Disjoint local π-bases and some selection properties

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Introduction

This is a joint work with Gary Gruenhage completed in May, 2023.

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The study of selection principles dates back to the work of Menger and Hurewicz in 1924–1926. The respective properties that nowadays bear their names are still important not only in general topology but also in functional analysis and topological algebra. Several dozens of selection properties were identified and classified by Scheepers in 1996–2003.

Tkachuk and Guerrero Sanchez discovered in 2017, in a completely unrelated context, that the σ-product $S = \{x \in \{0,1\}^{\omega_1} : |x^{-1}(1)| < \omega\}$ of the Cantor cube $\{0,1\}^{\omega_1}$ has the following nice selection property, which was later called *discrete selectivity*: if $\{U_n : n \in \omega\}$ is a family of non-empty open subsets of S, then it is possible to pick $x_n \in U_n$ for every $n \in \omega$ in such a way that the selection $\{x_n : n \in \omega\}$ is closed and discrete in S.

In 2017, Tkachuk proved that discrete selectivity takes place in $C_p(X, [0, 1])$ provided that X has countable tightness and $X \setminus A$ has a non-isolated point for any countable set $A \subset X$. This motivated the author to study discrete selectivity in a systematic way in 2018. Discrete selectivity is, evidently, present in any discrete space X but it becomes quite non-trivial in function spaces that usually have no isolated points. It was established by Tkachuk in 2018 that discrete selectivity holds in $C_p(X)$ for any uncountable space X.

However, there is still no characterization of discrete selectivity in spaces $C_p(X, [0,1])$ which can even be compact. It was established in 2018 that the space $C_p(X, [0,1])$ is discretely selective if an uncountable space X is either Lindelöf Σ or pseudocompact. Another result of the same paper of Tkachuk is a characterization, in terms of the topology of X, for the space $C_p(X, [0,1])$ to be discretely selective if X is an ω-monolithic space of countable tightness.

Tkachuk proved in 2023, that any non-metrizable locally convex space L has the *discrete shrinking property*, namely, any sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of L has a discrete shrinking, i.e., there exists a discrete family $\{V_n : n \in \omega\}$ of non-empty open subsets of L such that $V_n \subset U_n$ for every $n \in \omega$. It is clear that discrete shrinking property implies discrete selectivity. From the above-mentioned result it follows that the space $C_p(X)$ has the discrete shrinking property for any uncountable space X.

However, the situation is different for spaces $C_p(X, [0, 1])$ which are compact and hence not even discretely selective for discrete spaces X. Thus, the behavior of the discrete shrinking property is quite non-trivial in spaces $C_p(X, [0, 1])$. It was proved in 2022 that $C_p(X, [0, 1])$ has the discrete shrinking property if X is essentially uncountable (this means that for any countable set $A \subset X$, there exist disjoint countable subsets P and Q of $X \backslash A$ such that $\overline{P} \cap \overline{Q} \neq \emptyset$) and shown that there exist discretely selective spaces $C_p(X, [0, 1])$ which do not have the discrete shrinking property.

In this work we will show that the above-mentioned selection properties have interesting applications in the context of spaces with disjoint local π-bases.

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Compact spaces with disjoint local π-bases popped up naturally in a paper of Tkachuk and Wilson in 2019 where it was proved that, for a non-isolated point $p \in X$, the space $X \setminus \{p\}$ is not cellular-compact if and only if X has disjoint local π-base at he point p. It was established in the same paper that any compact ω-monolithic space of countable tightness has a disjoint countable π-base at every point. The same is true for sequential compact spaces under Luzin's Axiom; this was also proved in the above-mentioned paper so it is a very interesting question whether every compact space of countable tightness has a disjoint local π-base at every point.

a space X *weakly essentially uncountable* if for any countable set $A \subset X$, there exist disjoint countable sets $D, E \subset X \backslash A$ such that $\mathrm{cl}_{\beta X}(D) \cap \mathrm{cl}_{\beta X}(E) \neq \emptyset$. It is immediate that any essentially uncountable space is weakly essentially uncountable.

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Theorem 1.

If X a zero-dimensional weakly essentially uncountable space, then $C_p(X, \mathbb{D})$ has the discrete shrinking property.

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Theorem 1.

If X a zero-dimensional weakly essentially uncountable space, then $C_p(X, \mathbb{D})$ has the discrete shrinking property.

Corollary 2.

Let X be a zero-dimensional essentially uncountable space. Then $C_p(X, \mathbb{D})$ has the discrete shrinking property. In particular, $C_p(X, \mathbb{D})$ is discretely selective.

Corollary 3.

Let X be an uncountable zero-dimensional Lindelöf Σ-space. Then $C_p(X, \mathbb{D})$ has the discrete shrinking property; in particular, $C_p(X, \mathbb{D})$ is discretely selective.

Corollary 3.

Let X be an uncountable zero-dimensional Lindelöf Σ-space. Then $C_p(X, \mathbb{D})$ has the discrete shrinking property; in particular, $C_p(X, \mathbb{D})$ is discretely selective.

Corollary 4.

Let X be an uncountable zero-dimensional countably compact space. Then $C_p(X, \mathbb{D})$ has the discrete shrinking property and, in particular, $C_p(X, \mathbb{D})$ is discretely selective.

Theorem 5.

If X is an ω-monolithic zero-dimensional space of countable tightness, then the following conditions are equivalent:

(a) $C_p(X, \mathbb{D})$ is discretely selective;

(b) $C_p(X, \mathbb{D})$ has the discrete shrinking property;

(c) $C_p(X, \mathbb{I})$ is discretely selective;

(d) $C_p(X, \mathbb{I})$ has the discrete shrinking property;

(e) X is essentially uncountable;

(e) X is weakly essentially uncountable;

(f) $X \setminus D$ is uncountable if D is a clopen discrete subspace of X.

Theorem 6.

Let X be an uncountable space with a unique non-isolated point p. Then the following conditions are equivalent:

(a) $C_p(X, \mathbb{I})$ is discretely selective;

(b) $C_p(X, \mathbb{D})$ is discretely selective;

(c) $X \setminus A$ does not have the P-property for any countable $A \subset X \setminus \{p\}$.

Theorem 7.

Let X be an uncountable space with a unique non-isolated point p. The following conditions are equivalent:

(a) $C_p(X, \mathbb{I})$ has the discrete shrinking property;

(b) $C_p(X, \mathbb{D})$ has the discrete shrinking property;

(c) X is essentially uncountable.

Proposition 8.

Suppose that X is discretely selective and $f : X \to Y$ is a continuous bijection such that $f|A : A \to f(A)$ is a homeomorphism for any countable set $A \subset X$. Then Y is also discretely selective.

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Corollary 9.

Assume that X is a zero-dimensional space such that its Hewitt extension υX is also zero-dimensional and $C_p(\upsilon X, \mathbb{D})$ is discretely selective. Then $C_p(X, \mathbb{D})$ is discretely selective.

Corollary 10.

If X is an uncountable zero-dimensional pseudocompact space such that βX is also zero-dimensional, then $C_p(X, \mathbb{D})$ is discretely selective.

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Example 11.

Tkachuk proved in 2019 that there exists a pseudocompact dense subspace $X \subset \mathbb{D}^{\mathfrak{c}}$ such that every countable subset of X is closed and C^*-embedded in X. Thus X is uncountable and $\beta X = \mathbb{D}^{\mathfrak{c}}$; therefore both X and βX are zero-dimensional. Corollary 10 guarantees that $C_p(X, \mathbb{D})$ is discretely selective. However, $C_p(X, \mathbb{D})$ is easily seen to be pseudocompact so it cannot have the discrete shrinking property. Therefore $C_p(X, \mathbb{D})$ is a discretely selective non-metrizable topological group that does not have the discrete shrinking property.

Example 11 should be compared with a result of Tkachuk, proved in 2023 that discrete selectivity is equivalent to discrete shrinking property in any locally convex space.

Example 12.

The methods developed in this paper make it possible to give an easier construction, than in Example 11, of a zero-dimensional space X such that $C_p(X, \mathbb{D})$ is pseudocompact and discretely selective. Take a set A of cardinality ω_1 and a point $p \notin A$. Choose a family $\{A_n : n \in \omega\}$ of disjoint subsets of A such that $|A_n| = \omega_1$ for any $n \in \omega$ and $A = \bigcup_{n\in\omega} A_n$. Let $B_n = A_0 \cup \ldots \cup A_n$ for every $n \in \omega$; our space will be $X = A \cup \{p\}$ and all points of A will be isolated in X. We declare that a set $U \ni p$ belongs to $\tau(p, X)$ if there exists a countable set $D \subset A$ and $n \in \omega$ such that $U = X \backslash (B_n \cup D)$. It is straightforward that $X \backslash D$ is not P-space for any countable $D \subset X \backslash \{p\}$ and hence the space $C_p(X, \mathbb{D})$ is discretely selective by Theorem 6. Since all countable subsets of X are closed and C-embedded in X, the space $C_p(X, \mathbb{D})$ pseudocompact and hence we have one more example of a pseudocompact topological group which is discretely selective.

Our next step is to characterize discrete shrinking property in general classes of spaces and give some applications for topological groups. Another interesting application stems from the fact that if a space X has the disjoint shrinking property and X has a countable local π-base at a point $x \in X$, then there exists a disjoint local π-base at the point x. Since all compact spaces of countable tightness has countable π-character, it is a very interesting question whether they all have a disjoint local π-base at every point.

Definition 13.

Given a space X, suppose that κ is an infinite cardinal and $A \subset X$ is a non-empty set. We will say that $\pi\chi(A, X) \leqslant \kappa$ (or that the *outer π-character of A in X* does not exceed κ) if there exists a family $\mathcal{B} \subset \tau^*(X)$ (called an *outer π-base for A*) such that, for any $U \in \tau(A, X)$, there exists $B \in \mathcal{B}$ with $B \subset U$. If there is no outer π-base for A of cardinality $\leqslant \kappa$, we will say that $\pi\chi(A, X) > \kappa$. Observe that, given any $x \in X$, we have the inequality $\pi\chi(\{x\}, X) \leqslant \kappa$ if and only if $\pi\chi(x, X) \leqslant \kappa$.

Proposition 14.

If X is a discretely selective space and $K \subset X$ is a compact set with empty interior, then $\pi\chi(K, X) > \omega$.

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Corollary 15.

If X is a discretely selective space with a disjoint shrinking property, then $\pi\chi(K, X) > \omega$ for any compact set $K \subset X$.

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If X is a discretely selective space and $K \subset X$ is a compact set with empty interior, then $\pi\chi(K, X) > \omega$.

Corollary 15.

If X is a discretely selective space with a disjoint shrinking property, then $\pi\chi(K, X) > \omega$ for any compact set $K \subset X$.

Corollary 16.

If X has the discrete shrinking property, then $\pi\chi(K, X) > \omega$ for any compact $K \subset X$.

Theorem 17.

Suppose that X is a normal space with the following properties:

(a) $X = \bigcup_{n\in\omega} X_n$ and every X_n is closed in X;

(b) $\pi\chi(X_n, X) > \omega$ for every $n \in \omega$.

If, additionally, $\pi\chi(x, X) > \omega$ for any $x \in X$, then X has the discrete shrinking property.

Theorem 17.

Suppose that X is a normal space with the following properties:
(a) $X = \bigcup_{n\in\omega} X_n$ and every X_n is closed in X;
(b) $\pi\chi(X_n, X) > \omega$ for every $n \in \omega$.
If, additionally, $\pi\chi(x, X) > \omega$ for any $x \in X$, then X has the discrete shrinking property.

Proposition 18.

Suppose that X is a space and $\mathcal{C}$ is a closed cover of X for which there exists a countable network $\mathcal{N}$. If $\pi\chi(C, X) > \omega$ for every $C \in \mathcal{C}$, then the family $\mathcal{N}' = \{N \in \mathcal{N} : \pi\chi(N, X) > \omega\}$ is still a network for $\mathcal{C}$.

Theorem 19.

A Lindelöf Σ-space X has the discrete shrinking property if and only if $\pi\chi(K, X) > \omega$ for any compact subset $K \subset X$.

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Corollary 20.

A σ-compact space X has the discrete shrinking property if and only if $\pi\chi(K, X) > \omega$ for any compact subspace $K \subset X$.

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A Lindelöf Σ-space X has the discrete shrinking property if and only if $\pi\chi(K, X) > \omega$ for any compact subset $K \subset X$.

Corollary 20.

A σ-compact space X has the discrete shrinking property if and only if $\pi\chi(K, X) > \omega$ for any compact subspace $K \subset X$.

Theorem 21.

A space X with a countable network has the discrete shrinking property if and only if $\pi\chi(x, X) > \omega$ for every $x \in X$.

Corollary 22.

If G is a non-metrizable topological group with a countable network, then G has a discrete shrinking property.

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Example 23.

Let $G = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| \leqslant \omega\}$ be the Σ-product in the Cantor cube $\mathbb{D}^{\omega_1}$. Then G is a countably compact topological group such that $\pi\chi(K, G) > \omega$ for any compact set $K \subset G$. In particular, the fact that $\pi\chi(K, G) > \omega$ for any compact set $K \subset G$ need not imply that a group G is discretely selective.

In the rest of this talk we will deal with existence of disjoint local π-bases in compact spaces of countable tightness. This context appeared naturally in a paper of Tkachuk and Wilson published in 2019 where the authors proved that a compact space has a disjoint local π-base at a point $x \in X$ if and only if $X \setminus \{x\}$ is not cellular-compact. It was also established in the same paper that any ω-monolithic compact space of countable tightness has a disjoint local π-base at every point. Besides, under Luzin's Axiom, any sequential compact space has a disjoint local π-base at every point. Although it is still an open question whether every compact space of countable tightness has a disjoint local π-base at every point, we will present some progress in this direction. Interestingly, the new information we obtained was due to our techniques of work with shrinking properties.

The following fact was proved by Tkachuk and Wilson in 2019.

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Proposition 24.

Suppose that X is a compact space of countable tightness. If there is a dense set $D \subset X$ such that every point of D has a countable disjoint local π-base, then X has a countable disjoint local π-base at every point.

The next corollary gives a consistent answer to a question from the above-mentioned paper of Tkachuk and Wilson and provides a strong motivation for trying to find out whether it is true in ZFC that every compact space of countable tightness has a disjoint local π-base at every point.

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Corollary 25.

Under PFA, every compact space of countable tightness has a countable disjoint local π-base at every point.

Proposition 26.

Assume that X is a compact space of countable tightness. If every non-empty open subspace of X is non-separable, then X has a countable disjoint local π-base at every point.

Recall that the *W-game* $G(X, p)$ is played by opponents O and P at a point p of a space X. In the n-th round, Player O takes a set $U_n \in \tau(p, X)$ and P responds by picking a point $x_n \in U_n$. The play $\mathcal{L} = \{U_n, x_n : n \in \omega\}$ is a win for O if the sequence $\{x_n : n \in \omega\}$ converges to p. Otherwise P is the winner of the play $\mathcal{L}$. It is said that X is a *W-space* if Player O has a winning strategy in $G(X, p)$ for any point $p \in X$.

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Theorem 27.

If X is a compact W-space, then X has a countable disjoint local π-base at every point.

Corollary 28.

If X is a continuous image of a compact W-space, then it has a countable disjoint local π-base at every point. In particular, any continuous image of a first countable compact space has a countable disjoint local π-base at every point.

Observation 29.

It was proved by Tkachuk and Wilson in 2019 that if a compact space X with $t(X) \leqslant \omega$ has a dense set of points of countable character, then X must have a disjoint local π-base at every point. It is worth noting that Corollary 28 is not a consequence of this result, at least consistently, because Koszmider gave in 1999 a consistent example of a continuous image of a first countable compact space that has no points of first countability.

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Observation 30.

If $S = \{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| \leqslant \omega\}$ is the Σ-product in the Cantor cube $\mathbb{D}^{\omega_1}$, then S is a countably compact W-space which has no disjoint local π-base at any point. Indeed, since $c(S) \leqslant \omega$ such a π-base would be countable but $\pi\chi(x, S) > \omega$ for any $x \in S$. Therefore Theorem 27 cannot be proved for countably compact W-spaces.

Open Questions

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1. Question.

Suppose that G is a non-metrizable topological group of countable pseudocharacter. Must G be discretely selective?

2. Question.

Suppose that G is a non-metrizable Lindelöf topological group of countable pseudocharacter. Must G have the discrete shrinking property?

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Suppose that G is a non-metrizable topological Lindelöf Σ-group which is not locally compact. Must G be discretely selective?

5. Question.

Suppose that G is a non-metrizable hereditarily Lindelöf topological group. Must G be discretely selective?

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Suppose that G is a Lindelöf topological group for which the inequality $\pi\chi(K, G) > \omega$ holds for every compact set $K \subset G$. Must G be discretely selective?

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Suppose that X is a Lindelöf space for which the inequality $\pi\chi(K, X) > \omega$ holds for every compact set $K \subset X$. Must X be discretely selective?

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Suppose that G is a Lindelöf topological group for which the inequality $\pi\chi(K, G) > \omega$ holds for every compact set $K \subset G$. Must G be discretely selective?

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Suppose that X is a Lindelöf space for which the inequality $\pi\chi(K, X) > \omega$ holds for every compact set $K \subset X$. Must X be discretely selective?

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Suppose that X is a hereditarily Lindelöf space for which the inequality $\pi\chi(K, X) > \omega$ holds for every compact set $K \subset X$. Must X be discretely selective?

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10. Question.

Assume that X is a compact Fréchet–Urysohn space. Is true in ZFC that X has a countable disjoint local π-base at every point?

The last slide

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The very last slide

Thanks for your attention!!!