $X \in \mathcal{X}$, and an uncountable set $C \subset \omega_1$ and a function $\varphi : \mathcal{U} \to \omega_1$ such that for each $\gamma \in C$, the family
$$\{U \cap (\beta, \gamma) : \beta \in \gamma, U \in \mathcal{U}, \text{ and } \varphi(U) < \gamma\}$$
has the finite intersection property.

We leave it as an exercise the relationship of the above principles to $(\aleph_1, \aleph_0)$ Fréchet and sequentiality.

**Proposition 3.3.** *The principle $wP_1$ is equivalent to the statement that each compact space of weight less than $2^{\omega_1}$ is $(\aleph_1, \aleph_0)$-sequential. The principle $sP_1$ is equivalent to the statement that each space of character less $2^{\omega_1}$ is $(\aleph_1, \aleph_0)$-Fréchet.*

We will need an equivalent formulation of $P_1$ which can be found in [6, 7].

**Proposition 3.4** (CH)**.** *The principle $P_1$ is equivalent to the statement that if a poset $P$ is $\omega_1$-centered and each countable directed subset has a lower bound, then for any collection $\mathcal{D}$ of fewer than $2^{\aleph_1}$ dense sets there exists a $\mathcal{D}$-generic filter on $P$.*

Recall that, for a poset $P$, and a set $x$, $\check{x}$ denotes the canonical name for the set $x$ in the extension. Also, we may assume that a name, $\dot{X}$ of a subset of $\omega_1$ (i.e. $1 \Vdash \dot{X} \subset \check{\omega_1}$) has the following convenient form $\dot{X} \subset P \times \check{\omega_1}$. In addition, in the case where $P$ is $\mathrm{Fn}(I, 2)$ for any set $I$, we can assume that for each $\alpha \in \omega_1$, there is a countable set $I(\dot{X}, \alpha) \subset I$ such that for each $p \in P$, $(p, \check{\alpha}) \in \dot{X}$ if and only if $(p \restriction I(\dot{X}, \alpha), \check{\alpha}) \in \dot{X}$. We will use $p \perp q$ to denote the relation that $p$ and $q$ are incompatible in $P$.

We will say that a $\mathrm{Fn}(I, 2)$-name of a subset of $\omega_1$ is a $\mathrm{Fn}(J, 2)$-name if $J \subset I$ and for each $\alpha \in \omega_1$, $I(\dot{X}, \alpha) \subset J$.

**Definition 3.5.** Given a $P$-name $\dot{X}$ of a subset of $\omega_1$ as above and a condition $p \in P$ we introduce notation for some related names. Let $\dot{X}[p]$ denote the name $\{(q, \check{\alpha}) \in \dot{X} : q \leq p\}$, $(\omega_1 - \dot{X})$ will be $\{(q, \check{\alpha}) : q \Vdash \check{\alpha} \notin \dot{X}\}$, and $\dot{X}[p^\perp] = \{(q, \check{\alpha}) \in \dot{X} : q \perp p\}$. Finally, in the case that $P = \mathrm{Fn}(I, 2)$ for some $I$ and $J \subset I$, let $\dot{X}[\restriction J] = \{(q \restriction J, \check{\alpha}) : (q, \check{\alpha}) \in \dot{X}\}$.

The intended effect of $\dot{X}[p]$ and $\dot{X}[p^\perp]$ should be reasonably clear. The name $\dot{X}[\restriction J]$ is really just a projection of $\dot{X}$ to a $\mathrm{Fn}(J, 2)$-name with the property that a member, $q$, of $\mathrm{Fn}(J, 2)$ will force an $\alpha$ in $\dot{X}[\restriction J]$ if and only if there is a $p \in \mathrm{Fn}(I \setminus J, 2)$ such that $p \cup q$ forces $\alpha$ is in $\dot{X}$.

**Proposition 3.6.** *Suppose that $\dot{X}$ and $\dot{X}'$ are $Fn(I, 2)$-names of subsets of $\omega_1$. Let $p, q \in Fn(I, 2)$ with $p \perp q$ and $\alpha \in \omega_1$. Then*
$$p \Vdash \dot{X} = (\dot{X}[p] \cup \dot{X}'[p^\perp])$$

*and*

$$q \Vdash \dot{X}' = (\dot{X}[p] \cup \dot{X}'[p^\perp])$$

*and*

$$(p \Vdash \check{\alpha} \in \dot{X}) \text{ iff } (p \restriction I(\dot{X}, \alpha) \Vdash \check{\alpha} \in \dot{X}).$$

In the remainder of this section we prove the following two results. We prove Proposition 3.7 at the end.

**Proposition 3.7.** *If CH holds in the model $M$, and $G$ is $Fn(I,2)$-generic for any index set $I$ with $|I| \geq \aleph_2$, then $sP_1$ fails in any cardinal preserving extension of $M[G]$.*

**Theorem 3.8.** *Assume that CH and $P_1$ holds in the model $M$ and that $G$ is $Fn(I,2)$-generic over $M$ for any set $I \in M$. Then the principle $wP_1$ holds in $M[G]$.*

*Proof.* Let $\mathcal{X}$ be a collection of $\omega_2$ many $\mathrm{Fn}(I,2)$-names of subsets of $\omega_1$. Since $\mathcal{X}$ has cardinality $\omega_2$ and these are names of subsets of $\omega_1$, there is a subset $J$ of $I$ such that each $\dot{X} \in \mathcal{X}$ is a $\mathrm{Fn}(J,2)$-name. With no loss of generality then, we may assume that $J = \omega_2$ and for each $\lambda \leq \omega_2$ let $P_\lambda = \mathrm{Fn}(\lambda, 2)$. In fact, it is a simple consequence of Proposition 3.6 that we may assume that $I = \omega_2$ given the nature of the properties required in $wP_1$.

Since $2^{\omega_1} > \omega_2$, we will need to select a collection of $\omega_2$ many $P_{\omega_2}$-names of subsets of $\omega_1$ which will be closed under basic set-theoretic operations and contain the family $\mathcal{X}$. For this purpose, let $\theta$ be a sufficiently large regular cardinal and let $\mathcal{X}$ be a member and subset of an $\aleph_2$-sized elementary submodel $M$ of $H(\theta)$ such that $M^\omega \subset M$. With no loss, we may now assume that $\mathcal{X}$ is the set of all $P_{\omega_2}$-names of subsets of $\omega_1$ and (real) subsets of $\omega_1$ which are elements of $M$. Unless we explicitly mention to the contrary, we intend that every name of a subset of $\omega_1$ under discussion will be a member of $M$.

By induction on $\lambda < \omega_2$, we construct a family $\mathcal{F}_\lambda$ of $P_\lambda$-names in $M$ such that

1. for each $p \in P_\lambda$ and $P_\lambda$-name $H \in \mathcal{X}$, there is a $q < p$ and an $F \in \mathcal{F}_\lambda$, $q \Vdash H \in \{F, \omega_1 \setminus F\}$;
2. for each $\{F_i : i < n\} \subset \mathcal{F}_\lambda$, $1 \Vdash \bigcap F_i$ is uncountable.

We let $\mathcal{F}_0$ be any uniform filter which is maximal over $\mathcal{X} \cap \wp(\omega_1)$. If $\lambda = \alpha + 1$ and $\mathcal{F}_\alpha$ has been chosen, then $F \in \mathcal{F}_\lambda$ if there are $H_0, H_1 \in \mathcal{F}_\alpha$ such that $p \Vdash F = H_i$ iff $p(\alpha) = i$. It is easily seen that $\mathcal{F}_\lambda$ will satisfy the above conditions.

Now suppose that $\lambda$ is a limit ordinal and let $\mathcal{F}^0_\lambda$ equal $\bigcup\{\mathcal{F}_\alpha : \alpha < \lambda\}$. Let $\{(p_\xi, X_\xi) : \xi \in \omega_2\}$ enumerate all pairs $(p, X)$ such that $p \in P_\lambda$ and $X \in \mathcal{X}$ is a $P_\lambda$-name of a subset of $\omega_1$.

We construct $\mathcal{F}^\xi_\lambda$ by induction on $\xi < \omega_2$ and we assume, inductively, that if $\{F_i : i < n\} \subset \mathcal{F}^\xi_\lambda$, then the canonical name for $\bigcap_{i<n} F_i$ is in $\mathcal{F}^\xi_\lambda$

If there is a $q_\xi < p_\xi$ and an $F \in \mathcal{F}^\xi_\lambda$ such that $q_\xi \Vdash F \cap X_\xi = \emptyset$, then there's really nothing to do: $\mathcal{F}^{\xi+1}_\lambda$ is all finite intersections from $\mathcal{F}^\xi_\lambda \cup ((\omega_1 - F)[q_\xi] \cup (\check{\omega}_1[q^\perp_\xi]))$. Otherwise, let $F_\xi = (X_\xi[p_\xi]) \cup (\omega_1[p^\perp_\xi])$ and let $\mathcal{F}^{\xi+1}_\lambda$ be the collection of all names $F \cap F_\xi$ for $F \in \mathcal{F}^\xi_\lambda$. (Note that $(\omega_1 \setminus \gamma) \in \mathcal{F}_0$ for all $\gamma \in \omega_1$).

We have to check that condition (2) holds in the latter case. So let $\{F_i : i < n\} \subset \mathcal{F}^\xi_\lambda$ and $p \in P_\lambda$, we must show that $p \Vdash F_\xi \cap \bigcap\{F_i : i < n\}$ is uncountable. It is easily seen that this will suffice to verify condition (2) is maintained. Let $F$ be the name $\bigcap_{i<n} F_i$. In case $p \perp p_\xi$, then $p \Vdash F_\xi = \omega_1$, hence the result follows by induction. In case $p \not\perp p_\xi$, then we may assume that $p < p_\xi$ and the result follows since we know that every extension of $p_\xi$ forces that $F \cap X_\xi$ is uncountable.

Observation: For each $F \in \mathcal{F}_\lambda$, for each $\alpha < \lambda$ and each $p \in P_\lambda$ with $\mathrm{dom}(p) \cap \alpha = \emptyset$,

$$(F[p])[\restriction \alpha] = \{(q \restriction \alpha, \xi) : p \subset q \text{ and } q \Vdash \xi \in F\} \in \mathcal{F}_\alpha \ . \tag{3.1}$$

This follows easily from the maximality condition on $\mathcal{F}_\alpha$ and the inductive construction of $\mathcal{F}_\lambda$. Indeed, if there were some $H \in \mathcal{F}_\alpha$ and a condition $q \in P_\alpha$ such that $q \Vdash H \cap (F[p])[\restriction \alpha]$ is empty, then $q \cup p$ would force that $F \cap H$ is empty.

For the remainder, fix a sequence $\mathcal{G} = \{g_\lambda : \lambda \in \omega_2\}$ such that for each $\lambda \in \omega_2$, $g_\lambda$ is a function from $\omega_1$ onto $\lambda$.

**Lemma 3.9.** *Suppose that $\{M_i : i \in n\}$ are elementary submodels of $H(\theta)$ such that they all have the same transitive collapse (i.e. they are pairwise isomorphic) and that these maps all send the pair $\{\mathcal{G}, \mathcal{F}_{\omega_2}\} \in M_i$ to the same object. Suppose further that $F_i \in M_i \cap \mathcal{F}_{\omega_2}$ for each $i < n$. Let $\delta = M_0 \cap \omega_1$ (which also equals $M_i \cap \omega_1$ for each $i < n$). Then $1 \Vdash \bigcap_{i<n} F_i \cap \delta$ is not empty (and thus is cofinal in $\delta$).*

*Proof.* (of Lemma 3.9) We proceed by induction on $n$. For $n = 1$ it follows by elementarity and Proposition 3.6. Assume (and the fact that $P_{\omega_2}$ is ccc) that $1 \Vdash F_0 \cap \delta$ is not empty.

Now suppose $n > 1$ and fix any condition $p \in P_{\omega_2}$. It suffices to show that there is a $\beta \in \delta$ and a $q < p$ such that $q \Vdash \beta \in F_i$ for each $i < n$.

For each $i < n$, let $\lambda_i$ be minimal such that $F_i \in \mathcal{F}_{\lambda_i}$. Also let $f_i$ be the standard transitive collapsing function on $M_i$. It is well-known (see [3, III.5.9-14]) that $f_i$ is an isomorphism and, therefore, $f_j^{-1} \circ f_i$ is an isomorphism from $M_i$ to $M_j$ which is the identity on $\{\mathcal{G}, \mathcal{F}_{\omega_2}\}$ and on $M_i \cap \omega_1$. Additionally, if $\lambda \in M_i \cap M_j \cap \omega_2$, then $M_i \cap \lambda = M_j \cap \lambda$ since each is equal to $g_\lambda[\delta]$.

The lexicographic order on $\omega_1^n$ is defined by $\langle \alpha_i : i \in n \rangle < \langle \beta_i : i \in n \rangle$ if $\alpha_i < \beta_i$ where $i$ is minimal such that $\alpha_i \neq \beta_i$. Since $\omega_1$ is well-ordered, this defines a well-ordering on $\omega_1^n$.

We may assume that $\{M_i : i \in n\}$ is enumerated so that $f(\lambda_i) \geq f(\lambda_j)$ for $i \leq j$, and we then proceed by induction on $\langle f_i(\lambda_i) : i \in n \rangle$ in the lexicographic ordering.

For each $i$ with $0 < i < n$, let $\mu_i = \sup(M_i \cap M_0 \cap \lambda_0)$. Recall that $M_i \cap \mu_i = M_0 \cap \mu_i$.

Fix any $i$ such that $\mu_i$ is maximal, and there's no loss of generality if we assume that $i = 1$ for notational convenience. We proceed in cases according to whether or not $M_0 \cap [\mu_1, \lambda_0)$ is empty.

In case there is some $\mu \in M_0$ with $\mu_i \leq \mu < \lambda_0$, then set $H_0 = (F_0[p \restriction [\mu, \lambda_0]])[\restriction \mu]$. It follows that $H_0 \in M_0$ and that $H_0$ is a $P_\mu$-name, and by Equation 3.1 above, that $H_0 \in \mathcal{F}_{\omega_2}$. Therefore, by the inductive hypotheses applied to the sequence $\{H_0, F_1, \ldots, F_{n-1}\}$, there is a $q < p$ and a $\beta \in \delta$ such that $q \Vdash \beta \in H_0 \cap \bigcap_{0<i<n} F_i$. Recall that we may assume that $\mathrm{dom}(q) \subset \mathrm{dom}(p) \cup I(H_0, \beta) \cup \bigcup_{0<i<n} I(F_i, \beta)$, hence, in particular, we may assume that $\mathrm{dom}(q) \cap [\mu, \lambda_0) \subset \mathrm{dom}(p)$. Since $H_0 = (F_0[p \restriction [\mu, \lambda_0]])[\restriction \mu]$ and $q \restriction \mu \Vdash \beta \in H_0$, it follows that there is a $q' \in \mathrm{Fn}(\lambda_0 \setminus \mu, 2) \cap M_0$ such that $q' \supset p \restriction [\mu, \lambda_0]$ and $q \restriction \mu \cup q' \Vdash \beta \in F_0$. Therefore, it follows that $q$ is compatible with $q'$ and $q \cup q' \Vdash \beta \in \bigcap_i F_i$ as required.

Now we assume that there are no elements of $M_0$ in the interval $[\mu_1, \lambda_0)$. Therefore, $p \restriction \lambda_0 = p \restriction \mu_1$. In this case, we set $F_0' = f_1^{-1} \circ f_0(F_0)$ and note that $F_0' \in M_1 \cap \mathcal{F}_{\omega_2}$. Now the name for $F_0' \cap F_1$ is in $M_1 \cap \mathcal{F}_{\omega_2}$, and we apply the inductive hypothesis to the $(n-1)$-length sequence $(F_0' \cap F_1), F_2, \ldots, F_{n-1}$. Again, fix any $q < p$ and $\beta \in \delta$ so that $q \Vdash \beta \in (F_0' \cap F_1)$ and $q \Vdash \beta \in F_i$ for $2 \le n$. We finish by showing that $q \Vdash \beta \in F_0$. Note that $J = I(F_0, \beta) \subset M_0 \cap \lambda_0$ and so is a subset of $M_0 \cap \mu_1$. Therefore, $f_1^{-1} \circ f_0[J] = J$, and $J = I(F_0', \beta)$. It follows that $q \restriction J \Vdash \beta \in F_0'$ and, by the isomorphism $f_0^{-1} \circ f_1$ applied to each item, $q \restriction J \Vdash \beta \in F_0$. □

Now we continue with the proof of Theorem 3.6.

We will define a poset and a family of dense sets as in Proposition 3.4. Say that a family $\mathcal{M}$ of countable elementary submodels of $H(\theta)$ are pairwise compatible if they satisfy the condition in Lemma 3.9. For a countable elementary submodel $M$ of $H(\theta)$, let $f_M$ denote the transitive collapsing function. We will need the fact that $f_M(M)$, the transitive collapse, is a member of the $\aleph_1$-sized set $H(\omega_1)$ (see [3, VI.Ex 4]).

Conditions in our poset $P$ are pairs $(C_p, \mathcal{M}_p)$ where $C_p$ is a countable subset of $\omega_1$ and $\mathcal{M}_p$ is a countable collection of elementary submodels of $H(\theta)$ each containing $\mathcal{G}$ and $\mathcal{F}_{\omega_2}$. In addition, $\mathcal{M}_p$ can be expressed as the union of the pairwise compatible non-empty families $\mathcal{M}_{p,\delta}$ for $\delta \in C_p$ where $M \cap \omega_1 = \delta$ for each $M \in \mathcal{M}_{p,\delta}$. Finally, if $\delta < \delta'$ are both in $C_p$ and $M \in \mathcal{M}_{p,\delta}$, then there is some $M' \in \mathcal{M}_{p,\delta'}$ such that $M \in M'$. The poset $P$ is ordered as follows: $p < p'$ if $C_p \cap \sup C_{p'} = C_{p'}$ and for each $\delta \in C_{p'}$, $\mathcal{M}_{p,\delta} \supset \mathcal{M}_{p',\delta}$.

It is routine to show that countable directed subsets of $P$ are bounded below by the condition which is basically the union. In addition, it is completely trivial that $P$ is $\aleph_1$-centered since for each $p \in P$, the family

$$A_p = \{p' \in P : C_{p'} = C_p \text{ and } f_{M'}(M') \in \{f_M(M) : M \in \mathcal{M}_p\} \text{ for each } M' \in \mathcal{M}_{p'}\}$$

is centered and $A_p$ is determined by the element $\{f_M(M) : M \in \mathcal{M}_p\}$ of $H(\omega_1)$.

For each $F \in \mathcal{F}_{\omega_2}$ and $\gamma \in \omega_1$, the set $D_{F,\gamma} = \{p \in P : (\exists M \in \mathcal{M}_p)\ \gamma, F \in M\}$ is easily seen to be dense (given any $p \in P$, find $M \prec H(\theta)$ such that $\gamma, p, F \in M$ and take $(C_p \cup (M \cap \omega_1), \mathcal{M}_p \cup \{M\})$).

Therefore there is a filter $G \subset P$ such that $G \cap D_{F,\gamma}$ is not empty for each $F \in \mathcal{F}_{\omega_2}$ and $\gamma \in \omega_1$. We will set $C$ to be the (uncountable) union of all $C_p$ such that $p \in G$.

The hard part to this proof was accomplished in Lemma 3.9, because that is what will allow us to show that we can define $\varphi(F) \in \omega_1$ for $F \in \mathcal{U} = \mathcal{F}_{\omega_2}$ and have the condition in Definition 3.2.2 holding. For each $F \in \mathcal{F}_{\omega_2}$, set $\varphi(F)$ to be the minimum $\delta$ such that there is a $p \in G$ and an $M \in \mathcal{M}_{p,\delta}$ with $F \in M$. Note that the definition of $P$ and the directedness of $G$ guarantees that for each $\gamma \in C \setminus (\delta+1)$, there is $p' \in G$ such that $M \in M' \in \mathcal{M}_{p',\gamma}$, hence $F \in M'$. It now follows directly from Lemma 3.9 that $wP_1$ will hold. □

Now we prove Theorem 3.7.

*Proof.* (of Theorem 3.7.) Let $T$ denote the tree $2^{<\omega_1}$ as computed in the model $M$. Rather than work directly with $Fn(\omega_2, 2)$, we simplify notation by letting $G$ be $Fn(T \cup \omega_2, 2)$-generic over $M$. Since CH holds in $M$, $T$ has cardinality $\omega_1$. We fix (still in $M$) a collection $\{b_\xi : \xi \in \omega_2\}$ of maximal branches of $T$, i.e. $b_\xi \in 2^{\omega_1}$ for each $\xi \in \omega_2$.

For each $\xi \in \omega_2$, we define a set $X_\xi$ in $M[G]$ as follows:

$$\alpha \in X_\xi \text{ iff } G(b_\xi \restriction \alpha) = 1$$

and note that $X_\xi$ is an uncountable subset of $\omega_1$ whose complement is also uncountable by the genericty of $G$.

Consider any finite family $\xi_0 < \xi_1 < \cdots < \xi_{n-1} < \omega_2$, and fix any $\delta \in \omega_1$, such that $b_{\xi_i} \restriction \delta \neq b_{\xi_j} \restriction \delta$ for $i < j < n$. Let $p \in \mathrm{Fn}(T \cup \omega_2, 2)$ be any condition. Note that there is a $\beta \in \omega_1$ such that for each $t \in \mathrm{dom}(p) \cap T$, $t \in 2^{<\beta}$. Therefore if $\alpha \in \omega_1$ is any ordinal larger than each of $\beta$ and $\delta$, $p$ can be extended to force that $\alpha$ is a member of $X_{\xi_i}$ exactly for $i$ in any specified $I \subset n$. That is to say, the family $\{X_\xi : \xi \in \omega_2\}$ is an independent family (mod countable).

Next, for each $\xi \in \omega_2$, set $F_\xi = X_\xi$ if $G(\xi) = 1$ and $F_\xi = \omega_1 \setminus X_\xi$ if $G(\xi) = 0$. Since the family was independent mod countable, it follows that the family $\{F_\xi : \xi \in \omega_2\}$ generates a filter of uncountable sets. We show that in any ccc forcing extension $M'$ of $M[G]$, there is no uncountable $C$ and $\varphi : \{F_\xi : \xi \in \omega_2\} \to \omega_1$ as in the definition of $sP_1$. Indeed, given any such $M'$, $\varphi$ and unbounded $C$, since it is a ccc forcing extension over $M$, there is a function $\psi \in M$ such that $\psi : \omega_2 \to \omega_1$ and $\varphi(F_\xi) < \psi(\xi)$ for each $\xi \in \omega_2$. Similarly, there is a cub $C' \in M$ such that $C$ is cofinal in $\gamma$ for each $\gamma \in C'$ (see [3, VII]). Since CH holds in $M$, there is a $t \in T$ and a set $S \subset \omega_2$ of cardinality $\omega_2$, such that $b_\xi \restriction \psi(\xi) = t$ for all $\xi \in S$. Similarly, given any $\gamma \in C'$ such that $\psi(\xi) < \gamma$ for all $\xi \in S$, it is easily seen that there are $\xi < \eta$ both in $S$ so that $b_\xi \restriction \gamma = b_\eta \restriction \gamma$, $F_\xi = X_\xi$ and $F_\eta = \omega_1 \setminus X_\eta$. However, since $b_\xi \restriction \gamma = b_\eta \restriction \gamma$, it follows that $X_\xi \cap \gamma = X_\eta \cap \gamma$. Then, since $G(\xi) \neq G(\eta)$, it follows that $F_\xi \cap F_\eta \cap \gamma$ is empty. Clearly then for each $\delta \in C' \cap \gamma \setminus \psi(\xi)$, we see a failure of the statement of $sP_1$, completing the proof. □

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