COMPLETELY SEPARATED IN THE RANDOM AND COHEN MODELS

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Abstract. It is shown that in the model obtained by adding supercompact many random reals every C^*-embedded subset of a first countable space (even with character smaller than $\mathfrak{c}$) is C-embedded. It is also proved that if two ground model sets are completely separated after adding a random real then they were completely separated originally while CH implies that the Cohen poset does not have this property.

1. Introduction

Ohta and Yamazaki asked [7] if every C^*-embedded subset of a first countable space is C-embedded. It is shown in [4] that a counterexample can be derived from the assumption $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ and that if the Product Measure Extension Axiom (PMEA) holds then the answer is affirmative in some special cases.

We will show in Section 3 that in the model obtained by adding supercompact many random reals Ohta and Yamazaki's question has a positive answer with no extra assumptions needed. It is well known that this model satisfies PMEA and therefore this result improves the one from [4].

One of the key devices in Section 3 is that adding random reals does not introduce a continuous real-valued function that separates two ground model sets that were not so separated to begin with. In view of the results from [1] concerning separation of sets by disjoint open sets (namely, that if one can separate ground model subsets with open sets after adding random or Cohen reals then those sets were separated by open sets in the ground model) it is natural to ask if the Cohen poset behaves in the same way. Section 4 provides a construction (assuming CH) which shows that this is not the case.

2. Preliminaries

The purpose of this section is to establish the basic terminology. Our primary sources are [2] for topology; [6] and [3] for forcing and set theory (large cardinals and elementary embeddings).

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Let X be a topological space. A subset $A \subseteq X$ is C-embedded if every continuous real-valued function with domain A can be extended continuously to X. If every continuous function from A into $[0,1]$ has a continuous extension to X then A is C^*-embedded in X.

A zero-set in X is a set of the form $f^{-1}(0)$ for some continuous $f : X \to [0,1]$. Two sets $A, B \subseteq X$ are completely separated if there is a continuous $f : X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$; equivalently, A and B are contained in disjoint zero-sets.

If j is a function whose domain is transitive we will denote by $j(a)$ the value that j assigns to the element $a \in \operatorname{dom} j$ and $j''a$ will be used to represent $\{j(x) : x \in a\}$.

Let κ be a cardinal. We say that X has character less than κ (in symbols, $\chi(X) < \kappa$) if any point in X has a local base of cardinality $< \kappa$. 2^κ denotes the set of all functions from κ into $2 = \{0,1\}$. For each $\alpha < \kappa$ the set $a_\alpha := \{f \in 2^\kappa : f(\alpha) = 0\}$ is a clopen subset of the topological product 2^κ.

Let $\mathcal{B}$ be the σ-algebra generated by $\{a_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$ define $\mu(a_\alpha) = \mu(2^\kappa \setminus a_\alpha) = 1/2$. One can extend μ to obtain a probability measure on $\mathcal{B}$. This μ is called the Haar measure on 2^κ.

$2^{<\omega}$ is the set of all functions whose domain is an integer. Observe that when $\kappa = \omega$, $\mathcal{B}$ is generated by $\{[t] : t \in 2^{<\omega}\}$, where $[t] := \{f \in 2^\omega : t \subseteq f\}$, i.e. all the functions that extend t. Each $[t]$ will be called a basic clopen set for 2^ω.

$\mathbb{M}_\kappa$ is the poset obtained by identifying two members of $\mathcal{B} \setminus \{\emptyset\}$ if the measure of their symmetric difference is zero. $\mathbb{M}_\kappa$ is ccc and complete, i.e. if $S \subseteq \mathbb{M}_\kappa$ is not empty then S has a supremum in $\mathbb{M}_\kappa$, denoted by $\bigvee S$. In particular, if Φ is a formula and $\sigma_1, \dots, \sigma_n$ are names so that $a \Vdash \Phi(\sigma_1, \dots, \sigma_n)$, for some $a \in \mathbb{M}_\kappa$, then we define

$$[\![\Phi(\sigma_1, \dots, \sigma_n)]\!] := \bigvee\{b \in \mathbb{M}_\kappa : b \Vdash \Phi(\sigma_1, \dots, \sigma_n)\}.$$

If S is a non-empty subset of $\mathbb{M}_\kappa$ and has a lower bound in $\mathbb{M}_\kappa$ then S has an infimum which will be denoted by $\bigwedge S$.

If τ is a topology for X and $\mathbb{P}$ is any forcing notion then it could be the case that, in the generic extension, τ is no longer a topology for X due to the presence of new subsets of τ but τ will always be a base for some topology for X. Hence, whenever we refer to the topological space (X, τ) (or simply X) we will be referring to the topology on X that has τ as a base.

3. Consistency Modulo a Supercompact Cardinal

We start this section with an auxiliary result which is itself of significant interest.

Theorem 3.1. *Let κ be a cardinal. If X is a topological space and $A, B \subseteq X$, then the following are equivalent.*

(1) *A and B are completely separated.*
(2) $\mathbb{M}_\kappa \Vdash$ "*A and B are completely separated*"

Proof. To show that (1) implies (2) note that any continuous function from the ground model remains continuous in the generic extension.

Now assume (2) and let $\dot{f}$ be a name for a real-valued continuous function on X so that $\mathbb{M}_\kappa \Vdash$ "$\dot{f}[A] \subseteq \{0\} \wedge \dot{f}[B] \subseteq \{1\} \wedge \dot{f}[X] \subseteq [0,1]$".

For each $0 < r < 1$ define $U_r := \{x \in X : \mu([\![\dot{f}(x) < r]\!]) > 1 - r\}$. We show below that $\{U_r : r \in (0,1)\}$ is a family of open sets satisfying $\overline{U_r} \subseteq U_s$ whenever $s < t$, and $A \subseteq U_r \subseteq X \setminus B$ for every r. And therefore the map $h : S \to [0,1]$ given by $h(x) := \inf(\{1\} \cup \{r \in (0,1) : x \in U_r\})$ is continuous, $h[A] \subseteq \{0\}$ and $h[B] \subseteq \{1\}$.

Let r be arbitrary. If $x \in U_r$ and $b := [\![\dot{f}(x) < r]\!]$ then there exists a name for an open set $\dot{W}$ so that $b \Vdash$ "$x \in \dot{W} \wedge \dot{f}[\dot{W}] \subseteq [0,r)$". Fix an antichain $\{b_n : n < \omega\}$ and a family $\{W_n : n < \omega\}$ of open sets from the ground model so that $b = \bigvee\{b_n : n < \omega\}$ and $b_n \Vdash x \in W_n \subseteq \dot{W}$. Since $\sum_{n<\omega} \mu(b_n) = \mu(b) > 1 - r$ there is an integer m for which $\sum_{n<m} \mu(b_n) > 1 - r$. Define $a := \bigwedge\{b_n : n < m\}$ and $O := \bigcap\{W_n : n < m\}$. Hence $a \Vdash \dot{f}[O] \subseteq [0,r)$ and therefore $1 - r < \mu(a) \leq \mu([\![\dot{f}(y) < r]\!])$ for each $y \in O$. Clearly $x \in O \subseteq U_r$ so U_r is open.

To prove that $A \subseteq U_r \subseteq X \setminus B$ observe that $\mu([\![\dot{f}(x) = 0]\!]) = 1$ and $\mu([\![\dot{f}(y) = 1]\!]) = 1$ for all $x \in A$ and $y \in B$.

To finish the proof assume that $r < s$ and let $x \in \overline{U_r}$ be arbitrary. Let $\mathcal{W}$ be the collection of all open sets from the ground model that contain x. For each $W \in \mathcal{W}$ the condition $b_W := \bigvee\{[\![\dot{f}(y) < r]\!] : y \in W \cap U_r\}$ satisfies $b_W \Vdash$ "$\dot{f}[W] \cap [0,r) \neq \emptyset$"and $\mu(b_W) > 1 - r$. Set $b := \bigwedge\{b_W : W \in \mathcal{W}\}$. Since $\{b_W : W \in \mathcal{W}\}$ is closed under finite intersections, we obtain $\mu(b) \geq 1 - r > 1 - s$. We also have that $b \Vdash \dot{f}(x) \leq r$ which implies $1 - s < \mu(b) \leq \mu([\![\dot{f}(x) < s]\!])$. Thus $x \in U_s$. □

Assume that $\nu : \mathbb{M}_\kappa \to [0,1]$ is a probability measure. Note that the argument given above shows that if $\dot{f}$ is an $\mathbb{M}_\kappa$-name for a continuous real-valued function with domain X, then $h : X \to [0,1]$ given by

$$h(x) := \inf(\{1\} \cup \{r \in (0,1) : \nu([\![\dot{f}(x) < r]\!]) > 1 - r\})$$

is continuous.

Before proving the main theorem let us discuss a simplification that will be used: Any real-valued continuous function f can be expressed as $f = (f^+ + 1) - (f^- + 1)$, where both, f^+ and f^-, are continuous and non-negative. This simple remark shows that a set A is C-embedded in X iff any continuous function from A into $[1,\infty)$ has a continuous extension to X.

Theorem 3.2. *Let κ be a supercompact cardinal. In the model obtained by adding κ many random reals, every C^*-embedded subset of space whose character $< \kappa$ is C-embedded.*

Proof. Let $\dot{X}$, $\dot{\tau}$, $\dot{A}$ and $\dot{f}$ be $\mathbb{M}_\kappa$-names so that $\mathbb{M}_\kappa \Vdash$ "$\chi(\dot{X},\dot{\tau}) < \kappa$, $\dot{A}$ is C^*-embedded in $\dot{X}$ and $\dot{f} : \dot{A} \to [1,\infty)$ is continuous." As remarked above, it is enough to show that $\dot{f}$ has a continuous extension to $\dot{X}$. In order to do this we may assume that $\dot{X}$ and $\dot{A}$ have been decided, i.e. there are two sets (in fact ordinals) X and A from the ground model satisfying $1 \Vdash$ "$\dot{X} = \check{X} \wedge \dot{A} = \check{A}$."

Let G be an $\mathbb{M}_\kappa$-generic filter. Working in $V[G]$ we observe that $1/f$ is a function from A to $[0,1]$ and so can be extended to a continuous map $h : X \to [0,1]$. Note that we only have to prove that A and $Z(h)$ are completely separated. Indeed, if $s : X \to [0,1]$ is a continuous function so that $s[A] \subseteq \{0\}$ and $s[Z(h)] \subseteq \{1\}$ then $1/(s+h)$ extends f.

In $V[G]$ let ρ be a name for the canonical random real added by $\mathbb{M}_\omega$. In other words, $\mathbb{M}_\omega \Vdash \rho : \omega \to 2$ and $\mu([\![\rho(n) = i]\!]) = 1/2$ for all $n \in \omega$ and $i < 2$, where μ is the Haar measure on 2^ω described before. Also let $\dot{g}$ be an $\mathbb{M}_\omega$-name for the piecewise linear extension of ρ on $[1,\infty)$, i.e. $\dot{g} \upharpoonright [n, n+1]$ is the line segment that connects the points $(n, \rho(n))$ and $(n+1, \rho(n+1))$ for each positive integer n.

Fix a cardinal $\lambda > \max\{|X|, \mathfrak{c}\}$. Since κ is supercompact, there exists an elementary embedding $j_0 : V \to M$, where M is a transitive class closed under λ-sequences, so that $j_0(\alpha) = \alpha$ for each $\alpha < \kappa$ and $j_0(\kappa) > \lambda$. Therefore (see [1]) G can be extended to G^*, an $\mathbb{M}_{j(\kappa)}$-generic filter over M, and j_0 can be exentended to an elementary embedding $j : V[G] \to M[G^*]$ in such a way that $V[G]$ and $M[G]$ have exactly the same sets of rank $< \lambda$. As a consequence of this we obtain that $j(A)$ is a C^*-embedded subspace of $(j(X), j(\tau))$ and $j(f)$ is a continuous function from $j(A)$ into $[1,\infty)$. Since $\dot{g}$ can be interpreted as an $\mathbb{M}_{\kappa+\omega}$-name and $\kappa + \omega < j(\kappa)$ we get that $g := \mathrm{val}(\dot{g}, G^*)$ is a continuous function from $[1,\infty)$ into $[0,1]$. Hence $g \circ j(f)$ has a continuous extension $\psi : j(X) \to [0,1]$.

Elementarity, the fact that (X,τ) has character $< \kappa$, and our choice of λ imply that $j \upharpoonright X : X \to j''X$ is a homeomorphism (proof of Lemma 2.4 of [1]) where $j''X$ is considered as a subspace of $j(X)$. Thus the function $\varphi_0 : X \to [0,1]$ given by $\varphi_0(x) = \psi(j(x))$ is continuous. To show that it extends $g \circ f$ we only have to observe that if $x \in A$ then $j(f)(j(x)) = j(f(x))$ by elementarity and that $j(f(x)) = f(x)$ because $f(x)$ is a real number.

The argument given above proves that there is an $\mathbb{M}_{j(\kappa)}$-name, $\dot{\varphi}_0$, for a continuous extension of $\dot{g} \circ \dot{f}$. Using the fact that $\mathbb{M}_{j(\kappa)}$ is ccc and assuming that $\dot{\varphi}_0$ is a nice name we can find an ordinal α for which $\dot{\varphi}_0$ is an $\mathbb{M}_{\kappa+\omega} * \mathbb{M}_\alpha$-name and $\kappa + \alpha + \omega < j(\kappa)$.

Since $\mathbb{M}_{\kappa+\omega} * \mathbb{M}_\alpha$ and $\mathbb{M}_{\kappa+\alpha} * \mathbb{M}_\omega$ are forcing equivalent we can arrange things in such a way that $\dot{\varphi}_0$ is an $\mathbb{M}_{\kappa+\alpha} * \mathbb{M}_\omega$-name, G is extended to $\overline{G}$, an $\mathbb{M}_{\kappa+\alpha}$-generic filter over V, and, in $V[\overline{G}]$, ρ is an $\mathbb{M}_\omega$-name for the canonical random real added by $\mathbb{M}_\omega$. The rest of the argument takes place in $V[\overline{G}]$.

Let $\dot{\varphi}_1$ and $\dot{\varphi}_2$ be names for the maps $1 - \dot{\varphi}_0$ and $|\dot{\varphi}_0 - 1/2|$, respectively. If $b \in \mathbb{M}_\omega$ then $\mu_b : \mathbb{M}_\omega \to [0,1]$ defined by

$$\mu_b(a) = \frac{\mu(a \wedge b)}{\mu(b)},$$

where $a \wedge b$ is the infimum of $\{a, b\}$, is a probability measure and therefore (see the remark following Theorem 3.1) the function $\psi_{b,i} : X \to [0,1]$ given by

$$\psi_{b,i}(x) = \inf(\{1\} \cup \{r \in (0,1) : \mu_b([\![\dot\varphi_i(x) < r]\!]) > 1 - r\})$$

is continuous for all $i < 3$.

We claim that if b is a basic clopen set then there is an integer n_b so that $f^{-1}[n_b, \infty) \subseteq \psi_{b,i}^{-1}[1/3, 1]$ for all $i < 3$. To see that this is true let $t \in 2^{<\omega}$ be such that $b = [t]$ and let $n_b \in \omega \setminus \operatorname{dom} t$ be arbitrary. The arguments needed for each individual i are similar so we will present here only the case $i = 0$. Start with arbitrary $x \in f^{-1}[n_b, \infty)$ and $r \in (0, 1/3)$. Define $c := [\![\dot\varphi_0(x) < r]\!]$ and fix integers $m \geq n$ and $k < 3$ such that $m + k/3 \leq f(x) \leq m + (k+1)/3$. If $k = 0$ or $k = 2$ we obtain $c = [\![\rho(m) = 0]\!]$ or $c = [\![\rho(m+1) = 0]\!]$, respectively, and therefore $\mu_b(c) = 1/2$. When $k = 1$, $c = [\![\rho(m) = \rho(m+1) = 0]\!]$ and hence $\mu_b(c) = 1/4$. In any case, $\mu_b(c) < 2/3 < 1 - r$ which implies that $\psi_{b,i}(x) \geq 1/3$.

For each basic clopen set b and each integer $i < 3$ define $Z(b, i) := h^{-1}[1/n_b, 1] \cup \psi_{b,i}^{-1}[1/3, 1]$ to obtain a zero-set in X that contains A. We will show that $Z(h)$ and $\bigcap\{Z([t], i) : t \in 2^{<\omega} \wedge i < 3\}$ are disjoint and thus A and $Z(h)$ are completely separated (recall that two sets are completely separated iff they can be separated by disjoint zero-sets).

Given $z \in Z(h)$ let $a \in \mathbb{M}_\omega$ and $0 \leq k \leq 3$ be so that $a \Vdash k/4 \leq \dot\varphi_0(z) \leq (k+1)/4$. There is an integer $i < 3$ depending entirely on k so that $a \Vdash \dot\varphi_i(z) \leq 1/4$. Fix a real number $1/4 < r < 1/3$. Since μ is the Haar measure on 2^ω we can apply the analogue to Lebesgue Density Lemma for μ (see Section 17.B of [5]) and claim the existence of a basic clopen set b for which $1 - r < \mu_b(a)$. On the other hand, $a \leq [\![\dot\varphi_i(z) < r]\!]$ and therefore $1 - r < \mu_b([\![\dot\varphi_i(z) < r]\!])$. Clearly $\psi_{b,i}(z) < 1/3$ and hence $z \notin Z(b, i)$.

We just showed that, in $V[G]$, A and $Z(h)$ are completely separated after adding (an additional) $\alpha + \omega$ many random reals. According to Theorem 3.1 this implies that A and $Z(h)$ are completely separated in $V[G]$ and this finishes the argument.

□

4. A Distinction Between Random and Cohen

Does Theorem 3.2 remain true if we replace random with Cohen? Theorem 4.2 shows, at least, that the method used above does not work for the Cohen poset.

For any two sets A and B, B^A denotes the set of all functions from A into B. In particular, if α is an ordinal, $B^{<\alpha} := \bigcup\{B^\beta : \beta < \alpha\}$ and similarly for $B^{\leq\alpha}$.

We will consider every integer as an ordinal and therefore 2^n, $2^{<n}$ and $2^{\leq n}$ make sense. For each $t \in \omega^{<\omega}$ and $k < 2$ we define $t^\frown i := t \cup \{(\operatorname{dom} t, k)\}$ (note that we are considering functions as collections of ordered pairs).

For the next two results we will use the poset $\mathbb{P} = 2^{<\omega}$ ordered by $s \le t$ if s extends t.

Lemma 4.1. *There is a family $\{U(t,i) : t \in \mathbb{P} \wedge i < 2\}$ of subsets of ω so that*

(1) *If $s < t$ then $U(t,i) \subseteq U(s,i)$ for all $i < 2$.*
(2) *$U(t,0) \cap U(t,1) = \emptyset$ for all t.*
(3) *If F is a finite antichain in $\mathbb{P}$ and $f : F \to 2$ then $\bigcap\{U(t,f(t)) : t \in F\}$ is infinite.*
(4) *$n \in U(t,0) \cup U(t,1)$ for all $n \in \omega$ and each $t \in 2^n$.*

Proof. We will use induction on the levels of $\mathbb{P}$ to construct the family. For level 1 we only have to nodes: $\mathbf{0}$ and $\mathbf{1}$. Partition ω into five infinite parts, a_0, a_1, b_0, b_1 and c. Now define $U(\mathbf{0},0) = a_0 \cup a_1$, $U(\mathbf{0},1) = b_0 \cup b_1$, $U(\mathbf{1},0) = a_0 \cup b_0$ and $U(\mathbf{1},1) = a_1 \cup b_1$.

Assume that for $n \in \omega$ we have constructed $\{U(t,i) : t \in 2^{\le n} \wedge i < 2\}$ in such a way that conditions (1), (2) and (4) from the lemma hold and the following are also true.

(i) If f is a binary function whose domain is an antichain contained in $2^{\le n}$ then $\bigcap\{U(t,f(t)) : t \in \operatorname{dom} f\}$ is infinite.
(ii) $\omega \setminus \bigcup\{U(t,i) : t \in 2^{\le n} \wedge i < 2\}$ is infinite.

Let $\{t_k : k < 2^n\}$ be an enumeration of 2^n. For each $k < 2^n$ let $\{f^k_\ell : \ell < m\}$ be an enumeration of all binary functions whose domain is an antichain in $2^{\le n}$ and no element of its domain is compatible with t_k. Using induction on $\ell < m$ we obtain four pairwise disjoint infinite sets a^0_k, a^1_k, b^0_k and b^1_k so that each one of them intersects $\bigcap\{U(t,f^k_\ell(t)) : t \in \operatorname{dom}(f^k_\ell)\}$ in an infinite set for all $\ell < m$.

To finish the construction fix a partition $\{c^i_k : k < 2^n \wedge i < 2\} \cup \{d\} \subseteq [\omega]^\omega$ of $\omega \setminus \bigcup\{U(t,i) : t \in 2^{\le n} \wedge i < 2\}$ and define $U(t_k{}^\frown i, 0) := U(t_k,0) \cup a^i_k \cup c^0_k$ and $U(t_k{}^\frown i, 1) := U(t_k,1) \cup b^i_k \cup c^1_k$ (and if the integer $n+1$ is not an element of $U(t_k{}^\frown i, 0) \cup U(t_k{}^\frown i, 1)$ then add it to exactly one of them). □

Given two sets A and B we say that $A \subseteq^* B$ if $B \setminus A$ is finite. If S is an infinite set and for each $A \in \mathcal{A}$ we have $S \subseteq^* A$ then S is a *pseudointersection* of $\mathcal{A}$.

$[\omega]^\omega$ and $[\omega]^{<\omega}$ denote the set of all infinite subsets of ω and all finite subsets of ω, respectively.

For an infinite set $S \subseteq \omega$ and a function $h : \omega \to [0,1]$ we will say that $h[S]$ *converges to* p (in symbols, $h[S] \to p$) if the sequence $\langle h(x_n) : n \in \omega\rangle$ converges to p, where $S = \{x_n : n \in \omega\}$ and $x_n < x_{n+1}$ for each $n < \omega$.

Theorem 4.2. CH *implies that there exist a first countable Tychonoff zero-dimensional space X and two sets $A_0, A_1 \subseteq X$ which are not completely separated but after a Cohen real they are completely separated.*

Proof. Let $\mathbb{P}$ and $\{U(t,i) : t \in \mathbb{P} \wedge i < 2\}$ be as in the previous lemma. Use CH to fix $\{h_\alpha : \omega \leq \alpha < \omega_1\}$, an enumeration of all functions from ω into the interval $[0,1]$.

The strategy is to define a topology on ω_1 in such a way that ω is open discrete and each $\omega \leq \alpha < \omega_1$ will have a neighborhood base of the form $\{\{\alpha\} \cup S_\alpha \setminus n : n \in \omega\}$ for some $S_\alpha \in [\omega]^\omega$.

We will obtain this topology by induction. To be accurate, at stage α we will get three sets: D_α, a maximal antichain in $\mathbb{P}$; x_α, a function from D_α into 2; and $S_\alpha \in [\omega]^\omega$ satisfying the following.

(1α) $S_\alpha \subseteq^* U(t, x_\alpha(t))$ for each $t \in D_\alpha$.
(2α) $\beta < \alpha$ implies $|S_\beta \cap S_\alpha| < \omega$.
(3α) One of the following conditions holds.
  (a) $h_\alpha[S_\xi]$ does not converge for some $\xi \leq \alpha$.
  (b) There exist $k < 2$ and $\xi \leq \alpha$ so that $x_\xi \equiv k$ (i.e. $x_\xi(t) = k$ for all $t \in D_\alpha$) and $h_\alpha[S_\xi]$ does not converge to k.
(4α) For each binary function x whose domain is a finite antichain in $\mathbb{P}$ the set $\bigcap\{U(t,x(t)) : t \in \operatorname{dom} x\} \setminus \bigcup\{S_\xi : \xi \in a\}$ is infinite for each $a \in [\alpha \setminus \omega]^{<\omega}$.

Before going over the details of the induction let us show that a sequence satisfying all the given requirements provides us with the required space. Indeed, for each $k < 2$ define $A_k := \{\alpha < \omega_1 : x_\alpha \equiv k\}$ and assume that $h : X \to [0,1]$ is continuous. Let $\alpha < \omega_1$ be so that $h \restriction \omega = h_\alpha$. h's continuity implies that condition (3α-a) fails and therefore (3α-b) must hold. Hence there exists $\xi \leq \alpha$ so that $x_\xi \equiv k$, for some $k < 2$, and $h_\alpha[S_\xi]$ does not converge to k. Clearly $\xi \in A_k$ and $h(\xi) \neq k$. Therefore A_0 and A_1 cannot be separated by a continuous function.

On the other hand, if G is a $\mathbb{P}$-generic filter, let $g := \bigcup G$ and define $U_k := \bigcup\{U(g \restriction n, k) : n \in \omega\}$ for each $k < 2$. Observe that if $\alpha \in X \setminus \omega$ then $g \restriction m \in D_\alpha$ for some integer m and therefore

$$S_\alpha \subseteq^* U(g \restriction m, x_\alpha(g \restriction m)).$$

This property and the fact that $\alpha \in \overline{U_0}$ if and only if $S_\alpha \cap U_0$ is infinite imply that $\overline{U_0} \cap \overline{U_1} = \emptyset$ (recall item (2) from Lemma 4.1). The same property implies that if $\alpha \in A_k$ then $\alpha \in \overline{U(g \restriction m, k)} \subseteq \overline{U_k}$; in other words, $A_k \subseteq \overline{U_k}$. Therefore A_0 and A_1 are forced to be completely separated.

The only thing left is to construct the sequence. To do this let us assume that we are at stage α and we have defined $\{(D_\beta, x_\beta, S_\beta) : \omega \leq \beta < \alpha\}$ so that conditions (1β)-(4β) are satisfied for all β. For each binary function x for which $\operatorname{dom} x$ is a maximal antichain let

$$\mathcal{F}(x) := \{U(t,x(t)) \setminus I : t \in \operatorname{dom} x \wedge I \in \mathcal{I}\},$$

where $\mathcal{I} := \{\bigcup\{S_\xi : \xi \in a\} : a \in [\alpha \setminus \omega]^{<\omega}\}$. Observe that $\mathcal{F}(x)$ is countable and henceforth it has pseudointersections.

Seeking a contradiction, which comes at the end of the proof, we assume that for every maximal antichain D, for all functions $x : D \to 2$ and for every pseudointersection S of $\mathcal{F}(x)$ the set $h_\alpha[S]$ converges and if $x \equiv k$ for some $k < 2$ then $h_\alpha[S] \to k$.

Note that if S and S' are pseudointersections of $\mathcal{F}(x)$ then $S \cup S'$ is also a pseudointersection of $\mathcal{F}(x)$ and therefore $h_\alpha[S \cup S']$ converges to some real number r. Thus $h_\alpha[S] \to r$ and $h_\alpha[S'] \to r$. Hence, if x is a binary function for which $\operatorname{dom} x$ is a maximal antichain in $\mathbb{P}$ then there is a real number $\varphi(x)$ so that $h_\alpha[S] \to \varphi(x)$ for any pseudointersection S of $\mathcal{F}(x)$.

We claim that if D is a maximal antichain and $x \in 2^D$, the map $x \mapsto \varphi(x)$ is continuous, where 2^D is equipped with the product topology. Since D is countable, we only have to prove that if $\{x_n : n \in \omega\} \subseteq 2^D$ converges to $x \in 2^D$ then $\varphi(x_n) \to \varphi(x)$. If this were not the case then we would be able to find $\varepsilon > 0$ so that infinitely many n's satisfy $|\varphi(x_n) - \varphi(x)| > \varepsilon$. Without loss of generality let us assume that this happens for all $n \in \omega$. Now let H_n be a pseudointersection of $\mathcal{F}(x_n)$. By removing finitely many elements from H_n we can assume that $|h_\alpha(k) - \varphi(x)| > \varepsilon$ for all $k \in H_n$.

Write D as an increasing union of finite sets, $D = \bigcup_n F_n$, and enumerate $\mathcal{I} = \{I_n : n \in \omega\}$. Let $S = \{k_n : n \in \omega\}$ be a sequence of integers satisfying $k_n \in H_n \cap \bigcap\{U(t, x_n(t)) : t \in F_n\} \setminus (k_{n-1} \cup I_n)$. Observe that for each $t \in D$ and $n \in \omega$ there exists $m > n$ so that $t \in F_m$ and $x_i \restriction F_m = x \restriction F_m$ for all $i \geq m$. Hence $\{k_i : i \geq m\} \subseteq U(t, x(t)) \setminus I_n$ (recall that $F_m \subseteq F_i$). This proves that S is a pseudointersection of $\mathcal{F}(x)$ and therefore $h_\alpha[S] \to \varphi(x)$. In particular, there is an $n \in \omega$ so that $|h_\alpha(k_n) - \varphi(x)| < \varepsilon$, but $k_n \in H_n$. This contradiction shows that $\varphi \restriction 2^D$ is continuous.

For any set $Y \subseteq \mathbb{P}$ define $Y^{\downarrow} := \{t \in \mathbb{P} : \exists s \in Y (t < s)\}$.

Claim: Let E_0 and E_1 be maximal antichains. If $y_0 : E_0 \to 2$ and $y_1 : E_1 \to 2$ *agree on cones* (i.e. $y_0(s) = y_1(t)$ whenever $s \in E_0$ and $t \in E_1$ are comparable) then $\varphi(y_0) = \varphi(y_1)$.

Proof of the Claim: To show that $\varphi(y_0) = \varphi(y_1)$ we only have to prove that $\mathcal{F}(y_0)$ and $\mathcal{F}(y_1)$ have a common pseudointersection.

Let us start by proving that $E := (E_0 \setminus E_1^{\downarrow}) \cup (E_1 \setminus E_0^{\downarrow})$ is a maximal antichain. Given $s, t \in E$ there are two cases: Both belong to the same E_i (so they are incompatible) or (without loss of generality) $s \in E_0 \setminus E_1^{\downarrow}$ and $t \in E_1 \setminus E_0^{\downarrow}$. In the second case we obtain $s \not\leq t$ and $t \not\leq s$ and therefore s and t are incompatible. To prove maximality let $t \in \mathbb{P}$ be arbitrary. Since E_i is maximal, there exists $t_i \in E_i$ which is incompatible with t for each $i < 2$. If, for example, $t_0 \in E_1^{\downarrow}$ then $t_0 < s$ for some $s \in E_1$ and thus s and t are compatible, which gives $s = t_1$. Clearly, $t_1 \notin E_0^{\downarrow}$ so $t_1 \in E$.

The function $y := y_0 \restriction (E_0 \setminus E_1^{\downarrow}) \cup y_1 \restriction (E_1 \setminus E_0^{\downarrow})$ has domain E and agrees on cones with y_0 and y_1.

Let S be a pseudointersection of $\mathcal{F}(y)$ and let $i < 2$ be arbitrary. In order to prove that S is a pseudointersection of $\mathcal{F}(y_i)$ let F be a finite subset of E_i. For each $t \in F$ there exists $t' \in E$ so that $t \leq t'$. Therefore

$$S \subseteq^* \bigcap\{U(t', y(t')) : t \in F\} \setminus I \subseteq \bigcap\{U(t, y_i(t)) : t \in F\} \setminus I,$$

for all $I \in \mathcal{I}$. Which finishes the proof of the Claim.

Fix a sequence of positive real numbers $\{\varepsilon_n : n \in \omega\}$ so that $\sum_{n<\omega} \varepsilon_n < 1/3$.

The fact that $\omega^{<\omega}$ embeds densely in $2^{<\omega}$ and vice versa implies that everything we have done so far can be coded for $\omega^{<\omega}$ via the embedding. To simplify the next arguments we will switch to $\mathbb{P} = \omega^{<\omega}$ and keep the notation we developed for $2^{<\omega}$.

For each $n \in \omega$ the set D_n of all functions from n into ω is a maximal antichain in $\mathbb{P}$ and therefore $\varphi \restriction 2^{D_n}$ is continuous. Moreover, 2^{D_n} is compact and therefore φ is uniformly continuous so there exists a finite set $F_n \subseteq D_n$ so that for all $x, y \in 2^{D_n}$ if $x \restriction F_n = y \restriction F_n$ then $|\varphi(x) - \varphi(y)| < \varepsilon_n$. By enlarging F_n we can assume that there is an integer m_n so that F_n is the set of all functions from n into m_n and $m_n < m_{n+1}$.

The set $D^0 := \{t^\frown i : \exists n \in \omega(t \in D_n) \wedge \forall k < n(t(k) < m_k) \wedge m_n \leq i < \omega\}$ is a maximal antichain in $\mathbb{P}$. Let F^0 be a finite subset of D^0 so that $x \restriction F^0 = y \restriction F^0$ implies $|\varphi(x) - \varphi(y)| < 1/3$ for all $x, y : D^0 \to 2$. Let $\ell < \omega$ be large enough so that $F^0 \subseteq \omega^{<\ell}$.

For all $1 \leq k \leq \ell$ define $x_k : D_k \to 2$ by $x_k(t) = 1$ iff $x \restriction i \in F^0$ for some $i \leq k$. Also let $y_k : D_k \to 2$ be given by $y_k(t) = x_k(t \restriction (k-1)^\frown 0)$. If $t \in F_k$ then t and $t \restriction (k-1)$ have the same predecessors and since $F_k \cap F^0 = \emptyset$ we obtain $x_k \restriction F_k = y_k \restriction F_k$ which implies $|\varphi(x_k) - \varphi(y_k)| < \varepsilon_k$.

On the other hand, if $s \in D_{k-1}$ and $t \in D_k$ satisfy $t < s$ then $y_k(t) = x_{k-1}(s^\frown 0)$ and therefore x_{k-1} and y_k agree on cones. Hence $\varphi(y_k) = \varphi(x_{k-1})$.

The two previous paragraphs show that $|\varphi(x_\ell) - \varphi(x_1)| < \sum_{k=1}^{\ell} \varepsilon_k < 1/3$. Note that $x_1 \equiv 0$ and thus $\varphi(x_1) = 0$ which gives $\varphi(x_\ell) < 1/3$.

For each $t \in D^0$ fix a $t' \in D_\ell$ which is compatible with t. The function $z : D^0 \to 2$ defined by $z(t) = x_\ell(t')$ agrees on cones with x_ℓ and hence $\varphi(z) = \varphi(x_\ell)$. If $y : D^0 \to 2$ satisfies $y \equiv 1$ then $z \restriction F^0 = y \restriction F^0$ so $|\varphi(z) - \varphi(y)| < 1/3$ and since $\varphi(y) = 1$ we conclude that $\varphi(x_\ell) = \varphi(z) > 1/3$, a contradiction. This ends the proof of the theorem. $\square$

As mentioned in the introduction, it is consistent that Ohta and Yamazaki's question has a negative answer. More precisely, the counterexample is a pseudocompact first countable space that contains ω as a discrete C^*-embedded subspace but not C-embedded. It was also shown in [4] that under PMEA there is no such a space. The following result shows that the same is true if we add supercompact many Cohen reals.

Theorem 4.3. *Let κ be a supercompact cardinal. In the model obtained by adding κ many Cohen reals no pseudocompact space of character $< \kappa$ contains an infinite discrete C^*-embedded subspace.*

Proof. $\mathrm{Fn}(I, 2)$ denotes the set of partial functions from I into 2 ordered by $p \leq q$ iff $q \subseteq p$.

We will prove the theorem by contrapositive. Assume that $\dot{X}$ and $\dot{\tau}$ are names so that $\mathrm{Fn}(\kappa, 2) \Vdash$ "$(\dot{X}, \dot{\tau})$ is a topological space that contains a C^*-embedded copy of ω." Without loss of generality we can assume that there is an ordinal X so that $\mathrm{Fn}(\kappa, 2) \Vdash$ "$\dot{X} = \check{X}$ and ω is discrete and C^*-embedded in $(\check{X}, \dot{\tau})$"

Working in $V[G]$, where G is $\mathrm{Fn}(\kappa, 2)$-generic over V, let $\dot{g}$ be the canonical name for the Cohen real added by $\mathrm{Fn}(\omega, 2)$, i.e. $p \Vdash p \subseteq \dot{g}$ for all $p \in \mathrm{Fn}(\omega, 2)$. Proceeding as in the proof of Theorem 3.2 we obtain an ordinal α and an $\mathrm{Fn}(\kappa + \omega, 2) * \mathrm{Fn}(\alpha, 2)$-name $\dot{f}$ for a continuous function from X into $[0, 1]$ that extends $\dot{g}$. Moreover, we can arrange things in such a way that $\dot{f}$ is an $\mathrm{Fn}(\kappa + \alpha, 2) * \mathrm{Fn}(\omega, 2)$-name, G is extended to $\overline{G}$, an $\mathrm{Fn}(\kappa + \alpha, 2)$-generic filter over V, and, in $V[\overline{G}]$, $\dot{g}$ is an $\mathrm{Fn}(\omega, 2)$-name for the canonical Cohen real added by $\mathrm{Fn}(\omega, 2)$. We will work in $V[\overline{G}]$ for the rest of the argument.

Lemma 1.1 of [1] guarantees the existence of a family $\mathcal{L}$ of finite subsets of $\mathbb{P} := \mathrm{Fn}(\omega, 2)$ such that the following holds.

(1) For each maximal antichain $A \subseteq \mathbb{P}$ there is $L \in \mathcal{L}$ such that $L \subseteq A$.
(2) For any element $p \in \mathbb{P}$ with $|\operatorname{dom} p| \leq 3$ and for any collection $\{F_i : i < 3\} \subseteq \mathcal{L}$ there exists $q_i \in F_i$ $(i < 3)$ such that the set $\{p\} \cup \{q_i : i < 3\}$ has a lower bound.

Given $x \in X$ there exists a name $\dot{U}_x$ so that $\mathbb{P} \Vdash$ "$x \in \dot{U}_x \in \tau$ and the diameter of $\dot{f}[\dot{U}_x]$ is $< 1/4$." Let A_x be a maximal antichain in $\mathbb{P}$ and let $\{W_x(p) : p \in A_x\} \subseteq \tau$ be such that $p \Vdash \check{W}_x(p) = \dot{U}_x$, for each $p \in A_x$. Fix $L_x \in \mathcal{L}$ satisfying $L_x \subseteq A_x$ and define $W_x := \bigcap\{W_x(p) : p \in L_x\}$.

We will show that $\{W_n : n \in \omega\}$ is a discrete family (recall that $\omega \subseteq X$) and therefore X is not pseudocompact. Assume that $W_x \cap W_n \neq \emptyset$ for some $x \in X$ and $n \in \omega$. Let $m \in \omega \setminus \{n\}$ be arbitrary and set $p_0 := \{(m, 0), (n, 1)\}$. There exist $p_1 \in L_m$, $p_2 \in L_n$, $p_3 \in L_x$, and $q \in \mathbb{P}$ so that $q \leq p_i$ for all $i < 4$. Therefore $q \Vdash$ "$\dot{f}(m) = 0 \wedge \dot{f}(n) = 1 \wedge W_y \subseteq \dot{U}_y$" for each $y \in \{m, n, x\}$. This implies that $q \Vdash$ "$\dot{f}[\dot{U}_n] \cap \dot{f}[\dot{U}_x] \neq \emptyset \wedge 0 \in \dot{f}[\dot{U}_m] \wedge 1 \in \dot{f}[\dot{U}_n]$" and since $\dot{f}[\dot{U}_x]$ is forced to have diameter smaller than $1/4$ we get $q \Vdash \dot{U}_m \cap \dot{U}_x = \emptyset$ and whence $W_m \cap W_x = \emptyset$. □

5. Questions

(1) Is the large cardinal assumption needed in the proof of Theorem 3.2?
(2) Is it a ZFC result that Cohen fails to preserve not completely separated?
(3) Is Theorem 3.2 true if we replace random with Cohen?

References

[1] Alan Dow, Franklin D. Tall and William A.R. Weiss, *New Proofs of the Consistency of the Normal Moore Space Conjecture*, Topology and its Applications **37** (1990) 33-51.

[2] L. Gillman and M. Jerison, *Rings of Continuous functions*, D. Van Nostrand (1960).

[3] T. Jech, *Set Theory. The Third Millennium Edition, Revised and Expanded*, Springer-Verlag, 2006.

[4] J. Kulesza, R. Levy, and P. J. Nyikos, *Extending discrete-valued functions*, Trans. Amer. Math. Soc. 324 (1991), no. 1, 293-302.

[5] A. S. Kechris, *Classical Descriptive Set Theory*, New York: Springer-Verlag, 1995.

[6] K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, 1980.

[7] H. Ohta and K. Yamazaki, *Extension Problems of real-valued continuous functions*, Open Problems in Topology II - Edited by E. Pearl, Elsevier (2007).

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