

DENSELY k -SEPARABLE COMPACTA ARE DENSELY SEPARABLE

ALAN DOW AND ISTVÁN JUHÁSZ

ABSTRACT. A space has σ -compact tightness if the closures of σ -compact subsets determines the topology. We consider a dense set variant that we call densely k -separable. We consider the question of whether every densely k -separable space is separable. The somewhat surprising answer is that this property, for compact spaces, implies that every dense set is separable. The path to this result relies on the known connections established between π -weight and the density of all dense subsets, or more precisely, the cardinal invariant $\delta(X)$.

1. INTRODUCTION

In [1], Arhangel'skiĭ and Stavrova explored generalizations of the usual notion of tightness. In particular, they introduced the cardinal invariant $t_k^*(X)$ of a space X , which, in the countable case has been called σ -compact tightness. It was proven that in the class of compact spaces this invariant corresponded to the tightness, $t(X)$, of X . They raised the question of whether σ -compact tightness implied tightness in all Hausdorff spaces. Some partial results were obtained in [3] but the question remains open. This, and other generalizations of tightness were considered in [6]. The authors of [6] formulated a related concept, that we will call densely k -separable, and posed (unpublished) the problem of whether every compact space that was densely k -separable was, in fact, separable. We answer this question and discover the, unexpectedly strong, answer that such spaces will actually have countable π -weight. The definition (1.1) is analogous to $t_k^*(X)$ but we adopt the notation $\delta_k(X)$ because it also bears similarities to the cardinality invariant $\delta(X)$ which is the supremum of the densities of dense subsets, and the proof of our main result uses the main result from [7] that $\delta(X) = \pi(X)$ for compact spaces X (see also [4, 8]). In the next section we establish some basic properties of $\delta_k(X)$ that we will need for the proof of the main result. In the final section we only consider the case that $\delta_k(X)$ is countable and we prove the main result. Throughout the paper, when we refer to a topological space we intend that the space be a regular Hausdorff space. A subset W of a space X is regular closed if W is equal to the closure of its interior $\text{int}_X(W)$. We will work with regular closed sets rather than their interiors, i.e. regular open sets, since regular closed subsets of compact spaces are compact.

For a space X , the compact covering number, $\text{cov}_k(X)$, is the least cardinal τ such that X can be covered by τ many compact sets. In [1], the value of $t_k^*(X)$ is the least cardinal τ satisfying that the closure of every subset A of X is equal to

the union of the closures of subsets Y of A satisfying that $\text{cov}_k(Y) \leq \tau$. Following that theme we introduce dense set variations.

Definition 1.1. For a space X we set

$$d_k(X) = \min\{\text{cov}_k(Y) : Y \text{ is a dense subset of } X\}$$

and then

$$\delta_k(X) = \sup\{d_k(Y) : Y \text{ is a dense subset of } X\}.$$

Then the natural problem to ask is under what conditions on a space X do we have $\delta_k(X) = \delta(X)$? It is easy to find examples of σ -compact non-separable spaces which witness $d_k(X) < d(X)$ can hold. We do not know any example of a space X for which $\delta_k(X) < \delta(X)$ and raise it as an open problem.

Question 1. Is there a regular Hausdorff space X such that $\delta_k(X) < \delta(X)$?

It is natural to call a space X *k-separable* if $d_k(X) \leq \aleph_0$ and *densely k-separable* if $\delta_k(X) \leq \aleph_0$. The question, posed by Jan van Mill, that motivated this paper was whether every compact space X with $\delta_k(X) = \aleph_0$ is separable. We succeed in establishing the stronger result that compact densely k -separable spaces have countable π -weight. The π -weight, $\pi(X)$, of a space X is the minimum cardinality of a π -base for X . A π -base for a space X is a family of non-empty open sets that has the property that every non-empty open subset of X includes a member of that family. A local π -base at a point x is a family of non-empty open sets with the property that every neighborhood of the point includes a member of that family. One of the main elements of the proof is the connection to $\delta(X)$ mentioned above. It is shown in [7] that for compact Hausdorff space X , $\delta(X) = \pi(X)$ and we will need the following stronger result from [7].

Proposition 1.2 ([7]). For each compact space X , there is a dense subset Y of X such that $d(Y) = \pi(X)$.

A subset Y of a space X is G_δ -dense if Y meets every non-empty G_δ subset of X .

Corollary 1.3. For each compact space X there is a G_δ -dense subset Y of X such that $d(Y) = \pi(X)$.

Proof. Applying Proposition 1.2 we choose a dense set $Y \subset X$ so that no subset of Y of cardinality less than $\pi(X)$ is dense. Now let $\text{cl}_{\aleph_0}(Y) = \bigcup\{\bar{A} : A \subset Y \text{ and } |A| = \aleph_0\}$. For every infinite cardinal $\mu \leq |Y|$, each subset of $\text{cl}_{\aleph_0}(Y)$ of cardinality μ is contained in the closure of a subset of Y of cardinality μ . Therefore $\text{cl}_{\aleph_0}(Y)$ has no dense subset of cardinality less than $\pi(X)$. Since X is compact, $\text{cl}_{\aleph_0}(Y)$ is countably compact and dense. It is now immediate that $\text{cl}_{\aleph_0}(Y)$ is G_δ -dense in X . \square

2. ELEMENTARY PROPERTIES OF $\delta_k(X)$

A map f from a space X to a space Y is said to be irreducible if f is onto and the images of proper closed subsets of X are proper subsets of Y . When f is continuous and irreducible, it follows that any subset of X that maps to a dense subset of Y must itself be dense in X . Each continuous irreducible map is quasi-open. A map f is quasi-open if the image of every non-empty open set has non-empty interior.

Proposition 2.1. *If f is a continuous irreducible map from a space X onto Y , then $\delta(X) = \delta(Y)$, $\pi(X) = \pi(Y)$, and $\delta_k(X) \leq \delta_k(Y)$.*

Proof. Let f, X and Y be as in the statement. If $D \subset Y$ is dense, then the density of D is at most the density of $f^{-1}(D)$. Similarly, if D is a dense subset of X , then $f(D)$ is dense in Y and so there is a dense $E \subset f(D)$ of cardinality at most $\delta(Y)$. Any subset E' of D that maps onto E will be dense in X , hence the density of D is at most $\delta(Y)$. If \mathcal{B} is a π -base for X , then $\{\text{int}_Y(f(B)) : B \in \mathcal{B}\}$ is a π -base for Y . Conversely, if \mathcal{B} is a π -base for Y , then the family $\{X \setminus f^{-1}(Y \setminus B) : B \in \mathcal{B}\}$ is a π -base for X . If D is a dense subset of Y , then $f^{-1}(D)$ is a dense subset of X . Fix a family $\{A_\alpha : \alpha < \delta_k(X)\}$ of compact subsets of $f^{-1}(D)$ whose union is dense in X . It follows that $\{f(A_\alpha) : \alpha < \delta_k(X)\}$ is a family of compact subsets of D whose union is dense in Y . This proves that $\delta_k(Y) \leq \delta_k(X)$. \square

Proposition 2.2. *The following are equivalent for a continuous map from a space X onto Y :*

- (1) *for each open $U \subset X$, $\text{int}_Y(f(U))$ is dense in $f(U)$,*
- (2) *f is quasi-open,*
- (3) *the pre-image of every dense set is dense*

Proof. The proofs of the implications (1) implies (2) and (2) implies (3) are trivial. Assume that (3) holds and let U be a non-empty open subset of X . Since $f^{-1}(Y \setminus f(U))$ is not dense, it follows from (3) that $Y \setminus f(U)$ is not dense. This implies that $f(U)$ has non-empty interior. \square

Corollary 2.3. *If Y is the continuous quasi-open image of a space X , then $\delta_k(Y) \leq \delta_k(X)$.*

For a space X , the cellularity $c(X)$ is the supremum of the cardinalities of cellular (pairwise disjoint) families of open subsets of X . The first inequality in this next result follows easily from the fact that a compact subset of the union of a pairwise disjoint family of open sets will be contained a union of finitely many of them. The second is a consequence of the fact that $\delta(X) \leq \pi(X)$.

Proposition 2.4. *For all X , $c(X) \leq \delta_k(X) \leq \pi(X)$.*

Similarly, this next statement is easily proven but will be useful.

Proposition 2.5. *For all X and all regular closed $U \subset X$, $c(U) \leq c(X)$, $\delta_k(U) \leq \delta_k(X)$, and $\pi(U) \leq \pi(X)$.*

Say that a compact space X has uniform π -weight if every non-empty regular closed subset has π -weight equal to $\pi(X)$.

Proposition 2.6. *Each compact space X , with $\delta_k(X) < \pi(X)$, has a regular closed subset Y having uniform π -weight and satisfying that $\delta_k(Y) < \pi(Y)$.*

Proof. For each non-empty regular-closed subset U of X , let \mathcal{R}_U denote the set of non-empty regular closed subsets of $\text{int}_X(U)$. For each $U \in \mathcal{R}_X$, let δ_U be the minimum element of $\{\pi(W) : W \in \mathcal{R}_U\}$ and choose any $W_U \in \mathcal{R}_U$ with $\pi(W_U) = \delta_U$. Evidently $\pi(W) = \pi(W_U)$ for all regular closed subsets W of W_U , that is W_U has uniform π -weight and each element of \mathcal{R}_{W_U} also has uniform π -weight. We may now choose a maximal family, \mathcal{B} , of pairwise disjoint members of \mathcal{R}_X that have uniform π -weight. By Proposition 2.4, $|\mathcal{B}| \leq \delta_k(X)$. If $\mathcal{B}_U \subset \mathcal{R}_U$

is a π -base for each $U \in \mathcal{B}$, then $\bigcup\{\mathcal{B}_U : U \in \mathcal{B}\}$ is a π -base for X . It follows that there is a $Y \in \mathcal{B}$ such that $\pi(Y) > \delta_k(X)$. By Proposition 2.5, we have that $\delta_k(Y) \leq \delta_k(X) < \pi(Y)$ as required. \square

This next result is a crucial step in the proof of our main result. In this proof, and others, we will use the notion of elementary submodels. Most readers will be familiar enough with these notions and they are surveyed in [2]. In particular, for a regular cardinal θ , $H(\theta)$ denotes the set of all sets whose transitive closure has cardinality less than θ . An elementary submodel M (of cardinality less than θ) of $H(\theta)$ can be thought of as an element of $H(\theta)$ with the property that anything true in $H(\theta)$ of finitely many elements of M is also true in M . The classical Löwenheim-Skolem theorem ensures that every infinite element x of $H(\theta)$ is a subset of an elementary submodel M of $H(\theta)$ with $|M| = |x|$. Tarski proved that the union of an increasing chain of fewer than θ many elementary submodels of $H(\theta)$ is an elementary submodel of $H(\theta)$.

Lemma 2.7. *If X is a compact space with $\delta_k(X) = \kappa < \pi(X)$, then there is a quasi-open continuous image Y of X with $\delta_k(Y) \leq \kappa < \pi(Y) \leq w(Y) \leq 2^\kappa$.*

Proof. Let θ be the weight of X and identify X with a closed subset of $[0, 1]^\theta$. Let M be an elementary submodel of $H(\theta^+)$ such that $X \in M$, $M^\kappa \subset M$, and $|M| = 2^\kappa$. Now let pr_M denote the projection mapping from $[0, 1]^\theta$ onto $[0, 1]^{M \cap \theta}$, and let $Y = \text{pr}_M(X)$. Evidently $\pi(Y) \leq w(Y) \leq |M|$. If \mathcal{B} is a family of at most κ many basic open subsets of $[0, 1]^{M \cap \theta}$, then $\mathcal{C} = \{\text{pr}_M^{-1}(B) : B \in \mathcal{B}\}$ is an element of M . Since $\{C \cap X : C \in \mathcal{C}\}$ is not a π -base for X , it follows by simple elementarity that $\{B \cap Y : B \in \mathcal{B}\}$ is not a π -base for Y .

Now we prove that $\text{pr}_M \upharpoonright X$ is quasi-open. By Proposition 2.4, the cellularity of X is at most κ . Since Y is a continuous image of X , the cellularity of Y is also at most κ . By Proposition 2.2, it suffices to prove that the pre-image of each dense open subset of Y is a dense subset of X . Assume that D is a dense open subset of Y and choose any family \mathcal{B} of basic open subsets of $[0, 1]^{M \cap \theta}$ satisfying that the members of $\{B \cap Y : B \in \mathcal{B}\}$ are non-empty and pairwise disjoint subsets of D and so that the union of $\{B \cap Y : B \in \mathcal{B}\}$ is a dense subset of D . Since $c(Y) \leq \kappa$, the family \mathcal{B} has cardinality at most κ . Now we have that the family $\mathcal{C} = \{\text{pr}_M^{-1}(B) : B \in \mathcal{B}\}$ is a subset of M and, since $M^\kappa \subset M$, \mathcal{C} is an element of M . We observe that for every basic open subset U of $[0, 1]^{M \cap \theta}$ that meets Y , there is a $B \in \mathcal{B}$ such that $U \cap B \cap Y$ is not empty. Equivalently, for every basic open subset $U \in M$ of $[0, 1]^\theta$, there is a $C \in \mathcal{C}$ such that $U \cap C \cap X$ is not empty. Therefore, by elementarity, we have that for every basic open subset U of $[0, 1]^\theta$, there is a $C \in \mathcal{C}$ such that $U \cap C \cap X$ is not empty. This implies that the union of the family $\{C \cap X : C \in \mathcal{C}\}$ is a dense subset of X . Since $C \subset \text{pr}_M^{-1}(D)$ for all $C \in \mathcal{C}$, this completes the proof. \square

In the investigation of $t_k^*(X)$ in [3], it was fruitful to consider left-separated sequences of compact nowhere dense G_δ sets. A set is nowhere dense if its closure has empty interior. We also find that mixing the notion of left-separated sets with coverings by compact nowhere dense G_δ sets will be useful. A left-separated transfinite sequence in a space X is a sequence $\{x_\alpha : \alpha < \mu\}$ indexed by an ordinal μ that has the property that x_α is not in the closure of $\{x_\beta : \beta < \alpha\}$ for all $\alpha \in \mu$.

Lemma 2.8. *Assume that $\{x_\alpha : \alpha < \mu\}$ is a left-separated subset of a space X , and for each $\alpha < \mu$, there is a compact G_δ subset Z_α of X such that $x_\alpha \in Z_\alpha$ and $x_\beta \notin Z_\alpha$ for all $\alpha \neq \beta < \mu$. Then each compact subset K of $\{x_\alpha : \alpha < \mu\}$ is scattered and countable.*

Proof. This lemma is really two separate results combined into one. Namely, the first is that a compact subset of a left-separated sequence is scattered ([5]), and the second is that a compact first-countable scattered space is countable. We prove each statement. First assume that K is a compact subset of the left-separated sequence $\{x_\alpha : \alpha < \mu\}$. Since a space is scattered if every one of its closed subspaces has an isolated point, we may assume that K has no isolated point and obtain a contradiction. It is well-known that such a space K has a mapping, f , onto $[0, 1]$. To see this directly, assume that K is embedded into $[0, 1]^\theta$ where θ is the weight of K . Choose any countable elementary submodel M of $H(\theta^+)$ such that $K \in M$. Now $\text{pr}_M(K)$ is a compact metrizable space which, by elementarity, has no isolated points. If $\text{pr}_M(K)$ is totally disconnected, it is a copy of the Cantor space and otherwise $\text{pr}_M(K)$ has a non-trivial connected component. Any continuous real-valued function on $\text{pr}_M(K)$ that is not constant on such a component, will include an interval in its range. Therefore there is a mapping of $\text{pr}_M(K)$ onto $[0, 1]$. In either case, $\text{pr}_M(K)$ maps onto $[0, 1]$, and therefore, so does K . Now choose $\alpha < \mu$ minimal so that $f(K \cap \{x_\beta : \beta < \alpha\})$ has a crowded subset S . Each crowded subset of $[0, 1]$ has a countable crowded subset, and so the cofinality of α is countable. Since $\{x_\beta : \beta < \mu\}$ is left-separated, it follows that $K \cap \{x_\beta : \beta < \alpha\}$ maps onto the (compact) closure \bar{S} of S . However \bar{S} is uncountable, while $f(K \cap \{x_\beta : \beta < \gamma\})$ is countable for all $\gamma < \alpha$. This contradiction proves that K is scattered.

Now with K any compact scattered subset of $\{x_\alpha : \beta < \mu\}$, we have that K is first-countable since the family $\{Z_\alpha \cap K : \alpha < \mu\}$ witnesses that each point of K is a relative G_δ . Since K is compact it is first-countable. For each $x \in K$, let $\rho_K(x)$ denote the scattering level of K that contains x . For each $x \in K$, there is a relatively clopen subset W_x of K satisfying that $\rho_K(y) < \rho_K(x)$ for all $x \neq y \in W_x$; that is, W_x is a relatively clopen set witnessing that x is an isolated point of $K \setminus \{y \in K : \rho_K(y) < \rho_K(x)\}$. Since K is compact, a finite subcollection of $\{W_x : x \in K\}$ will cover K . If K is uncountable, then $\{\rho_K(x) : x \in K \text{ and } |W_x| > \aleph_0\}$ has a minimum element δ . Choose $x \in K$ such that $\rho_K(x) = \delta$ and $|W_x| > \aleph_0$. Note that W_y is countable for all $y \in W_x$. We now have a contradiction because $W_x \setminus \{x\}$ is σ -compact and yet the family $\{W_y : y \in W_x \setminus \{x\}\}$ is an open cover with no countable subcover. \square

3. ON DENSELY k -SEPARABLE COMPACT SPACES

This next result is similar to an old result of Malychin ([4, 3.17]) showing that an uncountable compact T_1 -space has cardinality at least \mathfrak{c} if each point of the space is a G_δ . In particular, if one constructed the family \mathcal{Z} in the statement of the Lemma to be upper semi-continuous then the conclusion of the Lemma would follow from Malychin's result. We instead prove the Lemma directly since the construction of the family \mathcal{Z} is then more natural and simpler.

Lemma 3.1. *Let X be a compact ccc space with no isolated points, then there is a partition \mathcal{Z} of X consisting of nowhere dense compact G_δ 's and satisfying that*

for all non-empty regular closed subsets U of X , the set $\{Z \in \mathcal{Z} : U \cap Z \neq \emptyset\}$ has cardinality \mathfrak{c} .

Proof. Let θ be the weight of X and for convenience we regard it as a compact subset of $[0, 1]^\theta$. Fix a continuous elementary chain $\{M_\alpha : \alpha \in \omega_1\}$ of countable elementary submodels of $H(\theta^+)$ so that for all $\beta < \alpha$, $X \in M_\beta \in M_\alpha$. For each countable partial function t from a subset of θ into $[0, 1]$, we let $[t]$ denote the G_δ subset of $[0, 1]^\theta$ consisting of all total functions that extend t . For such countable functions t , we also let $[t]_X$ denote the (possibly empty) set $[t] \cap X$.

For each α , set T_α equal to the set of all $t \in [0, 1]^{M_\alpha \cap \theta}$ such that $[t]_X$ is not empty. Also let $T_{\alpha,0} = \{t \in T_\alpha : \text{int}_X([t]_X) \neq \emptyset\}$. The members of the family $\{[t] : t \in T_\alpha\}$ are pairwise disjoint. Also, for $\beta < \alpha \in \omega_1$, the sets T_β and $T_{\beta,0}$ are elements of M_α . Since X is ccc and is in M_0 , $T_{\alpha,0}$ is countable for all $\alpha \in \omega_1$. For this reason we also have that $T_{\beta,0} \subset M_\alpha$ for all $\beta < \alpha \in \omega_1$.

The collection $T_{\omega_1,0} = \bigcup\{T_{\alpha,0} : \alpha \in \omega_1\}$ is a tree when ordered by \subset . We again note that $[t]_X \cap [t']_X$ is empty if t and t' have no common extension (incomparable) in $T_{\omega_1,0}$. Since X is ccc, this tree has no uncountable antichains. In fact, we now check that $T_{\omega_1,0}$ also has no uncountable chains. Assume that $t_\beta \subset t_\alpha$ where $\beta < \alpha$, $t_\beta \in T_{\beta,0}$, and $t_\alpha \in T_{\alpha,0}$. Note that for all $\gamma < \alpha$, $t_\alpha \upharpoonright M_\gamma$ is also a member of $T_{\gamma,0}$ since $\text{int}_X([t_\alpha \upharpoonright M_\gamma]_X)$ contains $\text{int}_X([t_\alpha]_X)$. Since X has no isolated points, the infinite open set $\text{int}_X([t_\alpha]_X)$ contains a disjoint pair of closed G_δ -subsets of X . In particular, there are incomparable countable extensions t_1, t_2 of t_α , functions from a countable subset of θ into $[0, 1]$, such that $[t_1]_X \cup [t_2]_X \subset \text{int}_X([t_\alpha]_X)$. Since $t_\beta \in M_{\beta+1}$, it then follows by elementarity that there are incomparable $t_1, t_2 \in T_{\beta+1}$ such that $[t_1]_X \cup [t_2]_X \subset \text{int}_X([t_\beta]_X)$. This shows that $\text{int}_X([t_\beta]_X) \setminus [t_\alpha]_X$ is not empty. Therefore, if $\{t_\alpha : \alpha < \omega_1\}$ is a chain in $T_{\omega_1,0}$, we can assume that $t_\alpha \in T_{\alpha,0}$ for each $\alpha \in \omega_1$, and we have that the existence of the family $\{\text{int}_X([t_\alpha]_X) \setminus [t_{\alpha+1}]_X : \alpha \in \omega\}$ contradicts that X is ccc.

Now we use $T_{\omega_1,0}$ to define a special antichain T in the tree $T_{\omega_1} = \bigcup\{T_\alpha : \alpha \in \omega_1\}$. A node t is in T if there is an $\alpha \in \omega_1$ such that $t \in T_\alpha \setminus T_{\alpha,0}$ and for all $\beta < \alpha$, $t \upharpoonright M_\beta \in T_{\beta,0}$. That is T is the set of all minimal nodes of $T_{\omega_1} \setminus T_{\omega_1,0}$. Fix any $x \in X$ and let $C_x = \{t \in T_{\omega_1} : x \in [t]_X\}$. Since C_x is an uncountable chain, it is not a subset of $T_{\omega_1,0}$. Therefore there is a minimal α such that there is a $t_x \in T_\alpha \setminus T_{\alpha,0}$ with $x \in [t_x]_X$. By the minimality of α , $t_x \in T$. This proves that $\mathcal{Z} = \{[t]_X : t \in T\}$ is a partition of X .

Now let U be a non-empty regular closed subset of X . We prove that the set $\{t \in T : U \cap [t]_X \neq \emptyset\}$ has cardinality \mathfrak{c} . For each non-empty regular closed W of U , let $\mu_W = \sup\{\alpha \in \omega_1 : (\exists t \in T_{\alpha,0}) W \cap \text{int}_X([t]_X) \neq \emptyset\}$. By passing to a regular closed subset of U with a minimum value of μ_U , we can assume that $\mu_W = \mu_U$ for all regular closed subsets W of U .

Assume first that there is a $t \in T_{\mu_U,0}$ such that $U \cap \text{int}_X([t]_X)$ is not empty and let $\alpha = \mu_U + 1$. Choose any regular closed subset W of U that is contained in $\text{int}_X([t]_X)$. Since $\mu_W = \mu_U$, it follows that W is covered by the family of compact nowhere dense sets $\{W \cap [t']_X : t \subset t' \in T_\alpha\}$. By the Baire category theorem, this family is uncountable. Also, the family $\{t' \in [0, 1]^{M_\alpha \cap \theta} : [t']_X \cap W \neq \emptyset\}$ is equal to $\text{pr}_{M_\alpha}(W)$ and so is an uncountable closed subset of the metric space $[0, 1]^{M_\alpha \cap \theta}$ and will therefore have cardinality \mathfrak{c} . All but countably many of the $t' \in T_\alpha$ with $W \cap [t']_X \neq \emptyset$ are in T . This completes the proof in this case.

The same proof as in the previous paragraph, with $W = U$, shows that if $\mu_U = 0$ and $U \cap \text{int}_X([t]_X)$ is empty for all $t \in T_{0,0}$, then $\{t \in T : U \cap [t]_X \neq \emptyset\}$ has cardinality \mathfrak{c} . Similarly, if μ_U is a successor ordinal, $\beta + 1$, then we let W be a non-empty regular closed subset of $U \cap \text{int}_X([t]_X)$ for any $t \in T_{\beta,0}$ such that $U \cap \text{int}_X([t]_X)$ is not empty, and proceed as above.

Now we consider the final case where μ_U is a limit ordinal and, if $\mu_U < \omega_1$, that $U \cap \text{int}_X([t]_X)$ is empty for all $t \in T_{\mu_U,0}$. Choose any $\alpha_0 < \mu_U$ and $t_0 \in T_{\alpha_0,0}$ such that $W \cap \text{int}_X[t_0]_X$ is not empty. We have started a recursive construction of choosing a strictly increasing sequence $\{\alpha_n : n \in \omega\} \subset \mu_U$ and a family $\{t_s : s \in 2^{<\omega} = \bigcup_n 2^n\}$ such that, for each $n \in \omega$

- (1) $U \cap \text{int}_X([t_s]_X)$ is not empty for each $s \in 2^n$,
- (2) $\{t_s : s \in 2^n\}$ are distinct elements of $T_{\alpha_n,0}$
- (3) $t_{s'} \subset t_s$ for all $s' = s \upharpoonright m$ and $m < n$.

Assume that $n \in \omega$ and that we have chosen $\alpha_n < \mu_U$ and the sequence $\{t_s : s \in 2^n\}$ so that the above conditions hold. For each $s \in 2^n$, choose a regular closed subset U_s of $U \cap \text{int}_X([t_s]_X)$. Since there are no uncountable chains in $T_{\omega_1,0}$, there is an $\alpha_s < \omega_1$ such that, for some $t'_s \in T_{\alpha_s,0}$, $U_s \cap \text{int}_X([t'_s]_X)$ is not empty and $\text{int}_X(U_s \setminus [t'_s])$ is not empty. Since $\mu_W = \mu_U$ for all regular closed subsets of U , there is also a $t''_s \in T_{\alpha_s,0}$ such that the open set $\text{int}_X(U_s \setminus [t'_s])$ meets $\text{int}_X([t''_s])$. Choose any $\alpha_{n+1} < \mu_U$ so that $\alpha_s \leq \alpha_{n+1}$ for all $s \in 2^n$. For each $s \in 2^n$, there are $t_{s \smallfrown 0}$ and $t_{s \smallfrown 1}$ in $T_{\alpha_{n+1},0}$ such that $t'_s \subset t_{s \smallfrown 0}$, $t''_s \subset t_{s \smallfrown 1}$, $U_s \cap \text{int}_X([t_{s \smallfrown 0}]_X)$ and $U_s \cap \text{int}_X([t_{s \smallfrown 1}]_X)$ are not empty. This completes the recursive construction. Let $\alpha = \bigcup_n \alpha_n$. Let ρ be any element of 2^ω , and let $t_\rho = \bigcup \{t_{\rho \upharpoonright n} : n \in \omega\}$. Since $M_\alpha = \bigcup \{M_{\alpha_n} : n \in \omega\}$, $t_\rho \in T_\alpha$ for each $\rho \in 2^\omega$. Since U is compact and $U \cap [t_\rho \upharpoonright M_{\alpha_n}]_X$ is not empty for each $n \in \omega$, we have that $U \cap [t_\rho]_X$ is not empty for each $\rho \in 2^\omega$. For each $\rho \in 2^\omega$ such that $t_\rho \notin T_{\alpha,0}$, it follows that $t_\rho \in T$ since $t_\rho \upharpoonright \alpha_n \in T_{\alpha_n,0}$ for all $n \in \omega$. This completes the proof of the Lemma. \square

Now we prove the main theorem.

Theorem 3.2. *A compact space is densely k -separable if and only if it has countable π -weight.*

Proof. It follows from Proposition 2.4 that $\delta_k(X) = \aleph_0$ for all compact spaces of countable π -weight. For the other direction assume that X is compact and that $\delta_k(X) = \aleph_0$. By Proposition 2.4, X is ccc. Now we assume that $\aleph_0 < \pi(X)$ and work towards a contradiction. By Lemma 2.7, we may replace X by a quasi-open continuous image and thereby assume that the weight of X is at most \mathfrak{c} . By Proposition 2.6, we can pass to a suitable regular closed subset of X and thereby assume that X has uniform π -weight. Now apply Lemma 3.1, choose a partition \mathcal{Z} of X by compact G_δ 's with the property that the set $\{Z \in \mathcal{Z} : W \cap Z \neq \emptyset\}$ has cardinality \mathfrak{c} for each non-empty regular closed subset W of X . By Corollary 1.3, we choose a G_δ -dense subset Y of X so that Y has no dense subset of cardinality less than $\pi(X)$. Of course this means that Y meets every member of \mathcal{Z} . Fix an enumeration $\{U_\alpha : \alpha < \pi(X)\}$ of a family of regular closed sets whose interiors form a π -base for X . Let y_0 be any element of $Y \cap \text{int}_X(U_0)$ and let $Z_0 \in \mathcal{Z}$ be such that $y_0 \in Z_0$. Assume, by induction that $\alpha < \pi(X)$ and we have chosen a sequence $\{y_\beta : \beta < \alpha\} \subset Y$ and a sequence $\{Z_\beta : \beta < \alpha\} \subset \mathcal{Z}$ so that for all $\beta < \alpha$

- (1) y_β is not in the closure of $\{y_\xi : \xi < \beta\}$,
- (2) $Z_\beta \notin \{Z_\xi : \xi < \beta\}$,

(3) $\{y_\xi : \xi \leq \beta\}$ meets $\text{int}_X(U_\beta)$.

We choose y_α and Z_α as follows. Since $\{y_\beta : \beta < \alpha\}$ is not dense, we may choose the minimal $\gamma_\alpha < \pi(X)$ such that $\text{int}_X(U_{\gamma_\alpha})$ is disjoint from $\{y_\beta : \beta < \alpha\}$. Choose δ_α so that U_{δ_α} is a subset of $\text{int}_X(U_{\gamma_\alpha}) \setminus \overline{\{y_\beta : \beta < \alpha\}}$. We note, by induction assumption (3), that if $\{y_\beta : \beta < \alpha\}$ is disjoint from $\text{int}_X(U_\alpha)$, then $\gamma_\alpha = \alpha$. Choose any $Z_\alpha \in \mathcal{Z} \setminus \{Z_\beta : \beta < \alpha\}$ so that $Z_\alpha \cap U_{\delta_\alpha}$ is not empty. Since Y is G_δ -dense, we can choose $y_\alpha \in Z_\alpha \cap \text{int}_X(U_{\gamma_\alpha})$. This completes the inductive construction of the left-separated sequence $\{y_\alpha : \alpha < \pi(X)\}$ together with the sequence $\{Z_\alpha : \alpha < \pi(X)\} \subset \mathcal{Z}$. By Lemma 2.8, each compact subset of $\{y_\alpha : \alpha < \pi(X)\}$ is countable. This proves that $\{y_\alpha : \alpha < \pi(X)\}$ does not have a σ -compact subset that is a dense subset of Y . However, we have ensured that $\{y_\beta : \beta \leq \alpha\} \cap \text{int}_X(U_\alpha)$ is not empty for all $\alpha < \pi(X)$, and so $\{y_\alpha : \alpha < \pi(X)\}$ is a dense subset of X . This contradicts that $\delta_k(X) = \aleph_0$. \square

REFERENCES

- [1] A. V. Arhangel'skiĭ and D. N. Stavrova, *On a common generalization of k -spaces and spaces with countable tightness*, *Topology Appl.* **51** (1993), no. 3, 261–268. MR1237392
- [2] Alan Dow, *An introduction to applications of elementary submodels to topology*, *Topology Proc.* **13** (1988), no. 1, 17–72. MR1031969
- [3] Alan Dow and Justin Moore, *Tightness in σ -compact spaces*, *Topology Proc.* **46** (2015), 213–232. MR3224176
- [4] István Juhász, *Cardinal functions in topology—ten years later*, 2nd ed., *Mathematical Centre Tracts*, vol. 123, Mathematisch Centrum, Amsterdam, 1980. MR576927
- [5] J. Gerlits and I. Juhász, *On left-separated compact spaces*, *Comment. Math. Univ. Carolinae* **19** (1978), no. 1, 53–62. MR0487982
- [6] I. Juhász and J. van Mill, *Variations on countable tightness*, *Acta Math. Hungar.* **153** (2017), no. 1, 75–82. MR3713564
- [7] I. Juhász and S. Shelah, *$\pi(X) = \delta(X)$ for compact X* , *Topology Appl.* **32** (1989), no. 3, 289–294. MR1007107
- [8] B. Šapirovskii, *Tightness and related notions*, *Topological spaces and their mappings* (Russian), *Latv. Gos. Univ., Riga*, 1979, pp. 119–131, 154 (Russian). MR569886