GENERALIZED SIDE-CONDITIONS AND MOORE-MRÓWKA

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Abstract. We prove that it is consistent (even with Martin's Axiom) that there is first-countable initially ω_1-compact space with cardinality greater than the continuum. We also prove that it is consistent with Martin's Axiom and $\mathfrak{c} = \omega_2$ that there is a compact space of countable tightness which is not sequential. It is known that neither statement is consistent with the Proper Forcing Axiom. We use an innovative new method of constructing proper posets with elementary submodels as side conditions introduced by Neeman.

Introduction

At the October 2011 Appalachian set-theory workshop, B. Velickovic presented *Proper Forcing Remastered* using a generalization of elementary submodels as side-conditions as introduced by Neeman [10]. We follow the notes [13]. One of the applications presented in [13] is the method by Baumgartner and Shelah for constructing a thin very tall superatomic Boolean algebra. This was masterfully generalized by M. Rabus to construct the first consistent example of a countably tight, initially ω_1-compact space which was not compact. A space is initially ω_1-compact if every open cover of cardinality at most ω_1 has a finite subcover. The one-point compactification of this space was also a consistent counterexample to the Moore-Mrowka problem because it is a compact space of countable tightness which is not sequential. The Moore-Mrowka problem asks if every compact Hausdorff space of countable tightness is sequential. It was shown that such a space can not exist under the proper forcing axiom, PFA, by Balogh [2] using elementary submodels as side conditions. Counterexamples to the Moore-Mrowka problem were constructed from $\diamondsuit$ by Ostaszewski [11] and Fedorchuk [6]. These Boolean algebras constructed by Baumgartner-Shelah, and Rabus are known as minimal Boolean algebras based along a linear (well-ordered) set. In a parallel development Koszmider [8] introduced a modification in which the ordered set underlying the structure of the algebra is instead a tree which gave an innovative method for constructing combinatorially complicated first countable spaces. A very recent paper, [4], improves the Rabus example by constructing a first-countable pre-image in the manner of [8]. This also gave an example of a first-countable initially ω_1-compact space which was not compact. This was in answer to a question by the author and van Douwen who had shown that CH and PFA implied there were no such examples. In this paper we explore the method of [13] to produce examples such as those by Rabus and Juhasz-Koszmider-Soukup. In so doing we are able to answer two additional questions. We show that the existence of such spaces is consistent with Martin's Axiom,

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we more explicitly use Koszmider's notion of T-algebra in the construction, and we produce an example of a first-countable initially ω_1-compact space with a compactification which is countably tight and is not sequential because of the stronger property that it has cardinality greater than $\mathfrak{c}$. One might view the Rabus and Juhasz-Koszmider-Soukup examples as generalizations of the Ostaszewski space, while the space constructed in this paper is similar to the Fedorchuk space. We also show that the example in this paper can be used to answer another question of Arhangelskii by showing it is consistent that there exists a first-countable initially ω_1-compact space of cardinality greater than $\mathfrak{c}$. Let us also mention that it is easily shown that the cardinality of such a space can not exceed $2^{\mathfrak{c}}$.

Theorem 0.1. *Each of the following are consistent with* $\mathrm{MA} + \mathfrak{c} = \omega_2$

(1) *there is a compact separable space of cardinality* $2^{\mathfrak{c}}$ *which has countable tightness,*
(2) *there is an initially* ω_1*-compact first-countable space which is not compact and has cardinality* $2^{\mathfrak{c}}$*,*
(3) *there is a compact separable first-countable space which fails to be Lindelof in a forcing extension by an* $\aleph_2$*-Souslin tree.*

Statement Theorem 0.1(3) (see Theorem 6.6 for the proof) is motivated by the paper [7] where it is shown that $\mathrm{MA}(\omega_1)$ implies that ccc posets preserve the Lindelöf property for compact spaces. Also it follows from PFA that each countably closed forcing preserves even the compactness property for compact spaces of countable tightness; although is not explicitly stated, it is a consequence of the proof of [3, 6.3]. Koszmider has already constructed a model of $\mathrm{MA}(\omega_1)$ in which there is a compact first-countable space which does not remain Lindelöf after forcing with the countably closed collapse of ω_2. That paper, [8], is one of the main influences for the present paper.

1. Preliminaries

1.1. **minimally generated algebras, T-algebras, and Topology.**

Definition 1.1. Let $(L,<)$ be a linear order. Let B be a subalgebra of $\mathcal{P}(L)$ with generators $\langle a_x : x \in L\rangle$. For any $K \subset L$, we let $B(K)$ denote the subalgebra of B that is generated by $\{a_x : x \in K\}$. We say that B is L-minimally generated by the sequence $\langle a_x : x \in L\rangle$ if, for each $x \in L$, $x \in a_x \subset \{y \in L : y \leq x\}$ and for $x < y \in L$, $a_x \cap a_y \in B(\{z \in L : z \leq x\})$.

Let us note that if L is finite, then the only requirement on the members of $\langle a_x : x \in L\rangle$ is that $x \in a_x \subset \{y \in L : y \leq x\}$.

Proposition 1.2. [5] *If* $(L,<)$ *is a well-ordering with no maximal element, and* B *is* L*-minimally generated by* $\langle a_x : x \in L\rangle$*, then* B *is superatomic. Moreover, the topology on* L *obtained by declaring each* a_x *to be clopen is locally compact scattered, and its one-point compactification is canonically homeomorphic to the Stone space of* B*.*

Now we recall Koszmider's notion of minimal T-algebras because it will produce a much larger Stone space while preserving considerable control over the structural properties of the space. We will specify a tree $T \subset 2^{<\omega_2}$ which is closed under initial segments, and for each $t \in T$, each of $t0$ and $t1$ are in T (where $t0$ denotes the function with domain $\mathrm{dom}(t)+1$ and takes on value 0 at $\mathrm{dom}(t)$). We let $\mathrm{Succ}(T)$ denote the family of $t \in T$ such that $\mathrm{dom}(t)$ is a successor, and for any $H \subset H$, we let $\mathrm{Succ}(H)$ abbreviate $H \cap \mathrm{Succ}(T)$. We will let bT denote the tree consisting of T together with all its maximal branches. For members x of $bT \setminus T$ we may interchangeably view x as a subset of T or as a member of $2^{\leq\omega_2}$. For $t \in bT$, let $t^{\downarrow} = \{s \in T : s \leq t\}$. For $x \in bT$, let $C_x = \{s \in Succ(T) : s \leq x\}$ and let $C_{<x}$ denote $C_x \setminus \{x\}$. For elements s, t of T, we let $s \wedge t$ denote the largest common predecessor in T. For any $t \in Succ(T)$, let $t^{\dagger}$ denote the element in T with the same domain such that $t \wedge t^{\dagger}$ is the predecessor of t (i.e. flip the last value of t). It will be convenient to let $t^{\dagger} = t$ for $t \in T \setminus Succ(T)$. If $t \in Succ(T)$, then $C_{<t}$ is equal to $C_t \cap C_{t^{\dagger}}$. For any $t \in T$, let $t^{+} = \{s \in bT : t \leq s\}$.

Definition 1.3. For a set $H \subset T$ (closed under the † operation and intersection) we say that the family $\{a_t : t \in Succ(T) \cap H\}$ is a T-system of minimally generated Boolean algebras if

(1) for $t \in Succ(T) \cap H$, $a_{t^{\dagger}}$ and a_t are disjoint and $a_{t^{\dagger}} = H \cap C_{t^{\dagger}} \setminus a_t$.
(2) for each $x \in H$, the sequence $\langle a_t : t \in H \cap C_x \rangle$ minimally generates the Boolean algebra $B(\langle a_t : t \in H \cap C_x \rangle)$.

Again, if H is finite, then the second condition only requires that $x \in a_x \subset H \cap C_x$. If $H = T$, then we have a T-algebra with generators $\{a_t : t \in Succ(T)\}$. For each branch $x \in bT$, we have the well-ordered set C_x and the C_x-minimally generated algebra B_x generated by $\langle a_t : t \in C_x \rangle$. Therefore we have a locally compact scattered topology defined on C_x with each a_t ($t \in C_x$) being a compact open subset. However these algebras fit together in a very special way. For distinct $x, y \in bT \setminus T$, B_x and B_y share the same generators up to $x \wedge y$, and then essentially have complementary sets at the next largest generator. In fact, if $t = x \wedge y$, $t0 \subset x$, and $t1 \subset y$, we are actually thinking of a_{t0} as the neighborhood of t along C_x and $a_{t1} = a_{(t0)^{\dagger}}$ as the neighborhood of t along C_y. More formally, for each $x \in bT$ (including T), let $\sigma_x : C_x \to x$ denote the predecessor operation along the branch x, i.e. for $\alpha \in \mathrm{dom}(x)$, $\sigma_x(x \restriction \alpha+1) = x \restriction \alpha$. Hence for $t \in Succ(T)$, $\sigma_t[a_t]$ and $\sigma_{t^{\dagger}}[a_{t^{\dagger}}]$ cover $(t \wedge t^{\dagger})^{\downarrow}$ and have only $t \wedge t^{\dagger}$ in their intersection. Now we have a homeomorphic topology, τ_x, induced by the bijection σ_x defined on the entire branch x, generated by the family $\{\sigma_x[a_t] : t \in C_x\}$. Let $(\hat{x}, \hat{\tau}_x)$ denote the one-point compactification of (x, τ_x) with the singleton x denoting the point at infinity.

Now we have a compact 0-dimensional topology on $X_T = bT \setminus T$ induced by $\{a_t : t \in Succ(T)\}$ defined as follows. We are simply giving an alternative (more convenient for forcing) presentation of the construction in [8].

Proposition 1.4. *Let $\{a_t : t \in Succ(T)\}$ be a T-system of minimally generated Boolean algebras. For each $t \in Succ(T)$, set $A_t = \{y \in bT \setminus T : y \cap t \in \sigma_t[a_t]\}$ and declare each such A_t to be a clopen subset of $X_T = bT \setminus T$. Then*

(1) *the resulting space is compact, Hausdorff, and 0-dimensional,*
(2) *for each $x \in bT \setminus T$, the map sending each $y \in bT \setminus T$ to $y \cap x$, is a continuous function onto the space $(\hat{x}, \hat{\tau}_x)$,*
(3) *the space X_T has countable tightness providing, for each $x \in X_T$, $\hat{x}$ has countable tightness at the single point x,*
(4) *the character of $x \in X_T$ is equal to the character of x in $\hat{x}$.*

Proof. Let us first note that $x \notin A_t$ for all $t \in C_x$. For distinct x, y and $t = \min(C_x \setminus y)$, we have that $y \in A_t$, $x \in A_{t^{\dagger}}$, and $A_t \cap A_{t^{\dagger}}$ is empty. Therefore

X_T is Hausdorff. The base for X_T is simply the family of finite intersections of the subbase $\{A_t : t \in Succ(T)\}$. To see that X_T is compact, assume that $\mathcal{A}$ is (by Zorn's Lemma and the Alexander subbase theorem [1]) a maximal cover by subbasic clopen sets which has no finite subcover. By maximality, we have that for each $t \in Succ(T)$, we have exactly one of $A_t, A_{t^\dagger}$ is in $\mathcal{A}$. It follows that there is a unique maximal chain $x \subset T$ such that $A_t \in \mathcal{A}$ for all $t \in C_x$. However this contradicts that the collection $\mathcal{A}$ is a cover since the point x is not in the union.

The map sending any $y \in bT \setminus T$ to $y \cap x$ satisfies that the pre-image of the basic clopen set $\sigma_x[a_t]$ is A_t for each $t \in C_x$. Therefore the map is continuous. The preimage of the single point $x \in \hat{x}$ is just the singleton $x \in X_T$. Since X_T is compact, the character of $x \in \hat{x}$ is equal to the character of $x \in X_T$. Furthermore, for any $\{x\} \cup Y \subset X_T$, x is a limit point of Y in X_T, if and only if, x is a limit point of $\{y \cap x : y \in Y\}$ in the space $\hat{x}$. Therefore the tightness claim is immediate as well. □

A consequence of Proposition 1.4 is that to construct the space desired in Theorem 0.1(1), we just have to ensure that bT has cardinality 2^c and each $(\hat{x}, \hat{\tau}_x)$ has countable tightness at x. Now we turn to the consideration of obtaining a suitable initially ω_1-compact subspace. We let $X(T)$ denote the subspace $X_T \cap 2^{<\omega_2}$ of X_T.

Lemma 1.5. *If T does not contain an isomorphic copy of $2^{\leq\omega}$, and if (x, τ_x) is initially ω_1-compact for each $x \in bT \cap 2^{\omega_2}$, then $X_T \cap 2^{\omega_2}$ contains no infinite compact subsets, and $X(T) = X_T \cap 2^{<\omega_2}$ is dense and initially ω_1-compact.*

Proof. Suppose that K is an infinite compact subset of $X_T \cap 2^{\omega_2}$. If K is scattered, then we may pass to a subset which is a simple converging sequence $\{x_n : n \in \omega\}$ and its limit x. For each n, let $t_n = x_n \wedge x$. If there is some $t \in x$ such that $t_n = \sigma_t(t)$ for infinitely many n, then A_t is a compact set not containing x which meets $\{x_n : n \in \omega\}$ in an infinite set. Thus we have that the set $\{t_n : n \in \omega\}$ is infinite and, since (x, τ_x) is assumed to be initially ω_1-compact, we can choose $t \in C_x$ so that $\sigma_t(t)$ is a limit of $\{t_n : n \in \omega\}$. That is, the set $\{n : t_n \in \sigma_t[a_t]\}$ is infinite, and therefore, again, we have that A_t is a compact set not containing x which meets $\{x_n : n \in \omega\}$ in an infinite set. On the other hand, if K is not scattered, then by shrinking K we can assume that K has no isolated points. Choose the maximum node $t_\emptyset \in T$ so that $t_\emptyset^+$ contains K (i.e. for each $x \in K$, $t_\emptyset \subset x$). The set $t_\emptyset^+ \cap X_T$ is equal to $\bigcap\{A_s : s^\dagger \in C_{t_\emptyset}\}$ and so is closed. Let $t_{\langle 0\rangle} = t_\emptyset 0$ and $t_{\langle 1\rangle} = t_\emptyset 1$. By the maximality, each of $K_{\langle 0\rangle} = K \cap A_{t_{\langle 1\rangle}} = K \cap (t_\emptyset 0)^+$ and $K_{\langle 1\rangle} = K \cap A_{t_{\langle 0\rangle}} = K \cap (t_\emptyset 1)^+$ are non-empty. Since each A_s is clopen, each of $K_{\langle 0\rangle}$ and $K_{\langle 1\rangle}$ are relatively clopen subsets of K, and so have no isolated points. Continue this recursion so that for $\psi \in 2^{<\omega}$, the choice of t_ψ and $K_\psi = t_\psi^+ \cap K$, we have that $K_{\psi 0} = K_\psi \cap A_{t_{\psi 1}}$ and $K_{\psi 1} = K_\psi \cap A_{t_{\psi 0}}$ is a non-trivial partition of K_ψ. Then we choose $t_{\psi 0}, t_{\psi 1}$ maximal extensions of $t_\psi 0, t_\psi 1$, respectively, so that $t_{\psi 0}^+$ contains $K_{\psi 0}$ and $t_{\psi 1}^+$ contains $K_{\psi 1}$.

Now we have that for each $r \in 2^\omega$, $\bigcap\{K_{r\restriction n} : n \in \omega\}$ is contained in $(\bigcup\{t_{r\restriction n} : n \in \omega\})^+$. By our assumption on T, we have that there is an $r \in 2^\omega$ so that $t_r \in bT \setminus T$, thus showing that K is not contained in $X_T \cap 2^{\omega_2}$.

Now we show that $X(T)$ is dense and initially ω_1-compact. Since $X_T \setminus X(T)$ contains no infinite compact sets, it is immedate that $X(T)$ is dense. Now let $Y \subset X(T)$ have cardinality ω_1, and let $K \subset X_T$ be the set of complete accumulation points of Y. We just have to show that K is not contained in 2^{ω_2}. Let us assume that it is, and is therefore finite, and obtain a contradiction. Let $x \in K$ be arbitrary. Since the mapping from X_T onto $\hat{x}$ is continuous, x is a limit point of the set $\{y \cap x : y \in Y\}$. If $t \in C_x$ is such that $Y_t = \{y \in Y : y \cap x = \sigma_x(t)\}$ is uncountable, then $(t^\dagger)^+$ is a compact set which contains a complete accumulation point of Y. Since K is finite, there are only finitely many such values of $t \in C_x$. By shrinking Y and using that $x \in K$, we may assume that Y_t is countable for each $t \in C_x$. Thus the family $Y_x = \{y \cap x : y \in Y\}$ is uncountable. Now, from the assumption that (x, τ_x) is initially ω_1-compact, there is a $t \in C_x$ such that $\sigma_x(t)$ is a complete accumulation point of Y_x. It is easily checked that $A_t \cap Y$ is uncountable. But $x \notin A_t$, and so $Y \setminus A_t$ is also uncountable. We can repeat this argument any finite number of times, which thus contradicts that K is finite. □

Now, to finish the proof of Theorem 0.1(2) we must produce a $T \subset 2^{<\omega_2}$ so that X_T has countable tightness and cardinality 2^{ω_2}, T has no embedded copies of $2^{\leq\omega}$, (x, τ_x) is initially ω_1-compact for each $x \in bT \cap 2^{\omega_2}$, and (surely the hardest) x has countable character in (x, τ_x) for each $x \in X_T \cap 2^{<\omega_2}$. We note that with this combination, we also have that X_T will have countable tightness. This is done by a series of many forcing related Lemmas and reformulated in Theorem 6.3. The main outline is that we define a poset following the methods of [10, 13] to construct a sequence $\{a_t : t \in Succ(T)\}$. We then show that X_T has the desired properties and that these properties are preserved by a ccc poset that will force Martin's Axiom. We also defer until the final section how we can then use this X_T to construct an initially ω_1-compact first countable space of the same cardinality.

For Theorem 0.1(3) we require all the same things of T but we impose the requirement that T contain an $\aleph_2$-Souslin tree and rather than having many ω_2-branches, we require that there should be none.

Throughout the paper we assume that $2^\omega < 2^{\omega_1} = \omega_2$ holds in the ground model. The reader can assume throughout that $T \subset 2^{<\omega_2}$ is any fixed tree in the ground model with no maximal nodes and no maximal branches with cofinality ω_1. In particular, for the results of Theorem 0.1 we require that T is either the full ground model $2^{<\omega_2}$ or, has no ω_2 branches but contains an $\aleph_2$-Souslin tree.

1.2. **forcing minimally generated Boolean algebras with finite conditions.** In order to compare minimally generated Boolean algebras with different base sets $K \subset L$ other than initial segments let the notation $B(\langle a_x : x \in L\rangle)$ (similarly $B(\langle c_x : x \in L\rangle)$ etc.) denote such an L-minimal algebra contained in $\mathcal{P}(L)$. Also let $B(\langle a_x : x \in L\rangle; K)$ denote the subalgebra of $B(\langle a_x : x \in L\rangle)$ that is generated by $\langle a_x : x \in K\rangle$.

Definition 1.6. For linearly ordered sets $(L, <)$ and suborder $(K, <)$, we say that $B(\langle a_x : x \in L\rangle)$ extends $B(\langle c_x : x \in K\rangle)$ if the embedding $c_x \mapsto a_x$ extends to an isomorphic embedding of $B(\langle c_x : x \in K\rangle)$ to $B(\langle a_x : x \in L\rangle; K)$.

This notion of extension is what is needed to ensure that a directed family will produce a minimally generated base.

Lemma 1.7. *Let $\mathcal{D}$ be a directed family of finite subsets of ω_2 and assume that $\bigcup \mathcal{D} = \omega_2$. Also suppose that $\mathcal{B} = \{B(\{a^L_x : x \in L\}) : L \in \mathcal{D}\}$ is a family of Boolean algebras such that each $B(\{a^L_x : x \in L\})$ is L-minimal for $L \in \mathcal{D}$ and that for $K \subset L$ both in $\mathcal{D}$, $B(\{a^L_x : x \in L\})$ is an extension of $B(\{a^K_x : x \in K\})$. Then* *the family* $\{a_x : x \in \omega_2\}$ *of subsets of* ω_2 *generates a superatomic Boolean algebra where* $a_x = \bigcup\{a^L_x : x \in L \in \mathcal{D}\}$.

Proof. According to Koppelberg [5] it is sufficient to simply verify that $B(\{a_x : x \in \omega_2\})$ is ω_2-minimal. Let $x < y \in \omega_2$. It follows immediately that $a_x \subset [0, x] \subset \omega_2$. Choose any $K \in \mathcal{D}$ such that $\{x, y\} \subset K$. For all $L \in \mathcal{D}$ such that $K \subset L$, we have that $B(\{a^K_z : z \in K\})$ embeds into $B(\{a^L_z : z \in L\})$. It follows easily then that $a_x \cap a_y$ is in the algebra generated by $\{a_z : z \in K\}$. □

Of course in the paper [12], the Baumgartner-Shelah Δ-function, f, is used to define a subposet $\mathbb{Q}_f$ of a special sub-family of all finite L-minimal ($L \in [\omega_2]^{<\omega}$) Boolean algebras so that $\mathbb{Q}_f$ is ccc. The generic filter G then defines a directed family as in 1.7, hence this is a forcing construction of an ω_2-minimal Boolean algebra (the ccc property implies that ω_2 is preserved). In this construction however, the induced locally compact scattered topology on ω_2 is not countably compact. Rabus adapts the argument to the full family of finite L-minimal Boolean algebras and is able to show that the induced topology on ω_2 is exactly as we require of our (x, τ_x) for $x \in bT \cap 2^{\omega_2}$. That is, the induced topology on ω_2 is initially ω_1-compact, and the one-point compactification has countable tightness.

Define the collection $\mathcal{H}_T \subset [T]^{<\omega}$ to be all $\emptyset \in H \in [T]^{<\omega}$ which are closed under †, intersections and each $H \in \mathcal{H}_T$ must also have the property that if $s, t \in H$ are incomparable, then each immediate successor of $s \cap t$ is in H. Note then that each $H \in \mathcal{H}_T$ is also closed under the operation of taking immediate predecessor.

Definition 1.8. For $H \subset F$, we say that $\langle a^F_t : t \in F \cap C_s\rangle$ is an extension of $\langle a^H_t : t \in H \cap C_s\rangle$ if the conditions on the algebras as in 1.6 are met.

Lemma 1.7 also holds for directed families of T-algebras. We have modified the presentation of a T-algebra from that of Koszmider to make it easier to force one with finite conditions.

1.3. **Two cardinal elementary submodels as side-conditions.** We assume $2^\omega < 2^{\omega_1} = \omega_2$ and we consider the structure $(H(\aleph_2), \in, \triangleleft)$ where $\triangleleft$ is a fixed well-ordering of $H(\aleph_2)$ in type ω_2. The well-ordering $\triangleleft$ ensures that if M, N are elementary submodels of this structure, then so is $M \cap N$. Of course for a regular cardinal θ, $H(\theta)$ denotes the set of sets whose transitive closure has cardinality less than θ. We may suppose that $\triangleleft$ has the property that it extends the tree ordering on $2^{<\omega_2}$ and the well-ordering on ω_2.

Definition 1.9. Let P be an elementary submodel of $H(\aleph_2)$ of size $\aleph_1$. P is internally approachable if it can be written as the union of an increasing continuous chain $\{P_\xi : \xi \in \omega_1\}$ of countable elementary submodels of $H(\aleph_2)$ such that $\langle P_\xi : \xi \le \eta\rangle \in P_{\eta+1}$, for every ordinal $\eta \in \omega_1$. We implicitly take $\{P_\xi : \xi \in \omega_1\}$ to be the $\triangleleft$-least chain witnessing that P is internally approachable.

Definition 1.10. Define $\mathcal{E}^2_0$ to be the collection of all countable elementary submodels of $(H(\aleph_2), \in, \triangleleft)$, and $\mathcal{E}^2_1$ the collection of all internally approachable elementary submodels. Set $\mathcal{E}^2 = \mathcal{E}^2_0 \cup \mathcal{E}^2_1$.

A finite set $\mathcal{M}$ is an $\in$-chain if it can be enumerated as $\langle M_i : i \le n\rangle$ so that $M_i \in M_{i+1}$ for all $i < n$. If $\mathcal{M}$ is a finite $\in$-chain which is closed under intersections and is an element of M in $\mathcal{E}^2$, then there is a smallest collection $\mathcal{M}^* \supset \{M\} \cup \mathcal{M}$ which is closed under intersections. This collection $\mathcal{M}^*$ is again an $\in$-chain, but $\in$

need not be transitive on this chain. See Lemma 1.8 of [13] for a proof of this and more discusion about the object $\mathbb{M}$, the set of finite $\in$-chains of models in $\mathcal{E}^2$ closed under intersections, as a forcing poset.

We will also need this important property of members of $\mathbb{M}$ from [13, Fact 3.2, 3.3].

Proposition 1.11. *For each $\mathcal{M} \in \mathbb{M}$ and $M, P \in \mathcal{M} \cap \mathcal{E}^2_0$, if $M \cap P \notin M$, then $M \cap \omega_1 \subset P \cap \omega_1$.*

Proof. The proof is by induction on $|\mathcal{M}|$. Since $\mathcal{M}$ is an $\in$-chain we may enumerate it, $\{M_i : i < n\}$, so that $M_i \in M_{i+1}$ for each $i < n-1$. If $M \cap P = M$, then we are done, so we may as well assume that $M_i = M \cap P$ is not equal to $M_{j_0} = M$. Note that the axiom of foundation implies that $i < j_0$. Since we are assuming that $M_i \notin M$, there must be a k with $M_k \in \mathcal{E}^2_1$ and $i < k < j_0$. Choose the maximal such k and note that $M_i \in M_k \in M_{j_0}$. Now fix $j_1 < k$ so that $M_k \cap M_{j_0} = M_{j_1}$. We still have that $M_i \notin M_{j_1}$, and so by induction, $M_{j_1} \cap \omega_1 \subset M_i$. Since $M \cap \omega_1 = M_{j_1} \cap \omega_1$ and $M_i \cap \omega_1 \subset P \cap \omega_1$, we have that $M \cap \omega_1 \subset P$. □

Proposition 1.12. *Let $\mathcal{M} \in \mathbb{M}$ and suppose that $M \cap P \notin M$ for $M, P \in \mathcal{M} \cap \mathcal{E}^2_0$. Then for any $t \in T \cap M \cap P$, $C_t \cap M \subset P$.*

Proof. Let g_t be the $\lhd$-least function from ω_1 onto C_t. It follows that $g_t \in M \cap P$. Also, by elementarity, g_t maps $M \cap \omega_1$ onto $C_t \cap M$. Therefore, by Proposition 1.11, $C_t \cap M \subset P$. □

As usual, if we have a poset $\mathbb{P}$ and $\mathbb{P} \in M$ for some elementary submodel (possibly uncountable) of $H(\theta)$ for some regular θ, then a condition $p \in \mathbb{P}$ is said to be $(M, \mathbb{P})$-generic if for each dense $D \subset \mathbb{P}$ which is in M and each $r \le p$, there is a $p_r \in D \cap M$ which is compatible with r. Loosely speaking, the method from [13] is to have finite $\in$-chains $\mathcal{M}_p$ as side-conditions for members p of $\mathbb{P}$. If $p \in M^* \prec H(\theta)$ and if M^* is either countable, or internally approachable of cardinality ω_1, then we require there is some condition $q < p$ such that $M^* \cap H(\aleph_2) \in \mathcal{M}_q$. We also require that this will ensure that q is $(M^*, \mathbb{P})$-generic. From this it follows that forcing with such a poset $\mathbb{P}$ will preserve $\aleph_1$ and $\aleph_2$.

2. The forcing

Definition 2.1. A condition $p \in \mathbb{P}$ (or $\mathbb{P}_T$) is any triple

$$\langle H^p, \langle a^p_t : t \in \mathrm{Succ}(H^p)\rangle, \mathcal{M}_p\rangle$$

where $H^p \in \mathcal{H}_T$ and $\langle a^p_t : t \in \mathrm{Succ}(H^p)\rangle$ is a T-system of minimal Boolean algebras (as in 1.3). $\mathcal{M}_p$ is a finite $\in$-chain of members of $\mathcal{E}^2$ which is closed under intersections. We place an additional restriction on our T-system. For each $P \in \mathcal{M}_p$ $\langle a^p_t : t \in \mathrm{Succ}(H^p)\rangle$ is an extension of the T-system $\langle a^p_t \cap P : t \in P \cap \mathrm{Succ}(H^p)\rangle$. We define the ordering on $\mathbb{P}$ by $p < q$ if $H^p \supset H^q$, $\mathcal{M}_p \supset \mathcal{M}_q$, and the T-system $\langle a^p_t : t \in \mathrm{Succ}(H^p)\rangle$ is an extension of $\langle a^q_t : t \in \mathrm{Succ}(H^q)\rangle$.

Notice that for any $H \in \mathcal{H}_T$, a triple $\langle H, \langle a_t : t \in \mathrm{Succ}(H)\rangle, \emptyset\rangle$ is in $\mathbb{P}$ so long as $t \in a_t \subset C_t \cap H$ for each $t \in \mathrm{Succ}(H)$.

For each $t \in \mathrm{Succ}(T)$, we let $\dot{a}_t$ denote the $\mathbb{P}$-name where $p \in \mathbb{P}$ with $t \in H^p$ forces that $\dot{a}_t \cap H^p = a^p_t$.

Definition 2.2. For any condition $p \in \mathbb{P}$, any set P, and $s \in P \cap \mathrm{Succ}(H^p)$, let $[s]^p_P$ denote the atom that contains the element s in the subalgebra generated by $\langle a^p_t : t \in P \cap H^p \cap C_s\rangle$. More concretely, $[s]^p_P = a^p_s \setminus \bigcup\{a^p_\rho : \rho \in P \cap H^p \cap C_{<s}\}$. For a condition $q \in \mathbb{P}$, let $[s]^p_q$ abbreviate $[s]^p_{H^q}$.

The next lemma is useful for producing extensions of members of $\mathbb{P}$.

Lemma 2.3. *Let $H \in \mathcal{H}_T$, let $\{a^p_t : t \in Succ(H\}$ be a T-system of minimal Boolean algebras and let $\mathcal{M} \in \mathbb{M}$. Then $p = \langle H, \langle a^p_t : t \in Succ(H)\rangle, \mathcal{M}\rangle$ is a member of $\mathbb{P}$ so long as for each $t \in P \cap Succ(H)$ and $s \in H \cap P \cap C_t$, $[s]^p_P$ is not split by a^p_t. Also, for any $q \in \mathbb{P}$, $p < q$ providing $H \supset H^q$, $\mathcal{M} \supset \mathcal{M}_q$, and for each $t \in \mathrm{Succ}(H^q)$, $a^p_t \cap H^q = a^q_t$, and for each $s \in C_t \cap H^q$, the set $[s]^p_q$ is not split by a_t.*

Proof. For the first part, we have to show that for $s \in H$ the embedding taking $\langle a^p_t \cap P : t \in P \cap H \cap C_s\rangle$ to $\langle a^p_t : t \in P \cap H \cap C_s\rangle$ lifts to an isomorphic embedding of $B(\langle a^p_t \cap P : t \in P \cap H \cap C_s\rangle)$ to $B(\langle a^p_t : t \in P \cap H \cap C_s\rangle; C_s \cap P)$. Of course it is immediate that for each $\rho < t \in H \cap C_s \cap P$, $\rho \in a^p_t \cap P$ if and only if $\rho \in a^p_t$. Since we are assuming that a^p_t does not split $[\rho]^p_P$, it follows that $a^p_t = \bigcup\{[\rho]^p_P : \rho \in a^p_t \cap P\}$. It also follows that, for each $\rho \in P \cap H \cap C_s$, $[\rho]^p_P$ is an atom in the algebra $B(\langle a^p_t \cap P : t \in P \cap H \cap C_s\rangle)$. It is then immediate that $(a^p_t \cap P) \mapsto a^p_t$ does generate the required isomorphism.

Now assume that q is as in the statement of the Lemma. We again have to show that for each $s \in H^q$, the mapping sending $a^q_t \mapsto a^p_t$ for $t \in H^q \cap C_s$ extends to an isomorphic embedding of $B(\langle a^q_t : t \in C_s \cap H^q\rangle)$ to $B(\langle a^p_t : t \in C_s \cap H^q\rangle)$. The proof is the same as in the first part. □

Definition 2.4. If $r \in \mathbb{P}$ and $M \in \mathcal{M}_r$, then

$$r \restriction M = \langle H^r \cap M, \langle a^r_t \cap M : t \in M \cap \mathrm{Succ}(H^r)\rangle, \mathcal{M}_r \cap M\rangle\ .$$

Lemma 2.5. *If $r \in \mathbb{P}$ and $M \in \mathcal{M}_r$, then $r \restriction M \in \mathbb{P}$ and $r < r \restriction M$.*

Proof. We show that $p = r \restriction M$ is a member of $\mathbb{P}$; the fact that $r < r \restriction M$ is then quite immediate. It is routine to check that $\mathcal{M}_p = M \cap \mathcal{M}_r$ is closed under finite intersections (see [13, 1.7]), and that $H^p = M \cap H^r$ is a member of $\mathcal{H}_T$. To apply Lemma 2.3, we check that if $P \in \mathcal{M}_r \cap M$, and $t \in P \cap M \cap \mathrm{Succ}(H^r)$ and $s \in M \cap H^r \cap P \cap C_t$, $[s]^p_P$ is not split by $a^p_t = a^r_t \cap M$. Since $P \cap M \in \mathcal{M}_r$, this follows from the fact that $r \in \mathbb{P}$, and so a^r_t does not split $[s]^r_{P\cap M}$. □

Lemma 2.6. *If $p \in \mathbb{P}$ and $p \in M \in \mathcal{E}^2$, then $p^* < p$ where $p^* = \langle H^p, \langle a^p_t : t \in \mathrm{Succ}(H^p)\rangle, \mathcal{M}_{p^*}\rangle$, and $\mathcal{M}_{p^*}$ is $(\mathcal{M}_p \cup \{M\})^*$, the smallest family of models which is closed under intersections (see* [13]*).*

Proof. If $M \in \mathcal{E}^2_1$, there is really nothing to check. If $M \in \mathcal{E}^2_0$, then it must be shown that for each $P \in \mathcal{M}_{p^*}$, $\langle a^p_t : t \in \mathrm{Succ}(H^p)\rangle$ is an extension of $\langle a^p_t \cap P : t \in P \cap \mathrm{Succ}(H^p)\rangle$. This is a straightforward application of elementarity since $p \in M$ implies that $\mathcal{M}_{p^*} \setminus \{M\} \subset M$. □

For this next result, let us remind the reader that $C_t = t^\downarrow \cap Succ(T)$, hence $t^\downarrow \setminus C_t$ is the predecessors of t that are on limit levels together with the root of T.

Lemma 2.7. *If $H \in \mathcal{H}_T$, $t \in T$ and $v = \max(t^\downarrow \setminus C_t)$, then the smallest $H(t) \in \mathcal{H}_T$ containing $H \cup \{t\}$ is equal to $H \cup \{\rho, \rho^\dagger : v \leq \rho \leq t\} \cup \{\psi, \psi^\dagger : \bar{v} \leq \psi \leq \bar{s}\}$, where $\bar{t} = \max\{s \leq t : (s^\dagger)^+ \cap H \neq \emptyset\}$, and $\bar{v} = \max(\bar{t}^\downarrow \setminus C_{\bar{s}})$. It can happen that $\bar{v} = v$.*

Lemma 2.8. *For each $q \in \mathbb{P}$ and $t \in T$, there is $p \leq q$ such that $H^p = H^q(t)$, $\mathcal{M}_p = \mathcal{M}_q$. In addition,*

(1) *if, for each $s \in H^q$ and $P \in \mathcal{M}_q \cap \mathcal{E}^2_0$, $t \in Succ(T) \setminus (C_s \cap P)$, then p may be chosen so that $H^q \cap C_t \subset a^p_t$,*
(2) *if $x \in H^q$ is arbitrary, then we may arrange that $t \notin a^p_\rho$ for all $\rho \in H^p \cap C_x$,*

Proof. Let $q \in \mathbb{P}$ and $t \in T \setminus H^q$. We proceed by induction on $\mathrm{dom}(t)$. The limit case is trivial and so we assume that $\mathrm{dom}(t)$ is not a limit; equivalently, we assume that $t \neq t^\dagger$. Condition (2) is vacuous unless $x \in t^+$ and if $x \in t^+$ we can assume that it is maximal. Let $v = \max(t^\downarrow \setminus C_t)$ as in Lemma 2.7.

Since $H^q \in \mathcal{H}_T$, we note that at least one of $t^+ \cap H^q$ and $(t^\dagger)^+ \cap H^q$ is empty. Let $\bar{t}$ and $\bar{v}$ be defined as in Lemma 2.7. If $v^+ \cap H^q$ is empty, then we have that $\bar{t} < v$. By the inductive hypothesis there is some $r < q$ with $\bar{t} \in H^r$ and $H^r \setminus H^q$ satisfying the required conditions. In addition, it is trivial to show that $\langle H^r \cup \{v\}, \langle a^r_s : s \in \mathrm{Succ}(H^r)\rangle, \mathcal{M}_q\rangle$ is also a suitable extension of q satisfying the requirements of the Lemma.

It follows then that we may simply assume that $v \leq \bar{t}$ and that $\bar{v} = v \in H^q$. Furthermore, applying the induction hypothesis again, we reduce to the assumption that $\bar{t}$ is t (if $t^+ \cap H^q$ is empty) or the predecessor of t. It is easy to check that after these reductions, we have that $H^p = H^q \cup \{t, t^\dagger\}$ is in $\mathcal{H}_T$. We must define a^p_s for all $s \in H^p$ and we will set $p = \langle H^p, \langle a^p_s : s \in \mathrm{Succ}(H^p)\rangle, \mathcal{M}_q\rangle$.

If we have, as in item (1), for all $s \in H^q$ and $P \in \mathcal{M}_q \cap \mathcal{E}^2_0$ that $t \notin C_s \cap P$, then we verify that we can make the assignments $a^p_t = C_t \cap H^p$ and $a^p_{t^\dagger} = \{t^\dagger\}$. If the hypothesis of item (1) fails then we note that $(t^\dagger)^+ \cap H^q$ is empty. So we also note that, in fact, so long as $(t^\dagger)^+ \cap H^q$ is empty, then we can reverse this assignment by setting $a^p_t = \{t\}$ and $a^p_{t^\dagger} = C_{t^\dagger} \cap H^p$.

For all $s \in H^q$ such that $t \notin C_s$, we set $a^p_s = a^q_s$. If $t^+ \cap H^q$ is also empty, then we are done, and the verifications that $p \in \mathbb{P}$ and $p < q$ are trivial. Otherwise there are members of H^q above t and we have chosen our maximal $x \in H^q$ which is above t. For each $s \in \mathrm{Succ}(H^p) \cap t^+$, we must arrange that t is in exactly one of $a^p_s, a^p_{s^\dagger}$. For each $\rho \in C_x \setminus C_t$, set $a^p_\rho = a^q_\rho$ and for each $s \in (\rho^\dagger)^+ \cap \mathrm{Succ}(H^q)$, define

$$a^p_s = \begin{cases} \{t\} \cup a^q_s & \rho^\dagger \in a^q_s \\ a^q_s & \rho^\dagger \notin a^q_s \end{cases}$$

It is routine to show that the T-system $\langle a^p_s : s \in \mathrm{Succ}(H^p)\rangle$ is an extension of $\langle a^q_s : s \in \mathrm{Succ}(H^q)\rangle$. There is no change to the L-system $\langle a^q_s : s \in C_x \cap \mathrm{Succ}(H^q)\rangle$. While for each $\rho \in H^q \cap C_x \setminus C_t$, and each $u \in H^q \cap (\rho^\dagger)^+$, we have, in effect, doubled the atom $\rho^\dagger$ in the L-system $\langle a^q_s : s \in C_u \cap \mathrm{Succ}(H^q)\rangle$.

In the case where $t \notin C_s \cap P$ for all $P \in \mathcal{M}_p \cap \mathcal{E}^2_0$ and $s \in H^q$, then it is immediate that $p \in \mathbb{P}$. Finally, we suppose that $t \in C_s \cap P$ for some $s \in \mathrm{Succ}(H^q)$ and $P \in \mathcal{M}_q \cap \mathcal{E}^2_0$. In this case, i.e. item (1) not holding, since we have that $a^p_t = \{t\}$, it follows that $a^p_s \cap a^p_t$ is one of $\{t\}, \emptyset$ and so is certainly in the algebra generated by $\langle a^q_\rho : \rho \in P \cap C_t\rangle$. ☐

Proposition 2.9. *For $s \subset t$ both in $\mathrm{Succ}(T)$ and $p \in \mathbb{P}$, with $s, t \in \mathrm{Succ}(H^p)$, and, with the finite set $L \subset C_{<s}$ defined as $H^p \cap C_{<s}$, we have that p forces*

(1) *$\dot{a}_s \cap \dot{a}_t$ contains $\dot{a}_s \setminus \bigcup\{\dot{a}_\rho : \rho \in L\}$ if $s \in \dot{a}_t$,*
(2) *$\dot{a}_s \cap \dot{a}_t$ is contained in $\bigcup\{\dot{a}_\rho : \rho \in L\}$ if $s \notin \dot{a}_t$.*

Proof. It follows from the definition of extension. □

This next lemma is proven in the next section (see Corollary 3.8).

Lemma 2.10. *If $\mathbb{P} \in M^*$ and $M^* \prec H(\aleph_3)$ is countable, then p is $(M^*, \mathbb{P})$-generic providing $M \in \mathcal{M}_p$ where $M = M^* \cap H(\aleph_2)$.*

Lemma 2.11. *If $\mathbb{P} \in M^*$ and $M^* \prec H(\aleph_3)$ is uncountable and internally approachable, then p is $(M^*, \mathbb{P})$-generic providing $M \in \mathcal{M}_p$ where $M = M^* \cap H(\aleph_2)$.*

Proof. Let $\{M_\alpha : \alpha \in \omega_1\}$ be the continuous chain of countable elementary submodels of $H(\aleph_3)$ with union equal to M^*. Let $D \in M^*$ be a dense open subset of $\mathbb{P}$ and let $r \in D$ with $r < p$. Choose a $\delta \in \omega_1$ so large that $D, \{P \cap M : P \in \mathcal{M}_r \cap \mathcal{E}_0^2\}$, and $H^r \cap M$ are each members of M_δ. Observe that $\mathcal{M} = \mathcal{M}_r \cup \{M_\delta \cap H(\aleph_2)\}$ is closed under intersections, and that $r^* = \langle H^r, \langle a_t^r : t \in \mathrm{Succ}(H^r)\rangle, \mathcal{M}\rangle$ is in $\mathbb{P}$ and is below r. Apply Lemma 2.10 to find a $q \in M_\delta \cap D$ which is compatible with r^*. □

3. $\mathbb{P}$ is proper

This is the most technically difficult (perhaps tedious) portion of the paper. We need a new notion and a technical lemma as a first stage of proving that $\mathbb{P}$ is proper.

Definition 3.1. Let $M \in \mathcal{E}_0^2$ and $r \in \mathbb{P}$; we write $L_r^M = \{\bigcup (C_t \cap M) : t \in H^r \setminus M\}$. We say that r is M-prepared if $M \in \mathcal{M}_r$ and

1. L_r^M is contained in H^r,
2. if $t \in L_r$ then there is a v_t in $M \cap H^r$ such that $v_t < t$ and $(v_t^+ \setminus t^+) \cap H^r = \{v_t\}$,
3. if $t \in L_r^M$, $P \in \mathcal{M}_r \setminus M$ and $v_t \in P$, then $M \cap C_t \subset P$,
4. if $t \in L_r^M$ is such that $t^+ \cap M \cap T \neq \emptyset$ then $u_t \in H^r$, where $u_t = \min(t^+ \cap M)$.

Lemma 3.2. *For each $r \in \mathbb{P}$, and countable $M \in \mathcal{M}_r \cap \mathcal{E}_0^2$ there is an $r' < r$ which is M-prepared and satisfies $L_{r'}^M = L_r^M$ and $\mathcal{M}_{r'} = \mathcal{M}_r$.*

Proof. The construction of r' will be obtained in a finite recursion. Note that, for any condition r, the definition of L_r^M as $\{\bigcup(C_t \cap M) : t \in H^r \setminus M\}$ is clear and consists only of limit nodes. Similarly, for each $t \in L_r^M$, it is routine to choose a limit v_t below t in M so that $v_t^+ \cap H^r \setminus t^+$ is empty. We may, by possibly increasing v_t, additionally ensure that $P \cap C_t \cap M \subset C_{v_t} \cap M$ for each $P \in \mathcal{M}_r$ for which $P \cap C_t \cap M$ is bounded in C_t. It follows from Proposition 1.12 that if $v_t \in P \in \mathcal{M}_r$, then $C_t \cap M \subset P$ as required in the definition of M-prepared.

We first prove that if $t \in L_r^M \setminus H^r$, there is an $r_1 < r$ such that $L_{r_1} = L_r$, $\mathcal{M}_{r_1} = \mathcal{M}_r$ and $H^{r_1} = H^r \cup \{t\}$. It then follows, by induction, that we may assume that $L_r^M \subset H^r$.

Fix any $t \in L_r^M \setminus H^r$ and choose the minimum $t_1 \in H^r \setminus M$ so that $t = \bigcup(t_1 \cap M)$. We first show that $H^{r_1} = \{t\} \cup H^r$ is in $\mathcal{H}_T$. Choose any $s \in H^r$ and consider $\bar{t} = t \cap s$. If $\bar{t} = t$, then of course we have that $\bar{t} \in H^{r_1}$. So assume that $\bar{t} < t$. Then, by the minimality of t_1, we have that $\bar{t} = t_1 \cap s$, and so, not only is it in H^r but so are both its immediate successors. This completes the proof that $H^{r_1} \in \mathcal{H}_T$, and it follows immediately that $r_1 < r$ is in $\mathbb{P}$ where $a_s^{r_1} = a_s^r$ for all $s \in \mathrm{Succ}(H^r)$, and $\mathcal{M}_{r_1} = \mathcal{M}_r$. It should be clear that $L_{r_1}^M$ is equal to L_r^M. With the exact same reasoning, we can ensure that v_t is in H^r without any other changes to r (other than enlarging H^r).

So now we simply assume that $(L_r^M \cup \{v_t : t \in L_r^M\}) \subset H^r$ and consider the process of ensuring that $u_t \in H^r$ for those $t \in L_r^M$ such that u_t exists. Let us note that, by elementarity, u_t is a limit node. Using that $H^r \in \mathcal{H}_T$, we have that if u_t is not in H^r then no finite extension of u_t is in H^r and at most one immediate successor of u_t has extensions in H^r. It now follows easily that if $u_t^+ \cap M$ is not empty, then $H^r \cup \{u_t\}$ is in $\mathcal{H}_T$. Similar to the case of ensuring $L_{r'}$ is contained in H^r we can trivially add u_t to H^r without making any other changes.

On the other hand, if $u_t^+ \cap H^r$ is empty, we may have to worry about $u_t \cap s$ for other values of $s \in t^+ \cap H^r$. Nevertheless, we can apply Lemma 2.8 so as to arrange (and now assume) that $u_t \in H^r$ without any changes to L_r^M, $H^r \cap M$, and $v_\rho^+ \cap H^r$ for any $\rho \in L_r^M \setminus \{t\}$.

Therefore, we may now also assume that each such $u_t \in H^{r'}$, which completes the construction of $r' < r$. □

Next we have a symmetric notion for identifying compatible members q which are in M.

Definition 3.3. If $r \in \mathbb{P}$ is M-prepared, then a condition $q \in \mathbb{P}$ is in (M, r)-good position if

(1) $q \in M$ and $q \leq r \restriction M$,
(2) $H^q \setminus H^r$ is contained in $\bigcup\{v_t^+ \setminus t^+ : t \in L_r^M\}$,
(3) for each $P \in \mathcal{M}_q \setminus \mathcal{M}_r$ and $t \in L_r^M$, if $v_t \in P$, then $H^r \cap M \cap C_{v_t} \subset P$.

If $r \in \mathbb{P}$ is M-prepared, then r itself is (M, r)-good. Nevertheless, it will be important to identify within M those elements of M that are in (M, r)-good position. Towards this we begin with a definition.

Definition 3.4. Suppose that $r \in \mathbb{P}$ is M-prepared for some $M \in \mathcal{M}_r \cap \mathcal{E}_0^2$ and let $\bar{r} \in \mathbb{P} \cap M$ denote $r \restriction M$. Also, using the notation of Definition 3.1, let $L_r^M(U) = \{(v_t, u_t) : t \in L_r \text{ and } t^+ \cap M \neq \emptyset\}$. Similarly set $\bar{L} = \{v_t : t \in L_r^M \text{ and } t^+ \cap M = \emptyset\}$. Define $\Gamma(r, M)$ to be the set of p satisfying

$$p < \bar{r}, \quad H^p \setminus H^{\bar{r}} \subset \bigcup\{v^+ : v \in \bar{L}\} \cup \bigcup\{v^+ \setminus u^+ : (v, u) \in L_r^M(U)\},$$
$$\text{and} \quad (\forall P \in \mathcal{M}_p \setminus \mathcal{M}_{\bar{r}})\ (\forall s \in P \cap H^p)(H^{\bar{r}} \cap C_s \subset P).$$

Lemma 3.5. *If $r \in \mathbb{P}$ is M-prepared, then $\Gamma(r, M)$ is non-empty element of M, and each q in $M \cap \Gamma(r, M)$ is in (M, r)-good position.*

Proof. Each of $\bar{r}$, $L_r^M(U)$, and $\bar{L}$ as defined in Definition 3.4 are members of M. A simple check of the definition of $\Gamma(r, M)$ shows that it is in M and has r as an element. Assume $q \in M \cap \Gamma(r, M)$ and let $s \in H^q \setminus H^r$. First assume that $s \in v^+$ for some $v \in \bar{L}$ and let $t \in L_r^M$ such that $v = v_t$. Since $s \in M$ and $t^+ \cap M$ is empty, we have that $s \in v_t^+ \setminus t^+$ as required in the definition of q being in (M, r)-good position. Otherwise, choose $(v, u) \in L_r^M(U)$ so that $s \in v^+ \setminus u^+$. Choose $t \in L_r^M$ so that $(v, u) = (v_t, u_t)$. Since $s \in M$ and u_t is the minimum of $t^+ \cap M$, we have that s is not above t. Therefore we again have that $s \in v_t^+ \setminus t^+$. Now consider any $P \in \mathcal{M}_q \setminus \mathcal{M}_r$ and $t \in L_r^M$. Since $v_t \in H^q$, we have that $H^{\bar{r}} \cap C_s \subset P$. Since $H^r \cap M = H^{\bar{r}}$, this completes the proof that q is in (M, r)-good position. □

Corollary 3.6. *Suppose that M^* is a countable elementary submodel of some $H(\theta)$ with $\mathbb{P} \in M^*$ and let $M = M^* \cap H(\omega_2)$. Also suppose that $r \in D$ for some $D \in M^*$*

and that r is M-prepared. Then there is a $q \in D \cap M$ which is in (M,r)-good position.

Proof. The condition r is a witness to the following valid existential statement: $D \cap \Gamma(r,M)$ is non-empty. Since $\Gamma(r,M)$ and D are in M^*, there is also some $q \in M^* \cap D \cap \Gamma(r,M)$. $\square$

The remainder of this section is devoted to proving

Theorem 3.7. *If $r \in \mathbb{P}$ is M-prepared and if $q \in M \cap \mathbb{P}$ is in (M,r)-good position, then r and q are compatible.*

Corollary 3.8. *The poset $\mathbb{P}$ is proper. Moreover, if $\mathbb{P} \in M^* \prec H(\theta)$, then r is $(M^*, \mathbb{P})$-generic providing $M^* \cap H(\aleph_2) \in \mathcal{M}_r \cap \mathcal{E}_0^2$.*

Borrowing from [12] we introduce an operation for amalgamating conditions q and r that we denote $q \oplus r$ and is called the minimal amalgamation. The word minimal refers to the idea that for t in the symmetric difference of H^q and H^r which have any elements of $H^q \cap H^r$ above them we add the minimal amount necessary to the new algebra element $a_t^{q\oplus r}$. The assumptions of r being M-prepared and q being in (M,r)-good position are designed so that the T-algebra $\langle a_t^{q\oplus r} : t \in \mathrm{Succ}(H^q \cup H^r)\rangle$ will behave properly with respect to being a common extension of each of $\langle a_t^q : t \in \mathrm{Succ}(H^q)\rangle$ and $\langle a_t^r : t \in \mathrm{Succ}(H^r)\rangle$, as well as proper behavior with respect to the models P in $\mathcal{M}_q \cup \mathcal{M}_r$. However there is a new challenge in that the set $H^q \cup H^r$ need not be a member of $\mathcal{H}_T$ and we will have to add steps to overcome that deficiency.

Definition 3.9. Suppose that $q, r \in \mathbb{P}$, $H^q \cup H^r \in \mathcal{H}_T$ and there is an $\bar{r} \in \mathbb{P}$ with $q, r \leq \bar{r}$ and $H^{\bar{r}} = H^q \cap H^r$. For each $t \in \mathrm{Succ}(H^q \cup H^r)$, we designate one of $t, t^\dagger$ to be primary. To make this canonical we assume that $t = (t \cap t^\dagger)0$ (i.e. t ends in a 0). If $t \in H^q \cap H^r$, then t is primary. If $t \in H^q \setminus H^r$ and $(t^\dagger)^+ \cap H^r$ is empty, then t is primary; otherwise $t^\dagger$ is primary. Similarly, if $t \in H^r \setminus H^q$ and $(t^\dagger)^+ \cap H^q$ is empty, then t is primary; otherwise $t^\dagger$ is primary. Let us recall that if $t \in H^q \setminus H^r$, then one of $t^+ \cap H^r$ or $(t^\dagger)^+ \cap H^r$ is empty.

We define $a_t^{q\oplus r}$ by recursion on the domain of t. Let us note that $[s]_q^{q\oplus r}$ and $[s]_r^{q\oplus r}$ have thereby also been defined for all $s \in C_{<t} \cap (H^r \cup H^q)$.

Now for the designated primary $t \in \mathrm{Succ}(H^q \cup H^r)$, we define

$$a_t^{q\oplus r} = \begin{cases} a_t^q \cup a_t^r & \text{if } t \in H^q \cap H^r \\ \{t\} \cup \bigcup\{[s]_q^{q\oplus r} : s \in C_{<t} \cap a_t^q\} & \text{if } t \in H^q \setminus H^r \\ \{t\} \cup \bigcup\{[s]_r^{q\oplus r} : s \in C_{<t} \cap a_t^r\} & \text{if } t \in H^r \setminus H^q \end{cases}$$

As usual, we define $a_{t^\dagger}^{q\oplus r}$ to be $(H^q \cup H^r) \cap C_{t^\dagger} \setminus a_t^{q\oplus r}$. Thus we have defined the collection $\langle a_t^{q\oplus r} : t \in \mathrm{Succ}(H^q \cup H^r)\rangle$. The definition of $\mathcal{M}_{q\oplus r}$ is the smallest collection containing $\mathcal{M}_q \cup \mathcal{M}_r$ which is closed under finite intersections.

It is immediate from the definition that if $t \in H^q \setminus H^r$ is primary, then $[t]_q^{q\oplus r}$ is simply $\{t\}$. However, this means that $[t^\dagger]_q^{q\oplus r}$ may contain many elements of $H^r \setminus H^q$. Analogous remarks hold for primary $t \in H^r \setminus H^q$.

Let us first check that $q \oplus r$ is a good algebraic extension of q and r. For any condition $p \in \mathbb{P}$, let p^- denote the condition we get by replacing $\mathcal{M}_p$ by the empty set. That is $p \leq p^-$, $H^{p^-} = H^p$ and $\mathcal{M}_{p^-} = \emptyset$. Note that with these assumptions we have that $a_t^{p^-} = a_t^p$ for each $t \in \mathrm{Succ}(H^p)$.

Lemma 3.10. *With q, r as in Definition 3.9, $q^- \oplus r^- \in \mathbb{P}$ is an extension of each q^- and r^-. Moreover, for each $s \in \mathrm{Succ}(H^q \cap H^r)$*

$$[s]_q^{q\oplus r} = [s]_{\bar r}^r \quad and \quad [s]_r^{q\oplus r} = [s]_{\bar r}^q \ .$$

Proof. Since $\mathcal{M}_{q^-\oplus r^-}$ is empty and $H^q \cup H^r \in \mathcal{H}_T$, and, for all $t \in H^q \cup H^r$, $a_t^{q^-\oplus r^-} \subset C_t \cap (H^q \cup H^r)$, it is immediate that $q^- \oplus r^-$ is in $\mathbb{P}$.

We apply Lemma 2.3 to show that $q^- \oplus r^-$ is below q^-. By symmetry we have that $q^- \oplus r^-$ is also below r^-. Choose $t \in \mathrm{Succ}(H^q)$, we must show $a_t^{q\oplus r} \cap H^q = a_t^q$ and that for each $s \in C_t \cap H^q$, $[s]_q^{q\oplus r}$ is not split by $a_t^{q\oplus r}$. Let us note that it is a triviality, that if s, x are distinct elements of $C_t \cap H^q$, then $[s]_q^{q\oplus r}$ and $[x]_q^{q\oplus r}$ are disjoint. It also follows that s is the only element of H^q in $[s]_q^{q\oplus r}$.

It is immediate that $a_t^q \subset a_t^{q\oplus r}$. If $t \in H^q \cap H^r$, then $a_t^r \cap H^q$ is contained in a_t^q, and so $a_t^{q\oplus r} \cap H^q = a_t^q$ as required. For the other inclusion when $t \in H^q \setminus H^r$, choose any $s \in a_t^q$ and consider $[s]_q^{q\oplus r} \subset a_t^{q\oplus r}$ as in the definition. We obviously have that $\{s\} = [s]_q^{q\oplus r} \cap H^q$ is contained in a_t^q.

Now we show that $a_t^{q\oplus r}$ does not split $[s]_q^{q\oplus r}$ for any $s \in C_{<t} \cap H^q$. We may assume that t is primary. The easy case is when $t \in H^q \setminus H^r$. If $s \in a_t^q$, then $[s]_q^{q\oplus r}$ is contained in $a_t^{q\oplus r}$ as in the definition. If $s \notin a_t^q$, then $[s]_q^{q\oplus r}$ is disjoint from $a_t^{q\oplus r}$ since $[x]_q^{q\oplus r}$ is disjoint from $[s]_q^{q\oplus r}$ for each $x \in a_t^q$.

Now we consider the case when $t \in H^q \cap H^r$ and so $a_t^{q\oplus r} = a_t^q \cup a_t^r$. Although $a_{t^\dagger}^{q\oplus r}$ is defined as $C_{t^\dagger} \cap (H^q \cup H^r) \setminus a_t^{q\oplus r}$, it is also the case that $a_{t^\dagger}^{q\oplus r} = a_{t^\dagger}^q \cup a_{t^\dagger}^r$. This follows directly from the fact that q and r are extensions of $\bar r$. That is, $a_t^q \cap H^r$ is equal to $a_t^r \cap H^q$. By this symmetry, it is enough to show that $a_t^{q\oplus r}$ does not split $[s]_q^{q\oplus r}$ for $s \in a_t^{q\oplus r} \cap H^q$. Again, since $a_t^r \cap H^q$ is contained in a_t^q, we have that $s \in a_t^q$. Since $t \in H^r$, we have that $s^+ \cap H^r$ is not empty. This implies that if $s \notin H^r$, then s is primary, and so $[s]_q^{q\oplus r}$ is simply $\{s\}$. Now we assume that $s \in H^q \cap H^r$, in which case $a_s^{q\oplus r} = a_s^q \cup a_s^r$. Since $r \leq \bar r$, we have that $[s]_{\bar r}^r \subset a_t^{q\oplus r}$. Therefore the proof that $a_t^{q\oplus r}$ does not split $[s]_q^{q\oplus r}$ is complete once we prove that $[s]_q^{q\oplus r} = [s]_{\bar r}^r$ (as stated in the Lemma).

If $x \in [s]_q^{q\oplus r} \setminus \{s\}$, then $x \in a_s^r \setminus a_\rho^r$ for all $\rho \in C_{<s} \cap H^q \cap H^r$. Therefore x is also in $[s]_{\bar r}^r$. Similarly if $s \neq x \in [s]_{\bar r}^r$, then $x \in a_s^r \setminus a_\rho^r$ for all $\rho \in C_{<s} \cap H^q \cap H^r$. Evidently, $x \notin H^{\bar r}$, in fact, x is not in $[\rho]_q^{q\oplus r}$ for all $\rho \in C_{<s} \cap H^q \cap H^r$. To show that $x \in [s]_q^{q\oplus r}$, it suffices to show that $x \notin a_\sigma^{q\oplus r}$ for all $\sigma \in C_{<s} \cap H^q \setminus H^r$. For each such σ, $s \in \sigma^+ \cap H^r$, and so σ is primary. Now x is not in $a_\sigma^{q\oplus r}$ since, by induction on such σ, x is not in $[\rho]_q^{q\oplus r}$ for all $\rho \in C_{<\sigma} \cap H^q$. □

The main lemma is the following.

Lemma 3.11. *If r is M-prepared, q is in (M, r)-good position, and $H^q \cup H^r \in \mathcal{H}_T$, then $q \oplus r$ is in $\mathbb{P}$ and is an extension of q and r.*

Proof. If $H^q \setminus H^r$ is empty, then $H^q = H^{r\restriction M}$. For $t \in H^q \cap H^r$, evidently $a_t^{q\oplus r} = a_t^r$. Also, for $t \in H^r \setminus H^q$ and $s \in H^q \cap a_t^r$, $[s]_{r\restriction M}^q = \{s\} \subset a_t^r$. Therefore, for all t, $a_t^{q\oplus r} = a_t^r$. It thus follows that for countable $P \in \mathcal{M}_r$, $\langle a_t^{q\oplus r} : t \in \mathrm{Succ}(H^{q\oplus r})\rangle$ is an extension of $\langle a_t^{q\oplus r} : t \in P \cap \mathrm{Succ}(H^{q\oplus r})\rangle$. Now we have to consider the countable models P in $\mathcal{M}_q \setminus \mathcal{M}_r$ which are all members of M. Let $\sigma < t$ both be in $P \cap \mathrm{Succ}(H^{q\oplus r})$. Since q is in (M, r)-good position, $H^r \cap M \cap C_t$ is contained in

P. But since $P \in M$, $H^r \cap M \cap C_t = H^r \cap P \cap C_t$. Since $M \in \mathcal{M}_r$, we have that $[\sigma]^r_M$ is not split by a^r_t. It follows then that $[\sigma]^r_P$ is not split by $a^{q\oplus r}_t = a^r_t$.

Now we will proceed by induction on the cardinality of $H^q \setminus H^r$. Choose any $\rho \in H^q \setminus H^r$ of maximum height. Notice that, by the maximality of ρ not being in H^r, neither immediate successor of ρ is in H^q. We define a condition q_1 so that $q < q_1$ and $\rho \notin H^{q_1}$.

If ρ is a limit node of T, then $H^{q_1} = H^q \setminus \{\rho\}$ is in $\mathcal{H}_T$. By induction, $q_1 \oplus r$ is in $\mathbb{P}$. Since ρ is a limit node, it is immediate that $q \oplus r = \langle H^q \cup H^r, \langle a^{q_1\oplus r}_t : t \in \mathrm{Succ}(H^{q_1} \cup H^r)\rangle, \mathcal{M}_{q_1\oplus r}\rangle$ is in $\mathbb{P}$. Similarly, the fact that $q \oplus r$ is below each of q and r follows immediately from the assumption that $q_1 \oplus r$ is below q_1 and r.

If ρ is not a limit node, then we may choose ρ so that $(\rho^\dagger)^+$ is disjoint from H^r. Let $\bar\rho$ be equal to $\rho \cap \rho^\dagger$. Since ρ was of maximum height, $\rho^\dagger$ is a maximal element of H^q. Choose $t_1 \in L^M_r$ so that $\rho \in (v_{t_1})^+ \setminus t_1^+$. Again let q_1 be the condition obtained by restricting q to $H^{q_1} = H^q \setminus \{\rho, \rho^\dagger\} \in \mathcal{H}_T$. That is, for each $t \in \mathrm{Succ}(H^{q_1})$, $a^{q_1}_t = a^q_t \setminus \{\rho, \rho^\dagger\}$. Applying the inductive hypotheses, we have that $q_1 \oplus r$ is in $\mathbb{P}$. It will be very helpful to understand the differences between $q \oplus r$ and $q_1 \oplus r$.

It is immediate from the definition of $a^{q\oplus r}_t$ that for each $t \in \mathrm{Succ}(H^q \cup H^r) \setminus \bar\rho^+$, $a^{q\oplus r}_t$ is equal to $a^{q_1\oplus r}_t$. More generally, by Lemma 3.10, for each $t \in \mathrm{Succ}(H^q)$, $\rho \in a^{q\oplus r}_t$ if and only if $\rho \in a^q_t$. In fact, by Definition 3.9 we have

Claim 3.11.1. For $t \in \mathrm{Succ}(H^q) \cap \rho^+$,

$$a^{q\oplus r}_t = \begin{cases} \{\rho\} \cup a^{q_1\oplus r}_t & \text{if } \rho \in a^q_t \\ a^{q_1\oplus r}_t & \text{if } \rho \notin a^q_t \end{cases}$$

For members of $H^r \setminus H^q$, the situation requires a definition. Let X_ρ be all the minimal elements of the set $\{x \in \mathrm{Succ}(H^q \cap H^r) : \rho \in a^q_x\}$.

Claim 3.11.2. For each $t \in \mathrm{Succ}(H^r)$

$$a^{q\oplus r}_t = \begin{cases} \{\rho\} \cup a^{q_1\oplus r}_t & \text{if } a^r_t \cap X_\rho \neq \emptyset \\ a^{q_1\oplus r}_t & \text{if } a^r_t \cap X_\rho = \emptyset \end{cases}$$

Proof of Claim 3.11.2. We prove this claim by induction on the level of t. First suppose that $a^r_t \cap X_\rho$ is not empty and fix (the unique) x in the intersection. We know that $\rho \in a^{q\oplus r}_x = a^q_x \cup a^r_x$. Since x is minimal, $\rho \notin a^{q\oplus r}_y$ for all $y \in C_{<x} \cap H^q \cap H^r$. Therefore, by Lemma 3.10, $\rho \in [x]^{q\oplus r}_r = [x]^q_{r\restriction M}$. Since $[x]^{q\oplus r}_r \subset a^{q\oplus r}_t$, we have that $\rho \in a^{q\oplus r}_t$. By the induction assumption, we have that $[s]^{q_1\oplus r}_r \subset [s]^{q\oplus r}_r \subset \{\rho\} \cup [s]^{q_1\oplus r}_r$ for each $s \in H^q \cap a^r_t$, and so $a^{q\oplus r}_t = \{\rho\} \cup a^{q_1\oplus r}_t$.

Now suppose that $a^r_t \cap X_\rho$ is empty. If $C_{<t} \cap X_\rho$ is empty then $\rho \notin [s]^{q\oplus r}_r$ for all $s \in C_{<t} \cap H^q \cap H^r$. Otherwise, there is a unique $x \in C_{<t} \cap X_\rho$ and we have, also by induction on t, that $\rho \notin [s]^{q\oplus r}_r$ for all $x \neq s \in C_{<t} \cap H^r$. By the inductive assumption then, we have that $[s]^{q\oplus r}_r = [s]^{q_1\oplus r}_r$ for all $s \in a^r_t$. It follows from Definition 3.9 that $a^{q\oplus r}_t = a^{q_1\oplus r}_t$. □

Now we let $P \in (\mathcal{M}_q \cup \mathcal{M}_r) \cap \mathcal{E}^2_0$. Fix any $\sigma \in C_t$ with both σ and P in $P \cap (H^q \cup H^r)$. By Lemma 2.3 it suffices to show that $a^{q\oplus r}_t$ does not split $[\sigma]^{q\oplus r}_P$. This will require considering various cases. By the induction hypothesis, we do have that if $\{\sigma, t\}$ is disjoint from $\{\rho, \rho^\dagger\}$, then $a^{q_1\oplus r}_t$ does not split $[\sigma]^{q_1\oplus r}_P$. Also, it suffices to restrict our attention to the case when t is designated primary in the definition of $q \oplus r$, and so we know that $\rho^\dagger$ is not equal to either σ or t. Therefore we may as well assume that $t \in \rho^+$.

Let us first consider the case where t is equal to ρ. Since q is in (M, r)-good position and $\rho \in P \cap v_{t_1}^+ \setminus t_1^+$, we have that $H^r \cap M \cap C_{v_{t_0}}$ is contained in P if $P \in \mathcal{M}_q$. On the other hand, if $P \cap M \notin \mathcal{M}_q$, then by Proposition 1.12, $M \cap C_{<\rho}$ is contained in P.

If $\sigma \in H^q \cap H^r$, then $[\sigma]_P^{q\oplus r} = (a_\sigma^q \cup a_\sigma^r) \setminus \bigcup\{a_x^{q\oplus r} : x \in P \cap C_{<\sigma} \cap (H^q \cup H^r)\}$. This is easily seen to be contained in $[\sigma]_P^q \cup [\sigma]_{r\restriction M}^r$. If $P \cap M \notin M$, then $[\sigma]_P^q = \{\sigma\}$, while if $P \in \mathcal{M}_q$ then a_ρ^q does not split $[\sigma]_P^q$. If $\sigma \in a_\rho^q$, then $[\sigma]_{r\restriction M}^r$ is contained in $a_\rho^{q\oplus r}$. If $\sigma \notin a_\rho^q$, then $[x]_q^{q\oplus r}$ is disjoint from $[\sigma]_q^{q\oplus r} = [\sigma]_{r\restriction M}^r$ for all $x \in a_\rho^q$. Therefore, if $\sigma \in a_t^q$, we have that $[\sigma]_P^{q\oplus r}$ is contained in $a_t^{q\oplus r}$, and if $\sigma \notin a_t^q$, then $a_t^{q\oplus r}$ is disjoint from $[\sigma]_P^{q\oplus r}$.

Now assume that $\sigma \in H^q \setminus H^r$. Since $H^r \cap a_\sigma^q$ is contained in P, we have that $[s]_q^{q\oplus r}$ is disjoint from $[\sigma]_P^{q\oplus r}$ for each $s \in H^r \cap a_t^q$. If $P \cap M \notin M$, then $[\sigma]_P^q$ is equal to $\{\sigma\}$ and so there is nothing to prove. If $P \cap M \in \mathcal{M}_q$, then we have that a_ρ^q does not split $[\sigma]_P^q$. Note that, since $q_1 \oplus r < q_1$, it follows from Lemma 2.3 that the family $\{[s]_{q_1}^{q_1\oplus r} : s \in C_\sigma \cap H^q\}$ is a partition of $C_\sigma \cap H^{q\oplus r}$. From this it follows that $[\sigma]_P^{q\oplus r} = [\sigma]_P^{q_1\oplus r}$ is equal to the union of the family $\{[s]_{q_1}^{q\oplus r} : s \in [\sigma]_P^q\}$. Also, by Lemma 3.10 and the Claims above, $a_\rho^{q\oplus r}$ is equal to $\{\rho\} \cup \bigcup\{[s]_{q_1}^{q_1\oplus r} : s \in C_{<\rho} \cap a_\rho^q\}$. Since a_ρ^q does not split $[\sigma]_P^q$ we now have that $a_\rho^{q\oplus r}$ does not split $[\sigma]_P^{q\oplus r}$.

The final case for $t = \rho$ is to consider $\sigma \in H^r \setminus H^q$. In this case, with $\sigma \in P$, we have that $P \notin \mathcal{M}_q$. Also, $P \cap M$ is not in M because $\rho \in P$. Therefore $C_{<\rho} \cap M \subset P$. By the induction hypothesis, $[s]_q^{q\oplus r}$ does not split $[\sigma]_P^{q\oplus r}$ for any $s \in C_{<\rho} \cap H^r \cap M$. Since $\sigma \notin a_\rho^q$, we have that $a_\rho^{q\oplus r}$ meets $[\sigma]_P^{q\oplus r}$ if and only if $[s]_q^{q\oplus r}$ meets it for some $s \in a_\rho^q \cap H^r$. Therefore, $a_\rho^{q\oplus r}$ does not split $[\sigma]_P^{q\oplus r}$.

Now we consider the cases when $\rho < t$. Notice that $t \in H^r$ since ρ was maximal in $H^q \setminus H^r$. If $\sigma < \rho$, then we can deduce that $a_t^{q\oplus r}$ does not split $[\sigma]_P^{q\oplus r}$ directly from the assumption that $a_t^{q_1\oplus r}$ does not split $[\sigma]_P^{q_1\oplus r}$. Now suppose that $\rho = \sigma$. We have that $H^r \cap a_\rho^q \subset P$ and, as seen above, $a_\rho^{q\oplus r}$ is equal to $\{\rho\} \cup \bigcup\{[s]_{q_1}^{q_1\oplus r} : s \in C_{<\rho} \cap a_\rho^q\}$. For $s \in H^r \cap a_\rho^q$, $[s]_{q_1}^{q_1\oplus r}$ is disjoint from $[\rho]_P^{q\oplus r}$ because $H^r \cap a_\rho^q \subset P$. For $s \in a_\rho^q \setminus H^r \subset H^q \setminus H^r$, s was designated as primary because $s < t$, and we have seen that $[s]_q^{q_1\oplus r}$ is simply $\{s\}$. Therefore we have shown that $[\rho]_P^{q\oplus r} = [\rho]_P^q$. If $t \in H^q \cap P$, then a_t^q does not split $[\sigma]_P^q$ and so we may assume that $t \in P \cap H^r \setminus H^q$. Then $M \cap C_\rho \subset P$, and so $[\sigma]_P^q = \{\sigma\}$. Now we consider the case that $\rho < \sigma$, and so $\sigma \notin H^q \setminus H^r$. Again, we have that $[\sigma]_P^{q_1\oplus r}$ is not split by $a_t^{q_1\oplus r}$. Certainly if $\rho \notin [\sigma]_P^{q\oplus r}$, then $a_t^{q\oplus r}$ will not split $[\sigma]_P^{q\oplus r}$. If ρ is in P, then ρ is not in $[\sigma]_P^{q\oplus r}$. Moreover, if $\rho \in a_\sigma^{q\oplus r}$, then either $\sigma \in H^q$ or $a_\sigma^r \cap X_\rho$ is non-empty. In either event, there is an $x \in X_\rho \cap C_\sigma \subset M$. First suppose that $P \cap M \in \mathcal{M}_r \cap M$, and therefore that σ and t are each in $H^r \cap M$. Let $Y = C_{<\sigma} \cap H^r \cap P$. In the case that $\sigma \notin a_t^r$ it follows that a_t^q is disjoint from $[\sigma]_P^q = a_\sigma^q \setminus \{a_y^q : y \in Y\}$. Since $a_t^r \cap H^q \subset a_t^q$, we also have that a_t^r is disjoint from $[\sigma]_P^q = a_\sigma^q \setminus \{a_y^q : y \in Y\}$. Similarly, each of a_t^r and a_t^q are disjoint from $[\sigma]_P^r = a_\sigma^r \setminus \{a_y^r : y \in Y\}$. An application of De Morgan laws demonstrates that $a_t^{q\oplus r} = a_t^q \cup a_t^r$ is disjoint from $(a_\sigma^q \cup a_\sigma^r) \setminus \bigcup\{(a_y^q \cup a_y^r) : y \in Y\} = [\sigma]_P^{q\oplus r}$. By exactly analogous reasoning, one can show that if $\sigma \in a_t^q \cap a_t^r$, then $a_t^{q\oplus r}$ will contain $[\sigma]_P^{q\oplus r}$.

Finally we may assume that $P \cap M \notin M \cap \mathcal{M}_r$ and that one of σ or t is not in $H^r \cap M$. From these assumptions it follows that $x \in P$ (recall that $\{x\} = a^q_\sigma \cap X_\rho$). Since $\rho \in a^{q\oplus r}_x$, it follows from Claim 3.11.2 that $\sigma = x$, i.e. if $x < \sigma$, then $a^{q\oplus r}_x$ is removed from $a^{q\oplus r}_\sigma$ in the definition of $[\sigma]^{q\oplus r}_P$. Also by Claim 3.11.2, we have that $\rho \in a^{q\oplus r}_t$ if and only if $x = \sigma \in a^{q\oplus r}_t$. So, if $\sigma \in a^{q\oplus r}_t$, then $[\sigma]^{q\oplus r}_P \subset a^{q\oplus r}_t$, and if $\sigma \notin a^{q\oplus r}_t$, then $a^{q\oplus r}_t$ is disjoint from $[\sigma]^{q\oplus r}_P$. This completes the proof of the Lemma. □

Then to complete the proof of Theorem 3.7, we have to handle the deficiency when $H^q \cup H^r$ is not in $\mathcal{H}_T$. This deficiency can be measured with L^M_r.

Definition 3.12. If r is M-prepared and q is in (M, r)-good position, let $L^M_r(q) = \{t \in L_r : (\exists s \in H^q)\ t \cap s \notin H^q \cup H^r\}$.

Lemma 3.13. *If r is M-prepared and q is in (M, r)-good position then $H^q \cup H^r \in \mathcal{H}_T$ so long as $L^M_r(q)$ is empty.*

Proof. We assume that $L^M_r(q)$ is empty. Let $s \in H^q \setminus H^r$ and $t \in H^r \setminus H^q$. Since q is in (M, r)-good position, there is a $t_1 \in L^M_r$ such that $s \in v^+_{t_1} \setminus t^+_1$. By assumption, we have that $s \cap t_1 \in H^q \cup H^r$. Obviously if $t_1 \le t$, then $s \cap t = s \cap t_1$ which is in $H^q \cup H^r$. Similarly, if $t_1 < t$, then again $t \cap s = t_1 \cap s$ which is in $H^q \cup H^r$. Finally we have that $\bar{t} = t \cap t_1 \in H^r$ is below v_{t_1}, and so $t \cap s = t \cap v_{t_1}$ which is in H^r. □

Then we show how to, in effect, reduce the size of $L^M_r(q)$.

Lemma 3.14. *If r is M-prepared and q is in (M, r)-good position and $t_0 \in L^M_r(q)$, then there is a condition $q_0 \le q$ such that q_0 is in (M, r)-good position and $L^M_r(q_0) = L^M_r(q) \setminus \{t_0\}$.*

Proof. Choose s_0 minimal in H^q so that $t_0 \cap s_0$ is not in $H^q \cup H^r$. There is a $t_1 \in L^M_r$ such that $s_0 \in (v_{t_1})^+ \setminus (t_1)^+$. We check that $t_1 = t_0$. If $t_0 < t_1$, then s_0 is a witness to the fact that u_{t_0} exists, and we would have that $t_0 \cap s_0$ is equal to $u_{t_0} \cap s_0 \in H^q$. Otherwise $t_0 \cap t_1$ is below v_{t_0}, and again we would have that $t_0 \cap s_0 = v_{t_0} \cap s_0 \in H^q$.

Let $\rho \in C_{s_0} \setminus C_{t_0}$ be chosen so that $\rho^\dagger \in C_{t_0}$. Since $H^q \in \mathcal{H}_T$, we have that $H^q \cap (\rho^\dagger)^+$ is empty. Set $\bar{\rho}$ to be the maximal limit below ρ and let $\bar{\psi}$ be the minimal element of $\{\psi : \bar{\rho} \subseteq \psi \subseteq \rho\} \setminus H^q$. We check that $\bar{H} = H^q \cup \{\psi, \psi^\dagger : \bar{\psi} \subseteq \psi \subseteq \rho\}$ is in $\mathcal{H}_T$. To show this it suffices to consider any $s_1 \in H^q$ and $\sigma \in \{\psi, \psi^\dagger : \bar{\psi} \subseteq \psi \subseteq \rho\}$ and to show that $s_1 \cap \sigma$ is in $\bar{H}$. If $s_0 \le s_1$, then $s_1 \cap \sigma = s_0 \cap \sigma$ is either σ or $\sigma \cap \sigma^\dagger$, and so is in $\bar{H}$. Similarly, if v_{t_0} is not below s_1, then $s_1 \cap \sigma$ is equal to $s_1 \cap v_{t_0}$, which is in H^q. So now we may assume that $v_{t_0} < t_1 = s_1 \cap t_0$. If t_1 is not below s_0, then $t_1 \cap s_0 = s_1 \cap s_0$ is in H^q. But also $t_1 \cap s_0$ would be equal to $t_0 \cap s_0$ which contradicts that $t_0 \cap s_0$ is not in H^q. Therefore t_1 is below s_0 which implies that $t_1 = t_1 \cap s_0 = (s_1 \cap t_0) \cap s_0 = s_1 \cap (t_0 \cap s_0) = (s_1 \cap s_0) \cap t_0$. Since s_0 is minimal, $(s_1 \cap s_0) \cap t_0$ must be in $H^q \cup H^r$. Since this value is above v_{t_0}, it must be in H^q.

It is immediate that if $q_0 < q$ is in $\mathbb{P}$ with $H^{q_0} = \bar{H}$, then $L^M_r(q_0) = L^M_r(q) \setminus \{t_0\}$. Now we detail how to define $a^{q_0}_t$ for $t \in \mathrm{Succ}(\bar{H})$. For each $\psi \in \{\psi : \bar{\psi} \subseteq \psi \subseteq \rho\}$, we set $a^{q_0}_\psi = \{\psi\}$. As required, $a^{q_0}_{\psi^\dagger} = \bar{H} \cap C_{\psi^\dagger}$, i.e. $a^{q_0}_{\psi^\dagger} = \bar{H} \cap C_{\psi^\dagger} \setminus a^{q_0}_\psi$.

Next we fix any maximal element x_0 of $H^q \cap s^+_0$, and set $a^{q_0}_y = a^q_y$ for all $y \in H^q \cap C_{x_0}$. This of course means that, for $\rho \in C_y$, $a^{q_0}_{y^\dagger} = a^q_{y^\dagger} \cup \{\psi : \bar{\psi} \subseteq \psi \subseteq \rho\}$.

For brevity, let $\Psi = \{\psi : \bar\psi \subseteq \psi \subseteq \rho\}$. Finally, for $s \in H^q \cap s_0^+ \setminus C_{x_0}$,

$$a_s^{q_0} = \begin{cases} a_s^q \cup \Psi & (\exists y \in a_s^q)\ y^\dagger \in C_{x_0} \\ a_s^q & (\exists y \in C_s \setminus a_s^q)\ y^\dagger \in C_{x_0}\ . \end{cases}$$

It may not be immediate that the two cases in the definition of $a_s^{q_0}$ are exhaustive, but they are, and the definition helps clarify the key idea. The key idea is that $\Psi \cup \{y\}$ is not split by a_s^q where $y \in C_s$ is the unique element such that $y^\dagger \in C_{x_0}$.

We set $\mathcal{M}_{q_0}$ to be $\mathcal{M}_q$ and to finish the proof, we have to consider any $P \cap \mathcal{M}_q \cap \mathcal{E}_0^2$, and show that if $\sigma < t \in \mathrm{Succ}(T) \cap P$, then $a_t^{q_0}$ does not split $[\sigma]_P^{q_0}$. It should be clear that this is immediate if σ is not above $\bar\rho$. In fact, it is still immediate if $\sigma \in \Psi$, since then $[\sigma]_P^{q_0}$ is simply $\{\sigma\}$ because if $\sigma \in P$, then $\Psi \subset P$.

Therefore we must consider the cases when $s_0 \le \sigma < t$. If $t \in C_{x_0}$, then also $\sigma \in C_{x_0}$ and we know that a_t^q does not split $[\sigma]_P^q$. Following the definitions, it is immediate that $a_t^{q_0} = a_t^q$ and $[\sigma]_P^{q_0} = [\sigma]_P^q$ and so $a_t^{q_0}$ does not split $[\sigma]_P^{q_0}$. Otherwise, let $y_t \in C_t$ be chosen so that $y_t^\dagger \in C_{x_0}$. If $\sigma < y_t$, then we still have that $\sigma \in C_{x_0}$ and so $[\sigma]_P^{q_0} = [\sigma]_P^q$ is not split by $a_t^q \cup \Psi$. The final case is that $y_t \le \sigma$ and so $y_t \in C_\sigma$. For all $x \in C_t \cap y_t^+$, we have that a_x^q does not split $\{y_t\} \cup \Psi$. Now if y_t is not in $[\sigma]_P^q$, then $[\sigma]_P^{q_0} = [\sigma]_P^q$ is not split by a_t^q. If y_t is in $[\sigma]_p^q$, then $[\sigma]_P^{q_0} = [\sigma]_P^q \cup \Psi$. If $y_t \in a_t^q$, then $a_t^{q_0}$ contains $[\sigma]_P^q \cup \Psi$, while if $y_t \notin a_t^q$, then $a_t^{q_0} = a_t^q$ is disjoint from $[\sigma]_P^q \cup \Psi$. □

4. Basic properties forced by $\mathbb{P}$

This next result should probably be called Rabus' Lemma. It is the key step in proving that (x, τ_x) is initially ω_1-compact for $x \in bT \cap 2^{\omega_2}$. We adapt the difficult proof from [12, 5.4].

Lemma 4.1 (CH)**.** *Let $x \in bT \cap 2^{\omega_2}$ and suppose that $\dot A$ is a $\mathbb{P}$-name of a countable subset of C_x and that there is a $p \in \mathbb{P}$ satisfying that for all $q < p$ there is an $r < q$ and a $t \in C_x$ with $r \Vdash t \in \dot A$, and $t \notin a_s^r$ for all $s \in H^q \cap C_x$. Then there is a $\lambda \in \omega_2$ and an $r < p$ with some countable $M_0 \in \mathcal{M}_r$ such that r forces that for each proper finite extension s of $x \restriction \lambda$, and each finite $H \subset C_{<s}$, $\dot A \cap M_0 \cap a_s \setminus \bigcup_{\sigma \in H} a_\sigma$ is infinite.*

Proof. Choose a countable $M_0^* \prec H(\aleph_3)$ which contains $\dot A, x, \lhd$ and p and choose an uncountable internally approachable $P_0^* \prec H(\aleph_3)$ with $M_0^* \in P_0^*$. Note that $\mathcal{E}^2 \in P_0^*$. Let $P_0 = P_0^* \cap H(\aleph_2) \in \mathcal{E}_1^2$, let $M_0 = M_0^* \cap H(\aleph_2)$ and set $\lambda = P_0 \cap \omega_2$. Let $r < p$ be any condition such that M_0, P_0 are both in $\mathcal{M}_r$. Since $\dot A$ is forced to be countable, we may suppose there is a countable set $\{\dot\psi_n : n \in \omega\} \in M_0^*$ of $\mathbb{P}$-names of elements of C_x such that r forces that $\dot A$ is equal to the sequence $\{\dot\psi_n : n \in \omega\}$ (by a standard abuse of notation).

Let s be any proper finite extension of $x \restriction \lambda$. It will suffice to find a condition $\bar q < r$ and a $\psi \in C_x \cap M_0$ so that $\bar q \Vdash \psi \in \dot A$ and $\psi \in a_s^{\bar q} \setminus \bigcup\{a_d^{\bar q} : d \in H^r \cap C_{<s}\}$. The main new step is that when verifying the good behavior of each $P \in \mathcal{M}_{\bar q}$ we will not have any assumptions about being in good position. Instead, using that $P_0 \in \mathcal{E}_1^2$ we will construct an auxiliary extension of r which will take care of the interactions between elements of $H^{\bar q}$.

We may assume that $C_s \setminus P_0$ is contained in H^r and that, by Lemma 2.8, there is a $\bar d \in H^r \cap P_0$ such that $C_x \cap H^r \cap P_0 \subset a_{\bar d}^r$. Now let $\{M_1, M_2, \ldots, M_{2\ell}\}$ (so that $M_{2i-1} = M_{2i}$ for each $1 < i \le \ell$) enumerate those $P \in (\mathcal{M}_r \setminus P_0) \cap \mathcal{E}_0^2$ such that $P \cap C_x \setminus P_0$ is not empty. Since we are assuming CH and P_0^* is internally approachable, it follows that P_0^* is closed under ω-sequences. Using elementarity, there is a condition $r_1 \in P_0^*$ so that $r_1 < r \restriction P_0$, the minimal element of $\mathcal{M}_{r_1} \setminus \mathcal{M}_r$ is some $P_1 \in \mathcal{E}_1^2$ and we have that for all $P \in \mathcal{M}_{r_1} \cap \mathcal{E}_0^2$, $P \cap P_1 \in \mathcal{M}_r$. We may further suppose that there is some $\bar{M}_1 \in \mathcal{M}_{r_1} \cap \mathcal{E}_0^2$ such that $\bar{M}_1 \cap C_x \setminus P_1$ is not empty, and $\bar{M}_1 \cap P_1 = M_1 \cap P_0$. Next we choose $r_2 \in \mathbb{P} \cap P_0^*$, so that again, $r_2 < r \restriction P_0$ and also so that there is a $P_2 \in \mathcal{M}_{r_2} \cap \mathcal{E}_1^2$ which is the minimal element of $\mathcal{M}_{r_2} \setminus \mathcal{M}_r$ and $r_1 \in P_2$. We also ensure that there is some $\bar{M}_2 \in \mathcal{M}_{r_2}$ so that $\bar{M}_2 \cap P_2 = M_2 \cap P_0$ and $\bar{M}_2 \cap C_x \setminus P_2$ is not empty. Continuing in this way we may choose such conditions $\{r_1, \ldots r_{2\ell}\}$. All we really need is the sequence $\{\mathcal{M}_{r_i} : 1 \leq i \leq 2\ell\}$ and, for each $1 \leq i \leq 2\ell$, $\bar{M}_i \in \mathcal{M}_{r_i}$ with $\bar{M}_i \cap P_i = M_i \cap P_0$. Notice that $\{P_1, \ldots, P_{2\ell}\} \in P_0^* \cap \mathcal{E}_1^2$ is a chain.

For each $1 \leq i \leq 2\ell$, choose the minimal element $y_i \in C_x \cap \bar{M}_i \setminus P_i$, and note that $y_i \wedge y_i^\dagger$ is on a limit level and not in P_i. Observe that if $y \in H^r \setminus P_0$ is such that $y \wedge x \in P_0$, then $y \wedge x \in P_1$. Let $S^r = \{x \restriction \lambda\} \cup \{\sigma \in H^r : \sigma \wedge x \in P_0\} = \{x \restriction \lambda\} \cup H^r \setminus (x \restriction \lambda)^+$ and note that $S^r \supset H^r \cap P_0$ and that $S^r \in \mathcal{H}_T$. Also note that $r \restriction S^r = \langle S^r, \langle a^r_\sigma : \sigma` \in \mathrm{Succ}(S^r)\rangle, \mathcal{M}_r\rangle$ is a condition in $\mathbb{P}$ and that $r < r \restriction S^r$.

We define a condition $\bar{r}$ extending $r \restriction S^r$. The collection $\mathcal{M}_{\bar{r}} = \mathcal{M}_r \cup \mathcal{M}_{r_1} \cup \cdots \cup \mathcal{M}_{r_{2\ell}}$ is already an $\in$-chain and is closed under intersections. We let $H^{\bar{r}} = S^r \cup \{y_i, y_i^\dagger, y_i \wedge y_i^\dagger : 1 \leq i \leq 2\ell\}$. It is routine to check that $H^{\bar{r}} \in \mathcal{H}_T$ (i.e. closed under intersection and $\dagger$). Notice that if $P \in \mathcal{M}_{\bar{r}}$ and distinct y_i, y_j are both in P then $P \in \mathcal{E}_1^2$ and $P_1 \subset P$. It is also true that for any $P \in \mathcal{M}_r \cap \mathcal{E}_0^2$, we have that $y_i \notin P$ for all $1 \leq i \leq 2\ell$.

Next we define $a^{\bar{r}}_s$ for all $s \in H^{\bar{r}}$. For $s \in \mathrm{Succ}(S^r)$, and so $s \wedge x \in P_0$, we let $a^{\bar{r}}_s = a^r_s$. For each $1 \leq i \leq 2\ell$, we define

(1) $a^{\bar{r}}_{y_i}$ to be $\{y_i\} \cup a^r_{\bar{d}}$,
(2) $a^{\bar{r}}_{y_i^\dagger}$ to be $\{y_i^\dagger\} \cup \{y_j : 1 \leq j < i\}$

If $1 \leq i < j \leq 2\ell$ are distinct and are members of $P \in \mathcal{M}_{\bar{r}}$, then each of $a^{\bar{r}}_{y_i} \cap a^{\bar{r}}_{y_j}$, and $a^{\bar{r}}_{y_i} \cap a^{\bar{r}}_{y_j^\dagger}$ are certainly in the algebra generated by $\langle a^{\bar{r}}_\rho : \rho \in P \cap C_{y_i} \cap H^{\bar{r}}\rangle$ since P will contain C_{y_j}. For each $s \in H^r \cap C_x \cap P_0$, $1 \leq i \leq 2\ell$ and $\rho \in \{y_i, y_i^\dagger\}$, $a^{\bar{r}}_s \cap a^{\bar{r}}_\rho$ is equal to one of $\{a^{\bar{r}}_s, \emptyset\}$. Therefore it is in the algebra generated by $\{a^{\bar{r}}_t : t \in P \cap H^{\bar{r}}\}$ for any $P \in \mathcal{M}_{\bar{r}}$ containing $\{\rho, s\}$. It is also straightforward to check that $a^{\bar{r}}_s \cap a^{\bar{r}}_t$ is in the algebra generated by $\{a^{\bar{r}}_y : y \in P \cap H^{\bar{r}}\}$ for all P containing $\{s, t\}$ for any pair $s, t \in C_x \cap H^{\bar{r}} \setminus P_1$. For other values of s, t we have $a^{\bar{r}}_s \cap a^{\bar{r}}_t = a^r_s \cap a^r_t$. It is also obvious that $\bar{r} < r \restriction S^r$.

Choose any $q < \bar{r}$ and $n \in \omega$ such that there is a $\psi \in H^q \cap C_x$ with $q \Vdash \psi = \dot{\psi}_n \in \dot{A}$, and $\psi \notin a^q_d$ for all $d \in H^r \cap P_0$. Now $\dot{\psi}_n, \dot{A} \in M_0^*$ and, by Corollary 3.8, q is $(\mathbb{P}, M_0^*)$-generic, so there is a condition $q' \in M_0$ compatible with q such that q' forces a value on $\dot{\psi}_n$. By elementarity then, we have that $\psi \in M_0$. By extending q we may assume that $q < q'$. Now it follows that $q \restriction P_1 \Vdash \psi \in \dot{A}$ and that $\psi \notin a^q_d$ for all $d \in C_x \cap H^r \cap P_0$.

We make the following key observation about $q \restriction P_1$. For each $1 \leq i \leq 2\ell$ and each $\rho \in H^q \cap C_x \cap \bar{M}_i \cap P_1$, each of $a^q_\rho \setminus a^q_{\bar{d}}$ and $a^q_\rho \cap a^q_{\bar{d}}$ are in the algebra, $B^q_{\rho,i}$, generated by $\langle a^q_s : s \in H^q \cap P_1 \cap C_\rho \cap \bar{M}_i\rangle$. The reason that this is true is that if we let i' be chosen so that $\{i', i\} = \{2j-1, 2j\}$ for some $1 \leq j \leq \ell$, then each of $a^q_\rho \cap a^q_{y_i}$ and $a^q_\rho \cap a^q_{y_{i'}}$ are in $B^q_{\rho,i}$. Of course the intersection of the two is $a^q_\rho \cap (a^q_{y_i} \cap a^q_{y_{i'}})$. But also the algebra generated by $\langle a_t^q : t \in H^{\bar{r}}\rangle$ is isomorphic to the algebra generated by $\langle a_t^{\bar{r}} : t \in H^{\bar{r}}\rangle$. In the latter algebra we have that $a_{y_i}^{\bar{r}} \cap a_{y_{i'}}^{\bar{r}} = a_{\bar{d}}^{r}$.

For each $s \in H^r \setminus S^r$, hence $C_x \cap P_0 \subset C_s$, let $P_s \in \mathcal{M}_r \cap \mathcal{E}_0^2$ be minimal such that $s \in P_s$. We define an auxiliary set b_s^q as follows

$$b_s^q = \bigcup\{[\rho]^q_{P_s \cap H^r} : \rho \in a_s^r \cap P_0\} \ .$$

Let φ denote the Boolean isomorphism lifting of the embedding of $\langle a_\rho^r : \rho \in H^r \cap C_{\bar{d}}\rangle$ into $\langle a_\rho^q : \rho \in H^r \cap C_{\bar{d}}\rangle$. $a_s^r \cap a_{\bar{d}}^r$ is the same as $a_s^r \cap P_0$ and so is equal to $\bigcup\{[\rho]^r_{P_s} : \rho \in a_s^r \cap P_0\}$. Since $q < r \restriction P_0$, for each $\rho \in a_s^r \cap P_0$, $\varphi([\rho]^r_{P_s})$ will equal $[\rho]^q_{P_s \cap H^r}$. This means that $\varphi(a_s^r \cap a_{\bar{d}}^r)$ is equal to b_s^q.

We are now ready to define our condition $\bar{q}$. We set $H^{\bar{q}} = (H^q \setminus (x \restriction P_1)^+) \cup (H^r \cap (x \restriction P_1)^+)$. It is useful to note that $H^{\bar{q}} \cap (x \restriction P_1)^+$ is equal to $H^r \cap (x \restriction \lambda)^+$. Since $x \restriction \lambda \in H^q$, it follows easily that $H^{\bar{q}} \in \mathcal{H}_T$. Recall that $H^r \cap (x \restriction P_1)^+$ is equal to $H^r \cap (x \restriction \lambda)^+$. Let $\mathcal{M}_{\bar{q}} = (\mathcal{M}_q \cap P_1) \cup (\mathcal{M}_r \setminus P_0)$ and define W to be $C_x \cap P_1 \cap H^q \setminus a_{\bar{d}}^q$. Notice that $\psi \in W$. For all $s \in H^{\bar{q}} \setminus (x \restriction P_1)^+$, define $a_s^{\bar{q}} = a_s^q$. Then we can define $a_s^{\bar{q}}$ for $s \in H^r \cap (x \restriction \lambda)^+$ as we did in Lemma 3.14. Choose any maximal $x_0 \in H^r \cap (x \restriction \lambda)^+$ so that $(s)^\dagger \in C_{x_0}$. Mimicking Lemma 3.14, and with W playing the role of Ψ, define, for $s \in H^r \cap (x \restriction \lambda)^+$

$$a_s^{\bar{q}} = \begin{cases} a_s^r \cup b_s^q \cup W & (\exists y \in a_s^r)\ y^\dagger \in C_{x_0} \\ a_s^r \cup b_s^q & (\exists y \in C_s \setminus a_s^r)\ y^\dagger \in C_{x_0}. \end{cases}$$

Notice that, since $s^\dagger \in C_{x_0}$, we have that $W \subset a_s^{\bar{q}}$. This means that $\psi \in a_s^{\bar{q}}$. Also, since $b_{s^\dagger}^q$ is equal to $a_{\bar{d}}^q \setminus b_s^q$, we have that $a_{s^\dagger}^{\bar{q}}$ is equal to $C_{s^\dagger} \cap H^{\bar{q}} \setminus a_s^{\bar{q}}$.

We need that $\bar{q}$ is below each of $q \restriction P_1$ and r. It is immediate that $a_s^{\bar{q}} \cap H^q = a_s^q$ for all $s \in \mathrm{Succ}(H^q) \cap P_1$. Given the definition of b_s^q, and following the approach in Lemma 3.14, it is straightforward to show that $a_s^{\bar{q}} \cap H^r = a_s^r$ for all $s \in \mathrm{Succ}(H^r)$.

We finish the proof by considering any countable $P \in \mathcal{M}_{\bar{q}}$, and let $\sigma < \tau$ be members of $P \cap \mathrm{Succ}(H^{\bar{q}})$. It will suffice to show that $a_\sigma^{\bar{q}} \cap a_\tau^{\bar{q}}$ is in the algebra generated by $\langle a_s^{\bar{q}} : s \in C_\sigma \cap H^{\bar{q}} \cap P\rangle$. First suppose that $\tau \in P_0$. Since $P \cap P_0 \in \mathcal{M}_q$, then the claim is immediate from the fact that $q \in \mathbb{P}$. Therefore we may assume that $\tau \in (x \restriction \lambda)^+$. In this case, there is an $i \leq 2\ell$ such that P is equal to M_i. If σ is also in $(x \restriction \lambda)^+$, then the verification that $a_\sigma^{\bar{q}} \cap a_\tau^{\bar{q}}$ is in $\langle a_s^{\bar{q}} : s \in C_\sigma \cap H^{\bar{q}} \cap P\rangle$ proceeds as in Lemma 3.14. If $\tau \in C_{x_0}$, then W is disjoint from $a_\sigma^{\bar{q}} \cup a_\sigma^{\bar{q}}$, and in this case, $a_\sigma^{\bar{q}} \cap a_\tau^{\bar{q}} = (a_\sigma^r \cup b_\sigma^q) \cap (a_\tau^r \cup b_\tau^q)$. Since each of P_σ, P_τ defined above are contained in M_i, b_σ^q and b_τ^q are in the algebra $\langle a_s^{\bar{q}} : s \in H^q \cap C_\sigma \cap M_i\rangle$. Similarly $a_\sigma^r \cap a_\tau^r$ is in the algebra $\langle a_s^r : s \in H^r \cap C_\sigma \cap M_i\rangle$. So if we calculate $a_\sigma^{\bar{q}} \cap a_\tau^{\bar{q}} \cap a_{\bar{d}}^{\bar{q}}$ we are in the algebra $\langle a_s^{\bar{q}} : s \in H^q \cap C_\sigma \cap M_i\rangle$. And, similarly $a_\sigma^{\bar{q}} \cap a_\tau^{\bar{q}} \setminus a_{\bar{d}}^{\bar{q}}$ is equal to $a_\sigma^r \cap a_\tau^r \setminus a_{\bar{d}}^{\bar{q}}$, and also equal to $a_\sigma^r \cap a_\tau^r \setminus (b_\sigma^q \cap b_\tau^q)$, and so is in the algebra $\langle a_s^{\bar{q}} : s \in H^q \cap C_\sigma \cap M_i\rangle$ because $M_i \in \mathcal{M}_r$. The argument for when there is a $y \in C_\tau$ with $y^\dagger \in C_{x_0}$ is similar since W does not really affect anything when σ is also in $(x \restriction \lambda)^+$.

Finally we consider the critical new case which is when $\sigma \in H^q \setminus (x \restriction \lambda)^+$. Now we have that σ is in $H^q \cap C_x \cap \bar{M}_i \cap P_1$, and we have the key property identified above, namely that each of $a_\sigma^q \setminus a_{\bar{d}}^q$ and $a_\sigma^q \cap a_{\bar{d}}^q$ are in the algebra $B^q_{\sigma,i}$ generated by $\langle a_s^q : s \in H^q \cap P_1 \cap C_\sigma \cap \bar{M}_i\rangle$. Recall also that $\bar{M}_i \cap P_1 = M_i \cap P_0$ and that $a_\sigma^q = a_\sigma^{\bar{q}}$. To show that $a_\sigma^{\bar{q}} \cap a_\tau^{\bar{q}}$ is in $B^q_{\sigma,i}$ we show that each of $(a_\sigma^{\bar{q}} \cap a_{\bar{d}}^{\bar{q}}) \cap a_\tau^{\bar{q}}$ and $(a_\sigma^{\bar{q}} \setminus a_{\bar{d}}^{\bar{q}}) \cap a_\tau^{\bar{q}}$ are in $B^q_{\sigma,i}$. Of course $(a_\sigma^{\bar{q}} \cap a_{\bar{d}}^{\bar{q}}) \cap a_\tau^{\bar{q}}$ is equal to $(a_\sigma^{\bar{q}} \cap a_{\bar{d}}^{\bar{q}}) \cap b_\tau^q$ and so in $B^q_{\sigma,i}$. Similarly, $(a^{\bar q}_\sigma \setminus a^{\bar q}_{\bar d}) \cap a^{\bar q}_\tau$ is empty if $W \subset a^{\bar q}_\tau$, otherwise it is equal to $a^{\bar q}_\sigma \setminus a^{\bar q}_{\bar d}$. In either case, this completes the proof that $a^{\bar q}_\sigma \cap a^{\bar q}_\tau$ is in $B^q_{\sigma,i}$. □

Lemma 4.2. *Forcing with $\mathbb{P}$ adds no new branches to T with uncountable cofinality.*

Proof. Suppose $\dot x$ is a $\mathbb{P}$-name and that some $p \in \mathbb{P}$ forces that $\dot x$ is a member of $bT \cap 2^\lambda$ where λ has uncountable cofinality. Let $p, \dot x, \mathbb{P}, \lambda$ be members of a countable elementary submodel M^* of $H(\aleph_3)$. Let $M = M^* \cap H(\aleph_2)$ and let $\delta = \sup(M \cap \lambda)$. Choose $r < p$ to be M-prepared and to force a value y on $\dot x \restriction \delta$. Choose $\beta_r < \lambda$ minimal such that r does not force a value on $\dot x(\beta_r)$; if there is no such value then r forces that $\dot x$ is not a new branch.

Now consider the family $\Gamma(r, M)$, from Lemma 3.5, which is in M^*. Since $r \in \Gamma(r, M)$, we apply elementarity to deduce that for each $\xi \in M \cap \lambda$, there is a $p \in \Gamma(r, M)$ which forces a value on $\dot x(\xi)$. For each $q \in \Gamma(r, M)$, let β_q denote the minimum value such that q does not force a value on $\dot x(\beta_q)$. Also choose $e_q \in \{0, 1\}$ such that there is some $p \in \Gamma(r, M)$ forcing that $\dot x(\beta_q)$ is equal to e_q. In fact, by elementarity, e_q is unique, since for each $q \in \Gamma(r, M) \cap M$, it is equal to the value $y(\beta_q)$ that r forces on $\dot x(\beta_q)$.

Choose any $r_1 < r$ which is M-prepared and forces that $\dot x(\beta_r) = 1 - e_r$. Applying elementarity, there is a $q \in \Gamma(r, M) \cap M$ with an extension $q_1 \in \Gamma(r_1, M) \cap M$ such that q_1 forces the value $1 - e_q$ on $\dot x(\beta_q)$. Now since $\beta_q \in M$, e_q is equal to $y(\beta_q)$. However this implies that q_1 is not compatible with r_1 since $r_1 < r$ also implies that $\dot x(\beta_q) = e_q$. This is a contradiction to Theorem 3.7 since, by Lemma 3.5, q_1 is in (M, r_1)-good position. □

These next two results will be used to prove X_T is forced to have countable tightness.

Lemma 4.3. *If $A \subset \omega_2$ is any infinite set with supremum $\mu < \omega_2$, then for each $x \in T \setminus T_{<\mu}$ and each $p \in \mathbb{P}$, there is a $q < p$ and an $\alpha \in A$ such that $q \Vdash x \restriction (\alpha{+}1) \in a^q_x \setminus a^q_s$ for all $s \in C_x \cap H^p \setminus \{x\}$.*

Proof. This follows from the density lemma. We may assume that $\{x, x^\dagger\} \subset H^p$. Choose $\alpha \in A$ so that $H^p \cap C_x \cap T_{<\lambda}$ is contained in $T_{<\alpha}$. Apply item 2 of Lemma 2.8 with $t = x \restriction (\alpha{+}1)$ and any maximal member of H^p extending $x^\dagger$. Since $x \restriction (\alpha{+}1)$ will not be in $a^q_{x^\dagger}$, it must be in a^q_x. □

Let us note that it follows from Lemma 4.3 that forcing with $\mathbb{P}_T$ will ensure that $\mathfrak{c} \geq |T|$, since it implies that the members of $\{a_t \cap 2^{<\omega} : t \in Succ(T)\}$ will all be distinct, and $|Succ(T)| = |T|$ since T has no maximal elements.

Lemma 4.4. *In the extension by $\mathbb{P}_T$, T does not contain an order-isomorphic copy of $2^{\leq\omega}$.*

Proof. If T is countable, there is nothing to prove because $2^{\leq\omega}$ is uncountable. Otherwise, choose any uncountable family $\{t_\alpha : \alpha \in \omega_1\} \subset Succ(T) \setminus 2^{<\omega}$. Assume that $\{\dot t_\psi : \psi \in 2^{<\omega}\}$ is a $\mathbb{P}_T$-name of an order-preserving embedding of $2^{<\omega}$ into T. For each $\alpha \in \omega_1$, let $\dot x_\alpha$ be the name for $\{n \in \omega : t_\alpha \restriction n + 1 \in a_{t_\alpha}\}$. Then let $\dot r_\alpha$ be the $\mathbb{P}_T$-name for the characteristic function of $\dot x_\alpha$. Let $M \prec H(\aleph_3)$ be a countable elementary submodel with $\lhd, T, \{\dot t_\psi : \psi \in 2^{<\omega}\} \in M$. Choose any $\alpha \in \omega_1$ with $(t_\alpha)^+$ disjoint from M and assume that q is $(M, \mathbb{P}_T)$-generic. We show that q does not forces that $\bigcup\{\dot t_\psi : \psi \subset \dot r_\alpha\}$ is in T. Choose any $\bar t \in T$ and we show that q does not force that $\bar t$ is equal to $\bigcup\{\dot t_\psi : \psi \subset \dot r_\alpha\}$.

Choose any $k \in \omega$ so that $H^q \cap 2^{<\omega} \subset 2^{<k}$. In fact, by extending q we can assume that $H^q \cap 2^{<\omega} = 2^{<k}$. Since q is $(M, \mathbb{P}_T)$-generic, we can choose $d \in M \cap \mathbb{P}_T$ which is compatible with q, $d < q \restriction M$, and such that d forces a value on $\dot{t}_\psi$ for all $\psi \in 2^{k+1}$. Of course we may assume that $2^{<k+2}$ is contained in H^d. We may as well assume there is a $\psi_{\bar{t}} \in 2^{k+1}$ such that $\dot{t}_{\psi_{\bar{t}}}$ is forced to be less than $\bar{t}$. Set $t = t_\alpha \restriction k+1$. We extend q to a condition p using Lemma 2.8(2) so that $t \notin a^p_{t_\alpha}$ if and only if $\psi_{\bar{t}}(k) = 1$. If $t \in a^p_{t_\alpha}$, then $p \Vdash \dot{r}_\alpha(k) = 1$, and if $t \in a^p_{t^\dagger_\alpha}$, then $p \Vdash \dot{r}_\alpha(k) = 0$. Thus, if $\psi_{\bar{t}}(k) = 1$, then set $x = t_\alpha$ in the application of Lemma 2.8(2), resulting in a condition $p < q$ with $t \in a^p_{t^\dagger_\alpha}$. Similarly, if $\psi_{\bar{t}}(k) = 0$, then set $x = t^\dagger_\alpha$ in the application of Lemma 2.8(2), resulting in $p < q$ and $t \in a^p_{t_\alpha}$. According to Lemma 2.8, $H^p \setminus H^q = \{t, t^\dagger\}$. The only change we make to the construction in Lemma 2.8 is that we define $a^p_t = a^d_t$. Since $d < q \restriction M$ and is compatible with q and $t \in P$ for all $P \in \mathcal{E}^2$, it is now easily checked that d is compatible with p. Now $d \Vdash \dot{t}_{\psi_{\bar{t}}} \subset \bar{t}$, and $p \Vdash \psi_{\bar{t}} \not\subset \dot{r}_\alpha$; hence q does not force that $\bigcup\{\dot{t}_\psi : \psi \subset \dot{r}_\alpha\}$ is in T. □

Lemma 4.5. *Let $\dot{A}$ be a $\mathbb{P}$-name of an uncountable set of successor ordinals in ω_2. Let $\lambda \in \omega_2$ have uncountable cofinality and suppose that $p \in \mathbb{P}$ forces that $\dot{A} \cap \lambda$ is cofinal in λ. For each $x \in bT$ and each $q < p$, there are $r < q$, finite $L \subset C_x$, and $\mu < \lambda$ such that either*

1. *for $\alpha \in \lambda \setminus \mu$ and $\bar{r} < r$, if $x \restriction \alpha \in H^{\bar{r}} \setminus \bigcup_{s \in L} a^{\bar{r}}_s$, then $\bar{r} \Vdash \alpha \notin \dot{A}$; or*
2. *for each $\bar{r} < r$, there is a $\bar{q} < \bar{r}$ and an $\alpha \in \mu$ such that $\bar{q} \Vdash \alpha \in \dot{A}$, and $x \restriction \alpha \in H^{\bar{q}} \setminus \bigcup\{a^{\bar{q}}_s : s \in C_{<x} \cap H^{\bar{r}}\}$.*

Proof. Let $x \in bT$ and $q < p$ be elements of a countable $M^* \prec H(\aleph_3)$. Set $M = M^* \cap H(\aleph_2)$, $\mu = \sup(M \cap \omega_2)$, and choose $r < q$ so that $M \in \mathcal{M}_r$. Assume there is no $\bar{r} < r$ satisfying case (1) and that $\bar{r}$ is any element below r as in case (2). Since we are assuming that case (1) fails, it is evident that C_x is not contained in $T_{<\lambda}$, and so we may assume that $H^{\bar{r}} \cap C_x$ is not contained in $T_{<\lambda}$. We may further extend $\bar{r}$ so that there is some α_0 in the interval (μ, λ) that $\bar{r}$ forces to be in $\dot{A}$ and so that that $x \restriction \alpha_0 \in H^{\bar{r}} \setminus \bigcup_{s \in L} a^{\bar{r}}_s$. With yet another extension of $\bar{r}$ we may suppose that $\bar{r}$ is M-prepared. Notice that $t = x \restriction \mu$ is in $L^M_{\bar{r}}$.

By Lemma 3.6 there is a $\bar{p} \in M$ which is in $(M, \bar{r})$-good position and so that there is some $\alpha \in M$ so that $\bar{p} \Vdash \alpha \in \dot{A}$ and with $x \restriction \alpha \in H^{\bar{p}} \cap v^+_t$. Of course this also means that $x \restriction \alpha$ is not in a^r_s for any $s \in H^{\bar{r}}$ nor in $a^{\bar{p}}_s$ for any $s \in H^{\bar{p}} \cap H^{\bar{r}}$. By Lemma 3.14, we can choose $\bar{p}$ so that $H^{\bar{p}} \cup H^{\bar{r}}$ is in $\mathcal{H}_T$. It follows immediately from the definition then that $x \restriction \alpha$ is not in $a^{\bar{p} \oplus r}_s$ for any $s \in C_x \cap H^{\bar{r}}$. This completes the proof that (2) holds with $\bar{q} = \bar{p} \oplus \bar{r}$. □

Lemma 4.6. *For each $x \in bT \cap 2^{\omega_2}$, if $\dot{A}$ is a $\mathbb{P}$-name of an uncountable subset of C_x, then there is a condition p and an element $t \in C_x$ such that p forces that $\dot{A} \cap \dot{a}_t$ is uncountable.*

Proof. Assume that $\dot{A}$ is forced by some p to be an uncountable but bounded subset of C_x. Let $P_0 \in \mathcal{E}^2_1$ be chosen so that $\dot{A}$ and p are in P_0. Let $\lambda = P_0 \cap \omega_2$. Let $r < p$ with $P_0 \in \mathcal{M}_r$. We show that r forces that $t = x \restriction \lambda + 1$ has the desired property. Assume that r_1 is any extension of r and $\gamma < \lambda$ is arbitrary. We show that there is an extension $\bar{q}$ of r_1 that forces that there is some $s \in \dot{A} \cap C_x \setminus T_{<\gamma}$ such that $s \in a^{\bar{q}}_t$. By possibly increasing γ, we may assume that γ is a limit, $H^{r_1} \cap P_0 \subset T_{<\gamma}$, and $P \cap P_0 \cap C_x \subset T_{<\gamma}$ for all countable $P \in \mathcal{M}_{r_1}$. By Lemma 2.11, r forces that

$\dot{A} \cap C_x$ contains a cofinal subset of $C_x \cap T_{<\lambda}$. Choose any extension $q_1 \in \mathbb{P}$ of r_1 with the property that there is an $s \in H^{q_1} \cap C_x \setminus T_{<\gamma}$ with $q_1 \Vdash s \in \dot{A}$. Since $\dot{A}$ is in P_0, there is no loss to assume that $q = q_1 \restriction P_0$ also forces that $s \in \dot{A}$.

Now we prove that there is an extension $\bar{q}$ of q and r_1 such that $s \in a_t^{\bar{q}}$ as required. First we note that $\bar{H} = H^q \cup H^{r_1}$ is in $\mathcal{H}_T$. This follows easily from the facts that $H^{q_1} \in \mathcal{H}_T$ and $H^q = H^{q_1} \cap T_{<\lambda}$. Define an auxiliary condition $r_2 < r_1$ as follows. Let $H^{r_2} = H^{r_1} \cup (\bar{H} \setminus (x \restriction \gamma)^+)$. Since $H^{r_1} \cap P_0 \subset T_{<\gamma}$ and γ is a limit, it is routine to check that $H^{r_2} \in \mathcal{H}_T$. For each $s \in H^{r_2}$, define $a_s^{r_2}$ to be $a_s^{q_1} \cap H^{r_2}$. Since $q_1 < r_1$, we certainly have that $a_s^{r_2} \cap H^{r_1} = a_s^{r_1}$ for all $s \in H^{r_1}$. We let $\mathcal{M}_{r_2} = \mathcal{M}_{r_1} \cup \mathcal{M}_q$. Since $\mathcal{M}_q = \mathcal{M}_{q_1} \cap P_0$, the family $\mathcal{M}_{r_2}$ is closed under intersections. To finish the proof that $r_2 < r_1$, we consider any countable $P \in \mathcal{M}_{r_2}$ and a pair $\sigma < \psi$ in $P \cap \mathrm{Succ}(H^{r_2})$. By possibly switching to $\psi^\dagger$, we may assume that $\sigma \in a_\psi^{r_1}$.

If $\sigma \notin (x \restriction \gamma)^+$, then we have that $[\sigma]_P^{r_2} = [\sigma]_P^{q_1}$ and $a_\psi^{q_1}$ does not split it. It is immediate that $a_\psi^{r_2} = a_\psi^{q_1} \cap \bar{H}$ also does not split $[\sigma]_P^{r_2}$. If σ is above $x \restriction \gamma$, then both σ and ψ are in H^{r_1}. Let us first show that $[\sigma]_P^{r_2} \cap H^{r_1}$ is contained in $a_\psi^{r_2}$. Since $q_1 < r_1$, we have that $\sigma \in a_\psi^{r_1}$ and so we have that $[\sigma]_P^{r_1} \subset a_\psi^{r_1}$. Therefore, if $y \in H^{r_1}$ is any element of $[\sigma]_P^{r_2} = a_\sigma^{r_2} \setminus \bigcup\{a_\rho^{r_2} : \rho \in P \cap C_{<\sigma} \cap H^{r_2}\}$, then y will be in $a_\sigma^{r_1} \setminus a_\rho^{r_1}$ for each $\rho \in P \cap C_{<\sigma} \cap H^{r_1}$. That is, y will be in $[\sigma]_P^{r_1}$ which is contained in $a_\psi^{r_1}$. Now assume that $y \in C_{<\sigma} \cap H^{r_2} \setminus H^{r_1}$ is not in $a_\psi^{r_2}$. We will show that $y \notin [\sigma]_P^{r_2}$. If $y \notin a_\sigma^{q_1}$, then this is immediate. Otherwise using that $q_1 < r_1$, and y is not in in $a_\sigma^{q_1} \cap a_\psi^{q_1}$, we have that there must be a ρ in $H^{r_1} \cap P \cap C_{<\sigma}$ such that y is in $a_\rho^{q_1}$. This implies that y is not in $[\sigma]_P^{r_2}$.

Let $\bar{r} = r_2 \restriction P_0$. Notice that $q < \bar{r}$. At this stage we could set $\bar{q}$ to be $q \oplus r_2$ with an inessential change to the definition, namely by designating $t^\dagger$ rather than possibly t to be primary. However, it is equivalent and easier to proceed as in Lemma 3.14. Set $\Psi = H^q \cap C_x \cap (x \restriction \gamma)^+$ and $\mathcal{M}_{\bar{q}} = \mathcal{M}_{r_2}$.

Fix any x_0 in H^{r_1} which is maximal and satisfies that $t^\dagger \in C_{x_0}$. For each $s \in \bar{H} \cap P_0$, define $a_s^{\bar{q}}$ to be a_s^q. For each $s \in H^{r_1} \setminus (P_0 \cup (x \restriction \gamma)^+)$, set $a_s^{\bar{q}}$ to be $a_s^{r_2}$. Finally, for each $s \in \bar{H} \cap (x \restriction \lambda)^+$,

$$a_s^{\bar{q}} = \begin{cases} a_s^{r_2} \cup \Psi & (\exists y \in a_s^{r_1})\ y^\dagger \in C_{x_0} \\ a_s^{r_2} & (\exists y \in C_s \setminus a_s^{r_1})\ y^\dagger \in C_{x_0}\ . \end{cases}$$

Let us observe that $\Psi \subset a_t^{\bar{q}}$ since $t^\dagger \in C_{x_0}$. The verification that $\bar{q}$ is in $\mathbb{P}$ proceeds as in Lemma 3.14 and is even easier because if $P \cap \Psi$ is not empty for some countable $P \in \mathcal{M}_{\bar{q}}$, then $P \in P_0$. In other words, if $P \in \mathcal{M}_{\bar{q}} \cap P_0$, and $\sigma < \tau$ are in $\mathrm{Succ}(T) \cap P$, then $a_s^{\bar{q}}$ is equal to $a_s^{r_2}$ for all $s \in C_\tau \cap H^{\bar{q}}$. On the other hand, if $P \in \mathcal{E}_0^2 \cap \mathcal{M}_{\bar{q}} \setminus P_0$, then $P \in \mathcal{M}_{r_1}$ and $P \cap T \cap P_0$ is contained in $T_{<\gamma}$. If $\sigma \in T_{<\gamma}$, then $a_\tau^{\bar{q}}$ and $a_\tau^{r_2}$ have the same intersection with $[\sigma]_P^{\bar{q}} = [\sigma]_P^{r_2}$. Finally, we consider the case $\sigma \in P \setminus P_0$ and proceed exactly as in Lemma 3.14. Of course we have that $a_\tau^{r_1}$ does not split $[\sigma]_P^{r_1}$. If $\sigma \in C_{x_0}$, then $[\sigma]_P^{\bar{q}} = [\sigma]_P^{r_1}$ is disjoint from Ψ and so is not split by $a_\tau^{\bar{q}}$. Otherwise, fix the unique $y \in C_{x_0}$ such that $y^\dagger \in C_{x_0}$. If $y \notin a_\sigma^{r_1}$, then we again have that $[\sigma]_P^{\bar{q}} = [\sigma]_P^{r_1}$ is disjoint from Ψ and is not split by $a_\tau^{\bar{q}}$. Finally, we assume that $y \in a_\sigma^{r_1}$ and have that if $y \in [\sigma]_P^{r_1}$ then $[\sigma]_P^{\bar{q}} = [\sigma]_P^{r_1} \cup \Psi$ and otherwise $[\sigma]_P^{\bar{q}} = [\sigma]_P^{r_1}$. The only case that needs checking then is when $y \in [\sigma]_P^{\bar{q}}$ and $a_t^{r_1}$ meets $[\sigma]_P^{r_1} \cup \Psi$. $a_t^{r_1}$ will meet $[\sigma]_P^{r_1} \cup \Psi$ if and only if $y \in a_\tau^{r_1}$. Therefore $\Psi \subset a_\tau^{r_1}$ and $[y]_P^{r_1} = [\sigma]_P^{r_1}$ is contained in $a_\tau^{r_1}$. This proves $a_\tau^{\bar{q}}$ does not split $[\sigma]_P^{\bar{q}}$. $\square$

5. Properties of X_T in the forcing extension by $\mathbb{P}$

Throughout this paper we assume that $2^\omega < 2^{\omega_1} = \omega_2$ holds in the ground model in which T and $\mathbb{P}$ were defined and that G is a $\mathbb{P}$-generic filter and we note some fundamental properties of our space X_T. Naturally we will let $\{a_t : t \in \mathrm{Succ}(T)\}$ denote the family of subsets of $\mathrm{Succ}(T)$ added by $\mathbb{P}$. Let us note that in further forcing extensions beyond that by $\mathbb{P}$, the family $\{a_t : t \in \mathrm{Succ}(T)\}$ is unchanged, but the collection of points in X_T or $X(T)$ may grow since new branches of T may be added. In fact we take great care to ensure that all new branches are in $X(T)$.

This first definition is just a reformulation of a point x being a limit point, or a complete accumulation point, of $\sigma_x(A)$ for some set $A \subset C_x$ in the space (x, τ_x). This is towards proving that (x, τ_x) is initially ω_1-compact.

Definition 5.1. For $x \in bT \setminus T$ and $A \subset C_x$, say that A is x-large (respectively x - ω_1-large) if $A \setminus \bigcup\{a_t : t \in L\}$ is infinite (respectively uncountable) for all finite $L \subset C_x$. In case $x \in T$, we say that $A \subset \mathrm{Succ}(T)$ is x-large (respectively x - ω_1-large) if $A \cap a_x \setminus \bigcup\{a_s : s \in L\}$ is infinite (respectively uncountable) for all finite $L \subset C_{<x}$.

Proposition 5.2. *If $A \cup A_1 \subset C_x$, for some $x \in bT$, are such that A is t-large (respectively t - ω_1-large) for each $t \in A_1$ and if A_1 is x-large, then A is x-large (respectively x - ω_1-large).*

Let us recall again the connection between the topology on X_T and the topologies $(\hat{x}, \hat{\tau}_x)$ and the induced topology on C_x for $x \in X_T$. Let Π_x denote the continuous map sending each $y \in X_T$ to $y \wedge x$ in $(\hat{x}, \hat{\tau}_x)$. And recall that the topology τ_x is induced by σ_x mapping C_x onto the entire branch x.

Proposition 5.3. *For $\{x\} \cup Y \subset bT \setminus T$ with $x \notin Y$, we have that x is in the closure of Y if and only if $\sigma_x^{-1}[\Pi_x[Y]\,]$ is x-large.*

Lemma 5.4. *if $x \in bT \cap 2^{\omega_2}$ and $A \subset C_x$ is x-large (respectively x - ω_1-large) then there is a $\lambda \in \omega_2$ such that A is t-large (respectively t - ω_1-large) for all $t \in C_x \setminus 2^{<\lambda}$.*

Proof. First we assume that A is x-large. By Lemma 4.5, we may assume that A is countable. By Lemma 4.1, there is some λ such that A is $x \restriction (\lambda + k)$-large for each integer $k > 0$. Let $A_1 = \{x \restriction \lambda + k : 0 < k \in \omega\}$. By Lemma 4.3, A_1 is t-large for all $t \in C_x \setminus 2^{<\lambda+\omega}$. Therefore, by Lemma 5.2, we have that A is t-large for all $t \in C_x \setminus 2^{<\lambda}$.

Now we suppose that A is uncountable and let A_1 be the set of $t \in C_x$ such that A is t - ω_1-large. By Lemma 5.2 and the first part of the proof, it suffices to check that A_1 is x-large. Let L be any finite subset of C_x and set $A_2 = A \setminus \bigcup\{a_\rho : \rho \in L\}$. Since A is x - ω_1-large, we have that A_2 is uncountable. By Lemma 4.6, there is a $t \in A_1$ such that A_2 is t - ω_1-large. Since $a_t \cap \bigcup\{a_\rho : \rho \in L\}$ is disjoint from A_2, it follows from Lemma 2.9 that $t \notin \bigcup\{a_\rho : \rho \in L\}$. This shows that A_1 is x-large. □

Corollary 5.5. *Each of the following are true of X_T:*

1. *if $x \in bT \cap 2^{\omega_2}$, then (x, τ_x) is initially ω_1-compact;*
2. *the previous statement is preserved in any forcing extension by a ccc poset of cardinality at most ω_1;*
3. *the space X_T has countable tightness;*
4. *$X(T)$ is normal, initially ω_1-compact, and C^*-embedded in X_T;*
5. *compact subsets of $X_T \setminus X(T)$ are finite;*

(6) *X_T is separable and has a dense set of points with countable character.*

It therefore follows that the cardinality of non-compact initially ω_1-compact separable spaces of countable tightness can exceed 2^{ω_1}.

Proof. Part (1) is immediate by Lemmas 5.4 and 5.3. Part (3) also follows from Lemma 5.3 and Lemma 4.5. We proved in Lemma 4.4 that T does not contain a copy of $2^{\leq\omega}$. Therefore part (5) follows from Part (1) and Lemma 1.5. It follows from Lemma 4.3, that for each $x \in X_T$, x is in the closure of $C_x \cap 2^{<\omega}$ in the space $(\hat{x}, \hat{\tau}_x)$. Choose any countable subset $D \subset 2^\omega \setminus V$ which is dense in the Cantor set topology. Now each $d \in D$ is also a (first-countable) member of $X(T)$, and for each $x \in X_T$, $\{d \cap x : d \in D\}$ has x in its closure in $(\hat{x}, \hat{\tau}_x)$. It follows that X_T is separable and so Part (6) is verified.

For Part (2), we consider a ccc poset $\mathbb{Q}$ with $|\mathbb{Q}| \leq \omega_1$. We know that $\mathbb{Q}$ will add no new members of $bT \cap 2^{\omega_2}$, so we may fix any $x \in bT \cap 2^{\omega_2}$, and suppose that $\dot{Y}$ is a $\mathbb{Q}$-name of a set of (possibly new) members of X_T. By Lemma 5.3, we will consider the $\mathbb{Q}$-name $\dot{A}$ of $\sigma_x^{-1}[\Pi_x[\dot{Y}]]$. If we have some $q \in \mathbb{Q}$ which forces that x is in the closure of $\dot{Y}$, then we have that q forces that $\dot{A}$ is x-large. We first show that q does not force that x is the only limit point of $\dot{Y}$. If so, then for each $\lambda \in \omega_2$, there is a finite set $L_\lambda \subset C_x \cap 2^{<\lambda}$ and a condition $q_\lambda < q$ such that $q_\lambda \Vdash a_{x\restriction\lambda+1} \cap A$ is contained in $\bigcup\{a_s : s \in L_\lambda\}$. By the pressing down lemma, there is a single finite set $L \subset C_x$ and a single condition $\bar{q} < q$ such that $L_\lambda = L$ and $q_\lambda = \bar{q}$ for all $\lambda \in S$ for some stationary set $S \subset \omega_2$. Let $A_1 = \{t \in C_x : (\exists r < \bar{q})\ r \Vdash t \in \dot{A} \setminus \bigcup\{a_s : s \in L\}\ \}$. Since $\bar{q}$ forces that $\dot{A}$ is x-large, we must have that A_1 is x-large. By Lemma 5.4, there is a $\lambda \in S$ such that A_1 is t_λ-large where $t_\lambda = x\restriction\lambda+1$. Since $\bar{q} = q_\lambda$ forces that $\dot{A} \cap a_{t_\lambda} \subset \bigcup\{a_s : s \in L\}$, we have our contradiction.

The proof in the case that q forces that $\dot{Y}$ has cardinality ω_1 and that x is the unique complete accumulation point breaks into two cases. In case some q forces that $\dot{A}$ has cardinality ω_1 proceed just as above by replacing x-large with x - ω_1-large. Otherwise, define $\dot{A}$ to be the set of $s \in C_x$ such that q forces that the set $\{y \in \dot{Y} : s^\dagger \subset y\}$ is uncountable. It is easy to see that q forces that $\dot{A}$ is x-large, and we again proceed as in the x-large case.

Finally we show that Part (2) holds. The fact that $X(T)$ is initially ω_1-compact follows from Part (1) and Lemma 1.5. To show that $X(T)$ is normal and C^*-embedded in X_T, it suffices to show that disjoint closed subsets of $X(T)$ have disjoint closures in X_T. To show this, assume that Y_1 and Y_2 are subsets of $X(T)$ which have x in their closure for some $x \in bT \cap 2^{\omega_2}$. Let $A_1 = \sigma_x^{-1}[\Pi_x[Y_1]]$ and $A_2 = \sigma_x^{-1}[\Pi_x[Y_2]]$ be the corresponding x-large sets. By Lemma 5.4, there is a $\lambda \in \omega_2$ so that each of A_1 and A_2 are t-large for all $t \in C_x \setminus 2^{<\lambda}$. By Lemma 4.3 and Lemma 5.2, each of A_1 and A_2 are y-large for all $y \in bT$ which extend $x\restriction\lambda+\omega$. Since $\mathbb{P}$ adds new countable sets, there are members y of $bT \setminus T$ in $2^{\lambda+\omega+\omega}$ which extend $x \restriction \lambda+\omega$. Therefore $y \in X(T)$, and by Lemma 5.3, y is in the closure of Y_1 and Y_2. This shows that Y_1 and Y_2 do not have disjoint closures in $X(T)$. □

We prove in the next section that $X(T)$ can be first-countable and initially ω_1-compact.

6. Martin's Axiom and Moore-Mrowka

We again consider the forcing extension by $\mathbb{P}_T$ and the space X_T for specific trees T. But first we prove results towards identifying the dense set of points of countable character.

Lemma 6.1. *Let $p \in \mathbb{P}$ and $M \in \mathcal{M}_p \cap \mathcal{E}_0^2$, so that p is $(M, \mathbb{P})$-generic. For each $d \in M \cap \mathrm{Succ}(T)$ and $s \in C_d$, p forces that the collection*

$$\dot{T}_{M,d,s} = C_d \cup \{t \in M \cap T : d \in C_t \text{ and } s \notin \bigcup \{a_\rho : \rho \in M \cap C_t \setminus C_d\}\}$$

is a chain. Furthermore, if some $r < p$ forces that $\bigcup \dot{T}_{M,d,s} = \dot{x} \in bT \setminus T$ and r is $(M_0, \mathbb{P})$-generic for some $M_0 \in \mathcal{M}_r$ with $\{M, s\} \in M_0$, then r forces that $C_{\dot{x}}$ is covered by $\{\dot{a}_\rho : \rho \in M_0 \cap \dot{x}\}$.

Proof. Let t_0, t_1 be members of M such that $d \in C_{t_0} \cap C_{t_1}$. Suppose that $r < p$ and $\{s, t_0, t_1\} \subset H^r$. Since $H^r \in \mathcal{H}_T$, we have that $t = t_0 \wedge t_1$ and each of $t0, t1$ are in H^r. By renaming, we may assume that $t0 \subseteq t_0$ and $t1 \subseteq t_1$. Since $a^r_{t0} \cup a^r_{t1} \supset C_t \cap H^r$, we have that one of $t0, t1$ is not in $\dot{T}_{M,d,s}$. Since $\dot{T}_{M,d,s}$ is forced to be closed downwards, it follows that r forces that one of t_0, t_1 is not in $\dot{T}_{M,d,s}$. We have therefore shown that p forces that $\dot{T}_{M,d,s}$ is a chain.

Now assume that M_0, r are as in the statement of the Lemma and consider any $\bar{s}$ such that $r \Vdash \bar{s} \in C_{\dot{x}}$. By extending r we may assume that $s, \bar{s}$ are each in H^r and our goal is to find an extension q and some $t \in M_0$ such that $\bar{s} \in a^q_t$ and $q \Vdash t \in \dot{x}$. We first show that we may assume that there is maximal element t of H^r such that $t \in M$ and $r \Vdash t \in \dot{x}$.

Choose the largest $t_0 \in H^r$ with the property that $r \Vdash t_0 \in \dot{x}$. Observe that $\bar{s}$ must be in C_{t_0}. By extending r, we may assume that $t_0 \in M$. To see this, notice that since r forces that t_0 is in $\dot{x}$, we must have that $(t_0)^+ \cap M$ is not empty. If t_0 is not in M, then the limit node $u_{t_0} = \min((t_0)^+ \cap M)$ is in M. It should be clear that r also forces that u_{t_0} is in $\dot{x}$. Simply adding u_{t_0} to H^r results in an extension in $\mathbb{P}$. Since this extension is trivially compatible with every extension of r, we now have that u_{t_0} is the maximal member of H^r which is forced to be in $\dot{x}$. So now we are assuming that t_0 is in M. It then follows that neither immediate successor of t_0 is in H^r. This is because $s \in a^r_{t0} \cup a^r_{t1}$ and so r would decide which successor of t_0 is in $\dot{T}_{M,d,s}$ and thus contradict the maximality of t_0. If t_0 has no successors at all in H^r then $t = t_0$ has the desired properties. Otherwise, let x_1 be the unique minimal element of H^r which is above t_0. Choose $t_2 \leq x_1$ to be minimal such that r does not force that $t_2 \in \dot{x}$. By the minimality of t_2, it is immediate that t_2 is a successor. Also, by the minimality of x_1, $(t_2^\dagger)^+ \cap H^r$ is empty. Since r does not force that $t_2 \in \dot{x}$, we must have that $(t_2^\dagger)^+ \cap M$ is not empty. If $(t_2)^+ \cap M$ is empty, then r forces that $t_2^\dagger$ is in $\dot{x}$. So again, the limit node $u = \min(M \cap (t_2^\dagger)^+)$ can be added to H^r and we can set $t = u$. So now we consider the case that $(t_2)^+ \cap M$ is non-empty. Since we also have that $(t_2^\dagger)^+ \cap M$ is non-empty, then we have that $\{t_2 \wedge t_2^\dagger, t_2, t_2^\dagger\} \subset M$. Also, there is an $r_1 < r$ forcing that $t_2 \notin \dot{x}$. Again by the minimality of t_2 and since $t_2 \in M$, we must have that $s \in a^{r_1}_{t_2}$. It is clear that we may shrink r_1 so that $t_2^\dagger$ is a maximal element of H^{r_1} and still have that r_1 forces that $t = t_2^\dagger \in \dot{x}$.

So, as we said, we assume that t is a maximal element of H^r which is in M and which is forced by r to be in $\dot{x}$. Any condition q extending r which has $\{t0, t1\} \subset H^q$ and with $s \in a^q_{t0}$, will force that $t1 \in \dot{x}$. We define such a condition q.

Choose $P_0 \in \mathcal{M}_r \cap \mathcal{E}^2_0$ so that $t \in P_0$ and so that there is a $\tau \in P_0 \cap C_t \cap H^r$ with $s \in a^r_\tau$. Choose P_0 so that $\delta_0 = \delta_{P_0} = P_0 \cap \omega_1$ is the smallest possible. Since M_0 satisfies these requirements (with $\tau = s$), there is such a P_0 and $\delta_0 \leq \delta_{M_0}$. Let τ denote the minimal member of $H^r \cap C_t \cap P_0$ so that $s \in a^r_\tau$, hence $s \in [\tau]^r_{P_0}$. Let g_t denote the $\lhd$-least function from ω_1 onto C_t. Recall that if $t \in P \in \mathcal{E}^2_0$ and $\delta_P < \delta_0$, then P_0 contains $P \cap C_t$. Similarly, if $t \in P$ and $\delta_0 \leq \delta_P$, then $P_0 \cap C_t \subset P$. In particular we have that $\tau \in M_0$.

We extend r by adding the pair $\{t0, t1\}$ to H^r and defining a_{t0} to be $\{t0\} \cup [\tau]^r_{P_0}$. The resulting condition q will force that $t1$ is in $\dot{x}$ and, that $\bar{s} \in a_\tau \cup a_{t1}$, which is all that is required. To check that q is a condition which is below r we apply Lemma 2.3. Recall that there are no members of H^r above t, so it suffices to simply focus on a^q_{t0}. It is a triviality that a^q_{t0} does not split $[\rho]^p_r$ for any $\rho \in C_t \cap H^r$ since they are all singletons. Now consider $t \in P \in \mathcal{M}_r \cap \mathcal{E}^2_0$ and $\psi \in C_{<t} \cap P$ and we show that a^q_{t0} does not split $[\psi]^q_P$. First assume that $\delta_P < \delta_0$. Since $t \in P$, it follows by the minimality of δ_{P_0} that $s \notin a^r_\rho$ for all $\rho \in P \cap C_t$. In particular, $s \notin a^r_\psi$ and $P \cap C_t \subset P_0$. Since τ, ψ are both in $P_0 \cap H^r$, it follows that $[\tau]_{P_0}$ is not split by a^r_ψ. Since $s \notin a^r_\psi$, we have that a^r_ψ and $[\psi]^r_P$ are disjoint from a_{t0}, and thus not split. Now suppose that $\delta_0 \leq \delta_P$. In this case, we have that $C_\tau \cap P_0 \subset P$ and so a^r_τ does not split $[\psi]^r_P$. And therefore, neither $[\tau]^r_{P_0}$ nor a^q_{t0} splits $[\psi]^r_P$. □

We will consider forcing by ccc Souslin-free posets but this next result is needed even in the case when the poset $\dot{Q}$ is the trivial poset. A forcing poset is Souslin-free (see [8] for more details) if there is no Souslin tree completely embedded in its completion. A ccc Souslin-free poset will not add any new branches to $2^{<\omega_2}$ which have uncountable cofinality. In the forcing extension by $\mathbb{P}_T$ over a model of GCH, we will have that $\mathfrak{c} = 2^{\omega_1} = \omega_2$. As explained in [8], there is a ccc poset of cardinality ω_2 which will force Martin's Axiom to hold and is constructed as a finite support iteration with factors that are forced to be ccc and Souslin-free posets of cardinality $\aleph_1$.

Lemma 6.2. *Let $\dot{Q}$ be the $\mathbb{P}$-name of a ccc Souslin-free poset of cardinality at most ω_1. Suppose that $\dot{x}$ is a $\mathbb{P} * \dot{Q}$-name and that some $(p, q) \in \mathbb{P} * \dot{Q}$ forces that $\dot{x}$ is a maximal chain of T which is not in V. Then there is a condition $(r, q') < (p, q)$ and a countable set M such that (r, q') forces that for each $s \in C_{\dot{x}}$ there is a $t \in M \cap \dot{x}$ with $s \in \dot{a}_t$.*

Proof. We can choose (p, q) to decide μ such that $\dot{x} \in 2^\mu$. Since μ has countable cofinality, it suffices to prove that for any successor $\gamma \in \mu$, we can extend (p, q) to decide $\dot{x} \restriction \gamma = d$ and have a countable set M as in the Lemma covering C_d. Assume otherwise, and suppose that (p, q) decides $d = \dot{x} \restriction \gamma$ is the witness that C_d can not be covered by a countable subset of $\{a_\rho : \rho \in \dot{x}\}$.

Let (p, q), $\dot{Q}$ and d be elements of a countable $M^* \prec H(\aleph_3)$. As usual, let $r < p$ be any extension of p such that $M = M^* \cap H(\aleph_2)$ is in $\mathcal{M}_r$ and so that r is $(M, \mathbb{P})$-generic (as per Lemma 2.10). Assume first that we can find a condition $(\bar{p}, \bar{q})$ below (r, q) so that there is an $s \in C_d$ so that $(\bar{p}, \bar{q})$ forces that $\dot{x} = \bigcup \dot{T}_{M,d,s}$ (as in Lemma 6.1). Of course this means $(\bar{p}, \bar{q})$ forces that the required countable set exists by Lemma 6.1.

Now assume there is no such extension and corresponding element $s \in C_d$. Simply pass to the generic extension by $\mathbb{P} * \dot{Q}$ with (p, q) in the generic filter G. For each $s \in C_d$, choose minimal $t \in val_G(\dot{T}_{M,d,s}) \setminus val_G(\dot{x})$. Of course this means that $t, t^\dagger \in M$, $a_{t^\dagger} \in val_G(\dot{x})$, and $s \in a_t$. □

Collecting all this work into one statement, we have the main result.

Theorem 6.3. *For each $T \subset 2^{<\omega_2}$ such that T has no maximal nodes and no branches of cofinality ω_1, there is a $\mathbb{P}_T$-name $\dot{Q}$ of a ccc poset such that $X(T)$ is a first-countable initially ω_1-compact space in each of the forcing extensions by $\mathbb{P}_T$ and by $\mathbb{P}_T * \dot{Q}$. In addition, Martin's Axiom holds in the forcing extension by $\mathbb{P}_T * \dot{Q}$.*

Proof. As discussed above, $\dot{Q}$ will be a $\mathbb{P}_T$-name of a length ω_2 finite support iteration of Souslin-free ccc posets of cardinality ω_1. By Lemma 4.2, $\mathbb{P}_T$ adds no new branches with cofinality ω_1, and by [8], $\mathbb{P}_T * \dot{Q}$ also does not add branches to T which have uncountable cofinality. Since $\dot{Q}$ is a finite support iteration of ccc posets of cardinality at most $\aleph_1$, each $\aleph_1$-sized subset of bT will be added by a completely embedded size $\aleph_1$ subposet of $\dot{Q}$. Therefore it follows from Corollary 5.5, $X(T)$ is initially ω_1-compact. It remains only to prove that $X(T)$ is first-countable in each of the relevant forcing extensions. To do so, we show that for each $x \in (bT \setminus T) \setminus 2^{\omega_2}$, there is a countable $D \subset C_x$ such that C_x is covered by $\bigcup\{a_d : d \in D\}$. It follows easily that this implies that $\{\mathcal{D}_x\}$ is the only point in $S(B_T) \setminus \bigcup\{A_d : d \in D\}$, hence that $\mathcal{D}_x$ has a countable base.

If x is not in the forcing extension by $\mathbb{P}_T$, then the existence of such a countable $D \subset C_x$ is proven in Lemma 6.2. Now assume that $\dot{x}$ is a $\mathbb{P}_T$-name for x and let $\dot{x} \in M^*$ for some countable elementary submodel M^* of $H(\aleph_3)$. With $M = M^* \cap H(\aleph_2)$, let p, M be as in Lemma 6.1. If some extension of p forces that C_x is covered by $\{a_t : t \in M \cap C_x\}$ then we are done. Therefore we may assume that there is some $s \in C_x$ such that p forces that $s \notin a_t$ for all $t \in M \cap C_x$. Fix any $d \in M \cap C_x$ so that $s \in C_d$. According to Lemma 6.1, we have that p forces that $\dot{x}$ is equal to $\bigcup \dot{T}_{M,d,s}$. Therefore, still by Lemma 6.1, there is an $r < p$ and a countable M' so that r forces that C_x is covered by $\{a_\rho : \rho \in M' \cap C_x\}$. □

Using Proposition 7.1 and Theorem 6.3 applied to the ground model $T = 2^{<\omega_2}$, we have the following two applications.

Corollary 6.4. *It is consistent with Martin's Axiom and $\mathfrak{c} = \omega_2$ that there is a compact space of countable tightness which is not sequential.*

Corollary 6.5. *It is consistent with Martin's Axiom and $\mathfrak{c} = \omega_2$ that there is first-countable initially ω_1-compact space with a compactification which has countable tightness and cardinality greater than $\mathfrak{c}$.*

If we apply Theorem 6.3 over a ground model in which there is a $\aleph_2$-Souslin tree we obtain the following application.

Corollary 6.6. *It is consistent with Martin's Axiom and $\mathfrak{c} = \omega_2$ that there is a compact first-countable space with the property that it can be forced to be non-Lindelof by the usual countably closed collapse, but also by forcing with an $\aleph_2$-Souslin tree.*

Proof. Let S be an $\aleph_2$-Souslin tree. Let $T \supset bS$ be the tree obtained by extending all branches of uncountable cofinality in such a way that all branches of T have countable cofinality. Let $\dot{Q}$ be the $\mathbb{P}_T$-name of the finite support iteration of ccc size $\aleph_1$-posets as in Theorem 6.3. Assume that forcing with $\mathbb{P}_T$ preserves that S is an $\aleph_2$-Souslin tree. It is well-known that $\dot{Q}$ will then also preserve that S is an $\aleph_2$-Souslin tree. Let G be a generic for $\mathbb{P}_T * \dot{Q}$ and work in $V[G]$. Since T has no cofinal branches, our space $X(T)$ constructed in §2 with the poset $\mathbb{P}_T$ will be compact. This is our desired model and our desired compact first-countable space for the statement of the Corollary. Forcing with S will add an $\aleph_2$-branch to T and the space $X(T)$ will no longer be compact. Additionally, $X(T)$ as calculated in $V[G]$ is the same space as $X(T)$ as calculated in the further forcing extension by S. Since S adds no new countable sets, $X(T)$ remains countably compact, and so it is no longer Lindelöf.

Now we work in the ground model, and show that S remains $\aleph_2$-Souslin after forcing with $\mathbb{P}_T$. Let $\dot{A}$ be the $\mathbb{P}$-name of a maximal antichain of S. Let $p \in \mathbb{P}_T$ be any condition and choose an elementary submodel $M \prec H(\aleph_3)$ so that $S, p, \mathbb{P}$ and $\dot{A}$ are all members of M, $M^\omega \subset M$, and M has cardinality $\aleph_1$. Choose any $r < p$ so that $M \cap H(\aleph_2) \in \mathcal{M}_r$, we will show that r forces that $\dot{A} \subset M$.

Let $\lambda = M \cap \omega_2$ and let s be any member of S_λ. By extending r, we may assume that $r \Vdash a \in \dot{A}$ for some $a \in S$ compatible with s. It suffices to show that $a \in M$. We may assume $r \restriction M \in \bar{P}$ for some countable $\bar{P} \in \mathcal{M}_r$. This also means that for all countable $P \in \mathcal{M}_r$, $P \cap M \subset \bar{P}$. Of course $\bar{P}$ is an element of M. Let $L = \{t \cap \bar{P} : t \in H^r \setminus M\}$.

Within M we can discuss those q which are extensions of $r \restriction M$, and which satisfy that, $L = \{t \cap \bar{P} : t \in H^q \setminus \bar{P}\} \subset H^q$, and that both $H^q \setminus (H^r \cap M)$ and $\mathcal{M}_q \setminus (\mathcal{M}_r \cap M)$ are disjoint from $\bar{P}$. Let $D \in M$ denote the set of all such q and note that r has an extension which is in D (obtained by simply adding L to H^r. If $q \in D$, then $H^q \cup H^r$ need not be in $\mathcal{H}_T$, but since for any $t_r \in H^r \setminus H^q$ and $t_q \in H^q \setminus H^r$, we have that if $t_r \cap t_q \in \bar{P}$, then $t_r \cap t_q \in H^q$. This is because there is some $t \in L$ such that $t_r \cap t_q$ is either above t or is equal to $t \cap t_q$. By the density lemma (Lemma 2.8) we can extend q to a condition $\bar{q}$ such that $H^{\bar{q}} \cup H^r$ is in $\mathcal{H}_T$ and $H^{\bar{q}} \cap \bar{P} = H^q$. It thus follows from Lemma 2.10, that each $q \in D$ is compatible with r.

Define A_D to be the set $\{a \in S : (\exists r \in D)\ a \in H^r \text{ and } r \Vdash a \in \dot{A}\}$. Let $A_D^\uparrow$ be the members of S which are above some member of A_D and let $A_D^\perp$ denote the members of S which are incompatible with every member of A_D. Then $A_D^\uparrow \cup A_D^\perp$ is dense open subset of S and is a member of M. Since s is not in $A_D^\perp$, it follows that there is some $a' \in A_D \cap M$ which is below s. Choose any $r \in D \cap M$ so that $r \Vdash a' \in \dot{A}$. Since r is compatible with q, we have shown that $a = a' \in M$. □

7. Cardinality of initially ω_1-compact spaces

It is interesting to note that any compactification of an initially ω_1-compact space of countable tightness will also have countable tightness. It suffices to prove that βX has countable tightness.

Proposition 7.1. *If X is initially ω_1-compact and has countable tightness, βX has countable tightness. Therefore, every compactification of X has countable tightness.*

Proof. By Sapirovskii's theorem, if βX does not have countable tightness, then it contains an uncountable free sequence $\{\mathcal{Z}_\alpha : \alpha \in \omega_1\}$. That is, for each $\alpha \in \omega_1$, $\mathcal{Z}_\alpha$ is an ultrafilter of zero sets of X, and for each $\alpha \in \omega_1$, there are disjoint zero sets

W_α, Z_α such that $W_\alpha \in \mathcal{Z}_\beta$ for all $\beta \leq \alpha$ and $Z_\alpha \in \mathcal{Z}_\beta$ for all $\beta > \alpha$. Since X is initially ω_1-compact, we may select $x_\alpha \in X \cap \bigcap\{W_\beta : \beta \geq \alpha\} \cap \bigcap\{Z_\beta : \beta < \alpha\}$. It follows easily that $\{x_\alpha : \alpha \in \omega_1\}$ is a free sequence in X. Since X has countable tightness, this implies that $\{x_\alpha : \alpha \in \omega_1\}$ has no complete accumulation point. This contradicts that X is initially ω_1-compact. □

Lemma 7.2. *A first countable initially ω_1-compact space will have cardinality at most $\mathfrak{c}$ if every closed separable subset X satisfies that βX has cardinality at most $\mathfrak{c}$.*

Proof. Let Y be a first-countable initially ω_1-compact space and suppose that βX has cardinality at most $\mathfrak{c}$ for every closed separable subset X of Y. We first show that for every zero-set ultrafilter $\mathcal{Z}$ on Y has a subcollection $\mathcal{Z}'$ of cardinality at most $\mathfrak{c}$ with empty intersection. Indeed, if this were not the case, we begin choosing points $y_\alpha \in Y$ and sets Z_α which are intersections of no more than $\mathfrak{c}$ many elements of $\mathcal{Z}$ so that Z_α is disjoint from the closure of $\{y_\beta : \beta < \alpha\}$ and y_α is chosen $\bigcap_{\beta\leq\alpha} Z_\alpha$. This recursion must fail at some stage $\alpha < \omega_1$ since otherwise we have constructed an uncountable free sequence in Y. The only reason for the induction to fail is that we can not choose Z_α. Since the closure of $\{y_\beta : \beta < \alpha\}$ has cardinality at most $\mathfrak{c}$, this implies that there is subcollection $\mathcal{Z}_\alpha \subset \mathcal{Z}$ whose intersection avoids this closure, and so the intersection must be empty.

Let M be an elementary submodel of some $H(\theta)$ so that $Y \in M$, $M^\omega \subset M$ and $|M| = \mathfrak{c}$. Of course $Y \cap M$ is a closed subset of Y. Suppose, for a contradiction, that there is some point $z \in Y \setminus M$. Let $\mathcal{W}_z$ denote the collection of all zero set subsets of Y which contain z and are members of M. It follows that the collection $\{W \cap M : W \in \mathcal{W}_z\}$ has the finite intersection property.

Choose any zero-set ultrafilter $\mathcal{Z}$ on Y which is in the closure in βY of $W \cap M$ for each $W \in \mathcal{W}_z$. Since βY has countable tightness there is a separable subset $X \in M$ such that $X \subset Y \cap M$ and the point $\mathcal{Z}$ is in the closure of X. Since βX has cardinality at most $\mathfrak{c}$, it follows that the closure in βY of X is contained in M. Therefore, the ultrafilter $\mathcal{Z}$ is an element of M. There will be a $Z \in \mathcal{Z} \cap M$ such that $z \notin Z$ since $\mathcal{Z}$ has a subfamily of cardinality at most $\mathfrak{c}$ (in M) with empty intersection. Since $X \setminus Z$ is a countable union of zero sets, there is also a zero set $W \in M$ such that $z \in W$. This contradicts that $\mathcal{Z}$ was chosen to extend $\mathcal{W}_z$. □

Lemma 7.3. *If X is a first-countable initially ω_1-compact space such that compact subsets of $\beta X \setminus X$ are finite and βX has cardinality greater than $\mathfrak{c}$, then there is a first-countable initially ω_1-compact space of cardinality greater than $\mathfrak{c}$.*

Proof. Let $D(\beta X)$ denote the standard Alexandroff double, that is, $D(\beta X) = \beta X \times \{0,1\}$. The points of $\beta X \times \{1\}$ are isolated, and for $x \in \beta X$, W is a neighborhood of $(x, 0)$ if and only if there is some open U of βX containing x such that W contains $U \times \{0,1\} \setminus (x, 1)$. Let $Y = (X \times \{0\}) \cup ((\beta X \setminus X) \times \{1\})$. Certainly Y is first-countable. If $A \subset (\beta X \setminus X)$ is infinite, then A has accumulation points in X since compact subsets of βX are finite. Therefore $A \times \{1\}$ has accumulation points in Y. Furthermore, if A has cardinality ω_1, let it be written as an increasing union of countable sets A_α $(\alpha \in \omega_1)$. We want to find a complete accumulation point $(x, 0)$ in Y. For each $\alpha \in \omega_1$, let $x_\alpha \in X$ be an accumulation point of $A \setminus A_\alpha$. If none of the x_α are complete accumulation points of A, then we may assume they are pairwise distinct. Let $x \in X$ be a complete accumulation point of $\{x_\alpha : \alpha \in \omega_1\}$.

It is clear that x is then a complete accumulation point of A. Therefore $(x, 0)$ is a complete accumulation point of $A \times \{1\}$. □

The final result follows from Corollary 6.5 and the previous Lemma.

Theorem 7.4. *It is consistent with Martin's Axiom and $\mathfrak{c} = \omega_2$ that there is a first-countable initially ω_1-compact space of cardinality greater than $\mathfrak{c}$.*

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