MORE TOPOLOGICAL PARTITION RELATIONS ON $\beta\omega$

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Abstract. W. W. Comfort, in collaboration with A. Kato and S. Shelah, explored the problem of determining which spaces Y satisfy that for every 2-coloring of $\beta\omega$ there will be a monochromatic copy of Y. We continue the exploration and answer some questions raised in their paper.

1. Introduction

In this paper we extend the work begun in [3]. More specifically we consider the instances of the topological partition relation $X \to (Y)^1_2$ where X is the space $\beta\omega$. The relation $X \to (Y)^1_2$ holds if for each partition $X_0 \cup X_1$ of X, there is a topological copy of Y contained in one of X_0 or X_1 (see [20]). As explained in the introduction of [3], the classical Bernstein subset of the reals and its complement serve to verify that $\mathbb{C} \not\to (\mathbb{C})^1_2$, and that, in the positive direction, the fact that $\mathbb{Q} \to (\mathbb{Q})^1_2$ is quite easily verified. Adopting the notation from [3], we let $Y \subseteq_h X$ denote the relation that there is a subspace of X that is homeomorphic copy of Y. Also, we let $A \approx B$ abbreviate that A and B are homeomorphic topological spaces. It is obvious that if $A \subseteq_h Y$ and $X \to (Y)^1_2$, then $X \to (A)^1_2$ also holds. Also if $Y \subseteq_h X_1 \subseteq_h X$ and $X_1 \to (Y)^1_2$, then $X \to (Y)^1_2$. Since $\beta\omega \subset_h \omega^* \subset_h \beta\omega$, it follows that, for all spaces Y, $\beta\omega \to (Y)^1_2$ is equivalent to $\omega^* \to (Y)^1_2$.

The main results from [3] that lead to this work are the following:

Proposition 1.1 ([3]). *(1) If $\beta\omega \to (Y)^1_2$, then $|Y| < 2^{\mathfrak{c}}$.*

(2) There is a non-discrete P-space Y such that $\beta\omega \to (Y)^1_2$.

(3) For each $p \in \omega^*$, $\beta\omega \not\to (\omega \cup \{p\})^1_2$.

Date: July 11, 2018.

1991 *Mathematics Subject Classification.* [2000] Primary: 54A25, 54D40, 05A17 Secondary: 54A10, 54D30, 04A20.

Key words and phrases. Topological Ramsey theory, Stone-Čech remainder .

The research was supported by the NSF grant No. NSF-DMS 1501506.

To the memory of my mathematical great-grandfather and dear friend.

A space is a P-space if countable intersections of open sets (G_δ-sets) are also open. A point of a space is P-point if each G_δ-set containing the point is a neighborhood of the point. Of course it follows from (3), that if $\beta\omega \to (Y)^1_2$ holds, then countable discrete subsets of Y are closed. A point in a space is called a weak P-point if it is not the limit point of any countable set ([15]) and a discrete weak P-point if it is not the limit of any countable discrete set ([17]).

The following questions and problems are raised in [3]

Question 1.1. (1) Characterize those Y such that $\beta\omega \to (Y)^1_2$.
(2) Does $\beta\omega \to (Y)^1_2$ for all P-spaces Y with $|Y| < 2^{\mathfrak{c}}$?
(3) If $\beta\omega \to (Y)^1_2$ must Y be a P-space?
(4) If Y is countable and $\beta\omega \to (Y)^1_2$, must Y be discrete?

We are not really any closer to an answer to (1), but we have some results about the other questions. Concerning question (2), we show that $\beta\omega \to (Y)^1_2$ for every P-space of weight and cardinality $\aleph_1$. This answers (2) in the affirmative if GCH holds. Next we discover that $\beta\omega \to (Y)^1_2$ does hold for some countable non-discrete spaces. These spaces are special examples of van Douwen's maximal crowded spaces [5]. We show that $\beta\omega \to (Y)^1_2$ holds for some but not all such spaces. In fact the only countable examples Y that we have for when $\beta\omega \to (Y)^1_2$ does hold are those that arise in studying $\beta\omega$ as a right topological semigroup ([14]). The question seems ripe for more exploration. The key tool in the investigation is Frolík's theorem concerning fixed points of self-embeddings of compact extremally disconnected spaces combined with the methods from [3] of building partitions of ω^* to be $\omega \cup \{p\}$-avoiding.

2. P-spaces

It was shown in [3] that $\omega^* \to (Y)^1_2$ for all P-spaces of the form $\{\kappa\} \cup \{\alpha+1 : \alpha < \kappa\}$ as subspaces of the usual compact ordinal space $\kappa+1$ (where κ has uncountable cofinality). This result depends on van Douwen's result (see [8]) that every P-space of weight at most $\mathfrak{c}$ can be embedded into $\beta\omega$. This suggests a seemingly related open question.

Question 2.1. Does $\beta\omega \to (Y)^1_2$ for all P-spaces $Y \subset_h \beta\omega$ that have at most one non-isolated point?

For any space X, X_δ is often used to denote the P-space obtained by endowing the set X with the G_δ-topology. For any index set I, we use 2^I to denote the product space $\{0,1\}^I$. Since the G_δ-topology modification of a space X of weight at most $\mathfrak{c}$ will also have weight at most $\mathfrak{c}$, it follows from van Douwen's result, that $(2^{\omega_1})_\delta \subset_h \beta\omega$. Since each non-empty G_δ subset of $(2^{\omega_1})_\delta$ contains a copy of $(2^{\omega_1})_\delta$. The fact that $\beta\omega \to (Y)^1_2$ for all P-spaces Y with cardinality and weight $\aleph_1$ follows from the next two lemmas. The proof of the second is similar to the proof that every countable first-countable space embeds into $\mathbb{Q}$.

Lemma 2.1. *If $X \subset \beta\omega$, then either X contains a copy of $(2^{\omega_1})_\delta$ or $\beta\omega \setminus X$ contains a copy of some dense subset of $(2^{\omega_1})_\delta$.*

Proof. Let X be a subset of $\beta\omega$ and assume that X contains no copy of $(2^{\omega_1})_\delta$. Let h be an embedding of $(2^{\omega_1})_\delta$ into $\beta\omega$. Since X contains no copy of $(2^{\omega_1})_\delta$, $h^{-1}[X]$ does not contain any non-empty open subsets of $(2^{\omega_1})_\delta$. This implies that $h[(2^{\omega_1})_\delta] \setminus X$ does contain a copy of a dense subset of $(2^{\omega_1})_\delta$. □

Lemma 2.2. *If Y is a P-space with cardinality and weight $\aleph_1$, then Y embeds in each dense subset of $(2^{\omega_1})_\delta$.*

Proof. Let X be any dense subset of $(2^{\omega_1})_\delta$ and let $\{y_\alpha : \alpha \in \omega_1\}$ be an enumeration of Y. For each $\alpha \in \omega_1$, let $\{U(\alpha, \xi) : \xi < \omega_1\}$ be an enumeration of descending clopen local base for y_α. Since Y is a P-space, we can choose, for each $\alpha \in \omega_1$, a value $\alpha \le \zeta_\alpha < \omega_1$ large enough so that the members of the family $\{U(\beta, \zeta_\alpha) : \beta \le \alpha\}$ are pairwise disjoint, and, for all $\gamma < \alpha$, each membber of $\{U(\beta, \zeta_\alpha) : \beta \le \alpha\}$ is contained in, or disjoint from, each member of $\{U(\xi, \zeta_\gamma) : \xi \le \gamma\}$. Choose $\zeta_0 = 0$ and assume, for convenience, that $U(0, 0) = Y$.

For each $x \in X$ and $\xi \in \omega_1$, let $[x \restriction \xi] = \{z \in X : x \restriction \xi \subset z\}$. We note that $[x \restriction \xi]$ is a G_δ-subset of 2^{ω_1}, and that $\{[x \restriction \xi] : \xi \in \omega_1\}$ is a descending clopen base for x in X. We choose $x_\alpha \in X$ by recursion on $\alpha \in \omega_1$ with the intention that the mapping sending y_α to x_α will be a homeomorphism. Choose $x_0 \in X$ arbitrarily and assume that we have chosen $\{x_\beta : \beta < \alpha\} \subset X$. Our inductive hypotheses are that for $\xi \le \gamma < \beta < \alpha$, if $y_\beta \in U(\xi, \zeta_\gamma)$, then $x_\beta \in [x_\xi \restriction (\gamma + \omega)]$, and if $y_\beta \notin U(\xi, \zeta_\gamma)$, then $x_\beta \notin [x_\xi \restriction (\gamma + \omega)]$.

Now we choose x_α. Let I_α denote the set of $\gamma < \alpha$ such that $y_\alpha \in \bigcup\{U(\xi, \zeta_\gamma) : \xi \le \gamma\}$. We have that I_α is not empty since our choice of $U(0, \zeta_0)$ ensures that $0 \in I_\alpha$. For each $\gamma \in I_\alpha$, choose the unique $\beta^\alpha_\gamma \le \gamma$ such that $y_\alpha \in U(\beta^\alpha_\gamma, \zeta_\gamma)$. If $\gamma < \delta$ are both in I_α, and if $\beta^\alpha_\gamma \ne \beta^\alpha_\delta$, then $\gamma < \beta^\alpha_\delta$, since $U(\beta^\alpha_\gamma, \zeta_\gamma) \cap U(\xi, \zeta_\gamma)$ is empty for $\beta^\alpha_\gamma \ne \xi \le \gamma$. If H is a finite subset of I_α and β is the maximum element of $\{\beta^\alpha_\delta : \delta \in H\}$, then y_β is an element of $U(\beta^\alpha_\gamma, \zeta_\gamma)$ for all $\gamma \in H$. Also if $\delta \in H$ is minimal such that $\beta = \beta^\alpha_\delta$, then by the first induction hypothesis, $x_\beta \in [x_{\beta^\alpha_\gamma} \restriction (\gamma + \omega)]$ for all $\gamma \in H$. This implies that the family $\{[x_{\beta^\alpha_\gamma} \restriction (\gamma + \omega)] : \gamma \in I_\alpha\}$ has non-empty intersection in the compact space 2^{ω_1} and so we may choose a point $x_\alpha \in X \setminus \{x_\beta : \beta < \alpha\}$ in this intersection. Moreover, if we let $\delta = \sup(I_\alpha)$, then using that X is dense, we can, and do, also ensure that $x_\alpha \notin \bigcup\{[x_\beta \restriction (\delta+\omega)] : \beta < \alpha\}$. The definition of the sequence $\{\beta^\alpha_\gamma : \gamma \in I_\alpha\}$ ensured that the first inductive hypothesis is maintained. We verify the second. Choose any $\xi \leq \gamma < \alpha$ and assume that $y_\alpha \notin U(\xi, \zeta_\gamma)$. If there is a $\beta^\alpha_\gamma \neq \xi$ such that $y_\alpha \in U(\beta^\alpha_\gamma, \zeta_\gamma)$, then it follows from the induction hypotheses that $x_{\beta^\alpha_\gamma} \restriction (\gamma + \omega) \neq x_\xi \restriction (\gamma + \omega)$. Otherwise, we have that $\gamma \notin I_\alpha$. If $\gamma \leq \sup(I_\alpha)$, then our choice of x_α ensured that $x_\alpha \notin \bigcup\{[x_\xi \restriction (\gamma+\omega) : \xi \leq \gamma\}$. Otherwise, let δ be the minimum element of $I_\alpha \setminus \gamma$ and note that $y_{\beta^\alpha_\delta} \notin \bigcup\{U(\xi, \zeta_\gamma) : \xi \leq \gamma\}$ since $y_\alpha \in U(\beta^\alpha_\delta, \zeta_\delta)$ witnesses that $U(\beta^\alpha_\delta, \zeta_\delta)$ it not contained in any member of $\{U(\xi, \zeta_\gamma) : \xi \leq \gamma\}$. Again, by the induction assumption, $x_{\beta^\alpha_\delta} \notin \bigcup\{[x_\xi \restriction (\gamma + \omega)] : \xi \leq \gamma\}$ and $x_\alpha \in [x_{\beta^\alpha_\delta} \restriction (\gamma + \omega)]$.

Finally, it should be clear that the inductive hypotheses have ensured that if $\xi < \gamma \in \omega_1$, then for $\xi < \alpha \in \omega_1$:

$$y_\alpha \in U(\xi, \zeta_\gamma) \text{ iff } x_\alpha \in [x_\xi \restriction (\gamma + \omega)] \ .$$

This proves that the map sending y_ξ to x_ξ is a homeomorphism. □

Corollary 2.3. *$\beta\omega \to (Y)^1_2$ for every P-space of cardinality and weight $\aleph_1$.*

3. COUNTABLE CROWDED SPACES

A space X is extremally disconnected if the closure of every open subset is open. Extremally disconnected spaces were defined in [13, Definition 16]. If an extremally disconnected space X is regular, then βX is also extremally disconnected. Therefore $\beta\omega$ is extremally disconnected. For a Tychonoff space X and a mapping from A into a compact space K, we let f^β denote the continuous extension of f to all of βX (see [12]). Here is Frolík's important theorem about fixed points of maps on extremally disconnected spaces.

Proposition 3.1. [10] *If f is an embedding of a compact extremally disconnected space X into itself, then the set of fixed points, fix(f), of f is a clopen subset of X.*

Now we recall some useful properties of countable subspaces of $\beta\omega$. The first is shown in [14, 3.40].

Proposition 3.2. *If A, B are σ-compact subsets of $\beta\omega$ such that $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty, then A and B have disjoint closures.*

Proof. Note that $\omega \cup A \cup B$ is a σ-compact (hence Lindelöf) space in which A and B have disjoint closures. Since this space is normal, there are disjoint open sets U_A, U_B such that $A \subset U_A$ and $B \subset U_B$. Since the closures of $(U_A \cap \omega)$ and $(U_B \cap \omega)$ are disjoint clopen subsets of $\beta\omega$ containing A and B respectively, we have that A and B have disjoint closures. $\square$

Corollary 3.3. *If A, B are countable subsets of $\beta\omega$ and $x \in \overline{A} \cap \overline{B}$, then there is clopen subset U of $\beta\omega$ such that $x \in U$ and either $U \cap B \subset \overline{A}$ or $U \cap A \subset \overline{B}$.*

Proof. Let $A_1 = A \setminus \overline{B}$ and $B_1 = B \setminus \overline{A}$. By Proposition 3.2, A_1 and B_1 have disjoint closures. By symmetry, assume that x has a clopen neighborhood U of x that is disjoint from $\overline{B_1}$. It follows that $U \cap B$ is contained in $\overline{A}$. $\square$

Proposition 3.4. *If A is a countable subspace of $\beta\omega$, then A is extremally disconnected and ι_A^β (where ι_A is the identity map on A) is a homeomorphism from βA to $\overline{A}$.*

Proof. It follows from Proposition 3.2 that disjoint open subsets of A have disjoint closures in A. This is equivalent to the definition of A being extremally disconnected. Similarly, disjoint open subsets of A have disjoint closures in $\beta\omega$ and this implies that i_A^β is a 1-to-1 map. $\square$

Corollary 3.5. *If A is a countable subspace of $\beta\omega$ and h is a homeomorphism from A into $\beta\omega$, then fix(h) is a clopen subset of A.*

Proof. Let $h : A \to B \subset \beta\omega$ be a homeomorphism and assume that $\text{fix}(h)$ is not empty. Now let $f = h^\beta$ be the continuous extension of h to $\beta A = \overline{A}$ as introduced above. Clearly, by continuity, $f[\overline{A}] = \overline{B}$. By Proposition 3.4, $\overline{B} = \beta B$ and since h is a homeomorphism, f is also an embedding. By Frolík's Theorem, Proposition 3.1, $\text{fix}(f)$ is a clopen subset of $\overline{A}$. Therefore $\text{fix}(h) = A \cap \text{fix}(f)$ is a clopen subset of A. $\square$

A space A is called a vD-space (for van Douwen space) if it is regular, countable, and the crowded subsets of A correspond to the open subsets of A. A space is crowded if it has no isolated points. Van Douwen constructed such spaces and proved that the topology on a vD-space is a maximal crowded topology (see [5]).

If f is a topological embedding of a countable space A into $\beta\omega$, then we saw in Proposition 3.4 that f^β is also an embedding. A point x is a far point of a space A if $x \in \beta A \setminus A$ and is not the limit of any closed discrete subset of A.

Proposition 3.6 ([5])**.** *If A is a vD-space, then*

(1) A is Tychonoff and extremally disconnected,
(2) A is nodec (nowhere dense subsets of A are closed)

(3) *$A \cup \{x\}$ is a vD-space for each far point x of A,*
(4) *there is a 1-to-1 function f from ω into ω^* such that $A \approx f[\omega]$ and, for each $n \in \omega$, $f(n)$ is the only point p of ω^* such that $f^\beta(p) = f(n)$.*

Furthermore, if there is a map f as in (4), then A is a vD-space.

Proof. We prove the last item first. Suppose that f is an embedding of ω into ω^* and let $A = f[\omega]$. Note that $f^\beta(a) = a$, for $a \in A$, and that $f^\beta[\omega^* \setminus A]$ is disjoint from A. We have to show that A has no isolated points and that every crowded subset is open. Since $\omega^* \cap (f^\beta)^{-1}(A)$ is equal to A, it is a nowhere dense subset of ω^* and this shows that A has empty interior in $\overline{A}$. Now suppose that $B \subset A$ is crowded. Since the compact subsets of A are finite, it follows that the closure of $\overline{B} \setminus A$ contains B. Now let U be the (unique) clopen subset of $\beta\omega$ such that $f[U \cap \omega] = B$. By continuity, $f^\beta[U \setminus \omega]$ contains $\overline{B} \setminus A$, and therefore also contains B. It then follows from the hypothesis on f that B is disjoint from $f^\beta[\omega^* \setminus U]$ and that $A \subset f^\beta[\omega^*]$. This proves that B is open in A since we have that the closure of $A \setminus B$ is disjoint from B.

Next we assume that A is a vD-space and verify properties (1)-(4) Since A is a vD-space, we have that A is countable, regular crowded and a subset of A is open if and only if it is crowded. The closure of a crowded set (in any space) is crowded, and so closures of open subsets of A are also open. Since A is regular, this also shows that A is Tychonoff.

Let D be a nowhere dense subset of A. To prove that A is nodec we simply have to prove that $(A \setminus D)$ is crowded. Since $A \setminus \overline{D}$ is open it is crowded, and since it is also dense, $(A \setminus \overline{D}) \cup \{x\}$ is crowded, hence open, for each $x \in A$.

If $x \in \beta A$ is a far point of A, then $A \cup \{x\}$ is clearly regular, countable, and crowded. It is immediate that open subsets of $A \cup \{x\}$ are crowded, so assume that x is an element of a crowded $B \subset A \cup \{x\}$. We know that $A \setminus B$ and $A \cap B$ have disjoint closures in βA, and so x is not in the closure of $A \setminus B$. It follows that B is a neighborhood of x.

Let g be any bijection from ω to A and so g^β is a continuous function from $\beta\omega$ onto βA. Using Zorn's Lemma, there is a compact subset K of $\beta\omega$ which is minimal with respect to the property that $g^\beta[K] = \beta A$. Since images of proper closed subsets are proper closed subsets of βA, images of open sets have relatively dense interior. Since βA is extremally disconnected, images of disjoint clopen sets are disjoint. Therefore g^β maps K bijectively to βA. It follows that $B = (g^\beta)^{-1}[A]$ is a copy of A. Let h denote the inverse function of $(g^\beta) \restriction K$. Now let $f = h \circ g$ be the function mapping ω bijectively to B. By continuity, it follows that $f^\beta = h \circ g^\beta$, and therefore that f^β is the identity on K. Fix any $p \in \omega^* \setminus K$ and let $x = f^\beta(p)$. Choose a clopen set $p \in U$ of $\beta\omega$ disjoint from K. It follows that $f^\beta[\omega \setminus U]$ is dense in $K = f^\beta[K] \subset f^\beta\overline{[\omega \setminus U]}$. Therefore $f^\beta[U \cap \omega]$ is not open in B, and so, by item (2), it is discrete and closed. It now follows that $f^\beta(p)$ is not a point in B. $\square$

We now answer Questions 1.1 (3) and (4) from above (and from [3]).

Theorem 3.7. *There is a countable crowded space Y such that $\beta\omega \to (Y)^1_2$.*

The proof follows from the next three. The first is surely folklore.

Proposition 3.8. *Every non-empty open subset of a countable crowded regular homogeneous space X is homeomorphic to X.*

Proof. Let X be a countable crowded regular homogeneous space. Let $U = \{u_n : n \in \omega\}$ be a non-empty open subset of X, and let $X = \{x_n : n \in \omega\}$ be an enumeration of X. By induction on n, choose a sequence $\{U_n, V_n : n \in \omega\}$ of clopen subsets of X such that, for each n,

(1) $U_n \approx V_n$,
(2) $u_n \in \bigcup_{k\leq n} U_k \subsetneq U$, and $x_n \in \bigcup_{k\leq n} V_k \subsetneq X$,
(3) $\{U_k : k \leq n\}$ and $\{V_k : k \leq n\}$ are each pairwise disjoint families.

Assume that we have chosen $\{U_k : k < n\}$ and $\{V_k : k < n\}$ as above. Let m be minimal such that $u_m \in U \setminus \bigcup_{k<n} U_k$ and let j be minimal such that $x_j \in X \setminus \bigcup_{k<n} V_k$. Let h be any homeomorphism of X such that $h(u_m) = x_j$. Choose a clopen U_n such that $u_m \in U_n \subsetneq U \setminus \bigcup_{k<n} U_k$ and $V_n = h[U_n]$ is a proper subset of $X \setminus \bigcup_{k<n} V_k$. Clearly it follows that $\bigcup_n U_n$ is homeomorphic to $\bigcup_n V_n$. $\square$

Lemma 3.9. *If A is a homogeneous vD-space, then $A \to (A)^1_2$.*

Proof. If $A = A_0 \cup A_1$ then at least one of A_0 or A_1 contains a crowded subset U. By Proposition 3.8, U is a copy of A. $\square$

Now we need an example of a homogeneous vD-space. A variety of vD-spaces were constructed in [7, 8] but none were constructed that were homogeneous. Homogeneous extremally disconnected spaces were discovered in [4] but they were not vD-spaces. Remarkably homogeneous vD-spaces arose in the study of the semi-group structure of $\beta\omega$. The interested reader is referred to the influential [14]. It will be more convenient to temporarily work with the discrete space $\mathbb{Z}$ (of the set of all integers) rather than ω. Briefly, the operation $+$ is defined on

$\beta\mathbb{Z}$ by extending the successor map σ on $\mathbb{Z}$ to all of $\beta\mathbb{Z}$ with σ^β. For an ultrafilter q on $\mathbb{Z}$ and $n \in \mathbb{Z}$, $n+q$ is defined to be $(\sigma^\beta)^n(q)$. This defines a function ρ_q from $\mathbb{Z}$ into $\beta\mathbb{Z}$ where $\rho_q(n) = n+q$, and for $p \in \mathbb{Z}^*$, $p+q = \rho_q^\beta(p)$. Since σ^β is an autohomeomorphism of $\mathbb{Z}^*$, it is clear that $\{n+q : n \in \mathbb{Z}\}$ is a homogeneous subspace of $\beta\mathbb{Z}$. A point $q \in \mathbb{Z}^*$ is an idempotent if $q+q=q$, and right maximal idempotents are discussed in [14, 2.12]. Finally, a right maximal idempotent q is strongly right maximal if $p+q=q$ implies that $p=q$. This is equivalent to having that $(\rho_q)^\beta$ is the extension of the 1-to-1 function from $\mathbb{Z}$ onto $A = \{n+q : n \in \mathbb{Z}\}$ that has the properties required in item (4) of Proposition 3.6. All of this is shown in [14, 9.11,9.15,9.17].

Proposition 3.10 ([14])**.** *There is a homogeneous vD-space. In fact, there are $2^{\mathfrak{c}}$ many pairwise non-homeomorphic homogeneous vD-spaces.*

Naturally this raises more questions about non-homogeneous vD-spaces. We only have results for non-homogeneous vD-spaces that contain a non-clopen copy of a homogeneous vD-space. In particular we investigate if $\beta\omega \to (E \cup \{x\})^1_2$ for x a far point of a homogeneous vD-space $E \subset \beta\omega$.

Definition 3.11. *For a homogeneous vD-space E, let $\mathcal{H}_E$ denote the set of embeddings of E into ω^*. For $A \subset \omega^*$, let $\mathcal{H}_E(A) = \{h \in \mathcal{H}_E : h[E] \subset A\}$.*

For any far point x of E, let $\mathcal{H}_{E,x}(A) = A \cup \{h^\beta(x) : h \in \mathcal{H}_E(A)\}$. By recursion on $0 < \alpha \leq \omega_1$, let $\mathcal{H}^\alpha_{E,x}(A) = \bigcup_{\eta<\alpha} \mathcal{H}_{E,x}(\mathcal{H}^\eta_{E,x}(A))$.

Say that a point x of a space X is an E-point of X if $x \in A \subset X$ for some $A \approx E$. For the remainder of this section E denotes a homogeneous vD-space. The first result is nearly immediate.

Theorem 3.12. *If x is a far point of E with the property that $h^\beta(x)$ is not an E-point of $\beta\omega$ for any $h \in \mathcal{H}_E$, then $Y = E \cup \{x\}$ is a vD-space satisfying that $\beta\omega \not\to (Y)^1_2$.*

Proof. Let X_0 be the set of E-points of $\beta\omega$ and let $X_1 = \beta\omega \setminus X_0$. Let $h \in \mathcal{H}_E$. By assumption $h[E] \subset X_0$ and $h^\beta(x) \notin X_0$. ☐

It is not known to follow from ZFC that each vD-space has far points. We postpone the discussion of the existence of far points of homogeneous vD-spaces until the next section. We make the following definition to help with the analysis of which far points x of a vD-space E might satisfy that $\beta\omega \to (E \cup \{x\})^1_2$.

Theorem 3.13. *If x is a far point of E such that there is an $A \approx E$ and an odd integer k such that $A \cap (\mathcal{H}^k_{E,x}(A) \setminus \mathcal{H}^{k-1}_{E,x}(A)) \neq \emptyset$, then $\beta\omega \to (E \cup \{x\})^1_2$.*

Proof. Choose A as in the statement and let c be any 2-coloring of $\overline{A}$. Since A is homogeneous, we may, by passing to a clopen subset of A, assume that c is constant on A. By symmetry assume that $c(A) = 0$. Note that if $c(y) = 0$ for any $y \in \mathcal{H}_{E,x}(A)$, then c is constant on a copy of $E \cup \{x\}$. Similarly, by induction on $j \leq k$, and $y \in (\mathcal{H}^{j+1}_{E,x}(A) \setminus \mathcal{H}^{j}_{E,x}(A))$, we must have that $c(y) = 1$ if and only if j is even. However, for the even number $j = k - 1$, we obtain an $a \in (\mathcal{H}^{j+1}_{E,x}(A) \setminus \mathcal{H}^{j}_{E,x}(A))$ with $c(a) = 0$. □

Say that a point y of ω^* is an (E, x)-point if there is an $h \in \mathcal{H}_E$ such that $y = h^\beta(x)$.

Definition 3.14. *A set $Y \subset \omega^*$ will be said to be (E, x)-open if for each (E, x)-point $y \in Y$, there is an $h \in \mathcal{H}_E(Y)$ such that $h^\beta(x) = y$. A coloring $c : Y \to 2$ is $(E \cup \{x\})$-free if there is no monochromatic copy of $E \cup \{x\}$.*

Proposition 3.15. *The union of any family of (E, x)-open sets is again (E, x)-open. In addition, if Y is (E, x)-open, then $\mathcal{H}_{E,x}(Y)$ is (E, x)-open.*

Corollary 3.16. *If Y is (E, x)-open, then $\mathcal{H}^{\omega_1}_{E,x}(Y)$ is also (E, x)-open.*

Lemma 3.17. *If $h_1, h_2 \in \mathcal{H}_E$ and $y = h_1^\beta(x) = h_2^\beta(x)$, then there is an $h \in \mathcal{H}_E$ such that $h^\beta(x) = y$ and $h[E] \subset h_1[E] \cap h_2[E]$.*

Proof. The map sending y to itself, and $h_1(e)$ to $h_2(e)$ (for $e \in E$) is an embedding of $h_1[E] \cup \{y\}$ to $h_2[E] \cup \{y\}$ and so, by Lemma 3.1, $U = \{y\} \cup \mathrm{fix}(h_2 \circ h_1^{-1})$ is a relative clopen neighborhood of y in $\{y\} \cup h_1[E]$. Choose any relatively clopen $W \subset U$ such that $y \in W$ and $U \setminus W$ is not empty. Then $h_1^{-1}[U] = h_2^{-1}[U]$ and $h_1^{-1}[W]$ is a proper clopen subset of E. There is an $h \in \mathcal{H}_E$ such that $h(e) = h_1(e)$ for all $e \in h_1^{-1}[W]$ and $h(e) \in U \setminus W$ for all $e \in E \setminus h_1^{-1}[W]$. Clear $h[E] \subset U \setminus \{y\} \subset h_1[E] \cap h_2[E]$ and $h^\beta(x) = y$ as required. □

Corollary 3.18. *If Y is (E, x)-open and if $y = h^\beta(x) \in Y$ for some $h \in \mathcal{H}_E$, then $\{y\} \cup (Y \cap h[E])$ contains a copy of $E \cup \{x\}$.*

Lemma 3.19. *If Y is (E, x)-open and c is an $E \cup \{x\}$-free 2-coloring of Y, then c extends uniquely to an $(E \cup \{x\})$-free 2-coloring of $\mathcal{H}^{\omega_1}_{E,x}(Y)$.*

Proof. We prove by induction on $\alpha < \omega_1$, that c extends uniquely to an $(E \cup \{x\})$-free 2-coloring, c_α, of $\mathcal{H}^{\alpha}_{E,x}(Y)$. If $\alpha = \eta + 1$, and $h \in \mathcal{H}_E(\mathcal{H}^\eta(Y))$, then, since $E \cup \{x\}$ is a vD-space, there is a clopen $U \subset E$ such that $x \in \overline{U}$ and c_η is constant on $h[U]$. If $y = h^\beta(x)$ is not in $\mathcal{H}^\eta(Y))$, then define $c_\alpha(y) = 1 - c_\eta(u)$ for any $u \in U$. It then follows by induction and Proposition 3.18 that c_α is $(E\cup\{x\})$-free. If α is a limit, then we just have to check that $c_\alpha = \bigcup_{\eta<\alpha} c_\eta$ is (E,x)-free. Suppose that $h\in\mathcal{H}_E$ and $h^\beta(x) = y \in \mathcal{H}^\alpha_{E,x}(Y)$, then, there is an $\eta<\alpha$ such that y is in the (E,x)-open set $\mathcal{H}^\eta_{E,x}(Y)$. By Proposition 3.18 and by induction assumption, c_η assumes 2 colors on $\{y\}\cup(h[E]\cap\mathcal{H}^\eta_{E,x}(Y))$. □

Lemma 3.20. *If each point $y\in\omega^*$ is an element of an (E,x)-open set that has an $(E\cup\{x\})$-free 2-coloring, then $\beta\omega\not\to(E\cup\{x\})^1_2$.*

Proof. Let $\{y_\alpha : \alpha\in 2^{\mathfrak{c}}\}$ be a well-ordering of ω^*. For each α, let $y_\alpha\in N_\alpha$ be an (E,x)-open set with an $E\cup\{x\}$-free 2-coloring d_α.

Let $c_0\supset d_0$ be the unique $(E\cup\{x\})$-free 2-coloring on $Y_0 = \mathcal{H}^{\omega_1}_{E,x}(N_0)$ as per Lemma 3.19. By Lemma 3.16, Y_0 is also (E,x)-open. Let $\alpha<2^{\mathfrak{c}}$, and assume that for $\eta<\alpha$, we have defined an (E,x)-open set Y_η and an $E\cup\{x\}$-free 2-coloring c_η of Y_η so that for $\gamma<\eta<\alpha$, $N_\gamma\cup\mathcal{H}^{\omega_1}_{E,x}(Y_\gamma)\subset Y_\eta$ and $c_\gamma\subset c_\eta$.

If α is a limit ordinal, then set $Y_\alpha=\bigcup_{\eta<\alpha} Y_\eta$ and $c_\alpha=\bigcup_{\eta<\alpha} c_\eta$. By Lemma 3.16, Y_α is (E,x)-open and, clearly, c_α is a 2-coloring. We have to check that c_α is $(E\cup\{x\})$-free. Assume that $h\in\mathcal{H}_E(Y_\alpha)$ and that $y=h^\beta(x)\in Y_\alpha$. Choose $\eta<\alpha$ so that $y\in Y_\eta$. Since Y_η is (E,x)-open, by Proposition 3.18, $\{y\}\cup(h[E]\cap Y_\eta)$ contains a copy of $E\cup\{x\}$ and so it is 2-colored by c_η.

Now assume that $\alpha=\eta+1$ and that Y_η and c_η have been chosen. We simply define Y_α to be $\mathcal{H}^{\omega_1}_{E,x}(Y_\eta)\cup N_\eta$, and we now have to define c_α. By Lemma 3.19, c_α extends uniquely to an $(E\cup\{x\})$-free 2-coloring, c'_α, on $\mathcal{H}^{\omega_1}_{E,x}(Y_\eta)$. Let $D_\eta = N_\eta\setminus Y'_\eta$ and define $c_\alpha = c'_\alpha\cup(d_\eta\restriction D_\eta)$. To check that c_α is (E,x)-free, we consider any $h\in\mathcal{H}_E(Y_\alpha)$. By Proposition 3.18, we may assume that $h^\beta(x)=d$ is in D_η. Since d is not in $\mathcal{H}^{\omega_1}_{E,x}(Y_\eta)$, it is not in the closure of $h[E]\cap\mathcal{H}^{\omega_1}_{E,x}(Y_\eta)$. Therefore then $\{d\}\cup(h[E]\cap D_\eta)$ contains a copy of $E\cup\{x\}$, which means it is 2-colored by d_η. □

Now we improve Theorem 3.12.

Theorem 3.21. *If E is a homogeneous vD-space and x is a far point of E which is not an E-point in $\overline{E}$, then $\beta\omega\not\to(E\cup\{x\})^1_2$.*

Proof. By Lemma 3.20, it suffices to prove that each $y\in\omega^*$ is an element of an (E,x)-open set, N_y, with an (E,x)-free 2-coloring. If y is not an (E,x)-point, then $N_y=\{y\}$ is such a set. Similarly, if there is an $h\in\mathcal{H}_E$ such that $h^\beta(x)=y$ and no point of $h[E]$ is an (E,x)-point, then $N_y=\{y\}\cup h[E]$ is such a set. Otherwise we inductively construct N_y as the union of a family $\{y\}\cup\{D_n : n\in\omega\}$. Let σ be a function from $\omega \setminus \{0\}$ onto $\omega \times \omega$ satisfying that $\sigma(n) \in n \times \omega$ for each $n > 0$ Let $D_0 = \{d(0,m) : m \in \omega\} \approx E$ be a sequence of (E,x)-points chosen so that $y \in \mathcal{H}_{E,x}(D_0)$. By induction on $n > 0$, we choose $D_n = \{d(n,m) : m \in \omega\} \approx E$ so that

(1) for $i < n$, $D_n \cap \overline{D_i}$ is empty,
(2) there is a unique $(i_n, j_n) \in n \times \omega$ such that $d(i_n, j_n) \in \mathcal{H}_{E,x}(D_n)$,
(3) for $(i_n, j_n) \neq (i,j) \in n \times \omega$, $d(i,j) \notin \overline{D_n}$
(4) if $d(\sigma(n))$ is an (E,x)-point, then there $i \leq n$ such that $d(i,j)$ is in $\mathcal{H}_{E,x}(D_i)$.

Note that it follows from item (2) and the fact that $y \in \overline{D_0}$, that y is not in $\overline{D_n}$ for $n > 0$. Assume we have chosen $\{D_i : i < n\}$ satisfying the inductive conditions and choose minimal $\ell_n \in \omega$ such that $d(\sigma(\ell_n))$ is an (E,x)-point while $d(\sigma(\ell_n)) \notin \mathcal{H}_{E,x}(D_i)$ for all $i < n$. There is such a value for ℓ_n by condition (2) and the fact that each point of D_0 is an (E,x)-point. Let $(i_n, j_n) = \sigma(\ell_n)$. By condition (1), choose a clopen set U of $\beta\omega$ such that $d(i_n, j_n) \in U$ and $U \cap \overline{D_i} = \emptyset$ for $i < i_n$. For each $i_n < i < n$, $d(i_n, j_n) \notin \overline{D_i}$ so we may also assume that $U \cap \overline{D_i}$ is empty for each $i_n < i$. When choosing $D_n \subset U$ this ensures that $d(i,j) \notin \overline{D_n}$ for all $(i,j) \in n \times \omega$ with $i \neq i_n$. Choose an $A \approx E$ so that there is an $h \in \mathcal{H}_E(A)$ with $h^\beta(x) = d(i_n, j_n)$. We may assume that $A = h[E] \subset U$. Since D_{i_n} is a copy of E and $(h^\beta)^{-1}(D_{i_n} \cap \overline{A})$ contains x, it follows from the assumption that x is not an E-point in $\overline{E}$ that $D_{i_n} \cap \overline{A}$ is discrete. By choosing $D_n \subset A$, we can arrange that $\overline{D_n} \cap D_{i_n} = \{d(i_n, j_n)\}$. This completes the construction.

It is straightforward to check that $N_y = \{y\} \cup \bigcup_n D_n$ is (E,x)-open. Let $c(y) = 1$ and $c[D_0] = 0$. We define a coloring $c \restriction D_n$ by induction on $n > 0$. For each n, (i_n, j_n) as in the construction satisfies that $i_n < n$. Define $c[D_n] = 1 - c(d(i_n, j_n))$. We show that c is (E,x)-free. Assume that $h \in \mathcal{H}_E(N_y)$ and that $h^\beta(x) \in N_y$. If $h^\beta(x) = y$, then $\{y\} \cup (h[E] \cap D_0)$ contains a copy of $E \cup \{x\}$. Otherwise, $h^\beta(x) = d(i,j) \in D_i$ for some $i \in \omega$, and $d(i,j)$ is evidently and (E,x)-point. With $\sigma(n) = d(i,j)$, there is, by condition (4), an $m \leq n$ such that $d(i,j) = d(i_m, j_m) \in \mathcal{H}_{E,x}(D_m)$. By Proposition 3.18, $h[E] \cap D_m$ is not empty and $\{d(i,j)\} \cup (h[E] \cap D_m)$ is 2-colored by c. □

Theorem 3.22. *If E is a homogeneous vD-space and x is a far point of E, then $\beta\omega \not\to (E \cup \{x\})^1_2$ if $\mathcal{H}_{E,x}(E)$ contains a nowhere dense copy of E.*

Proof. Again, by Lemma 3.20, it suffices to prove that each $y \in \omega^*$ is an element of an (E,x)-open set, N_y, with an (E,x)-free 2-coloring. If y is not an (E,x)-point, then $N_y = \{y\}$ is such a set. Otherwise, we show that we can choose an $A_x \approx E$ such that $x \in A_x$ and $\overline{A_x} \cap E = \emptyset$. By the hypothesis of the Theorem, there is an $A \approx E$ contained in $\mathcal{H}_{E,x}(E) \setminus E$ and such that $\overline{A}$ is nowhere dense in $\overline{E}$. We note that $D = \overline{A} \cap E$ is discrete and, since each point of $\mathcal{H}_{E,x}(E)$ is a far point of E, $\overline{D}$ is disjoint from A. Replace A by any crowded subset of A whose closure is disjoint from $\overline{D}$. There is a Fix any $x' \in A$ and $h \in \mathcal{H}_E(E)$ such that $h^\beta(x) = x'$.

Since $(E \cup \{x\}) \cap A_x = \{x\}$ it follows that $E \cup \{x\}$ is not homeomorphic to a subspace of E.

Claim 1. For each relatively clopen U of A_x, there is a relatively clopen E_U of E such that $\mathcal{H}_{E,x}(E_U) \cap A_x = U$.

Proof of Claim. By Proposition 3.2, U and $A_x \setminus U$ have disjoint closures in $\overline{E}$. Therefore we can choose a clopen $E_U \subset E$ such that $\overline{E_U} \cap A_x = U$. It is clear from the definition that $\mathcal{H}_{E,x}(E_U)$ is contained in $\overline{E_U}$, hence $\mathcal{H}_{E,x}(E_U) \cap A_x \subset U$. Since $\overline{A_x} \cap E$ is empty, we may choose a clopen $W \subset E_U$ such that $\overline{W} \cap A_x$ is empty. Fix any $u \in U$ and $h_u \in \mathcal{H}_{E,x}(E)$ such that $h_u^\beta(x) = u$. Since $\overline{E_U} \setminus \overline{W}$ is a clopen neighborhood of u in $\overline{E}$, $U_2 = E \cap (h^\beta)^{-1}(\overline{E_U} \setminus \overline{W})$ is a clopen subset of E with x in its closure. Also, $E \setminus U_2$ is homeomorphic to W so there is an $h \in \mathcal{H}_{E,x}(E_U)$ such that $h \restriction U_2 = h_u \restriction U_2$. This proves that $U \subset \mathcal{H}_{E,x}(E_U)$. □

Claim 2. If $D \approx E$ and each point of D is an (E, x)-point, then there is an $A \subset \mathcal{H}_{E,x}(E)$, such that $A \approx E$, $A \subset \overline{A} \subset (\overline{E} \setminus E)$, and there is an $h \in \mathcal{H}_E$ such that $D = h^\beta[A]$.

Proof of Claim. For each $d \in D$, choose an $h_d \in \mathcal{H}_E$ such that $h_d^\beta(x) = d$. By Proposition 3.1, we can, by simply shrinking the domain of h_d, arrange that $h_d[A_x] \subset D$ is a relatively clopen neighborhood of d in D. In addition, since there is no copy of $E \cup \{x\}$ in E, d is not in the closure of $h_d[E] \cap D$. Therefore we may also arrange that $h_d[E] \cap D$ is empty. Fix an enumeration $\{d_n : n \in \omega\}$ of D. For each n, Let $U_0 = h_0^\beta[A_x] \subset D$. Of course we have that $d_0 \in U_0$. Since U_0 and $D \setminus U_0$ have disjoint closures in $\beta\omega$, we can choose clopen $W_0 \subset \beta\omega$ such that $W_0 \cap D = U_0$. Set $D_{1,0} = W_0 \cap h_0[E]$. It follows easily that A_x is contain in the closure of $h_0^{-1}[D_{1,0}]$ (a clopen subset of E). By shrinking the domain of h_0, as in Claim 1, we can simply assume that $h_0[E] \subset W_0$ and still have that $U_0 = h_0^\beta[A_x]$. Choose n_1 minimal so that $d_{n_1} \notin U_0$. Working within $\beta\omega \setminus W_0$, we repeat the process and find $U_1, D_{1,1}, W_1$ such that W_1 is a clopen subset of $\beta\omega \setminus W_0$, $d_{n_1} \in U_1 = D \cap W_1$ and, with $h_1 = h_{d_{n_1}}$, $h_1[E] \subset W_1$. Continuing recursively, for at most ω steps, we have $\{W_k : k \in \omega\}$, pairwise disjoint clopen subsets of $\beta\omega$ whose union covers D, and for each k, an $h_k \in \mathcal{H}_E$ such that $h_k[E] \subset W_k$ and $h_k^\beta[A_x] = U_k = D \cap W_k$. It follows easily that $D_1 = \bigcup\{h_k[E] : k \in \omega\}$ is a copy of E and that $D \subset \mathcal{H}_{E,x}(D_1)$ and $\overline{D} \subset \overline{D_1} \setminus D_1$. Given any $h \in \mathcal{H}_E$ such that $h[E] = D_1$, simply set $A = (h^\beta)^{-1}[D]$. $\square$

We are ready to define N_y in Case 1. By recursion, choose a sequence $\{D_n : n \in \omega\}$ and $\{h_n : n \in \omega\} \subset \mathcal{H}_E$ so that, for each $n > 0$,

(1) $y = h_0^\beta(x)$ and $D_0 = h_0[E]$,
(2) D_n is a clopen subset of $h_n[E]$,
(3) D_n^0, the set of (E,x)-points of D_n, is a relatively clopen set,
(4) if D_n^0 is empty, then D_{n+1} is empty,
(5) $\overline{A_n} \subset \overline{E} \setminus E$ where $A_n = (h_n^\beta)^{-1}[D_{n-1}^0]$.

To begin the construction, choose any $h_0 \in \mathcal{H}_E$ so that $h_0^\beta(x) = y$. By removing a closed nowhere dense set, we may assume that D_0^0 is clopen (and change h_0 accordingly). Now let $n > 0$ and assume we have chosen $\{h_k, D_k : k < n\}$. If D_{n-1}^0 is empty, we may stop. Otherwise, we apply Claim 2 to $D = D_{n-1}^0$ and choose $h_n \in \mathcal{H}_E$ so that $(h_n^\beta)^{-1}[D_{n-1}^0] = A_n$ and $D_n = h_n[E]$ are as in the statement of Claim 2. Again, by possibly removing a nowhere dense subset of D_n, we may assume that D_n^0 is clopen.

We check that $c(y) = 1$, $c[\bigcup_k D_{2k}] = 0$, and $c[\bigcup_k D_{2k+1}] = 1$ is an (E,x)-free coloring of $N_y = \{y\} \cup \bigcup_k D_k$. Let $h \in \mathcal{H}_E(N_y)$ and assume that $h^\beta(x) \in N_y$. If $h^\beta(x) = y$, then, by Proposition 3.1, $h[E] \cap D_0$ is not empty, and so $h[E] \cup \{y\}$ is not mono-chromatic. Similarly, if $h^\beta(x) \in D_k$, then $h^\beta(x) \in D_k^0$, and again, by Proposition 3.1, $h[E] \cap D_{k+1}$ is not empty. It follows that $c(h^\beta(x)) \neq c(d)$ for $d \in h[E] \cap D_{k+1}$. $\square$

We complement Theorem 3.22 with this next result. We do not know if the assumption of being relatively closed can be dropped. However we do show in Lemma 4.10 that the hypothesis is not vacuous.

Theorem 3.23. *If E is a homogeneous vD-space and x is a far point of E such that, for each $A \approx E$, $A \cap \mathcal{H}_{E,x}(E)$ is discrete and relatively closed in $E \cup \mathcal{H}_{E,x}(E)$ then $\beta\omega \not\to (E \cup \{x\})^1_2$.*

Proof. By Lemma 3.20, it suffices to prove that each $y \in \omega^*$ is an element of an (E,x)-open set, N_y, with an (E,x)-free 2-coloring. If y is not an (E,x)-point, then $N_y = \{y\}$ is such a set. Otherwise choose any $D_0 = h_0[E]$ for some $h_0 \in \mathcal{H}_E$ such that $h_0^\beta(x) = y$. If no point of D_0 is an (E,x)-point, then $\{y\} \cup D_0$ is (E,x)-open.

Let $\{t_\ell : \ell \in \omega\}$ be an $<$-preserving enumeration of $\omega^{<\omega}$. For each $\emptyset \neq t \in \omega^{<\omega}$, let t^- denote the immediate predecessor of t. Also let $\prec$ be any well-ordering of E in type ω, and we assume whenever we choose any $D \approx E$, D acquires a similar well-ordering $\prec_D$. We inductively choose a family $\{D_t : t \in \omega^{<\omega}\}$ such that, for each t, D_t is empty or $D_t \approx E$. For brevity, when $D_t \approx E$, the ordering $\prec_{D_t}$ is denoted as $\prec_t$. We have chosen $D_\emptyset$, and let t denote t_1. We let d_t be the $\prec_\emptyset$ member of $D_\emptyset$ that is an (E,x)-point. Choose $D_t \approx E$ so that d_t is the unique point of $\mathcal{H}_{E,x}(D_t) \cap D_0$. In addition, d_t is not in the closure of $D_1 \cap \mathcal{H}_{E,x}(D_0)$, so we may simply assume that $D_1 \cap \mathcal{H}_{E,x}(D_0)$ is empty.

By induction on $j < \ell$, assume that we have chosen pairwise disjoint $\{D_{t_j} : j < \ell\}$ and points $\{d_{t_j} : j < \ell\}$ so that, for $i < j < \ell$,

(1) d_{t_j} is the $\prec_{(t_j^-)}$-least (E,x)-point of $D_{t_j^-} \setminus \{d_{t_k} : k < j\}$ if one exists, and $d_{t_j} = y$ otherwise,
(2) if $d_{t_j} \neq y$, then $D_{t_j} \approx E$ and $d_{t_j} \in \mathcal{H}_{E,x}(D_{t_j})$,
(3) if $d_{t_j} = y$, then $D_{t_j} = \emptyset$,
(4) $\mathcal{H}_{E,x}(D_{t_j}) \cap D_{t_i} \subset \{d_{t_j}\}$,
(5) $\mathcal{H}_{E,x}(D_{t_i}) \cap D_{t_j}$ is empty.

To choose D_{t_ℓ}, assume there is an (E,x)-point in $D_{t_\ell^-} \setminus \{d_{t_j} : j < \ell\}$, and let d_{t_ℓ} be the $\prec_{t_\ell}$-least. Let $h_\ell \in \mathcal{H}_E$ be chosen so that $h_\ell^\beta(x) = d_{t_\ell}$. Let $A_\ell = h_\ell[E]$. We can assume that $A_\ell \cap D_{t_j}$ is empty, for each $j < \ell$, because d_{t_ℓ} is not in $\mathcal{H}_{E,x}(D_{t_j})$. Since d_{t_ℓ} is a far point of A_ℓ, it is not in the closure of $A_\ell \cap \mathcal{H}_{E,x}(D_{t_j})$ for all $j < \ell$. Therefore, by shrinking the domain of h_ℓ we can ensure that $A_\ell \cap \mathcal{H}_{E,x}(D_{t_j})$ is empty for all $j < \ell$. Similarly, the discrete set $\mathcal{H}_{E,x}(A_\ell) \cap D_{t_j}$ does not have d_{t_ℓ} as a limit and so we can assume that $D_{t_\ell} \subset A_\ell$ is a copy of E with d_{t_ℓ} as limit and satisfies that $\mathcal{H}_{E,x}(D_{t_\ell}) \cap D_{t_j} \subset \{d_{t_\ell}\}$ for all $j < \ell$.

Having completed the construction, we define the coloring c of $N_\ell = \{y\} \cup \bigcup_\ell D_{t_\ell}$ where $c(y) = 1$ and $c[D_{t_\ell}] = 0$ if $|\operatorname{dom}(t_\ell)|$ is even, otherwise $c[D_{t_\ell}] = 1$. Suppose $h \in \mathcal{H}_E(N_y)$ and $h^\beta(x) \in D_{t_j}$. It follows from the construction that there is an ℓ with $d_{t_\ell} = h^\beta(x)$ and $t_\ell^- = t_j$. By Proposition 3.1, we have that $h[E] \cap D_{t_\ell}$ is not empty. Therefore $h^\beta(x) \cup h[E]$ is not a monochromatic copy of $E \cup \{x\}$. $\square$

4. FAR POINTS

Now we must discuss if there any far points of a vD-space E. It would be interesting to know more about βA for A a homogeneous vD-space. There is a natural, and useful, candidate for what its homeomorphism type. The space $E(2^{\mathfrak{c}})$ is defined as the unique extremally disconnected space that maps irreducibly onto the product space $2^{\mathfrak{c}}$. It is often called the Gleason cover or Iliadis absolute of $2^{\mathfrak{c}}$ ([11]). One can construct (a copy) of $E(2^{\mathfrak{c}})$ as a subspace of $\beta\omega$ as follows. Since $2^{\mathfrak{c}}$ is separable, there is a continuous map f of $\beta\omega$ onto $2^{\mathfrak{c}}$. By Zorn's Lemma, there is a closed subset K of $\beta\omega$ satisfying that $f[K] = 2^{\mathfrak{c}}$ while $f[J] \neq 2^{\mathfrak{c}}$ for all closed $J \subset K$. This subspace K is a copy of $E(2^{\mathfrak{c}})$. It was shown in [8] that there are vD-spaces that occur naturally as dense subsets of $E(2^{\mathfrak{c}})$. Moreover, it is known that if q is a minimal idempotent of ω^*, then the closure of $\{n + q : n \in \omega\}$ is homeomorphic to $E(2^{\mathfrak{c}})$ ([1]). We note that there are examples of vD-spaces A for which $\beta A \not\approx E(2^{\mathfrak{c}})$ (see [7]).

We recall the Rudin-Keisler (see [2, p410]) and the Rudin-Frolik relations on ultrafilters.

Definition 4.1. *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on ω.*

(1) $\mathcal{U} <_{RK} \mathcal{V}$ if there is a function $f \in \omega^\omega$ and an $e \in E$ such that $f^\beta(e) = \mathcal{V}$,

(2) $\mathcal{U} <_{RF} \mathcal{V}$ if there is an embedding f of ω into $\beta\omega$ such that $f^\beta(\mathcal{U}) = \mathcal{V}$.

Let $\mathcal{R}_E$ denote the set of ultrafilters $\mathcal{U}$ such that $\mathcal{U} <_{RK} e$ for some $e \in E$.

If S is a discrete subset of $\beta\omega$ and $\mathcal{U} \in \omega^*$, a point x is said to be a $\mathcal{U}$-limit of S if there is a 1-to-1 function $f : \omega \to S$ such that $f^\beta(\mathcal{U}) = x$. Of course, we then have $\mathcal{U} <_{RF} x$. It is well-known that if $\mathcal{U} <_{RF} \mathcal{V}$, then $\mathcal{U} <_{RK} \mathcal{V}$ and that $\mathcal{R}_E$ has cardinality at most $\mathfrak{c}$.

A point p of ω^* is a weak P-point if p is not the limit of any countable subset of ω^*. A subset K of ω^* is a weak P-set if $\overline{D} \cap K$ is empty for all countable subsets D of $\omega^* \setminus K$. More generally, a subset K of a space X is a weak P-set if $\overline{D}$ is disjoint from K for all countable $D \subset X$. We make some simple useful observations about weak P-sets.

Proposition 4.2. *If K is a weak P-set of a Tychonoff space X and $K_1 \subset K$ is a weak P-set of K, then K_1 is a weak P-set of X. If f is a continuous map from a compact space X onto a space Y, then the pre-image of each weak P-set of Y is a weak P-set of X.*

Proof. Let K be a weak P-set of X and let $K_1 \subset K$ be a weak P-set of K. Let D be a countable subset of $X \setminus K_1$. Now $\overline{D} \cap K_1$ is empty because each of $\overline{D \setminus X}$ and $\overline{D \cap X}$ are disjoint from K.

Now assume that f maps the compact space X onto Y and suppose that $K \subset Y$ is a weak P-set of Y. Let D be a countable subset of $\omega^* \setminus f^{-1}[K]$. Since $\overline{f[D]}$ is disjoint from K and $f[\overline{D}] \subset \overline{f[D]}$, it follows that $\overline{D}$ is disjoint from $f^{-1}[K]$. $\square$

We will need to use the techniques for constructing weak P-points from [15] and [16]. The reader may well wonder if the effort is worth it, but most of the extra effort would not be needed in the likely event that Theorem 3.23 could be improved (see Question 4.1). However once we started down the rabbit hole, it seemed important to have a result about an example of the form $E \cup \{x\}$ where x was an E-point. Kuen [15] proved that weak P-points exist and this was improved in [19] as follows.

Proposition 4.3. *The set of weak P-points of ω^* contains a set of cardinality $2^{\mathfrak{c}}$ consisting of pairwise $<_{RK}$-incomparable ultrafilters.*

Corollary 4.4. *There is a pairwise disjoint family $\mathcal{K}$ consisting of infinite compact weak P-set subsets of ω^*.*

Proof. Let $f : \omega \to \omega$ be any finite-to-one map with the property that $f^{-1}(n)$ has cardinality greater than n for all $n \in \omega$. As usual, f^β denotes the Stone-Čech extension mapping $\beta\omega$ onto $\beta\omega$. Evidently, $(f^\beta)^{-1}(p)$ is an infinite compact subset of ω^* for all $p \in \omega^*$. For each weak P-point $p \in \omega^*$, $(f^\beta)^{-1}(p)$ is easily seen to be a weak P-set. □

This next result is proven in [18, 4.1.5]

Proposition 4.5 (van Mill)**.** *Each compact extremally disconnected continuous image of ω^* is homeomorphic to some weak P-set in ω^*.*

The construction of weak P-points and weak P-sets utilizes the existence of $\mathfrak{c} \times \mathfrak{c}$-independent matrices as introduced in [15]. For more general topological constructions, we have seen that it is useful to reformulate this with the help of continuous mappings onto large powers of the 1-point compactification, $\mathfrak{c}+1$, of the discrete space $\mathfrak{c}$. For a function s from a finite subset of $\mathfrak{c}$ into $\mathfrak{c}$, we let $[s]$ denote the clopen subset of $(\mathfrak{c}+1)^{\mathfrak{c}}$ consisting of all those full functions that extend s.

This next result can be seen as a topological reformulation of the Boolean algebraic result [18, 4.1.2] (which we state without proof).

Proposition 4.6. *There is a mapping φ from ω^* onto $\omega^* \times (\mathfrak{c}+1)^{\mathfrak{c}}$.*

For the remainder of the section, we fix a map φ as in Proposition 4.6, and we let π denote the composition of φ with the projection onto the first coordinate copy of ω^*. Using the methods of [6, 18] we improve Proposition 4.5.

Lemma 4.7. *Suppose that $K \subset \omega^*$ is an infinite compact weak P-set. Then, for any separable extremally disconnected space E, there is a copy A of E such $\varphi[A] \subset K \times (\mathfrak{c}+1)^{\mathfrak{c}}$, $\overline{A}$ is a weak P-set, and no two points of $\overline{A}$ are RK-equivalent.*

Proof. Let $K \subset \omega^*$ be an infinite weak P-set. It follows from Proposition 4.2 that $K \times (\mathfrak{c}+1)^{\mathfrak{c}}$ is a weak P-set in $\omega^* \times (\mathfrak{c}+1)^{\mathfrak{c}}$. Let X be the φ pre-image of $K \times (\mathfrak{c}+1)^{\mathfrak{c}}$, and note that, by Proposition 4.2, X is a weak P-set in ω^*. Proposition 4.2 also shows that any subset of X that is a weak P-set in X will also be a weak P-set in ω^*. We will construct $A \subset X$ so that $\overline{A}$ is a weak P-set in X. Choose any copy E_1 of E contained in K. and fix an enumeration $\{y_n : n \in \omega\}$ of E_1. By induction on $\xi < \mathfrak{c}$, we choose a closed subset X_ξ of X and a set $I_\xi \subset \mathfrak{c}$ with $\bigcup_{\eta<\xi} I_\eta \subset I_\xi$ and $|I_\xi| \leq |\xi + \omega|$ such that φ maps X_ξ to a set that projects onto $\overline{E_1} \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_\xi}$. Of course $I_0 = \emptyset$ and X_0 is $X \cap \pi^{-1}(\overline{E_1})$. We ensure that the family $\{X_\xi : \xi < \mathfrak{c}\}$ is a descending sequence. At limit ξ, X_ξ is the intersection of the family $\{X_\eta : \eta < \xi\}$ and $I_\xi = \bigcup_{\eta<\xi} I_\eta$. By compactness, for limit ordinals ξ, $\varphi[X_\xi]$ will project onto $\overline{E_1} \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_\xi}$. When the construction is complete, we let $X_{\mathfrak{c}} = \bigcap\{X_\xi : \xi \in \mathfrak{c}\}$. In this recursive construction we have three distinct tasks so we let ι be any function from $\mathfrak{c}$ onto $\{1, 2, 3\}$ so that each of $\iota^{-1}(1), \iota^{-1}(2)$, and $\iota^{-1}(3)$ have cardinality $\mathfrak{c}$. We fix enumerations of the clopen subsets of X, ω-sequences of clopen subsets of X, and the set of all permutations, h, of ω. Let $\{W_\xi : \xi \in \iota^{-1}(1)\}$ enumerate $CO(X)$ (the clopen subsets of X). Let $\{h_\xi, U_\xi : \xi \in \iota^{-1}(2)\}$ enumerate the pairs (h, U) where h is a permutation on ω and U is a clopen subset of $\overline{E_1}$. Finally, let $\{\{U_n^\xi : n \in \omega\} : \xi \in \iota^{-1}(3)\}$ enumerate all countable families of pairwise disjoint elements of $CO(X)$ so that each such family is enumerated $\mathfrak{c}$ many times.

Now assume that $\{X_\gamma, I_\gamma : \gamma < \xi\}$ has been chosen as described above in such a way that these next three Claims govern how $X_{\gamma+1}$ was chosen according to the value of $\iota(\gamma)$. We recall that for limit γ, X_γ is defined to be $\bigcap\{X_\eta : \eta < \gamma\}$ and $I_\gamma = \bigcup\{I_\eta : \eta < \gamma\}$. We also reiterate that the inductive hypothesis that must be preserved is that $\varphi[X_\gamma]$ must project onto $\overline{E_1} \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_\gamma}$.

This next claim ultimately ensures that $\pi \restriction X_{\mathfrak{c}}$ is 1-to-1, and therefore a homeomorphism.

Claim 3. If $\iota(\xi) = 1$, then $X_{\xi+1}$ can be chosen so that for each $y \in \overline{E_1}$, $\pi^{-1}(y) \cap X_{\xi+1} \subset W_\xi$ or $\pi^{-1}(y) \cap X_{\xi+1}$ is disjoint from W_ξ.

Proof of Claim. For each $y \in \overline{E_1}$, let $F_{\xi,y} = \pi^{-1}(y) \cap X_\xi$. Assume s is a finite function from $\mathfrak{c} \setminus I_\xi$ into $\mathfrak{c}$ such that $F_{\xi,y}$ is disjoint from $W_\xi \cap \pi^{-1}[s]$ (respectively $\pi^{-1}[s] \setminus W_\xi$). Then there is a clopen $K_y \subset \overline{E_1}$ such that $y \in K_y$ and $K_y \times [s]$ is disjoint from $\pi[W_\xi]$ (respectively $\pi[X_\xi \setminus W_\xi]$). Choose n_0 minimal so that there is an s_0 such that $F_{\xi,y_{n_0}}$ is either disjoint from $W_\xi \cap \pi^{-1}[s_0]$ or from $\pi^{-1}[s_0] \setminus W_\xi$. Choose $n_1 > n_0$

minimal so that there is an s_1 extending s_0 such that such that $F_{\xi,y_{n_1}}$ is either disjoint from $W_\xi \cap \pi^{-1}[s_1]$ or from $\pi^{-1}[s_1] \setminus W_\xi$. Continue choosing the sequences $\{n_\ell : \ell \in \omega\}$ and $\{s_\ell : \ell \in \omega\}$. If we allow repetitions, we may assume, for convenience, that this sequence is infinite. Let L_0 be the set of ℓ such that $F_{\xi,y_{n_\ell}}$ is disjoint from $W_\xi \cap \pi_{-1}[s_{n_\ell}]$, and thus for $\ell \in L_1 = \omega \setminus L$, $F_{\xi,y_{n_\ell}}$ is disjoint from $\pi^{-1}[s_{n_\ell}] \setminus W_\xi$. For each $\ell \in L_0$, there is a clopen subset K_ℓ of K such that $y_{n_\ell} \in K_\ell$ and $\pi^{-1}[K_\ell \times [s_{n_\ell}]]$ is disjoint from W_ξ. Similarly, for $\ell \in L_1$, there is a K_ℓ such that $\pi^{-1}[K_\ell \times [s_{n_\ell}]]$ is contained in W_ξ. Since $\overline{E_1}$ is extremally disconnected, the closure, U_0, of $\bigcup\{K_\ell : \ell \in L_0\}$ is a clopen subset of $\overline{E_1}$. Let ρ be the function $\bigcup\{s_{n_\ell} : \ell \in \omega\}$ and set $X_{\xi+1}$ be the union of $U_0 \times [\rho]$ and $W_\xi \cap \big((\overline{E_1} \setminus U_0) \times [\rho]\big)$, where $[\rho] = \bigcap\{[s_{n_\ell}] : \ell \in \omega\}$. It is clear that, for all $y \in U_0$, $\pi^{-1}(y) \cap X_{\xi+1}$ is disjoint from W_ξ, and for $y \in \overline{E_1} \setminus U_0$, $\pi^{-1}(y) \subset W_\xi$. Now set $I_{\xi+1} = I_\xi \cup \mathrm{dom}(\rho)$ and we prove that $\pi[X_{\xi+1}]$ projects onto $\overline{E_1} \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_{\xi+1}}$. It suffices to prove that $\pi[X_{\xi+1}]$ projects onto the dense set $\bigcup_n \{y_n\} \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_{\xi+1}}$. A further reduction is that it suffices to show that, for each $n \in \omega$, $F_{\xi,y_n} \cap X_{\xi+1} \cap \pi^{-1}[s]$ is not empty for each finite function s from $\mathfrak{c} \setminus I_{\xi+1}$ into $\mathfrak{c}$ is not empty. We note that, by the induction hypothesis and compactness, $F_{\xi,y} \cap [\rho] \cap [s]$ is not empty. For $n = n_\ell$ ($\ell \in L_0$) $F_{\xi,y_n} \cap X_{\xi+1} \cap \pi^{-1}[s]$ is equal to $F_{\xi,y_n} \cap [\rho] \cap \pi^{-1}[s]$, and so is not empty. For $n \notin \{n_\ell : \ell \in L_0\}$, we have that, for any $\ell > n$, since $n \notin L_0$, $F_{\xi,y_n} \cap [s'] \cap W$ is not empty for all finite functions s' extending s_{n_ℓ}. Therefore it follows that $F_{\xi,y_n} \cap [\rho] \cap [s] \cap W_\xi$ is not empty. □

This next claim, for values of $\xi < \mathfrak{c}$ with $\iota(\xi) = 2$, is the step that will ensure that if h is a permutation of ω and $x \in X_{\mathfrak{c}}$ either $h^\beta(x) = x$ or $h^\beta(x) \notin X_{\mathfrak{c}}$.

Claim 4. If $\iota(\xi) = 2$ and then there is a choice for $X_{\xi+1}$ and $I_{\xi+1}$ such that $h^\beta_\xi[\pi^{-1}(U_\xi)] \setminus \pi^{-1}(U_\xi)$ is disjoint from $X_{\xi+1}$.

Proof of Claim. Let $h = h_\xi$ and $U = U_\xi$. Choose any clopen $\tilde{U} \subset \omega^*$ such that $\tilde{U} \cap \overline{E_1} = U$. Fix any $\zeta \in \mathfrak{c} \setminus I_\xi$ and for each $\alpha \in \mathfrak{c}$, let O_α be the clopen set $\varphi^{-1}(\tilde{U} \times [\langle \zeta, \alpha \rangle])$. Since h is a permutation, it follows that $\tilde{W}_\alpha = X_\xi \setminus h^\beta[\varphi^{-1}([\langle \zeta, \alpha \rangle])]$ is a clopen subset of X_ξ. We note that $h^\beta[\pi^{-1}(U) \cap \varphi^{-1}([\langle \zeta, \alpha \rangle]) \cap X_\xi] \setminus \pi^{-1}(U)$ is disjoint from $\tilde{W}_\alpha$.

For each $y \in \overline{E_1} \setminus U$, we again let $F_{\xi,y}$ denote the set $\pi^{-1}(y) \cap X_\xi$. Similar to Claim 3, we choose, if possible, values $n_0 \in \omega$, $\alpha_0 < \mathfrak{c}$, and a finite function s_{n_0} from $\mathfrak{c} \setminus (I_\xi \cup \{\zeta\})$ such that $F_{\xi,y_{n_0}} \cap \varphi^{-1}[s_{n_0}]$ is disjoint from $\tilde{W}_{\alpha_0}$. Continue choosing, when possible, $n_{\ell+1}$ with $y_{n_{\ell+1}} \notin U$, $\alpha_{\ell+1}$

and $s_{\ell+1} \supset s_{n_\ell}$, so that $F_{\xi,y_{n_{\ell+1}}} \cap \varphi^{-1}[s_{\ell+1}]$ is disjoint from $\tilde{W}_{\alpha_{\ell+1}}$. Let $\rho = \bigcup\{s_\ell : \ell \in \omega\}$ and $[\rho] = \bigcap\{[s_\ell] : \ell \in \omega\}$.

Now choose $\gamma \in \mathfrak{c} \setminus \{\alpha_\ell : \ell \in \omega\}$ and we show that $F_{\xi,y_n} \cap [\rho] \cap [s']$ meets $\tilde{W}_\gamma$ for all finite functions s' from $\mathfrak{c} \setminus (I_\xi \cup \{\zeta\} \cup \mathrm{dom}(\rho))$ and all $y_n \in K_1 \setminus U$. If such a set is empty, then there is an ℓ such that $n = n_\ell$. However, then $F_{\xi,y_n} \cap [\rho]$ is contained in $X_\xi \setminus \tilde{W}_{\alpha_\ell} \subset \tilde{W}_\gamma$. We are ready to define $X_{\xi+1}$ and $I_{\xi+1}$. We set $I_{\xi+1} = I_\xi \cup \{\zeta\} \cup \mathrm{dom}(\rho)$, and $X_{\xi+1}$ is $X_\xi \cap \varphi^{-1}(U \times [\langle\zeta,\gamma\rangle])$ union $X_\xi \cap \tilde{W}_\gamma \cap \pi^{-1}(K_1 \setminus U)$. It should be clear that $\varphi[X_{\xi+1}]$ projects onto $\overline{E_1} \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_{\xi+1}}$. As mention above, by the definition of $\tilde{W}_\gamma$, $h^\beta[X_\xi \cap \varphi^{-1}(U \times [\langle\zeta,\gamma\rangle])]$ is disjoint from $\tilde{W}_\gamma \cap \pi^{-1}(\overline{E_1} \setminus U)$. □

Now we adapt the steps from [6, 16] (fundamentally Kunen's original method [15] to construct what he called $\mathfrak{c}$-OK points) to intertwine in the recursive construction so as to ensure that $X_\mathfrak{c}$ is a weak P-set of X.

Claim 5. If $\iota(\xi) = 3$ and $\bigcup_{n\in\omega} U^\xi_n$ is disjoint from X_ξ, then $X_{\xi+1}$ can be chosen so that $\overline{D} \cap X_{\xi+1}$ is empty for any countable $D \subset \bigcup_{n\in\omega} U^\xi_n$.

Proof of Claim: If $\bigcup_{n\in\omega} U^\xi_n$ is not disjoint from X_ξ, then simply let $X_{\xi+1} = X_\xi$ and $I_{\xi+1} = I_\xi$. Otherwise, let L be the set $\ell \in \omega$ such that there is a pair $y_\ell \in \tilde{W}_\ell \in CO(\omega^*)$ and, finite function s_ℓ from $\mathfrak{c} \setminus I_\xi$ into $\mathfrak{c}$ so that $\varphi^{-1}(\tilde{W}_\ell \times [s_\ell])$ is disjoint from $X_\xi \cap \overline{\bigcup_n U^\xi_n}$. For each $\ell \in L$, choose $\tilde{W}_\ell$ and s_ℓ as indicated. For each $\ell \in L$, let $F_\ell = X_\xi \cap \varphi^{-1}(\tilde{W}_\ell \times [s_\ell])$. We note that we have that each of $\overline{\bigcup_n U_n} \cap \bigcup_{\ell\in L} F_\ell$ and $\bigcup_n U_n \cap \overline{\bigcup_{\ell\in L} F_\ell}$ is empty. By Lemma 3.2, the closed set $F_\xi = \overline{\bigcup_{\ell\in L} F_\ell}$ is disjoint from $\overline{\bigcup_n U^\xi_n}$. Let us also note that $\overline{\bigcup_{\ell\in L} \overline{E_1} \cap \tilde{W}_\ell}$ is a clopen subset of $\overline{E_1}$. Now we choose any countably infinite set $\{\alpha_n : n \in \omega\} \subset \mathfrak{c} \setminus I_\xi$ such that $\bigcup_{\ell\in L} \mathrm{dom}(s_\ell) \subset \{\alpha_n : n \in \omega\}$. For each $\eta \in \mathfrak{c}$, choose a clopen $Z_\eta \subset \beta\omega$ satisfying that $F_\xi \subset Z_\eta$, and, for all $n \in \omega$, $Z_\eta \cap U^\xi_n = U^\xi_n \cap (\bigcup_{k\le n} \varphi^{-1}([(\alpha_k,\eta)]))$. It follows that such a clopen set Z_η exists by applying Lemma 3.2 to the pair $A = F_\xi \cup \bigcup_n \big(U^\xi_n \cap (\bigcup_{k\le n} \varphi^{-1}([(\alpha_k,\eta)]))\big)$ and $B = \bigcup_n U^\xi_n \setminus \big(\bigcup_{k\le n} \varphi^{-1}([(\alpha_k,\eta)])\big)$. Now we define $X_{\xi+1}$ to be $X_\xi \cap \bigcap\{Z_\eta : \eta \in \mathfrak{c}\}$ and $I_{\xi+1} = I_\xi \cup \{\alpha_n : n \in \omega\}$. It remains to prove that $\varphi[X_{\xi+1}]$ projects onto $\overline{E_1} \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_{\xi+1}}$. Since $X_{\xi+1} \supset F_\xi$ and $\bigcup_{\ell\in L} \mathrm{dom}(s_\ell) \subset I_{\xi+1}$, we claim it is clear that, for each $\ell \in L$, this projection contains $\tilde{W}_\ell \times (\mathfrak{c}+1)^{\mathfrak{c}\setminus I_{\xi+1}}$. Now assume that W is clopen subset of ω^* that meets $\overline{E_1}$ and that is disjoint from $\bigcup_{\ell\in L} \tilde{W}_\ell$. Also let s be any finite function from $\mathfrak{c} \setminus I_{\xi+1}$ into $\mathfrak{c}$, and let H be any finite subset of $\mathfrak{c}$. We have to prove that $\varphi^{-1}[W \times [s]]$ meets $X_\xi \cap \bigcap_{\eta\in H} Z_\eta$. Let $\{\eta_i : i < m\}$ be any enumeration of H

and let y_j be any element of $W \cap E_1$. Let s_1 be the function from $\{\alpha_i : i < m\}$ onto H where $s_1(\alpha_i) = \eta_i$. Since $j \notin L$, we were not able to choose a pair $\tilde{W}_j, s_j$, hence it follows that $\varphi^{-1}[W \times ([s_1] \cap [s])]$ meets $\overline{\bigcup_n U_n^\xi} \cap X_\xi = \overline{\bigcup_{n>m} U_n^\xi} \cap X_\xi$. For each $n > m$, $\varphi^{-1}[[s_1]] \cap U_n^\xi$ is a subset of $U_n^\xi \cap \bigcap_{\eta \in H} Z_\eta$ and so $\varphi^{-1}[W \times ([s_1] \cap [s])]$ meets $X_\xi \cap \bigcap_{\eta \in H} Z_\eta$. Having completed the construction of $X_{\xi+1}$, we now check that it has the desired property. Let D be any countable subset of $\bigcup_n U_n^\xi$. For each $m \in \omega$, there is a countable subset S_m of $\mathfrak{c}$ such that, for all $\eta \in \mathfrak{c} \setminus S_m$, $\varphi^{-1}([(\alpha_m, \eta)]) \cap D$ is empty. Choose any $\eta \in \mathfrak{c} \setminus \bigcup_m S_m$, and observe that $Z_\eta \cap D$ is empty. $\square$

This completes the proof of the Lemma. $\square$

This next result shows that a homogeneous vD-space has far points.

Lemma 4.8. *If E is a homogeneous vD-space then there is an infinite discrete set $D \subset \overline{E}$ such that $E \cup D \approx E$ and every point of $\overline{D}$ is a far point of E.*

Proof. Let e be any point of E and note that $E \setminus \{e\}$ is homeomorphic to E, and by Proposition 3.6, e is a far point of $E \setminus \{e\}$. Therefore, by homogeneity, E itself has far points d satisfying that $E \cup \{d\} \approx E$. Fix any partition, $\{E_n : n \in \omega\}$ of E into non-empty clopen sets, and for each $n \in \omega$, let d_n be a far point of E_n such that $E_n \cup \{d_n\} \approx E$. Then $D = \{d_n : n \in \omega\}$ is a discrete subset of $\overline{E}$, $E \cup D \approx E$, and $\overline{D} \cap E = \emptyset$. We check that each point of $\overline{D}$ is a far point of E. Let S be any discrete subset of E and note that $\overline{S} \cap D$ is empty. Therefore, by Proposition 3.2, $\overline{S} \cap \overline{D}$ is empty. $\square$

Now we prove a result in connection to Theorem 3.21.

Lemma 4.9. *If E is a homogeneous vD-space, then there is a far point x of E that is not an E-point, and $\beta\omega \not\to (E \cup \{x\})^1_2$.*

Proof. Fix a discrete set $D \subset \overline{E}$ as in Lemma 4.8. Let $\mathcal{U}$ be a weak P-point of ω^* that is not an element of $\mathcal{R}_E$ (from Definition 4.1). Let x be a $\mathcal{U}$-limit of D. Assume that $A \approx E$ and that $x \in A$ and fix $h \in \mathcal{H}_E$ with $A = h[E]$. Since $\mathcal{U} \notin \mathcal{R}_E$, it follows that x is not a limit point of $\overline{A} \cap D$. This is because $e = h^{-1}(x)$ is not a $\mathcal{U}$-limit of the discrete set $(h^\beta)^{-1}[D \cap \overline{A}]$. But now by Proposition 3.2, we must have that x is a limit point of $\overline{D} \cap A$. This however contradicts that x is a weak P-point of $\overline{D} \setminus D$ and a far point of A. $\square$

Now we show that the hypothesis of Theorem 3.23 can hold.

Lemma 4.10. *If E is a homogeneous vD-space, then there is a far point x of E such that x is an E-point in $\overline{E}$ and $\beta\omega \not\to (E\cup\{x\})^1_2$.*

Proof. Fix a discrete set $D\subset \overline{E}$ as in Lemma 4.8. Also let $\{U_d : d\in D\}$ be a pairwise disjoint family of clopen subsets of $\overline{E}$ such that $d\in U_d$ for each $d\in D$. Since $\overline{D}\cap E$ is empty, we can assume that $E\subset\bigcup\{U_d : d\in D\}$. We will choose an x and $x\in A_x\approx E$ such that $A_x\subset\overline{D}$ and $\overline{A_x}$ is a weak P-set of $\overline{D}\setminus D$. Let $\mathcal{W}_x\in\omega^*$ denote the ultrafilter that will be used to select x as a $\mathcal{W}_x$-limit of D. In order to decide on our choice of A_x, and the resulting possible values for $\mathcal{W}_x$, we analyze the possible behavior of discrete subsets of $\mathcal{H}_{E,x}(E)$. We want to choose $\mathcal{W}_x$ in such a way that if $\{x_n : n\in\omega\}$ is a discrete subset of $\mathcal{H}_{E,x}(E)$, then the closure of $\{x_n : n\in\omega\}$ is disjoint from $E\cup D$. Since each x_n will be a $\mathcal{W}_x$-limit of a discrete subset of $\overline{E}$, we consider which choices of $\mathcal{W}_x$ may fail to have our desired property.

Assume that $e\in E\cup D$ is the limit of a discrete set $\{x_n : n\in\omega\}\subset\overline{E}$ and that, for some $\mathcal{W}$, each x_n is a $\mathcal{W}$-limit of a discrete $D_n\subset\overline{E}$. We may choose, $\{W_n : n\in\omega\}$, pairwise disjoint clopen subsets of $\overline{E}$ such that $x_n\in W_n$ and, by shrinking D_n, such that $D_n\subset W_n$. Similarly, for each n, there is a family $\{W_{n,m} : m\in\omega\}$ of pairwise disjoint clopen subsets of W_n such that, for each $d\in D_n$, there is an m such that $d\in W_{n,m}$. Evidently, the sequence $\{E\cap W_{n,m} : n,m\in\omega\}$ and the neighborhood trace of e determines $\mathcal{W}$ and so there are at most $\mathfrak{c}$ many such $\mathcal{W}$.

Let ρ be a bijection from ω onto D. We will use the family $\mathcal{K}$ from Corollary 4.4. For each $K\in\mathcal{K}$, we apply Lemma 4.7 to choose $A_K\approx E$ so that $\overline{A_K}$ is a weak P-set of ω^* such that $\pi[A_K]\subset K$ and such that no distinct points of $\overline{A_K}$ are RK-equivalent. For each $K\in\mathcal{K}$, let $\mathcal{W}(K)$ be any element of A_K (of course $\mathcal{W}(K)\in\omega^*$). Now we consider the homeomorphism ρ^β from ω^* onto $\overline{D}\setminus D$. Since $\mathcal{H}_E(E)$ has cardinality $\mathfrak{c}$, the set $Y=\bigcup\{h^\beta[D]\cup(h^\beta)^{-1}[D] : h\in\mathcal{H}_E(E)\}$ has cardinality $\mathfrak{c}$. Let $\mathcal{K}_Y$ be the set of cardinality $2^{\mathfrak{c}}$, consisting of those $K\in\mathcal{K}$ such that $\rho^\beta[\overline{A_K}]$ is disjoint from Y. Now choose $K\in\mathcal{K}_Y$ so that $\mathcal{W}(K)$ is not RK-equivalent to any of those $\mathfrak{c}$ many $\mathcal{W}$ that arise as a partition accumulating to some $e\in E\cup D$. Then $\rho^\beta(\mathcal{W}(K))=x$ is our desired far point and it is an element of $A_x=\rho^\beta[A_K]\approx E$. By our choice of $\mathcal{W}_x$, we have ensured that if $\{x_n : n\in\omega\}$ is a discrete subset of $\mathcal{H}_{E,x}(E)$, then the closure of $\{x_n : n\in\omega\}$ is disjoint from $E\cup D$.

Claim 6. For each $y\in\mathcal{H}_{E,x}(E)$ and $y\in A\approx E$, there is an $h\in\mathcal{H}_E(E)$ and a clopen $U\subset\overline{E}$ with $y\in U$ such that $U\cap A\subset h^\beta[A_x]$ and $U\cap h^\beta[A_x]$ is nowhere dense in $\overline{E}$.

Proof of Claim: Let $y \in A \approx E$ and let $h \in \mathcal{H}_E(E)$ be chosen so that $h^\beta(x) = y$. As stated just before the Claim, $y \notin E$. Let $A_y = h^\beta[A_x]$. Since $h[E]$ is a crowded subset of E, it is an open subset of E and $\overline{h[E]}$ is a clopen subset of $\overline{E}$. Since E is homogeneous, there is a homeomorphism g from A_y to A satisfying that $g(y) = y$. By Corollary 3.5, there is a clopen subset U of $\overline{E}$ such that $y \in U$ and every point of $U \cap A_y$ is a fixed point of g. This implies that $U \cap A_y$ is a relatively clopen subset of A. By shrinking U we can assume that $U \subset \overline{h[E]}$. Since $\overline{A_y}$ is disjoint from $h[E]$, it follows that $\overline{A_y \cap U}$ is a nowhere dense subset of $\overline{E}$. □

Claim 7. $\mathcal{H}_{E,x}(E) \cap \overline{A_x}$ is simply $\{x\}$.

Proof of Claim: Let $h \in \mathcal{H}_E(E)$ and assume that $x \neq h^\beta(x) = y \in \overline{A_x}$. Let $D_y = h^\beta[D]$, hence y is the $\mathcal{W}_x$-limit of D_y. Let $D_y = D_1 \cup D_2 \cup D_3$ where $D_1 = D_y \cap D$, $D_2 = D_y \cap (\overline{D} \setminus D)$ and $D_3 = D_y \setminus \overline{D}$. Since distinct points of $\overline{A_K}$ are not RK-equivalent, y is not the $\mathcal{W}_x$-limit of any subset of D, hence $y \notin \overline{D_1}$. Also $\overline{D_2} \cap \overline{A_x}$ is empty because $\overline{A_x}$ is a weak P-set of $\overline{D} \setminus D$ that was chosen to be disjoint from Y and D_2 is a subset of Y. Therefore it remains that y is in the closure of D_3. Let $D_4 = \overline{D_3} \cap D$. It follows from Corollary 3.3 that y is in the closure of D_4. Since $\overline{D_3} \subset h^\beta[\overline{E}]$ we may choose $D_5 \subset \overline{E}$ such that $h^\beta[D_5] = D_4$. Of course this means that $D_5 \subset (h^\beta)^{-1}[D] \subset Y$. Moreover, $D_5 \subset (\overline{D} \setminus D)$ since $h^\beta[D_5] \subset (\overline{h^\beta[D]} \setminus h^\beta[D])$. Also, x should be a limit point of D_5 since $y = h^\beta(x)$ is a limit point of $h^\beta[D_5]$. However, our contradiction is that x is not a limit of D_5 since D_5 is disjoint from the weak P-set $\overline{A_x}$. □

We finish the proof, using Theorem 3.23, by proving that, for each $A \approx E$, $\mathcal{H}_{E,x}(E) \cap A$ is relatively closed and discrete in $E \cup \mathcal{H}_{E,x}(E)$. Let $A \approx E$ and let $y \in \mathcal{H}_{E,x}(E) \cap A$. Choose $h \in \mathcal{H}_E(E)$ and clopen $U \subset \overline{E}$ as in Claim 6. Choose, by continuity, a clopen $W \subset \overline{E}$ such that $h[W \cap E] = U \cap h[E]$ and note that $x \in \overline{W \cap E}$. It also follows that $U \cap h^\beta[A_x] \subset \overline{U \cap h[E]} = h^\beta[W]$. We prove that $\mathcal{H}_{E,x}(E) \cap (U \cap A)$ is equal to $\{y\}$. Let z be any point in $\mathcal{H}_{E,x}(E) \cap A \cap U$ and choose $w \in W \cap A_x$ such that $h^\beta(w) = z$. Choose an $h_z \in \mathcal{H}_E(E)$ such that $h_z^\beta(x) = z$. Now, by Lemma 3.17, there is an $h_3 \in \mathcal{H}_E$ such that $h_3^\beta(x) = z$ and $h_3[E] \subset h[E] \cap h_z[E]$. Evidently, $h_4 = h^{-1} \circ h_3 \in \mathcal{H}_E(E)$ and $h_4^\beta(x) = w$. By Claim 7, $w = x$, and so $z = y$.

By Claim 6, we have now shown that $\mathcal{H}_{E,x}(E) \cap A$ is discrete. It is clearly closed if it is finite so assume that $\{x_n : n \in \omega\} = \mathcal{H}_{E,x}(E) \cap A$. By the statement just before Claim 6, we also know that $\{x_n : n \in \omega\}$

has no limit points in $E \cup D$. Now we show that $\{x_n : n \in \omega\}$ has no limit points in $\mathcal{H}_{E,x}(E)$ which will finish the proof. Let $y \in \mathcal{H}_{E,x}(E)$ and choose $h_y \in \mathcal{H}_E(E)$ such that $h_y^\beta(x) = y$. Again let $A_y = h_y^\beta[A_x]$ and $D_y = h_y^\beta[D]$. Since $h_y[E]$ is a crowded subset of E, it is an open subset of E. Let $U = \overline{h_y[E]}$ and note that U is a clopen subset of $\overline{E}$ that is a neighborhood of y. Since we want to prove that y is not a limit point of $\{x_n : n \in \omega\}$ we may as well assume that $\{x_n : n \in \omega\}$ is a subset of $U \setminus \{y\}$.

We first prove that $z_n = (h_y^\beta)^{-1}(x_n) \in \mathcal{H}_{E,x}(E)$ for all $n \in \omega$. Fix any n and choose $h_n \in \mathcal{H}_E(E)$ such that $h_n^\beta(x) = x_n$. Let $A = h_n[E] \cap h_y[E]$ and note that A is an open subset of $h_n[E]$. Therefore $h_n[E] \setminus \overline{A}$ and $h_y[E]$ are disjoint crowded, hence open, subsets of E. Since $x_n \in \overline{h_y[E]}$, it follows that $x_n \notin \overline{h_n[E] \setminus \overline{A}}$. This proves that $x_n \in \overline{A}$. Since x_n is a far point of $h_n[E]$ it is not in the closure of $h_n[E] \cap (\overline{A} \setminus A)$. Therefore we can choose a clopen $W \subset E$ such that $x \in \overline{W}$ and $h_n[W] \subset A$. As usual, we can, by shrinking W, arrange that $A \setminus h_n[W]$ is also homeomorphic to E. Now we can choose another $g_n \in \mathcal{H}_E(E)$ so that $g_n(e) = h_n(e)$ for $e \in W$ and $g_n[E \setminus W] \subset A \setminus h_n[W]$. Then $h_y^{-1} \circ g_n \in \mathcal{H}_E(E)$ and satisfies that $(h_y^{-1} \circ g_n)^\beta(x) = (h_y^\beta)^{-1}(x_n)$. This completes the proof that z_n is in $\mathcal{H}_{E,x}(E)$.

Therefore $\{z_n : n \in \omega\}$ is a discrete subset of $\mathcal{H}_{E,x}(E) \setminus \{x\}$ and so its closure is disjoint from $D \cup E$. Also $x \notin \overline{\{z_n : n \in \omega\}}$ and so, by Claim 7, $B = \overline{\{z_n : n \in \omega\}}$ is disjoint from $\overline{A_x}$. Since $\overline{B} \cap D$ is empty, it follows from Lemma 3.3 that x is not in the closure of $B \setminus \overline{D}$. However we also have that x is not in the closure of $B \cap (\overline{D} \setminus D)$ since $\overline{A_x}$ is a weak P-set of $\overline{D} \setminus D$ and $B \cap \overline{A_x}$ is empty. Now applying the homeomorphism h_y^β we have that y is not in the closure of $\{x_n : n \in \omega\}$ as required. □

We close with questions.

Question 4.1. Can Theorem 3.23 be proven when we drop the hypothesis of relative discreteness?

Question 4.2. If Y is a non-homogeneous vD-space that is the union of two homogeneous vD-spaces, does $\beta\omega \to (Y)^1_2$?

Question 4.3. Is there a non-homogeneous vD-space Y such that $\beta\omega \to (Y)^1_2$?

Question 4.4. If q is a strongly right maximal idempotent of $(\beta\omega, +)$, is the closure of $\{n + q : n \in \omega\}$ homeomorphic to $E(2^{\mathfrak{c}})$?

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