A NEW LINDELOF SPACE WITH POINTS G_δ

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ABSTRACT. We prove that $\diamondsuit^*$ implies there is a zero-dimensional Hausdorff Lindelöf space of cardinality $2^{\aleph_1}$ which has points G_δ. In addition, this space has the property that it need not be Lindelöf after countably closed forcing.

1. Introduction

The set-theoretic principle $\diamondsuit^*$ was formulated by Jensen ([2, p128] and [6, VI #16, p181]).

Definition 1.1. $\diamondsuit^*$ is the statement that there are countable $\mathcal{A}_\alpha \subset \mathcal{P}(\alpha)$, for $\alpha \in \omega_1$, such that for every $A \subset \omega_1$ there is a cub $C \subset \omega_1$ such that $A \cap \alpha \in \mathcal{A}_\alpha$ for all $\alpha \in C$.

Definition 1.2. [7] A Lindelöf space is *indestructible* if it remains Lindelöf after any countably closed forcing. A Lindelöf space is *destructible* if it is not indestructible.

Notice that $\diamondsuit^*$ implies CH but is consistent with $2^{\aleph_1}$ being arbitrarily large ([6, VII (H18)-(H20) p249]). As is well-known, Shelah proved, using forcing, that it is consistent with CH to have Hausdorff zero-dimensional Lindelöf spaces with points G_δ which had cardinality $\aleph_2$ (see [5]). In establishing the consistency with CH of there being no such spaces with cardinality strictly between $\aleph_1$ and $2^{\aleph_1}$, Shelah also established the relevance of the notion of a space being destructible (see [5]). I. Gorelič produced another forcing construction to establish the consistency of the existence of Lindelöf spaces with points G_δ which had cardinality $2^{\aleph_1}$ while allowing $2^{\aleph_1}$ to be as large as desired. F. Tall [7] points out that each of these examples is indestructible.

In this note we will prove

Theorem 1.3. $\diamondsuit^*$ *implies there is a space that is zero-dimensional Hausdorff Lindelöf destructible of cardinality* $2^{\aleph_1}$ *and that has points* G_δ.

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This is the first consistent example of a Lindelöf Hausdorff destructible space with points G_δ.

Question 1. Does every Lindelöf Hausdorff destructible space have cardinality at least $2^{\aleph_1}$?

2. A Lindelöf tree

We build our space X using the structure $2^{\leq\omega_1}$. For each $t \in 2^{\leq\omega_1}$ let $[t]$ denote the set $\{s \in 2^{\leq\omega_1} : t \subseteq s\}$. For any $t \in 2^{<\omega_1}$ such that $dom(t)$ is a successor, let $t^\dagger$ be the other immediate successor of the immediate predecessor of t, i.e. t and $t^\dagger$ are the two immediate successors of $t \cap t^\dagger$.

Let σ denote the standard topology on $2^{\leq\omega_1}$ that has the family

$$\{\emptyset\} \cup \{[\rho \restriction \xi + 1] : \xi \in \omega_1\} \cup \{[t \restriction \xi + 1] \setminus ([t^\frown 0] \cup [t^\frown 1]) : \xi \in dom(t), t \in 2^{<\omega_1}\}$$

as a subbase. Of course t is isolated and $[t]$ is clopen for all t such that $dom(t) \in \omega_1$ is not a limit.

This next lemma is very well-known but since it is crucial to our construction, we include a proof.

Lemma 2.1. *The topology σ on $2^{\leq\omega_1}$ is compact zero-dimensional and Hausdorff. Also, for each $\alpha \in \omega_1$, $2^{\leq\alpha}$ is a compact first-countable subspace.*

Proof. One standard method of proof is to construct a canonical embedding of $2^{\leq\omega_1}$ into $2^{2^{<\omega_1}}$ and show that the range is closed in the product topology. However we will give a more direct proof. Certainly σ is zero-dimensional since the members of the generating subbase are easily shown to also be closed. If s, t are distinct elements of $2^{\leq\omega_1}$, we show they have disjoint neighborhoods. If $t \subset s$, then, for any $\xi \in dom(t)$, $t \in [t \restriction \xi + 1] \setminus ([t^\frown 0] \cup [t^\frown 1])$ and $s \in ([t^\frown 0] \cup [t^\frown 1])$. Otherwise, we may assume that $y = s \cap t$ is strictly below each of s and t, and note that $[y^\frown 0]$ and $[y^\frown 1]$ are disjoint and each contains one of s, t.

Now assume that $\mathcal{U}$ is a cover by basic open sets. Let $T_\mathcal{U}$ denote the set of all $t \in 2^{<\omega_1}$ such that there is no finite subcollection of $\mathcal{U}$ whose union contains $[t]$. If $\emptyset \notin T_\mathcal{U}$ then $\mathcal{U}$ has a finite subcover. So assume that $T_\mathcal{U}$ is not empty. Observe that if $t \in T_\mathcal{U}$, then $t \restriction \xi \in T_\mathcal{U}$ for all $\xi \in dom(t)$. For each $\rho \in 2^{\omega_1}$, there is a $\xi \in \omega_1$ such that $[\rho \restriction \xi + 1] \in \mathcal{U}$, so we have that $T_\mathcal{U}$ is a subtree of $2^{<\omega_1}$ with no uncountable branch. Similarly, $T_\mathcal{U}$ has no maximal elements, since if each of $[t^\frown 0]$ and $[t^\frown 1]$ are covered by a finite union from $\mathcal{U}$, then certainly, $[t] = \{t\} \cup [t^\frown 0] \cup [t^\frown 1]$ is as well. Choose any maximal chain

$\{t_\xi : \xi \in \alpha\} \subset T_{\mathcal{U}}$ and let $t = \bigcup\{t_\xi : \xi \in \alpha\}$. Since T has no maximal elements, t is on a limit level and $\mathcal{U}$ contains a finite cover of $[t]$. But in addition, there is some $\xi < \alpha$ such that $[t_\xi] \setminus ([t^\frown 0] \cup [t^\frown 1])$ is in $\mathcal{U}$. This is a contradiction, since it shows that $\mathcal{U}$ has a finite cover of $[t_\xi]$ – contradicting that $t_\xi \in T_{\mathcal{U}}$.

It is obvious that $2^{\leq\alpha}$ is a closed subset of $2^{\leq\omega_1}$, and, for each non-isolated $t \in 2^{\leq\alpha}$, the collection $\{[t \restriction \xi+1] \setminus ([t^\frown 0] \cup [t^\frown 1]) : \xi \in dom(t)\}$ is a neighborhood base at t. □

Next we consider Lindelöf subspaces.

Lemma 2.2. *If $Y \subset 2^{<\omega_1}$ satisfies that $Y \cap 2^\alpha$ is countable for all $\alpha \in \omega_1$, then the complement of Y in $2^{\leq\omega_1}$ is Lindelöf in the topology induced by σ.*

Proof. Assume that $\mathcal{U}$ is a cover of $2^{\leq\omega_1} \setminus Y$ by basic clopen sets. Let us again set $T_{\mathcal{U}}$ to be the set of $t \in 2^{<\omega_1}$ such that $\mathcal{U}$ contains a countable cover of $[t] \setminus Y$. As in the proof of Lemma 2.1, $T_{\mathcal{U}}$ (if non-empty) is downwards closed, has no maximal elements, and no uncountable branches. Now let us show that $T_{\mathcal{U}}$ is branching. Suppose that $T_{\mathcal{U}} \cap [t]$ is a chain. Then it is a countable chain (with supremum in Y), and let $\{t_\gamma : \gamma \in \alpha\}$ be an enumeration in increasing order and let t_α denote the union. For each $\gamma \in \alpha$, we have that $t^\dagger_{\gamma+1}$ is not in $T_{\mathcal{U}}$, and so there is a countable $\mathcal{U}_\gamma \subset \mathcal{U}$ whose union covers $(\{t_\gamma\} \cup [t^\dagger_{\gamma+1}]) \setminus Y$. Furthermore there is a countable $\mathcal{U}_\alpha \subset \mathcal{U}$ that covers $[t_\alpha] \setminus Y$. It should be clear that $\bigcup\bigcup\{\mathcal{U}_\gamma : \gamma \leq \alpha\}$ covers $[t]$.

Now we have established that $T_{\mathcal{U}}$ is branching and has no maximal elements. Set $t_\emptyset = \emptyset$ and by recursion on $s \in 2^{<\omega}$, choose $t_s \in T_{\mathcal{U}}$ so that for $s \in 2^{<\omega}$, $t_s \subset (t_{s^\frown 0} \cap t_{s^\frown 1})$ and $t_{s^\frown 0} \perp t_{s^\frown 1}$. Let $\delta \in \omega_1$ so that $\{t_s : s \in 2^{<\omega}\} \subset 2^{<\delta}$. Choose any $x \in 2^\omega$ so that $t_x = \bigcup_n t_{x\restriction n} \in 2^{\leq\delta} \setminus Y$. By construction, $dom(t_x)$ is a limit ordinal. Choose any $\xi \in dom(t_x)$ so that $[t_x \restriction \xi + 1] \setminus ([t_x^\frown 0] \cup [t_x^\frown 1])$ is contained in some $U \in \mathcal{U}$. Fix n so that $\xi < dom(t_{x\restriction n})$, and choose any $s \in 2^{<\omega}$ so that $x \restriction n \subset s$ and $s \not\subset x$. Finally we can conclude that $T_{\mathcal{U}}$ must be empty, since we have that $[t_s] \subset U$. □

3. POINTS G_δ

Let $\{\mathcal{A}_\alpha : \alpha \in \omega_1\}$ be a sequence as in Definition 1.1 witnessing the statement $\diamondsuit^*$.

Definition 3.1. For each limit $\alpha \in \omega_1$ let $S_\alpha = \{t \in 2^\alpha : t^{-1}(1) \in \mathcal{A}_\alpha\}$. For $0 < \alpha$ not a limit, let S_α be the empty set, and let $S_0 = \{\emptyset\}$.

Lemma 3.2. *For each $\rho \in 2^{\omega_1}$, there is a cub $C_\rho \subset \omega_1$ such that $C_\rho \subset \{\alpha : \rho \restriction \alpha \in S_\alpha\}$.*

Proof. This is just a restatement of the fact that the sequence $\{\mathcal{A}_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit^*$ sequence. □

For each $\rho \in 2^{\omega_1}$ fix a cub C_ρ as in Lemma 3.2.

Proposition 3.3. *For each $\rho \in 2^{\omega_1}$, there is a countable-to-one function $f_\rho : \omega_1 \to 2^\omega$ so that for each $x \in 2^\omega$, there is a $\delta_x \in C_\rho \cup \{0\}$ and $\delta_x < \gamma_x \in C_\rho$ so that $f_\rho^{-1}(x)$ is equal to the interval $[\delta_x, \gamma_x)$.*

Proof. First let $\{\delta_x : x \in 2^\omega\}$ be any enumeration of $C_\rho \cup \{0\}$. For each $x \in 2^\omega$, define γ_x to be $\min(C_\rho \setminus [0, \delta_x])$. Assume that $\delta_x < \delta_y$. Then it is obvious that $\gamma_x \leq \delta_y$. Now define f_ρ so that $f_\rho([\delta_x, \gamma_x)) = \{x\}$ for all $x \in 2^\omega$. □

Now we are ready to prove our main theorem.

Proof of Theorem 1.3. Fix the sequence $\{S_\alpha : \alpha \in \omega_1\}$ as in Definition 3.1, and let Y equal the union of this family. Our space X will have as its base set $(2^{\omega_1} \times 2^\omega) \cup 2^{<\omega_1} \setminus Y$. We will use the fact (Lemma 2.2) that $2^{\leq\omega_1} \setminus Y$ is Lindelöf when using the topology σ. Recall that for each $\rho \in 2^{\omega_1}$ and $\xi \in \omega_1$, $[\rho \restriction \xi + 1] \setminus Y$ is a clopen set. In this proof, for any $s \in 2^{<\omega}$, we will use $[s]_{2^\omega}$ to denote the set $\{x \in 2^\omega : s \subset x\}$.

We define a clopen base for the topology τ. For each $t \in 2^{<\omega_1}$, we use the notation $[t]_X$ to denote

$$[t]_X = [t] \cap (2^{<\omega_1} \setminus Y) \quad \cup \quad ([t] \cap 2^{\omega_1}) \times 2^\omega \ .$$

Again, for each $\rho \in 2^{\omega_1}$ and each $\xi \in \omega_1$, the set $[\rho \restriction \xi + 1]_X$ is declared to be a clopen set in τ (i.e. $[\rho \restriction \xi + 1]_X$ and its complement are in τ). Let us observe that for $t \in Y$, $[t]_X$ is equal to $[t^\frown 0]_X \cup [t^\frown 1]_X$ and so is also clopen.

Next, for each $\rho \in 2^{\omega_1}$ and each $x \in 2^\omega$, let $f_\rho^{-1}(\{x\})$ be denoted as $[\delta_x^\rho, \gamma_x^\rho)$ as per Proposition 3.3. For $s \in 2^{<\omega}$, and $\gamma \in C_\rho$, we define

$$U(\rho, s, \gamma) = (\{\rho\} \times [s]_{2^\omega}) \cup \bigcup \{[\rho \restriction \delta_x^\rho]_X \setminus [\rho \restriction \gamma_x^\rho]_X : x \in [s]_{2^\omega} \text{ and } \gamma \leq \delta_x^\rho\} \ .$$

When the choice of ρ is clear from the context, we will use δ_x, γ_x as referring to $\delta_x^\rho, \gamma_x^\rho$. The topology τ will also contain each such $U(\rho, s, \gamma)$. Notice that, for each $\gamma \in C_\rho$ and each $n \in \omega$, the family $\{U(\rho, s, \gamma) : s \in 2^n\}$ is a partition of the clopen set $[\rho \restriction \gamma]_X$, and so each is clopen.

Claim 1. For each $t \in 2^{<\omega_1} \cap X$, the family

$$\{[t \restriction \xi + 1]_X \setminus ([t^\frown 0]_X \cup [t^\frown 1]_X) : \xi \in dom(t)\}$$

is a neighborhood base for t.

To show this we must consider some ρ, s, γ such that $t \in U(\rho, s, \gamma)$ and $\gamma \in C_\rho$. There is a unique $x \in 2^\omega$ such that $t \in [\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$. Since $\rho \restriction \delta_x \in Y$, we know that $t \neq \rho \restriction \delta_x$. Since $[t \restriction \delta_x + 1] \setminus ([t^\frown 0] \cup [t^\frown 1])$ is contained in $[\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$, we have proven the claim.

Claim 2. For each $\rho \in 2^{\omega_1}$ and $z \in 2^\omega$, the point (ρ, z) is the only element of the intersection of the family $\{U(\rho, z \restriction n, \gamma_z) : n \in \omega\}$.

It is clear that for any $\gamma \in C_\rho$, $U(\rho, s, \gamma) \cap (\{\rho\} \times 2^\omega)$ is equal to $\{\rho\} \times [s]_{2^\omega}$. Now suppose that $\psi \in 2^{\omega_1} \setminus \{\rho\}$ and $t \in X \cap 2^{<\omega_1}$. Let $\rho \restriction \xi_\psi = \psi \cap \rho$ and $\rho \restriction \xi_t = t \cap \rho$. Choose any $s \in 2^{<\omega}$ so that $z \in [s]_{2^\omega}$ and neither of $f_\rho(\xi_t)$, $f_\rho(\xi_\psi)$ are in $[s]_{2^\omega} \setminus \{z\}$. But now, if $\gamma_z \leq \xi$ then $f_\rho(\xi) \neq z$. Therefore, for all $x \in [s]_{2^\omega}$ with $\gamma_z \leq \gamma_x$, we have that $\{\xi_t, \xi_\psi\}$ is disjoint from $[\delta_x, \gamma_x)$, and therefore $[\rho \restriction \delta_x]_X \setminus [\rho \restriction \gamma_x]_X$ is disjoint from $\{t\} \cup (\{\psi\} \times 2^\omega)$. This completes the proof of the claim.

Let Φ be the canonical map from X (with topology τ) onto $2^{\leq\omega_1} \setminus Y$ (with topology σ). That is, $\Phi(t) = t$ for all $t \in X \cap 2^{<\omega_1}$, and $\Phi((\rho, x)) = \rho$ for all $\rho \in 2^{\omega_1}$ and $x \in 2^\omega$. It is evident that point preimages under Φ are compact. It is immediate that Φ is continuous since $\Phi^{-1}[t] = [t]_X$ for all $t \in 2^{<\omega_1}$. This is also useful to show that Φ is closed. By [3, 1.4.13] it is sufficient to show that if $U \subset X$ is an open set containing a fiber $\Phi^{-1}(t)$ for some $t \in 2^{\leq\omega_1} \setminus Y$, then there is a neighborhood W of t such that $\Phi^{-1}(W)$ is contained in U. Let then, $t \in 2^{\leq\omega_1} \setminus Y$ and suppose that $U \subset X$ is an open set containing $\Phi^{-1}(t)$. This is obvious if $t \in 2^{<\omega_1}$, so suppose that $t = \rho \in 2^{\omega_1}$. Since $\Phi^{-1}(\rho)$ is simply $\{\rho\} \times 2^\omega$, it is clear that there is $\gamma \in C_\rho$ and $n \in \omega$ such that $U(\rho, s, \gamma) \subset U$ for each $s \in 2^n$. As remarked above, this implies that $[\rho \restriction \gamma]_X$ is contained in U. Since $[\rho \restriction \gamma]$ is a neighborhood of ρ and, again, $[\rho \restriction \gamma]_X = \Phi^{-1}([\rho \restriction \gamma])$, this completes the proof that Φ is a closed mapping.

Now that we have established that there is a perfect map (continuous, closed, point-preimages compact) from X onto a Lindelöf space, we conclude [3, 3.8.8] that X is also Lindelöf.

Finally, it is immediate that the forcing notion $2^{<\omega_1}$ will introduce a new member ψ of $2^{<\omega_1}$. Since the forcing adds no new members to $2^{<\omega_1}$, the set $\{\psi \restriction \xi + 1 : \xi \in \omega_1\}$ is a subset of X and has no complete accumulation point in X. We conclude that X is not Lindelöf in the forcing extension. □

4. Remarks on Consistency

Let us consider the following principle which is evidently weaker than $\diamondsuit^*$.

Definition 4.1. $w\diamondsuit^*$ is the statement that there is a subset $Y \subset 2^{<\omega_1}$ such that

(1) for each $\alpha \in \omega_1$, $Y \cap 2^{\leq\alpha}$ contains no perfect set,
(2) for each $\rho \in 2^{\omega_1}$, there is a cub $C_\rho \subset \omega_1$ such that $\{\rho \restriction \gamma : \gamma \in C_\rho\}$ is contained in Y.

Say that the set Y is a $w\diamondsuit^*$ sequence.

The hypothesis "CH and $w\diamondsuit^*$" is sufficient to prove Theorem 1.3. It is probable that this is a weaker statement than $\diamondsuit^*$ but, just as a $\diamondsuit^*$ sequence is destroyed by forcing with $2^{<\omega_1}$ (see [6, p300 J5]), so too is a $w\diamondsuit^*$-sequence. This implies that $w\diamondsuit^*$ fails in the models in which it has been shown that any Lindelöf points G_δ space of cardinality greater than ω_1 must be destructible. In particular, such a model (see [7]) is obtained by countably closed forcing that collapses a supercompact cardinal to $\aleph_2$. It is a reasonable conjecture to hope that in that model Lindelöf spaces with points G_δ will have cardinality at most $\aleph_1$, and the approach till now has focussed on trying to show that there are (in ZFC) no destructible Lindelöf spaces with points G_δ. However there is a stronger property that such spaces must have which we now define.

Definition 4.2. Say that a regular Lindelöf space with points G_δ is *reconstructible* if it is destructible and, there is a countably closed poset so that in the forcing extension, it is no longer Lindelöf but it can be embedded into a regular Lindelöf space with points G_δ.

It may not be as natural, but there is a similar, but weaker, property which is the property we are really after. We use the word elementarily in reference to the set-theoretic notion of elementary extensions of models.

Definition 4.3. Say that a regular Lindelöf space X with points G_δ is *elementarily reconstructible* if there is a countably closed poset so that in the forcing extension, it is no longer Lindelöf and there is a regular Lindelöf space Y with points G_δ that has a dense subspace Z and a continuous mapping f from Z onto X and satisfies that f is a homeomorphism on the pre-image of the points with character at most ω_1.

Clearly an elementarily reconstructible space that has character at most ω_1 will be reconstructible. A reader of Tall's paper [7] will realize that in the forcing extension mentioned above, if there is a Lindelöf space with points G_δ and character at most ω_1 which has cardinality greater than ω_1 then that space will be reconstructible and that the cardinal collapsing poset described above will be the witness. It is also true, but not as easily checked, that each Lindelöf space with points G_δ and cardinality greater than ω_1 will be elementarily reconstructible.

On the other hand, not only does the poset $2^{<\omega_1}$ render our space to be non-Lindelöf, it also creates a subspace which can not be embedded into a Lindelöf space with points G_δ.

Proposition 4.4. *If $Y \subset 2^{<\omega_1}$ is a $w\diamondsuit^*$-sequence, then in the forcing extension by $2^{<\omega_1}$, there is a $\psi \in 2^{\omega_1}$ such that $T_\psi(Y) = \{\alpha : \psi \restriction \alpha \in Y\}$ is stationary.*

Since $\{\psi \restriction \alpha : \alpha \in T_\psi(Y)\}$, as a subspace of $2^{<\omega_1}$, is homeomorphic to $T_\psi(Y)$ as a subspace of the ordinal ω_1, this next proposition shows that our space X is not reconstructible.

Proposition 4.5. *If S is a stationary subset of ω_1, then S can not be embedded in a Lindelöf space with points G_δ.*

Proof. Assume that Z is a Lindelöf space with S as a subspace. Since S can not equal a union of non-stationary sets, and Z is Lindelöf, there is a point z of Z with the property that every neighborhood of z meets S in a non-stationary set. Let us show that z is not a G_δ-point. Let $\{U_n : n \in \omega\}$ be a family of open subsets of Z, each meeting S in a non-stationary set. Since S is a subspace, $S \setminus U_n$ is a closed subset of S that misses the stationary set U_n. Of course this implies that $S \setminus U_n$ is countable. This shows that each G_δ of Z that contains z will also contain many points of S. □

We close with the obvious question.

Question 2. Does CH imply there is a regular Lindelöf space with points G_δ that is elementarily reconstructible?

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