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ALL PAROVICHENKO SPACES MAY BE SOFT-PAROVICHENKO

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To the memory to Phil Zenor, one of founders of this journal

Abstract. It is shown that, assuming the Continuum Hypothesis, every compact Hausdorff space of weight at most $\mathfrak{c}$ is a remainder in a soft compactification of $\mathbb{N}$.

We also exhibit an example of a compact space of weight $\aleph_1$ — hence a remainder in some compactification of $\mathbb{N}$ — for which it is consistent that is not the remainder in a soft compactification of $\mathbb{N}$.

Introduction

A compactification, $\gamma\mathbb{N}$, of the discrete space $\mathbb{N}$ of natural numbers is said to be *soft* if for all pairs $\langle A, B\rangle$ of disjoint subsets of $\mathbb{N}$ the following holds: if $\operatorname{cl} A \cap \operatorname{cl} B \neq \emptyset$ then there is an autohomeomorphism h of $\gamma\mathbb{N}$ such that $h[A] \cap B$ is infinite and h is the identity on the remainder $\gamma\mathbb{N} \setminus \mathbb{N}$.

Banakh asked in [1] whether every Parovichenko space is soft-Parovichenko, where a Parovichenko space is defined to be a remainder in some compactification of $\mathbb{N}$ and, naturally, a soft-Parovichenko space is a remainder in some soft compactification of $\mathbb{N}$. Parovichenko's classic theorem, from [7], characterizes, assuming CH, the Parovichenko spaces as the compact Hausdorff spaces of weight at most $\mathfrak{c}$.

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Example 1. The Čech-Stone compactification, $\beta\mathbb{N}$, of $\mathbb{N}$ is soft, vacuously; hence $\beta\mathbb{N}\setminus\mathbb{N}$ is soft-Parovichenko.

At the other end of the spectrum the one-point compactification $\alpha\mathbb{N}$ is soft too as *every* permutation of $\mathbb{N}$ determines an autohomeomorphism of $\alpha\mathbb{N}$.

Remark 1. As remarked by Banakh in [1]: if $\delta\mathbb{N}$ is a compactification of $\mathbb{N}$ with the property that whenever $x\in\operatorname{cl}A$, where $x\in\delta\mathbb{N}\setminus\mathbb{N}$ and $A\subseteq\mathbb{N}$, there is a sequence in A that converges to x, then $\delta\mathbb{N}$ is a soft compactification.

Indeed, if $x\in\operatorname{cl}A\cap\operatorname{cl}B$ and S and T are subsets of A and B respectively that converge to x then one takes a permutation h of $\mathbb{N}$ that interchanges S and T and is the identity outside $S\cup T$. The extension of h by the identity on X is an autohomeomorphism.

In [2] the reader can find more information on the background of the problem of remainders in soft compactifications, including, in Theorem 9.5, some classes of compact spaces that are soft-Parovichenko:

- Parovichenko and of character less than $\mathfrak{p}$
- perfectly normal
- of weight less than $\mathfrak{p}$

In each case one obtains the stronger statement that every compactification with the space as its remainder is soft, because in each case the compactification satisfies the condition in Remark 1. The cardinal number $\mathfrak{p}$ is equal to the cardinal number $\mathfrak{t}$ discussed in Section 2 below.

1. Applying the Continuum Hypothesis

In this section we prove the statement in the abstract. The Continuum Hypothesis (CH) implies that every Parovichenko space is soft-Parovichenko.

Let X be compact Hausdorff and of weight $\aleph_1$. We may assume X is embedded in the Tychonoff cube $[0,1]^{\omega_1}$ and, for technical convenience, that $X\subseteq\{0\}\times[0,1]^{[1,\omega_1)}$.

Our aim will be to construct a sequence $\langle f_\alpha:\alpha<\omega_1\rangle$ of functions from $\mathbb{N}$ to $[0,1]$ such that the Čech-Stone extension βf of its diagonal map $f:\mathbb{N}\to[0,1]^{\omega_1}$ satisfies $\beta f[\beta\mathbb{N}\setminus\mathbb{N}]=X$. To make sure that $f[\mathbb{N}]$ is discrete we demand that $f_0(n)=2^{-n}$ for all n. In this way $f[\mathbb{N}]\cup X$ will be a compactification of $\mathbb{N}$ with X as its remainder.

Ensuring softness. To ensure softness of the compactification we take our inspiration from Remark 1.

Along with the functions f_α we construct an almost disjoint family $\mathcal{S}$ of subsets of $\mathbb{N}$ such that in the end every $S\in\mathcal{S}$ converges to a point x_S of X.

In addition we ensure that whenever $\langle A, B\rangle$ is a pair of disjoint subsets of $\mathbb{N}$ whose closures intersect then there will two sets S and T in $\mathcal{S}$ such that $S \cap A$ and $T \cap B$ are infinite, and $x_S = x_T$. As in Remark 1 a permutation of $\mathbb{N}$ that interchanges $S \cap A$ and $T \cap B$ and is the identity outside these sets gives an autohomeomorphism of the compactification as required.

The construction. We let $\bigl\langle\langle A_\alpha, B_\alpha\rangle : \alpha < \omega_1\bigr\rangle$ enumerate the set of ordered pairs of disjoint infinite subsets of $\mathbb{N}$. We shall construct

- a sequence $\langle f_\alpha : \alpha < \omega_1\rangle$ of functions from $\mathbb{N}$ to $[0, 1]$
- a sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ of subsets of $\mathbb{N}$
- a sequence $\langle x_\alpha : \alpha < \omega_1\rangle$ of points in X
- a sequence $\langle K_\alpha : \alpha < \omega_1\rangle$ of subsets of $\mathbb{N}$

For each δ we let $g_\delta : \mathbb{N} \to [0, 1]^\delta$ be the diagonal map of $\langle f_\alpha : \alpha < \delta\rangle$, and, for bookkeeping purposes, I will be the set of δ for which the closures of $g_\delta[A_\delta]$ and $g_\delta[B_\delta]$ intersect.

The sequences should satisfy the following conditions.

1. if $\alpha < \delta$ then the set $g_\delta[S_\alpha]$ converges to the point $x_\alpha \restriction \delta$
2. if $\delta \in I$ then there are $\alpha, \beta \leqslant \delta$ such that $x_\alpha \restriction \delta = x_\beta \restriction \delta$, and both intersections $S_\alpha \cap A_\delta$ and $S_\beta \cap B_\delta$ are infinite
3. for all δ the family $\{S_\alpha : \alpha < \delta\} \cup \{K_\delta\}$ is almost disjoint
4. if $\alpha < \beta$ then $K_\beta \subseteq^* K_\alpha$
5. if $\alpha \in \omega_1$ then $\beta g_\alpha[\mathbb{N}^*] = \beta g_\alpha[K_\alpha^*] = X \restriction \alpha$.

In condition 2 we do not exclude the possibility that $\alpha = \beta$.

At each stage δ we choose the set S_δ, construct the function f_δ, and determine the set $K_{\delta+1}$ as a subset of K_δ. This means that in case δ is a limit we must construct K_δ first.

Making K_δ if δ is a limit. Let $\langle \mathcal{U}_n : n < \omega\rangle$ be a sequence of finite families of basic open sets in $[0, 1]^\delta$ such that for all n we have $X \restriction \delta \subseteq \bigcup \mathcal{U}_n$ and $(X \restriction \delta) \cap U \neq \emptyset$ for all $U \in \mathcal{U}_n$, and such that for every open set O around $X \restriction \delta$ there is an n such that $\bigcup \mathcal{U}_n \subseteq O$. For every n there is a finite set F_n such that every member of $\mathcal{U}_n$ has its support in F_n. Let $\langle \delta_n : n < \omega\rangle$ be a strictly increasing sequence of ordinals that converges to δ and such that $F_n \subseteq \delta_n$ for all n.

The family $\mathcal{U}_n$ can also be considered to be a family of basic open sets in the product $[0, 1]^{\delta_n}$. The condition that $\beta g_{\delta_n}[K_{\delta_n}^*] = X \restriction \delta_n$ for all n translates into two things:

- for every n there is a natural number N_n such that $g_{\delta_n}(k) \in \bigcup \mathcal{U}_n$ for $k \in K_{\delta_n} \setminus N_n$
- for every $U \in \mathcal{U}_n$ the set $\{k \in K_{\delta_n} : g_{\delta_n}(k) \in U\}$ is infinite

But then the same holds with g_δ replacing g_{δ_n}.

Using this we determine a strictly increasing sequence $\langle M_n : n<\omega\rangle$ of natural numbers such that $M_n \geqslant N_n$ for all n, such that $K_{\delta_{n+1}} \setminus M_{n+1} \subset K_{\delta_n}$ for all n, and such that for every $U \in \mathcal{U}_n$ there is a $k \in K_{\delta_n} \cap [M_n, M_{n+1})$ such that $g_\delta(k) \in U$.

We let $K_\delta = \bigcup_{n<\omega}\bigl(K_{\delta_n} \cap [M_n, M_{n+1})\bigr)$. By construction $\beta g_\delta[K_\delta^*] = X$, and because $K_\delta \subseteq^* K_{\delta_n}$ for all n the set is also almost disjoint from all S_α for $\alpha < \delta$.

The actual construction. Now let $\delta \in \omega_1$ and assume that everything has been constructed up to and/or including δ.

If the closures of $g_\delta[A_\delta]$ and $g_\delta[B_\delta]$ intersect then we add δ to the set I and determine S_δ by considering a few cases.

First shrink A_δ and B_δ to infinite sets C and D such that the closures of $g_\delta[C]$ and $g_\delta[D]$ intersect in exactly one point of $X \restriction \delta$, this point is going to grow into x_δ, so we denote it $x_\delta \restriction \delta$. Note that the union $g_\delta[C] \cup g_\delta[D]$ converges to $x_\delta \restriction \delta$.

The cases that can occur are

- both C and D are almost disjoint from the S_α with $\alpha < \delta$; in this case we let S_δ be an infinite subset of $C \cup D$, that meets both C and D in an infinite set and is such that $K_\delta \setminus S_\delta$ contains an infinite set that converges to $x_\delta \restriction \delta$.
- C is almost disjoint from the S_α with $\alpha < \delta$, but D is not; in this case we have a $\beta < \delta$ such that $S_\beta \cap D$ is infinite, and so $x_\beta \restriction \delta = x_\delta \restriction \delta$. Now let S_δ be an infinite subset of C as in the previous case.
- D is almost disjoint from the S_α with $\alpha < \delta$, but C is not; in this case we have an $\alpha < \delta$ such that $S_\alpha \cap D$ is infinite, and so $x_\alpha \restriction \delta = x_\delta \restriction \delta$. Now let S_δ be an infinite subset of D as in the previous cases.
- neither C nor D is almost disjoint from the S_α with $\alpha < \delta$; this means that condition 2 is already met. We let S_δ be an infinite subset of K_δ that converges to some $x_\delta \restriction \delta$, again subject to the condition from the first three cases.

In all four cases let $K_{\delta+1} = K_\delta \setminus S_\delta$. Then $\beta g_\delta[K_{\delta+1}] = X$, because of the condition on S_δ. That condition is met automatically if $x_\delta \restriction \delta$ is not an isolated point of $X \restriction \delta$.

In case the closures do not intersect we choose an arbitrary point $x_\delta \restriction \delta$ of $X \restriction \delta$ and choose an infinite subset S_δ of K_δ that converges to this point and such that $K_\delta \setminus S_\delta$, which will be $K_{\delta+1}$, contains an infinite set that converges to $x_\delta \restriction \delta$ as well.

In all cases we still have $\beta g_\delta[K_{\delta+1}^*] = X$.

Before we proceed to the definition of f_δ we first choose the δth coordinates of the points x_α for $\alpha \leqslant \delta$. For each α we check whether there is a $\beta < \alpha$ such that $x_\alpha \restriction \delta = x_\beta \restriction \delta$. If that is the case then we must let $x_\alpha(\delta) = x_\beta(\delta)$. In the other case we ensure that $\langle x_\alpha \restriction \delta, x_\alpha(\delta)\rangle \in X \restriction (\delta+1)$. This then introduces the demand that $f_\delta[S_\alpha]$ converge to $x_\alpha(\delta)$.

To specify the function f_δ we proceed much as in the construction of K_δ for limit δ.

We take a sequence $\langle \mathcal{U}_n : n < \omega\rangle$ of finite families of basic open sets in $[0,1]^{\delta+1}$ as follows. First take an increasing sequence $\langle F_n : n < \omega\rangle$ of finite subsets of $\delta + 1$ such that $\delta \in F_0$ and $\bigcup_n F_n = \delta + 1$.

Next we let $\mathcal{B}_n$ be the family of all products $\prod_{\alpha\leqslant\delta} I_\alpha$, where I_α is an interval of the form $[0, 2^{-n})$, $\bigl(i \cdot 2^{-n}, (i + 1) \cdot 2^{-n}\bigr)$ or $(1 - 2^{-n}, 1]$ in $[0, 1]$ if $\alpha \in F_n$ and $I_\alpha = [0, 1]$ if $\alpha \notin F_n$.

We let $\mathcal{U}_n = \{B \in \mathcal{B}_n : B \cap (X \restriction (\delta + 1)) \neq \emptyset\}$ and we write every $U \in \mathcal{U}_n$ as $V_U \times I_U$, where V_u is in $[0, 1]^\delta$ and I_U is an interval in $[0, 1]$.

For every n we let $\mathcal{S}_n = \{S_\alpha : \alpha \in F_n\} \cup \{K_{\delta+1}\}$ and we take an N_n such that

- for all distinct X and Y in $\mathcal{S}_n$ the intersection $X \cap Y$ is contained in N_n
- for all $k \geqslant N_n$ we have $g_\delta(k) \in \bigcup\{V_U : U \in \mathcal{U}_n\}$
- for all $U \in \mathcal{U}_n$ and all $\alpha \leqslant \delta$: if $x_\alpha \restriction \delta \in V_U$ then $g_\delta(k) \in V_U$ for all $k \in S_\alpha \setminus N_n$

Because $\beta g_\delta[K^*_{\delta+1}] = X$ we know that for every $U \in \mathcal{U}_n$ the set $\{k \in K_{\delta+1} : g_\delta(k) \in V_U\}$ is infinite. Using this we take a strictly increasing sequence $\langle M_n : n < \omega\rangle$ of natural numbers such that $M_n \geqslant N_n$ for all n and we can define f_δ on $K_{\delta+1}$ such that for all n and $U \in \mathcal{U}_n$ there a $k \in K_{\delta+1} \cap [M_n, M_{n+1})$ such that $\langle g_\delta(k), f_\delta(k)\rangle \in U$.

We define f_δ on $S_\alpha \cap [M_n, M_{n+1})$ whenever $\alpha \in F_n$; because $M_n \geqslant N_n$ there will be no interference with the values that we specified on $K_{\delta+1}$ and between the different S_αs.

For each $\alpha \in F_n$ we can simply define $f_\delta(k) = x_\alpha(\delta)$ for $k \in S_\alpha \cap [M_n, M_{n+1})$.

For all $k \in [M_n, M_{n+1})$ that are not in $K_{\delta+1}$, nor in any of the S_α for some $\alpha \leqslant \delta$, we simply choose $f_\delta(k)$ in such a way that $\langle g_\delta(k), f_\delta(k)\rangle \in \bigcup \mathcal{U}_n$.

To see that $f_\delta[S_\alpha]$ converges to $x_\alpha(\delta)$ it suffices to observe that f_δ has the constant value $x_\alpha(\delta)$ on the intersection $S_\alpha \cap [M_n, \omega)$, where n is such that $\alpha \in F_n$.

2. Some examples of weight $\aleph_1$

One half of Parovichenko's characterization is a ZFC result: every compact space of weight $\aleph_1$ is a remainder of $\mathbb{N}$. Our proof is in the spirit of Błaszczyk and Szymanski's proof of this statement in [3]. The main difference is in the number of tasks that need to be done to construct the map $f : \mathbb{N} \to [0,1]^{\omega_1}$. To make X a remainder involves just $\aleph_1$ many tasks, whereas to make it a soft remainder seems to involve $\mathfrak{c}$ many tasks as there are that many pairs of subsets of $\mathbb{N}$, hence the need to assume CH.

In this section we present some examples to show that without CH not every compact space of weight $\aleph_1$ is automatically a soft remainder of $\mathbb{N}$.

The ordinal space $\omega_1 + 1$. This relatively simple space already offers some difficulties regarding softness. To be sure: it is a soft remainder, but the proof requires us to consider two cases, depending on the value of the small cardinal $\mathfrak{t}$.

- if $\mathfrak{t} > \aleph_1$ then *every* compactification of $\mathbb{N}$ with $\omega_1 + 1$ as a remainder is soft,
- if $\mathfrak{t} = \aleph_1$ then *some but not all* compactifications of $\mathbb{N}$ with $\omega_1 + 1$ as a remainder are soft.

All compactifications of $\mathbb{N}$ with $\omega_1 + 1$ as a remainder have roughly the same structure, as described by Franklin and Rajagopalan in [6].

Let $\gamma\mathbb{N}$ be such a compactification, where we assume that $\mathbb{N}$ and $\omega_1 + 1$ are disjoint. We apply normality to find, for every $\alpha < \omega_1$. open sets U_α and V_α with disjoint closures such that $[0,\alpha] \subseteq U_\alpha$ and $[\alpha+1,\omega_1] \subseteq V_\alpha$. Then $\gamma\mathbb{N} \setminus (U_\alpha \cup V_\alpha)$ is a compact subset of $\mathbb{N}$, and hence finite. By adding this finite set to U_α we can in fact assume that $U_\alpha \cup V_\alpha = \gamma\mathbb{N}$ for all α.

We let $T_\alpha = U_\alpha \cap \mathbb{N}$ for all α. The sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$ has the following property:

$$(*) \text{ if } \beta < \alpha \text{ then } T_\beta \setminus T_\alpha \text{ is finite and } T_\alpha \setminus T_\beta \text{ is infinite.}$$

Conversely, every such sequence determines a compactification of $\mathbb{N}$. To this end write $T_{\omega_1} = \mathbb{N}$ and then define a topology on $\mathbb{N} \cup (\omega_1 + 1)$ as follows: the points of $\mathbb{N}$ are isolated and if $\alpha \leqslant \omega_1$ then the sets

$$(\beta, \alpha] \cup \bigl(T_\alpha \setminus (T_\beta \cup F)\bigr)$$

where $\beta < \alpha$ and $F \subseteq \mathbb{N}$ is finite form a local base at α. To ensure that the ordinal 0 is not an isolated point we tacitly assume that T_0 is infinite. The sets $U_\alpha = T_\alpha \cup [0,\alpha]$ and $V_\alpha = (\mathbb{N} \setminus T_\alpha) \cup [\alpha+1,\omega_1]$ are as in the previous paragraph.

Before we investigate how the softness of $\gamma\mathbb{N}$ depends on the sequence $\langle T_\alpha : \alpha < \omega_1 \rangle$ we make a short digression on the cardinal number $\mathfrak{t}$ alluded to above.

A sequence $\langle T_\alpha : \alpha < \delta\rangle$ that satisfies property $(*)$ above is also called a tower. The cardinal number $\mathfrak{t}$ is defined to be the minimum ordinal δ for which there is a *complete tower*: a tower $\langle T_\alpha : \alpha < \delta\rangle$ with the additional property that if S is such that $T_\alpha \setminus S$ is finite for all α then $\mathbb{N} \setminus S$ is finite.

Thus, $\mathfrak{t} > \aleph_1$ means that *every* tower $\langle T_\alpha : \alpha < \omega_1\rangle$ is *incomplete*: there is an infinite set R such that $T_\alpha \cap R$ is finite, for all α, whereas $\mathfrak{t} = \aleph_1$ means that there is *some* tower $\langle T_\alpha : \alpha < \omega_1\rangle$ that is complete, that is, for which no such infinite set exists.

Now let A and B be disjoint subsets of $\mathbb{N}$ such that $\operatorname{cl} A \cap \operatorname{cl} B \neq \emptyset$ in $\gamma\mathbb{N}$. We consider a few cases.

Case 1: there is an $\alpha \in \omega_1$ that is in the closure of A and B. Then A and B have infinite intersections with U_α and we can take infinite subsets C of $A \cap U_\alpha$ and D of $B \cap U_\alpha$ such that $\operatorname{cl} C = C \cup \{\alpha\}$ and $\operatorname{cl} D = D \cup \{\alpha\}$. Take any permutation h of $\mathbb{N}$ that interchanges C and D and is the identity outside $C \cup D$. Then h extends to an autohomeomorphism of $\gamma\mathbb{N}$ that is the identity on $\gamma\mathbb{N} \setminus \mathbb{N}$. By construction $h[A] \cap B$ contains D.

Case 2: no such α can be found; hence $\operatorname{cl} A \cap \operatorname{cl} B = \{\omega_1\}$. In this case we know that for every α the intersections $V_\alpha \cap A$ and $V_\alpha \cap B$ are both infinite. Now we split into two subcases.

Subcase 2a: there are infinite sets subsets C of A and D of B such that $C \cup D \subseteq^* V_\alpha$ for all α. Then C and D converge to ω_1 and, as above, interchanging C and D will witness softness.

By the definition of $\mathfrak{t}$ this subcase occurs, no matter how $\gamma\mathbb{N}$ is constructed, if $\mathfrak{t} > \aleph_1$. This then proves the first statement at the beginning of this subsection.

Subcase 2b: one of A and B does not contain an infinite set as in Subcase 2a, say A to be definite. We claim that $\omega_1 \cap \operatorname{cl} A$ is closed and unbounded in ω_1. That it is closed is clear. To show unboundedness let $\alpha \in \omega_1$. The set $A \cap V_\alpha$ is infinite, but by assumption there is a β such that $A \cap V_\alpha \setminus V_\beta$ is infinite. It closure intersects ω_1 and is inside $V_\alpha \setminus V_\beta$, hence $\operatorname{cl} A$ intersects the interal $[\alpha + 1, \beta]$.

It now follows that $\omega_1 \cap \operatorname{cl} B$ is closed and *bounded* in ω_1; say $\omega_1 \cap \operatorname{cl} B \subseteq [0, \alpha]$. This means that $D = B \cap V_\alpha$ has ω_1 as its only accumulation point and hence that it converges to ω_1.

In this case the pair $\langle D, A\rangle$ witnesses non-softness: if $h : \gamma\mathbb{N} \to \gamma\mathbb{N}$ is a homeomorphism that is the identity on ω_1 then $h[D]$ converges to ω_1 and so $h[D] \subseteq^* V_\beta$ for all β and by our assumption on A this means that $h[D] \cap A$ is finite.

Note that there is no Subcase 2c: in that subcase $\omega_1 \cap \operatorname{cl} A$ and $\omega_1 \cap \operatorname{cl} B$ would both be closed and unbounded and this would bring us back to Case 1.

We shall show that under the assumption $\mathfrak{t} = \aleph_1$ one can construct two compactifications, $\gamma_1\mathbb{N}$ and $\gamma_2\mathbb{N}$, of $\mathbb{N}$ and with remainder $\omega_1 + 1$ such that in $\gamma_1\mathbb{N}$ Case 1 always occurs, so $\gamma_1\mathbb{N}$ is soft, and in $\gamma_2\mathbb{N}$ there is a pair $\langle A, B\rangle$ where Subcase 2b occurs, hence $\gamma_2\mathbb{N}$ is not soft.

By the assumption $\mathfrak{t} = \aleph_1$ there is a complete tower $\langle T_\alpha : \alpha < \omega_1\rangle$.

We let $\gamma_1\mathbb{N}$ be the compactification of $\mathbb{N}$ determined by this tower. Let A and B be disjoint subsets of $\mathbb{N}$ whose closures intersect and assume $\omega_1 \in \operatorname{cl} A \cap \operatorname{cl} B$. Because neither A nor B contains an infinite subset as in Subcase 2a the argument in Subcase 2b applies to A and B to show that $\omega_1 \cap \operatorname{cl} A$ and $\omega_1 \cap \operatorname{cl} B$ are closed and unbounded in ω_1. We find that we are in Case 1 and that $\gamma_1\mathbb{N}$ is a soft compactification of $\mathbb{N}$.

To construct the promised non-soft compactification $\gamma_2\mathbb{N}$ we take the sum $\gamma_1\mathbb{N} \oplus \alpha\mathbb{N}$ of $\gamma_1\mathbb{N}$ and the one-point compactification $\alpha\mathbb{N}$, and identify the points ω_1 and ∞.

We let A be the copy of $\mathbb{N}$ from $\gamma_1\mathbb{N}$ and B the copy of $\mathbb{N}$ from $\alpha\mathbb{N}$. This pair witnesses Subcase 2b above and thus shows that $\gamma_2\mathbb{N}$ is not a soft compactification of $\mathbb{N}$.

Remark 2. As mentioned above the cardinal number $\mathfrak{t}$ is equal to the number $\mathfrak{p}$ mentioned in the introduction.

A consequence of this is that we could have concluded right away that $\mathfrak{t} > \aleph_1$ implies that every compactification of $\mathbb{N}$ with remainder $\omega_1 + 1$ is soft. We believe the argument that we gave above is more instructive.

Nevertheless we do record here for future use that under the assumption $\mathfrak{t} > \aleph_1$ every compactification of $\mathbb{N}$ whose remainder has weight $\aleph_1$ or less is soft. This means that if we want to prove, in ZFC, that some specific compact space of weight $\aleph_1$ is soft-Parovichenko we can (or rather must) assume that $\mathfrak{t} = \aleph_1$.

The compact ordered space $\omega_1 + 1 + \omega_1^*$. The compact ordered space $K = \omega_1 + 1 + \omega_1^*$ is of weight $\aleph_1$ and hence is a Parovichenko space. It is only slightly more complicated than $\omega_1 + 1$ but, as we shall see, its softness is already undecidable.

We think of K as the quotient of $(\omega_1 + 1) \times 2$ obtained by identifying $\langle \omega_1, 0\rangle$ and $\langle \omega_1, 1\rangle$ to one point, which we call Ω.

Note that $(\omega_1 + 1) \times 2$ is a soft remainder, as a sum of two soft compactifications of $\mathbb{N}$ is again a soft compactification. This example will therefore show two things: a compact space of weight $\aleph_1$ need not be a soft remainder, and the continuous image of a soft remainder need not be a soft remainder itself.

It is easy to exhibit *some* compactification of $\mathbb{N}$ with remainder K that is not soft, assuming $\mathfrak{t} = \aleph_1$ of course: take the compactification $\gamma_1\mathbb{N}$ from the previous subsection. We take the quotient of $\gamma_1\mathbb{N} \times 2$ obtained by identifying $\{\omega_1\} \times 2$ to one point. The sets $A = \mathbb{N} \times \{0\}$ and $B = \mathbb{N} \times \{1\}$ witness non-softness of the resulting compactification: if $C \subseteq A$ is infinite then its closure contains at least one point of $\omega_1 \times \{0\}$; any homeomorphism that maps C into B will move that point to $\omega_1 \times \{1\}$.

We shall show that the principle (NT) from [4] implies that every compactification of $\mathbb{N}$ with remainder K contains two sets that behave like $\mathbb{N} \times \{0\}$ and $\mathbb{N} \times \{1\}$ in the above example.

To formulate (NT) we need to define the notion of a weakly σ-bounded family of infinite subsets of $\mathbb{N}$: given a family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ we let $\mathcal{A}^{\downarrow}$ denote the family of infinite sets X for which there is a member of $\mathcal{A}$ that contains it. We call $\mathcal{A}$ *weakly σ-bounded* if for every countable subfamily $\mathcal{X}$ of $\mathcal{A}^{\downarrow}$ there is an $A \in \mathcal{A}$ such that $A \cap X$ is infinite for all $X \in \mathcal{X}$.

The principle (NT) states the following:

> for each weakly σ-bounded subfamily $\mathcal{A}$ of $\mathcal{P}(\mathbb{N})$ and each subfamily $\mathcal{B}$ of $\mathcal{A}$ of cardinality at most $\aleph_1$ there is a subset C of $\mathbb{N}$ such that $C \cap B$ is infinite for all $B \in \mathcal{B}$ *and* for every infinite subset D of C there is an $A \in \mathcal{A}$ such that $A \cap D$ is infinite.

In [4] this principle was shown to be consistent with $\mathfrak{c} = \mathfrak{b} = \aleph_2$.

Now let $\delta\mathbb{N} = \mathbb{N} \cup K$ be a compactification, where K is the remainder. For every α we choose pairwise disjoint subsets sets L_α, M_α and R_α of $\mathbb{N}$ such that, with closures taken in $\gamma\mathbb{N}$

- $[0, \alpha] \times \{0\} \subseteq \operatorname{cl} L_\alpha$,
- $[\alpha + 1, \omega_1] \times 2 \subseteq \operatorname{cl} M_\alpha$,
- $[0, \alpha] \times \{1\} \subseteq \operatorname{cl} R_\alpha$, and
- the three closures are pairwise disjoint.

We apply the principle (NT) to the families $\mathcal{L} = \{L_\alpha : \alpha < \omega_1\}$ and $\mathcal{R} = \{R_\alpha : \alpha < \omega_1\}$, and the subfamilies $\mathcal{B}_L = \{L_{\alpha+1} \setminus L_\alpha : \alpha < \omega_1\}$ of $\mathcal{L}^{\downarrow}$ and $\mathcal{B}_R = \{R_{\alpha+1} \setminus R_\alpha : \alpha < \omega_1\}$ of $\mathcal{R}^{\downarrow}$ respectively.

The families $\mathcal{L}$ and $\mathcal{R}$ are clearly weakly σ-bounded: if $\mathcal{B}$ is a countable family of infinite sets such that for all $B \in \mathcal{B}$ there is an α_B with $B \subseteq L_{\alpha_B}$ then take $\alpha = \sup_B \alpha_B$; the set L_α is as required because $B \subseteq^* L_\alpha$ for all $B \in \mathcal{B}$. The same argument works for $\mathcal{R}$.

The families $\mathcal{B}_L$ and $\mathcal{B}_R$ are of cardinality $\aleph_1$ and refine $\mathcal{L}$ and $\mathcal{R}$, respectively. The principle (NT) then guarantees there are subsets C_L and C_R of $\mathbb{N}$ such that

- $B \cap C_L$ is infinite, for all $B \in \mathcal{B}_L$, and, likewise $B \cap C_R$ is infinite, for all $B \in \mathcal{B}_R$, and

- for every infinite subset D of C_L (or C_R) there is an $L \in \mathcal{L}$ (or an $R \in \mathcal{R}$) such that $D \cap L$ (or $D \cap R$) is infinite

We derive some consequences from this.

For every α the set $L_{\alpha+1} \setminus L_\alpha$ converges to the point $\langle \alpha+1, 0\rangle$, hence $\langle \alpha+1, 0\rangle \in \operatorname{cl} C_L$. It follows that the point in the middle, Ω, is in the closure of C_L. And by a symmetric argument $\Omega \in \operatorname{cl} C_R$ also.

The intersection $C_L \cap C_R$ is finite. For if it were infinite then by the second condition in (NT) there is an α such that $L_\alpha \cap C_L \cap C_R$ is infinite, and by a second application of that condition there is a β such that $L_\alpha \cap C_L \cap C_R \cap R_\beta$ is infinite. But $L_\alpha \cap R_\beta$ is finite, contradiction. So we may as well assume that C_L and C_R are disjoint.

Now let h be an autohomeomorphism of $\gamma\mathbb{N}$ with the property that $h[C_L] \cap C_R$ is infinite. Take an α such that $h[C_L] \cap C_R \cap R_\alpha$ is infinite. Then take a β such that $L_\beta \cap C_L \cap h^{\leftarrow}[C_R \cap R_\alpha]$ is infinite. Take $\gamma \leqslant \beta$ such that $\langle \gamma, 0\rangle$ is in the closure of the latter set; then $h(\gamma, 0)$ is in the closure of R_α. Hence certainly $h(\gamma, 0) \neq \langle \gamma, 0\rangle$.

The Cantor cube 2^{ω_1} and the Tychonoff cube $[0,1]^{\omega_1}$. Two natural compact spaces of weight $\aleph_1$ to consider are the Cantor and Tychonoff cubes 2^{ω_1} and $[0,1]^{\omega_1}$. We shall show that both are soft-Parovichenko. Since the ordered space $\omega_1 + 1 + \omega_1^\star$ is a subspace of both cubes this shows that subspaces of soft-Parovichenko spaces need not be soft-Parovichenko themselves.

As mentioned in Remark 2 we can assume $\mathfrak{t} = \aleph_1$.

We apply Theorem 3 from [5] and take a set $\mathcal{F}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ that is of cardinality $\mathfrak{c}$ and *independent*, which means that given f_1, …, f_k in $\mathcal{F}$ and n_1, …, n_k in $\mathbb{N}$ there is an $m \in \mathbb{N}$ such that $f_i(m) = n_i$ for all i. One readily checks that this is equivalent to the following: the image of the diagonal map $e : \mathbb{N} \to \mathbb{N}^{\mathcal{F}}$ of the family $\mathcal{F}$ is dense where we consider the product topology on $\mathbb{N}^{\mathcal{F}}$ and the discrete topology on $\mathbb{N}$.

We take an injective sequence $\langle f_\alpha : \alpha < \omega_1\rangle$ of elements of $\mathcal{F}$ and a complete tower $\langle T_\alpha : \alpha < \omega_1\rangle$ in $\mathbb{N}$.

We make a technical adjustment to these two sequences, as follows.

Let $N = \{\langle m, n\rangle \in \mathbb{N}^2 : n \leqslant m\}$. We define a new tower $\langle I_\alpha : \alpha < \omega_1\rangle$ and a new independent sequence $\langle F_\alpha : \alpha < \omega_1\rangle$ of functions on N. For every α let

- $I_\alpha = N \cap (T_\alpha \times \mathbb{N})$, and
- define F_α by

$$F_\alpha(m,n) = \begin{cases} f_\alpha(n) & \text{if } m \notin T_\alpha \text{ and } f_\alpha(n) \neq 0\text{, and} \\ 0 & \text{otherwise} \end{cases}$$

It is elementary to verify that $\langle I_\alpha : \alpha < \omega_1\rangle$ is also a complete tower and that the sequence $\langle F_\alpha : \alpha < \omega_1\rangle$ is again independent. We have additionally created some interplay between the two: if $\beta \geqslant \alpha$ then $I_\alpha \setminus F_\beta^\leftarrow(0)$ is finite.

We adjust the two sequences once more, this time without renaming them. We identify N with $\mathbb{N}$ via some bijection; and we let the codomains of the functions F_α be the set Q of rational numbers in $[0,1]$, via some bijection that sends 0 to 0.

We let $e : \mathbb{N} \to [0,1]^{\omega_1}$ be the diagonal map of the sequence $\langle F_\alpha : \alpha < \omega_1\rangle$, defined by $e(n)(\alpha) = F_\alpha(n)$. By the remark above the image set $e[\mathbb{N}]$ is dense in Q^{ω_1} where Q carries the discrete topology, so $e[\mathbb{N}]$ is certainly dense in $[0,1]^{\omega_1}$.

It follows that $\beta e : \beta\mathbb{N} \to [0,1]^{\omega_1}$ induces a surjection from $\beta\mathbb{N} \setminus \mathbb{N}$ onto $[0,1]^{\omega_1}$; this surjection determines a compactification of $\mathbb{N}$ with $[0,1]^{\omega_1}$ as its remainder.

The compactification can be visualised as a subspace of the Alexandroff double $A([0,1]^{\omega_1})$ of $[0,1]^{\omega_1}$: the underlying set of $A([0,1]^{\omega_1})$ is $[0,1]^{\omega_1} \times 2$ with $[0,1]^{\omega_1} \times \{0\}$ being the set of isolated points; our compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ then is

$$([0,1]^{\omega_1} \times \{1\}) \cup (e[\mathbb{N}] \times \{0\}).$$

We identify $\mathbb{N}$ with $e[\mathbb{N}] \times \{0\}$ and $[0,1]^{\omega_1}$ with $[0,1]^{\omega_1} \times \{1\}$. We can save on notation by observing for $x \in [0,1]^{\omega_1}$ and $A \subseteq \mathbb{N}$ we have $x \in \operatorname{cl} A$ iff x is an accumulation point of $e[A]$; and A converges to x iff $e[A]$ converges to x.

Let A and B be disjoint in $\mathbb{N}$ and let $x \in [0,1]^{\omega_1}$ be in the intersection of their closures in $\gamma\mathbb{N}$.

Let M be a countable elementary substructure of $H((2^{\aleph_1})^+)$ that contains our tower, our independent sequence of functions, and A, B and x. Let $\delta = M \cap \omega_1$.

If H is a finite subset of δ and $\varepsilon > 0$ then the basic open set

$$O(x,H,\varepsilon) = \bigl\{y : (\forall \alpha \in H)\bigl(|y_\alpha - x_\alpha| < \varepsilon\bigr)\bigr\}$$

meets $e[A]$ and $e[B]$ in infinite sets. By elementarity there is an $\alpha \in \delta$ such that I_α, and hence also I_δ, has infinite intersections with these infinite sets.

Write δ as the union of an increasing sequence $\langle H_n : n < \omega\rangle$ of finite sets. By the remark above we can find sequences $\langle a_n : n < \omega\rangle$ and $\langle b_n : n < \omega\rangle$ in A and B such that $e(a_n) \in O(x, H_n, 2^{-n}) \cap A \cap I_\delta$ and $e(b_n) \in O(x, H_n, 2^{-n}) \cap B \cap I_\delta$ for all n.

Let y be the point defined by $y \restriction \delta = x \restriction \delta$ and $y(\alpha) = 0$ for $\alpha \geqslant \delta$. We claim that $\langle a_n : n < \omega\rangle$ and $\langle b_n : n < \omega\rangle$ converge to y. Indeed, let H be a finite subset of ω_1 and $\varepsilon > 0$. Fix K such that $H \cap \delta \subseteq H_K$. Also, using the fact that $I_\delta \setminus F_\alpha^{\leftarrow}(0)$ is finite for $\alpha \in H \setminus \delta$ find $L \geqslant K$ such that $F_\alpha(a_n) = F_\alpha(b_n) = 0$ when $n \geqslant L$ and $\alpha \in H \setminus \delta$. Then $n \geqslant L$ implies $a_n, b_n \in O(y, H, \varepsilon)$.

Now take the permutation h of D that interchanges a_n and b_n for all n and leaves the other elements in their places.

This shows that $[0,1]^{\omega_1}$ is soft-Parovichenko.

To show that 2^{ω_1} is soft Parovichenko one can use basically the same argument as above. The only change that needs to be made is to the diagonal map e: let

$$e(n)(\alpha) = \begin{cases} 1 & \text{if } F_\alpha(n) \neq 0, \text{ and} \\ 0 & \text{if } F_\alpha(n) = 0 \end{cases}$$

3. Remarks and questions

The title of this paper uses the words ‘may be’ rather than the word ‘are’ and the previous two sections show why that is. Under the Continuum Hypothesis the ‘are’ is justified but not in general.

In ZFC all compact spaces of weight $\aleph_1$ are Parovichenko, but as we have seen the space $\omega_1+1+\omega_1^\star$ is a Parovichenko space that is consistently not soft-Parovichenko.

This state of affairs suggests various further questions about the nature of soft remainders of $\mathbb{N}$.

In exploring the possible parallels between the classes of Parovichenko spaces and soft-Parovichenko spaces we saw that the Continuum Hypothesis simply implies that there is no difference.

The class of Parovichenko spaces is closed under continuous images; the class of soft-Parovichenko spaces is not, consistently.

The class of Parovichenko spaces is, consistently, not closed under subspaces: in the Cohen model where $\mathfrak{c} = \aleph_2$ the ordinal space $\omega_2 + 1$ is not remainder of $\mathbb{N}$ even though it is a subspace of the Parovichenko space 2^{ω_2}. We have seen that the same holds for soft-Parovichenko spaces.

A fair number of known Parovichenko spaces is also soft-Parovichenko; see the list in the introduction. Notably absent in that list are the separable compact spaces, so that will be our first question:

Question 1. Is every separable compact space soft-Parovichenko?

There are a few spaces in this class that are worth singling out. Given that every space of weight less than $\mathfrak{t}$ is a soft remainder we ask in particular

Question 2. Are the cubes $2^{\mathfrak{t}}$ and $[0,1]^{\mathfrak{t}}$ soft-Parovichenko?

We can ask this for every cardinal in the interval $[\mathfrak{t},\mathfrak{c}]$ but instead we ask whether there is some relationship between these cardinals.

Question 3. If $\kappa < \lambda$ and 2^λ is a soft remainder is then 2^κ soft as well? Likewise for the Tychonoff cubes.

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