WEIGHT, NET WEIGHT, AND ELEMENTARY SUBMODELS

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Abstract. In this note we prove several theorems that are related to some results and problems from [6].

We answer two of the main questions that were raised in [6]. First we give a ZFC example of a *Hausdorff* space in $C(\omega_1)$ that has uncountable net weight. Then we prove that after adding any number of Cohen reals to a model of CH, in the extension every *regular* space in $C(\omega_1)$ has countable net weight.

In the last section we prove in ZFC the following two statements:
(i) If $S \subset \omega_1$ is stationary then for any *regular* topology on S of uncountable weight S has a non-stationary subset that has uncountable weight as well.
(ii) For any topology on ω_1, if all final segments of ω_1 have uncountable weight then ω_1 has a non-stationary subset of uncountable weight.
In contrast to this, it was shown in [6] that the analogous statements for net weight are not provable in ZFC.

It is remarkable that all our proofs of the above results make essential use of elementary submodels.

The authors dedicate this paper to the memory of Peter Nyikos, our long-time mathematical friend and colleague, and in recognition of his many significant contributions to topology.

1. Introduction

This article continues the investigation of some interesting strengthenings of the countable chain condition property which are known to follow from having a countable network. A key concept, introduced under a different name by Tkachenko [7], is that of a hereditarily good (HG) topological space Y, which means that whenever a set of $\aleph_1$-many points from Y together with an assignment of a neighborhood to each of these points is given, there are two points which are contained in each other's prescribed neighborhoods. This property is sometimes also called *pointed CCC*. Clearly such spaces are both hereditarily separable and hereditarily Lindelöf. The Sorgenfrey line is an example of a hereditarily separable and hereditarily Lindelöf space that is not HG and Tkachenko asked about the connection of HG to having countable net weight. Recall that a family $\mathcal{N}$ of subsets of a space X is a network if for each point x and neighborhood U, there is an $N \in \mathcal{N}$ with $x \in N \subset U$. The net weight $nw(X)$ of a space X is the smallest cardinality of a network for X.

The property of HG was further strengthened in [3] and re-named and studied in [6] as follows.

2020 *Mathematics Subject Classification.* 03E05, 03E10, 03E35, 54A25, 54A35, 54D45.
Key words and phrases. weight, net weight, Cohen real, stationary set, elementary submodel.

Definition 1. (1) *A space X is in $C(\omega_1)$ if for every partial neighborhood assignment U with $\mathrm{dom}(U) \subset X$ uncountable, there is an uncountable $Y \subset \mathrm{dom}(U)$ satisfying that $y \in U(z)$ for all $y, z \in Y$.*

(2) *A space X is in $N(\omega_1)$ if every size ω_1 subspace of it has a countable network.*

Clearly a space with a countable network is in $C(\omega_1)$, consequently we have

$$\{X : nw(X) = \omega\} \subset N(\omega_1) \subset C(\omega_1).$$

2. A Hausdorff example

Answering a question of Hart and Kunen, raised in both [3] and [4], it was proved in [6] Theorem 1.5 that under CH every regular space in $C(\omega_1)$ has a countable network, i.e. has countable net weight. (Actually, this was shown to follow from the principle *super stick*, a consequence of CH that is actually consistent with the negation of CH.) Problem (1) on p. 4 of [6] asks if in this result *regular* can be weakened to *Hausdorff*. This problem is solved in Theorem 2 below.

Theorem 2. *There is a refinement of the topology on the reals that is in $C(\omega_1)$ and does not have countable net weight.*

Proof. The topology τ on $\mathbb{R}$ will be quite simple. For each $r \in \mathbb{R}$, we will choose a set $a_r \in [\mathbb{R} \setminus \{r\}]^\omega$ that has r as its unique accumulation point, i.e. a_r "converges" to r. Then a neighborhood base in τ at $r \in \mathbb{R}$, will be the family

$$\{(q, s) \setminus a_r : (q, s \in \mathbb{Q}) \text{ and } q < r < s\}.$$

The family $\{a_r : r \in \mathbb{R}\}$ will be chosen to satisfy:

(*) For any countable subfamily $\mathcal{T}$ of $[\mathbb{R}]^\mathfrak{c}$ there is a subset X of $\mathbb{R}$ of cardinality $\mathfrak{c}$ such that whenever $r \in X$ is a complete accumulation point of $T \in \mathcal{T}$, the set a_r intersects T.

For any $T \in [\mathbb{R}]^\mathfrak{c}$, we shall denote by T° the set of all complete accumulation points of T. Note that we have $|T \setminus T^\circ| < \mathfrak{c}$ for any $T \in [\mathbb{R}]^\mathfrak{c}$.

We now check that having (*) will imply that τ does not have a countable network. Assume that $\mathcal{S}$ is any countable family of subsets of $\mathbb{R}$, moreover let $\mathcal{T} = \{T \in \mathcal{S} : |T| = \mathfrak{c}\}$. If $\mathcal{T} = \emptyset$ then, of course, $\mathcal{S}$ doesn't even cover $\mathbb{R}$, and so cannot be a network for any topology on $\mathbb{R}$. So, we assume that $\mathcal{T} \neq \emptyset$.

Let us put

$$Y = \bigcup\{S \in \mathcal{S} : |S| < \mathfrak{c}\} \cup \bigcup\{T \setminus T^\circ : T \in \mathcal{T}\}.$$

Then we clearly have $|Y| < \mathfrak{c}$. Now let $X \in [\mathbb{R}]^\mathfrak{c}$ be as in (*) for the family $\mathcal{T}$. We may then pick $r \in X \setminus Y$. Every $S \in \mathcal{S}$ that contains r is an element of $\mathcal{T}$ and r is a complete accumulation point of it. But then $a_r \cap S \neq \emptyset$, and so S is not a subset of the neighborhood $\mathbb{R} \setminus a_r$ of r. Hence $\mathcal{S}$ is not a network for our topology τ.

Now, to define the family $\{a_r : r \in \mathbb{R}\}$ satisfying (*) we first fix a well-ordering $\prec$ of $\mathbb{R}$ in order-type $\mathfrak{c}$. We will also use elementary submodels to guide our choices.

Let θ be a regular cardinal so that $\mathcal{P}(\mathbb{R}) \in H(\theta)$. Let $\mathfrak{M}$ denote the set of countable elementary submodels M of $H(\theta)$ with $(\mathbb{R}, \prec) \in M$.

For each $M \in \mathfrak{M}$, we define $\mathcal{S}(M)$ to be the family

$$\{T \cap M : T \in M \cap [\mathbb{R}]^\mathfrak{c}\}.$$

In other words, $\mathcal{S}(M)$ is the countable family consisting of those countable subsets of $\mathbb{R}$ which are the traces on M of the cardinality $\mathfrak{c}$ subsets of $\mathbb{R}$ that happen to be members of M. Note that then $M \cap \mathbb{R}$ is the largest element of $\mathcal{S}(M)$, moreover every such trace $T \cap M$ is a $\prec$-cofinal subset of $M \cap \mathbb{R}$.

The map sending each $M \in \mathfrak{M}$ to $\mathcal{S}(M)$ is decidedly not 1-to-1 since $\mathfrak{M}$ has cardinality at least $2^{\mathfrak{c}}$, while $\mathfrak{S} = \{\mathcal{S}(M) : M \in \mathfrak{M}\}$ obviously has cardinality $\mathfrak{c}$.

We now formulate and prove a technical lemma that will play a crucial role in establishing (*).

Lemma 3. *Let us fix $M \in \mathfrak{M}$ and $r \in \mathbb{R}$, moreover let $\mathcal{T}(M,r)$ be the family of those $T \in \mathcal{S}(M)$ for which r is a limit point of every $\prec$-final segment of $T \cap (q,s)$ for any pair of rationals with $q < r < s$. Then there is a countable subset a of $M \cap \mathbb{R}$ of $\prec$-order-type ω that converges to r and intersects every member of $\mathcal{T}(M,r)$.*

Proof. First we note that $\mathcal{T}(M,r)$ is (countably) infinite. Indeed, for any pair of rationals with $q < r < s$ the interval (q,s) is an element of M. It follows that the countable set $T = (q,s) \cap M$ is an element of $\mathcal{S}(M)$ that is $\prec$-cofinal in $\mathbb{R} \cap M$ together with all its $\prec$-final segments. Furthermore, for each $x \in \mathbb{R} \cap M$, r is a limit point of the final segment $\{y \in T : x \prec y\}$ of T. This clearly constitutes infinitely many members of $\mathcal{T}(M,r)$.

Let $\{T_n : n \in \omega\}$ enumerate all the elements of $\mathcal{T}(M,r)$. Note that then, for every $n \in \omega$, r is a limit point of the $\prec$-cofinal set $(r - 1/n, r + 1/n) \cap T_n$ as well.

It is now straightforward to construct, by an ω-length recursion, a sequence $\langle x_n : n < \omega \rangle$ in $\mathbb{R} \cap M$ such that $a = \{x_n : n \in \omega\}$ is as required. Indeed, start with $x_0 \in (r-1, r+1) \cap T_0$, and given x_n let x_{n+1} be the $\prec$-minimum of $\{y : x_{n-1} \prec y \text{ and } |y - r| < 2^{-n}\}$. □

We should emphasize that the above defined family $\mathcal{T}(M,r)$, and hence the set a that we chose, only depends on $\mathcal{S}(M)$, even though its definition used M.

We may now fix a listing $\{\mathcal{S}_r : r \in \mathbb{R}\}$ of $\mathfrak{S}$ that is $\mathfrak{c}$-abundant, i.e. for every $\mathcal{S} \in \mathfrak{S}$, the set $\{r \in \mathbb{R} : \mathcal{S}_r = \mathcal{S}\}$ has cardinality $\mathfrak{c}$.

Given any $r \in \mathbb{R}$, although there are many $M \in \mathfrak{M}$ with $\mathcal{S}(M) = \mathcal{S}_r$, we just fix one such M and then apply Lemma 3 to M and r to find a_r with the properties of a as described there. Again we note that the choice of a_r does not actually depend on this chosen M, but solely on the value of $\mathcal{S}(M) = \mathcal{S}_r$.

Next we show that with this choice for $\{a_r : r \in \mathbb{R}\}$ the topology τ satisfies (*), hence τ has uncountable net weight.

To see this, consider any countable subfamily $\mathcal{T}$ of $[\mathbb{R}]^{\mathfrak{c}}$ and choose any $\tilde{M} \in \mathcal{M}$ such that $\mathcal{T} \in \tilde{M}$. By construction we then have that $X = \{r : \mathcal{S}_r = \mathcal{S}(\tilde{M})\}$ has cardinality $\mathfrak{c}$. Choose any $r \in X$. To prove that (*) holds using Lemma 3 we must prove that for $T \in \mathcal{T}$ such that r is a complete accumulation point of T, $T \cap M \in \mathcal{T}(M,r)$. So assume that $T \in \mathcal{T}$ and that $r \in T^\circ$, recall that T° denotes the set of all complete accumulation points of T. Now we have that $T \cap T^o$ has cardinality $\mathfrak{c}$, in fact $T \setminus T^\circ$ has cardinality less than $\mathfrak{c}$. By elementarity, given that for all rationals $q < r < s$ the set $T \cap T^\circ \cap (q,r) \in M$ is $\prec$-cofinal, we then clearly have $T \cap T^\circ \cap M \in \mathcal{T}(M,r)$, consequently $a_r \cap T \neq \emptyset$.

Finally, we need to prove that $(\mathbb{R}, \tau)$ is in $C(\omega_1)$. To do this we consider any τ-neighborhood assignment U so that $\mathrm{dom}(U)$ is an uncountable subset of $\mathbb{R}$. By passing to an uncountable subset we may assume that there is a fixed pair $q < s \in \mathbb{Q}$

such that $r \in U(r) \supset (q,s) \setminus a_r$ for all $r \in \mathrm{dom}(U)$. We must then find an uncountable $Y \subset \mathrm{dom}(U)$ satisfying that $Y \subset (q,s) \setminus a_r$ for all $r \in Y$. Equivalently, it suffices to show that there is an uncountable $Y \subset \mathrm{dom}(U)$ satisfying that $y \notin a_r$ for all $r, y \in Y$. Fortunately, as the $\prec$-order type of each a_r is ω, this is an immediate consequence of the following instance (Proposition 4) of the classical Erdős-Specker theorem in [2]. □

Proposition 4. *If f is any function from a well-ordered set $(U, \prec)$ of order-type ω_1 such that $f(x) \subset U$ has $\prec$-order type at most ω for all $x \in U$, then there is an uncountable free set $Y \subset U$, meaning that $y \notin f(x)$ for all $x \neq y \in Y$.*

3. Going from CH to models obtained by adding Cohen reals to CH

Theorem 1.6 of [6] is a strengthening of the above mentioned Theorem 1.5 of [6]. It says that after adding any number of Cohen reals to a model of CH, in the extension every regular space in $C(\omega_1)$ belongs to a class called there $K(\omega_1)$, for which we have

$$N(\omega_1) \subset K(\omega_1) \subset C(\omega_1).$$

We do not give here the, somewhat involved, definition of $K(\omega_1)$ because we shall not need it. We only mention that, by [6], it is consistent to have regular spaces that belong to $K(\omega_1) \setminus N(\omega_1)$ or to $C(\omega_1) \setminus K(\omega_1)$.

Problem (2) on p. 4 of [6] asks if Theorem 1.6 of [6] can actually be strengthened to saying that after adding any number of Cohen reals to a model of CH, in the extension every regular space in $C(\omega_1)$ belongs even to $N(\omega_1)$. Our next result yields an affirmative answer to this problem, in fact it yields much more.

Theorem 5. *Assume that CH holds in our ground model V and κ is any cardinal. Then if we add κ Cohen reals, i.e. we force with $\mathrm{Fn}(\kappa, 2)$, then in the extension $V[G]$ every regular space in $C(\omega_1)$ has countable net weight.*

Proof. Let X be the base set for a topology and let τ denote a family of $\mathrm{Fn}(\kappa,2)$-names for subsets of X that is forced to be a basis for a regular topology on X that is in the class $C(\omega_1)$. Of course, we may assume that $X \in V$. We may also assume that the elements $\dot{W} \in \tau$ are nice names in the sense that they are subsets of $X \times \mathrm{Fn}(\kappa, 2)$ and, for every $x \in X$, the set $\{p : (x,p) \in \dot{W}\}$ is a (possibly empty) antichain. We may further assume that τ contains the set of all nice names $\dot{W}$ that have the property that $1 \Vdash \dot{W} \in \tau$. Naturally the evaluation, $\mathrm{val}_G(\dot{W})$, of such a name $\dot{W}$ by a generic filter G is the set $\{x : (\exists p \in G)\ (x,p) \in \dot{W}\}$.

Next we choose an elementary submodel M of $H(\theta)$, for any regular $\theta > 2^\kappa$, so that $\{X, \kappa, \tau\} \in M$. Choose M so that also $M^\omega \subset M$ and $|M| = \mathfrak{c} = \aleph_1$.

Let τ_M be the elements (names) of τ that are members of M (simply $\tau_M = \tau \cap M$). Let G_M be a generic filter for $\mathrm{Fn}(\kappa \cap M, 2) = M \cap \mathrm{Fn}(\kappa, 2)$. For each $\dot{W} \in \tau_M$, the set $\dot{W} \cap M$ is a $Fn(\kappa \cap M, 2)$-name of a subset of $X \cap M$. In this extension $V[G_M]$, we let

$$\sigma = \{\mathrm{val}_{G_M}(\dot{W} \cap M) : \dot{W} \in \tau \cap M\}.$$

We claim that σ generates a regular topology on $X \cap M$ in the extension $V[G_M]$. Let $x \in X \cap M$ and $\dot{U}$ be an element of $\tau \cap M$ and let $p \in G_M$ force that $x \in \dot{U}$. Since p forces the statement that there is a $W \in \tau$ with $x \in W \subset \overline{W} \subset \dot{U}$, we can apply the forcing maximum principle and our assumption on τ to deduce that there is a $\dot{W} \in \tau$ satisfying that $p \Vdash x \in \dot{W}$ and that $p \Vdash \mathrm{cl}_X(\dot{W}) \subset \dot{U}$. By elementarity, there is such a $\dot{W}$ in M. For each $y \in X \cap M$, each of the statements $p \Vdash y \in \mathrm{cl}_X(\dot{W})$ and $p \Vdash y \in \dot{U}$ are absolute between M and $H(\theta)$. This proves that the topology generated by σ is regular in the model $V[G_M]$. A similar, but even simpler, argument may be used to prove its Hausdorffness.

We now prove that, in the extension $V[G_M]$, the space $(X \cap M, \sigma)$ has a countable network consisting of closures of countable sets. Since CH holds in this extension, it will suffice for this, by [6, Theorem 1.5], to prove that $(X \cap M, \sigma)$ has the $C(\omega_1)$ property (in $V[G_M]$).

It will be simplest to work in V and to suppose that 1 forces $\dot{U}$ is a $\mathrm{Fn}(\kappa \cap M, 2)$-name (nice as usual) of a partial neighborhood assignment into $\tau \cap M$ and $\mathrm{dom}(\dot{U})$ is an uncountable subset of $X \cap M$. It should be clear that we may assume that the elements in the range of $\dot{U}$ are named by members of σ.

We use the assumption that 1 forces over $\mathrm{Fn}(\kappa, 2)$ that the space (X, τ) is in $C(\omega_1)$. So, there is a nice name, $\dot{Y}$, for an uncountable subset of $\mathrm{dom}(\dot{U})$ witnessing $C(\omega_1)$. By transfinite recursion we may then choose for $\alpha < \omega_1$ conditions $p_\alpha \in \mathrm{Fn}(\kappa, 2)$, distinct points $y_\alpha \in X \cap M$, and names $\dot{W}_\alpha \in \tau \cap M$, satisfying that p_α forces that $y_\alpha \in \dot{Y}$ and $\dot{U}(y_\alpha)$ is assigned to the name $\dot{W}_\alpha \cap M$. Let us note that $p_\alpha \cap M \in M$ forces that $y_\alpha \in \mathrm{dom}(\dot{U})$ and that $\dot{U}(y_\alpha) = \dot{W}_\alpha$.

Apply a standard Δ-system argument to find a root $\bar{p} \in \mathrm{Fn}(\kappa, 2)$ and an uncountable set $\Lambda \subset \omega_1$ satisfying that $\bar{p} \subset p_\alpha$ and $p_\alpha \setminus \bar{p}$ and $p_\beta \setminus \bar{p}$ have disjoint domains for all $\alpha \neq \beta \in \Lambda$.

Let us note that $p_\alpha \cup p_\beta$ forces that $y_\alpha \in \dot{W}_\beta$ for all $\alpha, \beta \in \Lambda$. Also, by elementarity we then have that, for all $\alpha, \beta \in \Lambda$,

(1) $(p_\alpha \cap M) \cup (p_\beta \cap M)$ forces that $y_\alpha \in \dot{W}_\beta$, and
(2) $(p_\alpha \cap M)$ forces that $\dot{U}(y_\alpha) = \dot{W}_\alpha \cap M$.

This completes the proof that $(X \cap M, \sigma)$ is forced by $\mathrm{Fn}(\kappa \cap M, 2)$ to have the $C(\omega_1)$ property.

It follows from [6] that $(X \cap M, \sigma)$ is therefore forced by $\mathrm{Fn}(\kappa \cap M, 2)$ to have a countable network. Since our space $(X \cap M, \sigma)$ is regular and hereditarily separable, it also has a countable network consisting of closed separable sets. Using that $M^\omega \subset M$, there is a list $\{\dot{S}_n : n \in \omega\} \in M$ of (nice) names for countable subsets of $X \cap M$ whose closures taken in $(X \cap M, \sigma)$ form a network for $(X \cap M, \sigma)$. However it now follows, by elementarity, that it is forced that $\{\mathrm{cl}_{(X,\tau)}(\dot{S}_n) : n \in \omega\}$ is a network for all of (X, τ). This is because, by the Tarski-Vaught criterion (see [5, Lemma IV.7.3]), $M[G]$ is an elementary submodel of $H(\theta)[G]$. □

Recall that a poset has property K if every uncountable subset of it has an uncountable subset that is linked. The interested reader can check that the only properties of the poset $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$ that were used in the proof are that both $\mathbb{P}$ and $\mathbb{P}/G_M$ have property K. These posets were similarly utilized in [1], where they were called *finally property K*.

4. Getting non-stationary sets of uncountable weight

Hart and Kunen constructed in [3] a consistent example of a first countable 0-dimensional, hence regular space of net weight ω_2 that is in $C(\omega_1)$. However, their example is also in $N(\omega_1)$.

In [6], Theorem 4.11 it was shown to be consistent that there is a 0-dimensional topology on ω_1 such that a subspace of it has countable net weight iff it is non-stationary. It is easy to see that such a space is in $C(\omega_1) \setminus N(\omega_1)$.

This led us to consider the natural question if an analogous result could be proved in which net weight is replaced with weight. Our next result shows that this is impossible for regular spaces.

Theorem 6. *If τ is a* regular *topology of uncountable weight on a stationary subset S of ω_1 then S has a non-stationary subset that has uncountable weight as well.*

Proof. If S has a countable subset of uncountable weight we are done because all countable sets are non-stationary. So we may assume that all countable subspaces have countable weight.

If τ is not hereditarily separable then S has an uncountable left-separated subspace, all of whose uncountable subspaces have uncountable weight. So, we are done again because every uncountable subset of S has an uncountable non-stationary subset. So, we may assume that τ is (hereditarily) separable

This, in turn, implies that τ is first countable. Indeed, let D be a countable τ-dense subset of S. Then $D \cup \{\xi\}$ is also τ-dense for every point $\xi \in S$. So, since τ is regular, we have

$$\chi(\xi, S) = \chi(\xi, D \cup \{\xi\}) \leq \omega$$

by our first assumption. We may thus fix for each $\xi \in S$ a countable τ-neighborhood base $\mathcal{B}_\xi$.

Now, let us fix a continuous increasing elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary submodels of $H(\theta)$ for a large enough regular cardinal θ such that S, τ, and the function sending $\xi \in S$ to $\mathcal{B}_\xi$ all belong to M_0. ($\theta = (2^{\omega_1})^+$ will suffice.) Moreover, let us put $\delta_\alpha = M_\alpha \cap \omega_1$ for each $\alpha < \omega_1$. It follows from our assumptions that $S \cap \delta_0$ is τ-dense in S.

As is well-known, then $C = \{\delta_\alpha : \alpha < \omega_1\}$ is closed and unbounded in ω_1, consequently $T = S \setminus C$ is non-stationary. Note that

$$T = (S \cap \delta_0) \cup \bigcup \{S \cap (\delta_\alpha, \delta_{\alpha+1}) : \alpha < \omega_1\}.$$

We shall complete the proof by showing that the τ-weight of T is uncountable.

To see this, let us put for each $\alpha < \omega_1$

$$\mathcal{A}_\alpha = \bigcup \{\mathcal{B}_\xi : \xi \in S \text{ and } \xi \leq \delta_\alpha\},$$

moreover set $\mathcal{A} = \bigcup \{\mathcal{A}_\alpha : \alpha < \omega_1\}$. Note that we have $\mathcal{B}_\xi \in M_\alpha$ whenever $\xi \in S \cap \delta_\alpha = S \cap M_\alpha$, consequently, we have $\mathcal{A}_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$.

Since $\mathcal{A}_\alpha$ is countable for each $\alpha < \omega_1$, it follows that $\mathcal{A}_\alpha$ is not a τ-base, hence there are some $\eta \in S$ and $V \in \mathcal{B}_\eta$ such that if $\eta \in B \in \mathcal{A}_\alpha$ then $B \setminus \overline{V} \neq \emptyset$. Here we use the regularity of τ again. Actually, by elementarity, there is such an $\eta \in S$ satisfying $\delta_\alpha < \eta < \delta_{\alpha+1}$, hence $\eta \in T$.

Since $S \cap \delta_0 \subset T$ is τ-dense in S, we have $T \cap (B \setminus \overline{V}) \neq \emptyset$ as well, hence the above chosen $\eta \in T$ is a witness for the fact that the family $\{B \cap T : B \in \mathcal{A}_\alpha\}$ is not a τ-base of T. On the other hand, $\{B \cap T : B \in \mathcal{A}\}$ clearly is a τ-base of T.

It follows that no countable subfamily of $\{B \cap T : B \in \mathcal{A}\}$ is a τ-base of T. But every base of any space X has a subfamily of size $w(X)$ that is a base of X, consequently we conclude that the τ-weight of T must indeed be uncountable. □

Of course, Theorem 6 implies that no *regular* topology on ω_1 can have the property that a subspace of it has countable weight iff it is non-stationary. Our last result implies this corollary of Theorem 6 for all topologies on ω_1 but with a considerably harder proof.

Theorem 7. *If τ is any topology on ω_1 such that every final segment $[\gamma, \omega_1)$ of ω_1 has uncountable weight then some non-stationary subset of ω_1 has uncountable weight as well.*

The remainder of this section is devoted to the proof of Theorem 7. Thus we suppose that τ is a topology on ω_1 and that every final segment has uncountable weight. Just as in the proof of Theorem 6, we may then assume that all countable subsets have countable weight, moreover that τ is hereditarily separable, because any uncountable left separated subspace has an uncountable non-stationary subset, which even has uncountable net weight.

Definition 8. *We say that $\alpha \in \omega_1$ is a* bad point *if for every countable collection $\mathcal{N} \subset \tau$ of neighborhoods of α there is a non-stationary subset $A \subset \omega_1$ with $\alpha \in A$ such that $\mathcal{N} \restriction A = \{U \cap A : U \in \mathcal{N}\}$ is not a local basis at α in the subspace A.*

Clearly, any bad point must have uncountable τ-character. We will prove Theorem 7 first in the case that there are no bad points, and secondly when there is a bad point.

Lemma 9. *Assume that no bad points exist. Then there is a non-stationary set of uncountable weight.*

Proof. By our assumption we may assign to every $\alpha \in \omega_1$ a countable collection of neighborhoods $\mathcal{N}_\alpha$ such that $\mathcal{N}_\alpha \restriction A$ is a local base at α in A whenever $\alpha \in A$ and A is a non-stationary subset of ω_1.

For any countable subset $\mathcal{U} \subset \tau$ and for any $\gamma < \omega_1$ we know that $\mathcal{U} \restriction [\gamma, \omega_1)$ is not a basis, hence there is a countable subset $I \subset [\gamma, \omega_1)$ of limit order type that witnesses this in the following way: There are $\alpha \in I$ and a neighborhood V of α so that $I \cap V \setminus U \neq \emptyset$ for all $U \in \mathcal{U}$. We denote by $\mathcal{W}(\mathcal{U}, \gamma)$ the set of all such witnesses $I \subset [\gamma, \omega_1)$.

We next define by recursion on $\xi < \omega_1$ countable sets I_ξ and ordinals γ_ξ such that the sequence $\langle \gamma_\xi : \xi < \omega_1 \rangle$ is strictly increasing and continuous, moreover $\bigcup\{I_\xi : \xi < \eta\} \subset \gamma_\eta$ for all $\eta < \omega_1$.

To begin with, we choose $I_0 \in [\omega_1]^\omega$ of limit order type arbitrarily, and then set $\gamma_0 = \sup I_0$. If $\{I_\xi : \xi < \eta\}$ and $\{\gamma_\xi : \xi < \eta\}$ have been so defined, then we let $A_\eta = \bigcup\{I_\xi : \xi < \eta\}$ and $\gamma_\eta = \sup A_\eta$, moreover $\mathcal{U}_\eta = \bigcup\{\mathcal{N}_\alpha : \alpha \in A_\eta\}$. Then we choose $I_\eta \in \mathcal{W}(\mathcal{U}_\eta, \gamma_\eta)$.

Clearly, this recursion goes through for all $\eta < \omega_1$, moreover $A = \bigcup\{I_\xi : \xi < \omega_1\}$ is non-stationary, for it is disjoint from the cub set $\{\gamma_\xi : \xi < \omega_1\}$. But then for $\mathcal{U} = \bigcup\{\mathcal{N}_\alpha : \alpha \in A\}$ we have that $\mathcal{U} \restriction A$ is a basis of A, while by our construction no countable subset of it is, hence A has uncountable weight. □

So, for the remainder of the proof of Theorem 7 we are assuming that there are bad points.

Lemma 10. *If $\alpha \in \omega_1$ is a bad point and if $\mathcal{U} \subset \tau$ is any countable collection of neighborhoods of α then for every $\alpha < \gamma < \omega_1$ there is a countable subset I of the final segment $[\gamma, \omega_1)$ such that $\alpha \notin \overline{I}$ but $U \cap I \neq \emptyset$ for all $U \in \mathcal{U}$.*

Proof. Since we know that $\gamma = [0, \gamma)$ has countable weight, by enlarging $\mathcal{U}$ if necessary, we may assume $\mathcal{U}$ is a filter base and satisfies that $\mathcal{U} \restriction \gamma$ is a local basis at α in the countable subspace γ. By definition, as α is a bad point, there is a non-stationary set A with $\alpha \in A$ such that $\mathcal{U} \restriction A$ is not a local basis at α in the subspace A. Therefore we may choose an open neighborhood W of α for which $A \cap U \setminus W \neq \emptyset$ for all $U \in \mathcal{U}$. Since there is a $V \in \mathcal{U}$ such that $A \cap \gamma \cap V \subset W$, we actually have that $(A \setminus \gamma) \cap U \setminus W \neq \emptyset$ for all $U \in \mathcal{U}$. Since $\mathcal{U}$ is countable, there is a countable set $I \subset A \setminus \gamma$ such that $I \cap U \setminus W \neq \emptyset$ for all $U \in \mathcal{U}$. Since $I \cap W = \emptyset$, we have that $\alpha \notin \overline{I}$. □

In the remaining part of our proof we shall make heavy use of the cub subsets of ω_1. We denote by $\mathcal{C}$ the family of all cub subsets of ω_1, moreover we make use of the following standard fact.

Lemma 11. *Let f be any function from $\mathcal{C}$ into ω_1. Then there is a $\gamma \in \omega_1$ such that for any $D \in \mathcal{C}$ there is $C \in \mathcal{C}$ with $f(C) = \gamma$ and $|C \setminus D| \leq \omega$.*

Proof. Suppose that, for each $\delta < \omega_1$, there is a cub C_δ satisfying that no member of $f^{-1}(\delta)$ is contained, mod countable, in C_δ. Let $\bar{C}$ denote the diagonal intersection $\{\gamma \in \omega_1 : \delta < \gamma \Rightarrow \gamma \in C_\delta\}$. Then $\bar{C} \in \mathcal{C}$, so we let $f(\bar{C}) = \bar{\delta}$. But $\bar{C} \setminus C_{\bar{\delta}}$ is countable, contradicting our choice of $C_{\bar{\delta}}$. □

Now we prove a stronger consequence of the existence of a bad point that was suggested by the referee.

Lemma 12. *Let $\alpha \in \omega_1$ be a bad point. Then there is a non-stationary set $A \subset \omega_1$ containing α such that the character of α in A is uncountable. So, a fortiori the set A has uncountable weight.*

Proof. For each $C \in \mathcal{C}$ we choose a countable elementary submodel M_C of a suitable $H(\theta)$ such that $\alpha, C, \tau \in M_C$ and then set $\delta_C = M_C \cap \omega_1$. Also, using that our topology τ is hereditarily separable, we choose, for every open set $U \in \tau$, a countable set $H(U)$ whose τ-closure is equal to $\omega_1 \setminus U$.

For every $C \in \mathcal{C}$, the set $A_C = \delta_C \cup (\omega_1 \setminus C)$ is non-stationary. We claim that there is a $C \in \mathcal{C}$ such that A_C is as required by the Lemma. We argue by contradiction and choose for every $C \in \mathcal{C}$ a countable family $\mathcal{U}_C$ of neighborhoods of α that traces as a local base at α in the subspace A_C.

Now we define a function f from $\mathcal{C}$ into ω_1 as follows:

$$f(C) = \min\{\gamma : \delta_C \leq \gamma \text{ and } H(U) \subset \gamma \text{ for all } U \in \mathcal{U}_C\} .$$

We then apply Lemma 11 to find $\zeta \in \omega_1$ so that for any $D \in \mathcal{C}$ there is $C \in \mathcal{C}$ with $C \setminus D$ countable and $f(C) = \zeta$. Let $\delta = \min\{\delta_C : C \in f^{-1}(\zeta)\}$; by the choice of the M_C, we know that $\alpha < \delta \leq \zeta$. Furthermore, let $\mathcal{B}$ be a countable family of open sets that determine a local base at α in the subspace ζ.

We construct, recursively, sequences $\langle \gamma_\eta : \eta < \omega_1 \rangle$ and $\langle I_\eta : \eta < \omega_1 \rangle$ as follows. Let $\gamma_0 = \zeta$ and apply Lemma 10 to find a countable subset I_0 of $\omega_1 \setminus \gamma_0$ such that $\alpha \notin \overline{I_0}$ but $B \cap I_0 \neq \emptyset$ for all $B \in \mathcal{B}$. Given $\langle I_\xi : \xi < \eta \rangle$, let γ_η be the minimum ordinal that contains $\bigcup\{I_\xi : \xi < \eta\}$, and then apply Lemma 10 to obtain a countable $I_\eta \subset (\gamma_\eta, \omega_1)$ such that $\alpha \notin \overline{I_\eta}$ but $B \cap I_\eta \neq \emptyset$ for all $B \in \mathcal{B}$.

The set $D = \{\gamma_\eta : \eta \in \omega_1\}$ is closed and unbounded and is disjoint from the union $A = \bigcup\{I_\eta : \eta < \omega_1\}$. There is a closed and unbounded C with $f(C) = \zeta$ and $C \setminus D$ countable. Since $C \cap A$ is countable, we have that $C \cap I_\eta$ is empty for all but countably many η.

Take any η such that $C \cap I_\eta$ is empty. Since $\delta \leq \delta_C$, A_C contains $\delta \cup (\omega_1 \setminus C)$, and so, $\alpha \notin \overline{I_\eta}$ implies that there is some $U \in \mathcal{U}_C$ such that $I_\eta \cap U$ is empty. Therefore $I_\eta \subset \overline{H(U)}$ and $H(U) \subset f(C) = \zeta$. Since $\mathcal{B}$ is a local base at α in ζ, there is a $B \in \mathcal{B}$ such that $B \cap \zeta \subset U$. But this means that $B \cap H(U)$ is empty, contradicting that $B \cap \overline{H(U)} \supset B \cap I_\eta$ is not empty. □

Acknowledgement The research on this paper was started when the second author visited the Department of Mathematics and Statistics of UNC Charlotte. He would like to thank this institution as well as the NKFIH grant no. 149211 for financially supporting this visit.

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