π-WEIGHT AND THE FRÉCHET-URYSOHN PROPERTY

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Abstract. We prove that there is a countable regular Fréchet-Urysohn space with uncountable π-weight.

1. Introduction

Juhasz: Is there a countable Fréchet-Urysohn space which has uncountable π-weight?

In 1978, Malyhin asked if every countable Fréchet-Urysohn group was metrizable, an important problem which remained unsolved until Hrusak and Ramos Garcia [3] established the independence in 2012. Since π-weight is the same as weight for a topological group, this led Juhasz to pose his problem. Malyhin certainly knew that if $\mathfrak{p} > \omega_1$, then any countable dense subgroup of 2^{ω_1} would be Fréchet. Gerlitz and Nagy [2] introduced γ-sets and proved that the existence of an uncountable γ-set implied the existence of a countable non-metrizable Fréchet-Urysohn group. Nyikos [4] proved that if $\mathfrak{p} = \mathfrak{b}$, then there was such a group, and Orenshtein and Tsaban [5] showed that this hypothesis also implied the existence of an uncountable γ-set.

With respect to Juhasz's question, Barman and the author [1] prove that if Cohen reals are added then countable Fréchet-Urysohn spaces may all have π-weight less than the continuum. On the other hand, in the model constructed by Hrusak and Ramos Garcia [3], there are no examples with uncountable π-weight less than the continuum.

The following question was asked by Justin Moore during the author's talk at the 2012 Summer Topology Conference in Makato.

Question 1. Is there a countable Fréchet-Urysohn space with π-weight equal to $\mathfrak{b}$?

This question remains open. The result in this paper shows there is a countable Fréchet-Urysohn space with π-weight at least $\mathfrak{b}$.

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More generally one may ask about the spectrum of cardinals κ for which there is a countable Fréchet-Urysohn space with π-weight κ. It is consistent with $\omega_1 = \mathfrak{b} < \mathfrak{c}$ that the π-weight can not be larger than $\mathfrak{b}$ [1]. And we again mention that it is consistent with $\mathfrak{b} = \mathfrak{c} > \omega_1$ that there is no countable Fréchet-Urysohn space with π-weight strictly between ω and $\mathfrak{b}$ [3].

2. Preservation

This paper began as a proof that $\mathfrak{b} = \mathfrak{c}$ implied there was a Fréchet-Urysohn topology on ω which had uncountable π-weight. We then explored ideas to make the space indestructible with respect to proper forcings that did not add dominating reals and realized that we should be using $\omega^{<\omega}$ as the base space and to take advantage of the tree structure. This led to the notion of a down-sequential or $\downarrow$-sequential topology on $\omega^{<\omega}$.

As usual, $\omega^{<\omega}$ is the set of all finite functions into ω whose domain is a finite ordinal. For each $t \in \omega^{<\omega}$, let $t^{\downarrow} = \{s \in \omega^{<\omega} : s \subseteq t\}$. Similarly for a set $I \subset \omega^{<\omega}$, let $I^{\downarrow} = \bigcup\{t^{\downarrow} : t \in I\}$. It will also be convenient to let, for a set A, $A^{\uparrow} = \bigcup\{[t] : t \in A\}$, where $[t] = t^{\uparrow} = \{s \in \omega^{<\omega} : t \subseteq s\}$.

Definition 2.1. A topology τ on $\omega^{<\omega}$ is $\downarrow$-sequential if, for each $t \in \omega^{<\omega}$,

(1) $[t]$ is in τ and the sequence $\{t^\frown j : j \in \omega\}$ τ-converges to t,
(2) if a set $I \subset \omega^{<\omega}$ converges to t, then so does $I^{\downarrow}$.

Let $\{t_k : k \in \omega\}$ be a listing of $\omega^{<\omega}$ satisfying the coherence condition that if $t_k \subset t_m$, then $k < m$. For a function $g \in \omega^\omega$ and $t_k \in \omega^\omega$, let $g(t_k) = g(k)$. Similarly, for any $I \subset \omega^{<\omega}$ and integer m, we abuse notation and assume that $I \cap m$ is equal to $I \cap \{t_k : k < m\}$.

Let $\{g_\alpha : \alpha \in \mathfrak{b}\}$ be an unbounded mod finite family of strictly increasing functions from ω^ω. Ensure that $id_\omega < g_\alpha <^* g_\beta$ for $\alpha < \beta$, where id_ω denotes the identify function.

We have a π-weight preserving device.

Lemma 2.2. *Assume $X = (\omega^{<\omega}, \tau)$ is $\downarrow$-sequential and that for each $\alpha \in \mathfrak{b}$ there is a non-empty $U \in \tau$ such that for each $t \in U$, there is a $k > g_\alpha(t)$ with $t^\frown k \notin U$. Then X has π-weight at least $\mathfrak{b}$.*

Proof. For each $\alpha \in \mathfrak{b}$, let U_α be selected for g_α as per the statement in the Lemma. Suppose that $\Gamma \subset \mathfrak{b}$ has cardinality $\mathfrak{b}$. We will prove that $\bigcap\{U_\alpha : \alpha \in \Gamma\}$ has empty interior. Since $\mathfrak{b}$ is a regular cardinal, this will show that the π-weight of τ can not be less than $\mathfrak{b}$. Assume that $W \in \tau$ is non-empty and contained in U_α for all $\alpha \in \Gamma$. Let us note that since τ is $\downarrow$-sequential, W is an infinite set. Choose any k so that the collection $\{g_\alpha(k) : \alpha \in \Gamma\}$ is unbounded, and therefore $\{g_\alpha(n) : \alpha \in \Gamma\}$ is unbounded for all $n \geq k$. By simply increasing k, we may assume that $t_k \in W$. It follows then that the set $\{j : t_k^\frown j \notin W\}$ is infinite. This contradicts that τ is $\downarrow$-sequential. $\square$

We present a preservation result which, ultimately, was too weak for our purposes. The needed strengthening is buried in the proof of Lemma 3.4. To formulate our preservation result, we generalize the well-known α_1 notion formulated by Arhangelskii. Recall that a space X is α_1 if for each $x \in X$ and family $\{I_n : n \in \omega\}$ of sequences converging to x, there is a converging sequence I which mod finite contains each I_n.

Definition 2.3. Say that a space X is α_1^+ if whenever a sequence $\langle x_n : n \in \omega\rangle$ converges to a point x, and, for each n, I_n is a countable sequence converging to x_n, there is a sequence $\langle J_n\rangle_n$ so that $I_n \setminus J_n$ is finite for each n, and, for any infinite set $I \subset \bigcup_n J_n$, I converges to x so long as $I \cap J_n$ is finite for all n.

Theorem 2.4. *Suppose that there is a $\downarrow$-sequential topology on $\omega^{<\omega}$ which is α_1, α_1^+, and has the property described in Lemma 2.2. If $\mathbb{P}$ is a proper poset which does not add a dominating real, then in the forcing extension by $\mathbb{P}$, τ can be extended to a $\downarrow$-sequential Fréchet-Urysohn topology of uncountable π-weight.*

Since the proof shares, and even generated, many of the ideas of the main theorem, we defer the proof until after Theorem 3.5.

3. THE MAIN CONSTRUCTION

Let $\vec{g}$ denote the family $\{g_\alpha : \alpha \in \mathfrak{b}\}$ as detailed for Lemma 2.2. We begin by simply choosing a family of sets $\{U_\alpha, W_\alpha : \alpha \in \mathfrak{b}\}$, and we will use this family to construct a topology $\tau_{\vec{g}}$ on $\omega^{<\omega}$.

Lemma 3.1. *There is a family $\{U_\alpha, W_\alpha : \alpha \in \mathfrak{b}\}$ of subsets of $\omega^{<\omega}$ so that, for each $\alpha \in \mathfrak{b}$,*

(a) $\emptyset \in U_\alpha = U_\alpha^\downarrow$, $W_\alpha = \omega^{<\omega} \setminus U_\alpha$, $W_\alpha = W_\alpha^\uparrow$,

(b) for each $t \in U_\alpha$, there is a $j > g_\alpha(t)$ such that $t^\frown j \in W_\alpha$,

(c) for each $t \in U_\alpha$, the set $\bigcup_{j\in\omega}[t^\frown j] \cap g_\alpha(t^\frown j)$ is almost contained in U_α (note that $t^\frown j \in [t^\frown j] \cap g_\alpha(t^\frown j)$).

Proof. Fix any $\alpha < \mathfrak{b}$. We define, by recursion, $U_{\alpha,n}, W_{\alpha,n}$ so that, for each n,

(1) $t_n \in U_{\alpha,n} \cup W_{\alpha,n}$,

(2) $U_{\alpha,n}$ and $W_{\alpha,n}$ are disjoint,
(3) for $m < n$, $U_{\alpha,m} \subseteq U_{\alpha,n}$ and $W_{\alpha,m} \subseteq W_{\alpha,n}$,
(4) $U_{\alpha,n} = U^{\downarrow}_{\alpha,n}$ and $W_{\alpha,n} = W^{\uparrow}_{\alpha,n}$,
(5) for each $k \geq n$ either $[t_k] \subset W_{\alpha,n}$ or $[t_k] \cap W_{\alpha,n}$ is empty and $[t_k] \cap U_{\alpha,n}$ is finite,
(6) if $t_n \in U_{\alpha,n}$, then $\bigcup\{[t_n^\frown j] \cap g_\alpha(t_n) : j \in \omega\}$ is almost contained in $U_{\alpha,n+1}$,
(7) there is a $j > g_\alpha(t_n)$ such that $[t_n^\frown j] \subset W_{\alpha,n+1}$.

The properties listed above essentially describe how to construct the family. Once we have constructed the family, we simply set $U_\alpha = \bigcup_n U_{\alpha,n}$ and $W_\alpha = \bigcup_n W_{\alpha,n}$. We define $U_{\alpha,0}$ to be the singleton set $\{t_0\}$ and $W_{\alpha,0}$ is empty.

Given that $U_{\alpha,n}, W_{\alpha,n}$ satisfy the inductive conditions we define $U_{\alpha,n+1}$ and $W_{\alpha,n+1}$ as follows.

If $t_n \in W_{\alpha,n}$, then define

$$U_{\alpha,n+1} = U_{\alpha,n} \text{ and } W_{\alpha,n+1} = \begin{cases} W_{\alpha,n} & \text{if } t_{n+1} \in U_{\alpha,n} \\ W_{\alpha,n} \cup [t_{n+1}] & \text{if } t_{n+1} \notin U_{\alpha,n} \end{cases} .$$

Note that if $t_{n+1} \notin U_{\alpha,n}$, then $[t_{n+1}] \cap U_{\alpha,n}$ is empty.

If $t_n \in U_{\alpha,n}$, then choose any $\ell > g_\alpha(t_n)$ such that $t_n^\frown \ell \notin U_{\alpha,n}$ and define

$$U_{\alpha,n+1} = U_{\alpha,n} \cup \bigcup \left\{ [t_n^\frown j] \cap g_\alpha(t_n^\frown j) : \ell \neq j \in \omega \right\}$$

and

$$W_{\alpha,n+1} = \begin{cases} W_{\alpha,n} \cup [t_n^\frown \ell] & \text{if } t_{n+1} \in U_{\alpha,n+1} \\ W_{\alpha,n} \cup [t_n^\frown \ell] \cup [t_{n+1}] & \text{if } t_{n+1} \notin U_{\alpha,n+1} \end{cases} .$$

It is evident that $U_{\alpha,n+1} \cap W_{\alpha,n+1}$ is empty. Similarly, it is immediate that $W_{\alpha,n+1} = W^{\uparrow}_{\alpha,n+1}$ and $t_{n+1} \in U_{\alpha,n+1} \cup W_{\alpha,n+1}$. Now choose any $t_m \in U_{\alpha,n+1} \setminus U_{\alpha,n}$, i.e. $t_m \in [t_n^j] \cap g_\alpha(t_n^\frown j)$ for some $j \neq \ell$. Then $t_m^{\downarrow} \cap [t_n] \subset U_{\alpha,n+1}$ since we have assumed that if $t_k \subset t_m$, then $k < m$. This implies that $U_{\alpha,n+1} = U^{\downarrow}_{\alpha,n+1}$. Suppose that $k \geq n+1$ and first suppose that $[t_k] \cap W_{\alpha,n+1}$ is not empty. If $[t_k] \cap W_{\alpha,n}$ is not empty, then $[t_k] \subset W_{\alpha,n+1}$ follows from the induction hypotheses. Otherwise we consider the two cases where $[t_k]$ meets either $[t_n^\frown \ell]$ or $[t_{n+1}]$. By the coherence of the indexing, we have that t_k is not a strict predecessor of either $t_n^\frown \ell$ or t_{n+1}. Thus, if $[t_k]$ meets either of these sets, it is contained in them. So, now we may assume that $[t_k]$ is disjoint from $W_{\alpha,n+1}$ and we have to show that $[t_k] \cap U_{\alpha,n+1}$ is finite. By induction, $[t_k] \cap U_{\alpha,n}$ is finite, and, again, since t_k is not a predecessor of t_n, we have that $[t_k]$ meets at most one set of the form $[t_n^\frown j]$. Thus it follows that $[t_k] \cap U_{\alpha,n+1}$ is finite. $\square$

Let τ_0 be the rational topology on $\omega^{<\omega}$ that has the family $\{[t], \omega^{<\omega}\setminus [t] : t \in \omega^{<\omega}\}$ as a subbase. We let $\tau_{\vec{g}}$ be the topology that is generated by the collection $\tau_0 \cup \{U_\alpha : \alpha \in \mathfrak{b}\}$. This topology will have the property from Lemma 2.2. It is useful to observe that, for each $\alpha \in \mathfrak{b}$, $W_\alpha \in \tau_{\vec{g}}$ because $W_\alpha = W_\alpha^\uparrow$. Let us check that this topology is $\downarrow$-sequential, although we note that it may not be Fréchet-Urysohn.

Lemma 3.2. *The topology $\tau_{\vec{g}}$ is $\downarrow$-sequential.*

Proof. Since the family $\tau_0 \cup \{U_\alpha : \alpha \in \mathfrak{b}\}$ forms a subbase, property c of Lemma 3.1 ensures that $\{t^\frown j : j \in \omega\}$ converges to t for all $t \in \omega^{<\omega}$. Now suppose that some $I \subset \omega^{<\omega}$ $\tau_{\vec{g}}$-converges to t. To show that $I^\downarrow$ also converges, it suffice to show that $I^\downarrow \setminus U_\alpha$ is finite for all $\alpha \in \mathfrak{b}$ such that $t \in U_\alpha$. Since $U_\alpha = U_\alpha^\downarrow$ for all $\alpha < \mathfrak{b}$, it is evident that $(I \cap U_\alpha)^\downarrow \subset U_\alpha$ for any α. If $t \in U_\alpha$, then $(I \setminus U_\alpha)^\downarrow$ is finite and so it follows that $I^\downarrow$ is almost contained in U_α. This completes the proof. □

We need a definition and a key Lemma before proving the main theorem.

Definition 3.3. For each $t \in \omega^{<\omega}$, let $\mathcal{I}_t$ denote the family of infinite subsets I of $\omega^{<\omega}$ which $\tau_{\vec{g}}$-converge to t. For $A \subset \omega^{<\omega}$, define $A^{(1)}$ to be the set $A \cup \{t : \ (\exists I \in \mathcal{I}_t) \ \ I \subset A\}$.

Lemma 3.4. *In $\tau_{\vec{g}}$, for each $A \subset \omega^{<\omega}$, the set $\left(A^{(1)}\right)^{(1)}$ is equal to $A^{(1)}$.*

Proof. Suppose that $\{x_n : n \in \omega\} \subset A^{(1)}$ and is in $\mathcal{I}_t$. If $\{x_n : n \in \omega\} \cap A$ is infinite, then $t \in A^{(1)}$, so we may assume that each x_n is not in A. For each n, there is an $I_n \subset A$ such that $I_n \in \mathcal{I}_{x_n}$. We may assume that $\{x_n : n \in \omega\}$ is contained in $[t] \setminus \{t\}$. For each n, choose j_n so that $t^\frown j_n \subseteq x_n$. We may assume, by passing to a subsequence, that $j_n < j_m$ for $n < m$. Let $B = \{\beta \in \mathfrak{b} : t \in U_\beta\}$, and for each $\beta \in B$, fix a function $f_\beta \in \omega^\omega$ so that $I_n \setminus f_\beta(n) \subset U_\beta$ for all but finitely many $n \in \omega$. Since $\beta < \mathfrak{b}$, we may choose the f_β's by recursion and arrange that for all $\gamma \in B \cap \beta$, $f_\gamma <^* f_\beta$.

Choose any $\alpha_0 \in \mathfrak{b}$ large enough so that $L_0 = \{n : I_n \cap g_{\alpha_0}(t^\frown j_n) \neq \emptyset\}$ is infinite. Now choose α_1 large enough so that

$$L_1 = \{n \in L_0 : I_n \cap g_{\alpha_1}(t^\frown j_n) \setminus (f_{\alpha_0}(n) + g_{\alpha_0}(t^\frown j_n)) \neq \emptyset\}$$

is also infinite. By recursion, similarly choose $\alpha_{\ell+1}$ so that

$$L_{\ell+1} = \{n \in L_\ell : I_n \cap g_{\alpha_{\ell+1}}(t^\frown j_n) \setminus (f_{\alpha_\ell}(n) + g_{\alpha_\ell}(t^\frown j_n)) \neq \emptyset\}$$

is infinite.

Now set $\mu = \sup_\ell \alpha_\ell$ and choose any infinite $L \subset L_0$ that is mod finite contained in each L_ℓ. For each $n \in L$, let a_n be the element of

$I_n \cap g_\mu(t^\frown j_n)$ with maximum index, and let $I = \{a_n : n \in L\}$. We show that $I \in \mathcal{I}_t$, and conclude that $t \in A^{(1)}$.

Suppose that $\beta \in B \cap \mu$. Choose ℓ so that $\beta < \alpha_\ell$. We have that there is some m_β such that $U_\beta \supset I_n \setminus g_{\alpha_\ell}(t^\frown j_n)$ for each $n \in L \setminus m_\beta$. Similarly, there is an m_ℓ so that $g_{\alpha_\ell}(t^\frown j_n) < g_\mu(t^\frown j_n)$ for all $n > m_\ell$. Thus, it follows that $I \subset^* U_\beta$.

Now suppose that $\mu \leq \beta$ and that $\beta \in B$. Choose m so that $g_\mu(t^\frown j) \leq g_\beta(t^\frown j)$ for all $j > m$. In this case, our construction of U_β, see Lemma 3.1.c, has ensured that, for all but finitely many n with $j_n > m$, U_β contains $I_n \cap g_\mu(t^\frown j_n)$. Thus, U_β almost contains I. □

Theorem 3.5. *There is a Fréchet-Urysohn $\downarrow$-sequential topology τ on $\omega^{<\omega}$ with π-weight at least $\mathfrak{b}$.*

Proof. For each set $A \subset \omega^{<\omega}$, let $W_A = \bigcup\{[t] : t \in A^{(1)}\}$. Observe that $W_A = W_A^\uparrow = (A^{(1)})^\uparrow$. Also, let $U_A = \omega^{<\omega} \setminus W_A$ and observe that $U_A = U_A^\downarrow$. The topology τ has the family $\tau_{\vec{g}} \cup \{U_A : A \subset \omega^{<\omega}\}$ as a subbase.

We first check that if $I \subset \omega^{<\omega}$ τ-converges to t, then so does $I^\downarrow$. Since each W_A is open in $\tau_{\vec{g}}$, we consider an A with $t \in U_A$. Therefore $I \setminus U_A$ is finite, and also $(I \setminus U_A)^\downarrow$ also finite. But now, since $U_A = U_A^\downarrow$, we obviously have that $(I \cap U_A)^\downarrow \subset U_A$. This shows that $I^\downarrow \setminus U_A$ is finite.

Next we prove that for each $t \in \omega^{<\omega}$ and each $I \in \mathcal{I}_t$, we have that I will τ-converge to t. It will then follow that τ is $\downarrow$-sequential and, by Lemma 2.2, has π-weight at least $\mathfrak{b}$. To show that I will τ-converge to t it suffices to show that $I \setminus U_A$ is finite for any A such that $t \in U_A$. Assume that $t \in U_A$, and therefore that $t \notin A^{(1)}$. Since $t \notin A^{(1)} = (A^{(1)})^{(1)}$ and $I^\downarrow$ converges to t, we have that $I^\downarrow \cap A^{(1)}$ is finite. By removing a finite set from I (hence with no loss of generality) we may assume that $I^\downarrow \cap A^{(1)}$ is empty. This is equivalent to saying that $I \cap W_A$ is empty, and therefore we have shown that I is (mod finite) contained in U_A.

Finally we make the easy observation that τ is Fréchet-Urysohn. Assume that, for some $t \in \omega^{<\omega}$ and $A \subset [t]$, we have that $t \notin A$ and no sequence contained A τ-converges to t. Since each $\tau_{\vec{g}}$-converging sequence remains τ-converging, we have that $t \notin A^{(1)}$. Therefore t is not in the closure of A since $t \in U_A$ and $U_A \cap A = \emptyset$. □

We finish the paper with a proof of Theorem 2.4

Proof of Theorem 2.4. In the ground model, let $\mathcal{I}_t$ denote the family of sequences that τ-converge to the point $t \in \omega^{<\omega}$. In the forcing extension we define, for $A \subset \omega^{<\omega}$, the set

$$A^{(1)} = A \cup \{t \in \omega^{<\omega} : \ (\exists I \in \mathcal{I}_t) \ \ |I \cap A| = \omega\} \ .$$

For each $A \subset \omega^{<\omega}$, we let $W_A = \bigcup\{[t] : t \in A^{(1)}\}$ and $U_A = \omega^{<\omega} \setminus W_A$.

We let $\tilde{\tau}$ be the topology that is generated by $\tau \cup \{U_A : A \subset \omega^{<\omega}\}$. We will show that it is Fréchet-Urysohn and $\downarrow$-sequential.

The key property is to again show that $(A^{(1)})^{(1)}$ is equal to $A^{(1)}$ for each $A \subset \omega^{<\omega}$. To do so, assume that $t \in (A^{(1)})^{(1)} \setminus A$. Choose $\{x_n : n \in \omega\} \in \mathcal{I}_t$ so that $A^{(1)} \cap \{x_n : n \in \omega\}$ is infnite. Since we are trying to proof that $t \in A^{(1)}$, we may as well assume that $A \cap \{x_n : n \in \omega\}$ is empty.

We will use the fact that infinitely many of the x_n are in $A^{(1)}$ to choose a collection of sequences from the corresponding $\mathcal{I}_{x_n}$. However we must now be more careful about the fact that we are in a (proper) forcing extension. We will use the well-known property that every countable subset of the ground model is contained in a countable set from the ground model. By this property, we have, in the ground model, a sequence $\{I(n,m) : n,m \in \omega\}$ so that $\{I(n,m) : m \in \omega\} \subset \mathcal{I}_{x_n}$ for each n, and which has the property that for each n such that $x_n \in A^{(1)}$, there is an m such that $I(n,m) \cap A$ is infinite. By applying the α_1-property, we can find, for each n, a single $I_n \in \mathcal{I}_{x_n}$ so that $I(n,m) \subset^* I_n$ for all m. We do so in the ground model, and so we may have that $\{I_n : n \in \omega\}$ is also in the ground model, and that the elements are pairwise disjoint.

Next, by applying the α_1^+-property (in the ground model) we may assume that any infnite set $I \subset \bigcup_n I_n$, from the ground model, such that $I \cap I_n$ is finite for all n, will be a member of $\mathcal{I}_t$. Finally, a simple application of the fact that $\mathbb{P}$ does not add a dominating real shows that A will meets some such I in an infinite set. This completes the proof that $t \in A^{(1)}$.

Now we can conclude, as in the proof of Theorem 3.5, that for each $t \in \omega^{<\omega}$ and $I \in \mathcal{I}_t$, I will $\tilde{\tau}$-converge to t. It follows from this that $\tilde{\tau}$ is $\downarrow$-sequential and, by Lemma 2.2, has uncountable π-weight (although $\mathbb{P}$ may collapse cardinals it does preserve the property of being uncountable).

The proof that it is Fréchet-Urysohn is certainly immediate. If $A \subset [t]$ and $t \notin A^{(1)}$, then t has the neighborhood U_A which is disjoint from A. □

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