SOME IDENTITIES INVOLVING CONVOLUTIONS OF DIRICHLET CHARACTERS AND THE MÖBIUS FUNCTION

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ABSTRACT. In this paper we present some identities involving convolutions of Dirichlet characters and the Möbius function, which are related to a well known identity of Ramanujan, Hardy and Littlewood.

1. Introduction

In the present paper we consider a class of sums which involve Dirichlet characters and the Möbius function convolved with itself. We are interested in potential identities which involve such sums together with sums over zeros of the Riemann zeta-function, but which do not involve the zeros of the given Dirichlet L-function.

Let us consider the arithmetic functions $a_1(n)$ and $a_2(n)$ given by the convolutions

$$a_1 := \chi * \mu * \mu, \tag{1.1}$$

and

$$a_2 := \chi_3 * \chi_2\mu * \chi_1\mu, \tag{1.2}$$

where μ denotes the Möbius function and $\chi, \chi_1, \chi_2, \chi_3$ are Dirichlet characters. Alternatively one may first define $a_1(n)$ on prime powers by letting

$$a_1(p^k) = \begin{cases} \chi(p) - 2 & \text{if } k = 1, \\ \chi(p^{k-2})(\chi(p) - 1)^2 & \text{if } k \geq 2, \end{cases} \tag{1.3}$$

and then extend the definition of $a_1(n)$ by multiplicativity. Similarly one may define $a_2(n)$ to be the unique arithmetic function which is multiplicative and which is defined on prime powers by

$$a_2(p^k) = \begin{cases} \chi_3(p) - \chi_2(p) - \chi_1(p) & \text{if } k = 1, \\ \chi_3(p^{k-2})(\chi_3(p) - \chi_2(p))(\chi_3(p) - \chi_1(p)) & \text{if } k \geq 2. \end{cases} \tag{1.4}$$

Ramanujan (see [2, 3, 4, 13, 14]) communicated an identity to Hardy and Littlewood during his stay in Cambridge, which was missing the contribution of the non-trivial zeros of the Riemann zeta-function. The corrected version, established by Hardy and Littlewood in [11], is as follows:

2010 *Mathematics Subject Classification.* 11M06, 11M26.

Key words and phrases: Riemann zeta-function, Möbius function, Dirichlet L-functions.

Let α and β be two positive numbers such that $\alpha\beta = \pi$. Assume that the series $\sum_\rho \left(\Gamma\left(\frac{1-\rho}{2}\right)/\zeta'(\rho)\right)\beta^\rho$ converges, where ρ runs through the non-trivial zeros of $\zeta(s)$, and that the non-trivial zeros of $\zeta(s)$ are simple. Then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha^2/n^2}-\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\beta^2/n^2}=-\frac{1}{2\sqrt{\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\beta^{\rho}. \tag{1.5}$$

For more work related to this identity, the reader is referred to Berndt [4, p. 470], Bhaskaran [5], Paris and Kaminiski [12, p. 143] and Titchmarsh [16, p. 219, Section 9.8]. In [7], Dixit obtained the following character analogue of the Ramanujan-Hardy-Littlewood identity (1.5):

Let χ be a primitive character modulo q, and α and β be two positive numbers such that $\alpha\beta = 1$. Assume the series $\sum_\rho \frac{\pi^{\rho/2}\beta^\rho\Gamma((1+a-\rho)/2)}{q^{\rho/2}L'(\rho,\bar{\chi})}$ converges, where ρ denotes a non-trivial zero of $L(s,\bar{\chi})$ and that the non-trivial zeros of the associated Dirichlet L-function are simple. Then

$$\begin{aligned}\alpha^{a+\frac{1}{2}}\sqrt{\epsilon_\chi}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1+a}}e^{-\frac{\pi\alpha^2}{qn^2}}&-\beta^{a+\frac{1}{2}}\sqrt{\epsilon_{\bar{\chi}}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^{1+a}}e^{-\frac{\pi\beta^2}{qn^2}}\\&=-\frac{\sqrt{\epsilon_{\bar{\chi}}}}{2\sqrt{\beta}}\left(\frac{q}{\pi}\right)^{\frac{1+a}{2}}\sum_{\rho}\frac{\Gamma\left(\frac{1+a-\rho}{2}\right)}{L'(\rho,\bar{\chi})}\left(\frac{\pi}{q}\right)^{\frac{\rho}{2}}\beta^{\rho}.\end{aligned} \tag{1.6}$$

Here

$$a=\begin{cases}0 & \text{if } \chi(-1)=1,\\ 1 & \text{if } \chi(-1)=-1\end{cases} \tag{1.7}$$

and ϵ_χ denotes the Gauss sum

$$\epsilon_\chi := \sum_{n=1}^{q}\chi(n)e^{\frac{2\pi i n}{q}},$$

which satisfies

$$|\epsilon_\chi| = \sqrt{q} \text{ and } \epsilon_\chi\epsilon_{\bar{\chi}} = q\chi(-1),$$

for a primitive character χ mod q.

In [8], Dixit obtained a one-variable generalization of (1.5), and in [9] and [10] Dixit and two of the authors obtained a one variable generalization of (1.6) and respectively analogues of these identities to Hecke forms.

We remark that it is not necessary to assume convergence of the series on the right-hand side of (1.5) and (1.6), instead one can bracket the terms of the series as explained in [11, p. 158] and [16, p. 220].

It is worth mentioning that the proofs of the above results in (1.5) and (1.6) are quite sensitive to the type of functional equation satisfied by the L-functions involved in those identities. Also, specific knowledge of a certain Mellin transform obtained from those functional equations is needed. One new class of identities where one can handle the corresponding functional equations and where one does have knowledge of the needed Mellin transform is discussed in the present paper.

We will prove the following result.

Theorem 1. *Let χ be an even primitive Dirichlet character mod q. Assume that the zeros of $\zeta(s)$ are simple, and distinct from the zeros of $L(s,\chi)$. Let α and β be two positive real numbers such that $\alpha\beta = q\pi$. Then,*

$$\sqrt{\frac{\alpha}{\epsilon_{\bar\chi}}}\sum_{n=1}^{\infty}\frac{\tilde a_1(n)}{n}e^{\frac{-\alpha^2}{n^2}} - \sqrt{\frac{\beta}{\epsilon_{\chi}}}\sum_{n=1}^{\infty}\frac{a_1(n)}{n}e^{\frac{-\beta^2}{n^2}} = -\frac{1}{2\sqrt{\beta\epsilon_\chi}}\sum_{\rho} S_\rho, \tag{1.8}$$

where $\tilde a_1(n) = (\bar\chi * \mu * \mu)(n)$, $\rho = \tau + i\nu$ runs through the non-trivial zeros of $\zeta(s)$, and S_ρ is given by

$$S_\rho = \frac{\beta^\rho L(\rho,\chi)\Gamma\left(\frac{1-\rho}{2}\right)}{\left(\zeta'(\rho)\right)^2}\left(\log\beta - \frac{\zeta''(\rho)}{\zeta'(\rho)} + \frac{L'(\rho,\chi)}{L(\rho,\chi)} - \frac{1}{2}\psi\left(\frac{1-\rho}{2}\right)\right),$$

where ψ denotes the digamma function. The sum over ρ involves bracketing the terms so that the terms for which

$$|\nu - \nu'| < \exp(-A_1\nu/\log\nu) + \exp(-A_1\nu'/\log\nu'),$$

where A_1 is a suitable positive constant, are included in the same bracket.

Let us remark that, while the Dirichlet L-function $L(s,\chi)$ and its derivative do appear in the above identity, the summation on the right-hand side of (1.8) is over the zeros of the Riemann zeta-function only. The exact location of the zeros of $L(s,\chi)$ is irrelevant for the above identity, as long as these zeros differ from the zeros of $\zeta(s)$. This is widely believed to be so. In fact, by the Grand Simplicity Hypothesis (see Rubinstein and Sarnak [15]), combined with the Generalized Riemann Hypothesis, these zeros should not only be simple and distinct, but they should all lie on the critical line and their imaginary parts should be linearly independent over $\mathbb{Q}$. The same remark applies to the hypotheses from the statement of Theorem 2 below. Returning to Theorem 1, it is worth mentioning that numerical computations show that, despite the complicated shape of the sum over the zeros of the Riemann zeta-function on the right-hand side of the identity, in practice this sum is rapidly convergent. For example, in Table 1 below it was enough to take the first 100 zeros in order for the first six digits on both sides of the identity to coincide.

By the same method one can prove the following result.

Theorem 2. *Let χ_1, χ_2 and χ_3 be primitive Dirichlet characters of conductors q_1, q_2 and q_3 respectively. Assume that the zeros of $L(s,\chi_1)L(s,\chi_2)$ are simple and distinct from the zeros of $L(s,\chi_3)$. Let α and β be two positive numbers such that $\alpha\beta = \frac{\pi q_3}{q_1q_2}$. Let $\tilde a_2(n) = (\bar\chi_3 * \bar\chi_1\mu * \bar\chi_2\mu)(n)$. Then,*

a) If χ_1, χ_2 and χ_3 are all even, we have

$$\sqrt{\frac{\alpha\epsilon_{\bar\chi_1}\epsilon_{\bar\chi_2}}{\epsilon_{\bar\chi_3}}}\sum_{n=1}^{\infty}\frac{\tilde a_2(n)}{n}e^{\frac{-\alpha^2}{n^2}} - \sqrt{\frac{\beta\epsilon_{\chi_1}\epsilon_{\chi_2}}{\epsilon_{\chi_3}}}\sum_{n=1}^{\infty}\frac{a_2(n)}{n}e^{\frac{-\beta^2}{n^2}}$$

$$= -\frac{\sqrt{\epsilon_{\chi_1}\epsilon_{\chi_2}}}{2\sqrt{\beta\epsilon_{\chi_3}}}\left[\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)L(\rho,\chi_3)}{L'(\rho,\chi_1)L(\rho,\chi_2)}\beta^{\rho}+\sum_{\rho'}\frac{\Gamma\left(\frac{1-\rho'}{2}\right)L(\rho',\chi_3)}{L(\rho',\chi_1)L'(\rho',\chi_2)}\beta^{\rho'}\right]. \tag{1.9}$$

b) If χ_1, χ_2 and χ_3 are all odd, we have

$$\alpha\sqrt{\frac{\alpha\epsilon_{\bar{\chi}_1}\epsilon_{\bar{\chi}_2}}{\epsilon_{\bar{\chi}_3}}}\sum_{n=1}^{\infty}\frac{\tilde{a}_2(n)}{n^2}e^{\frac{-\alpha^2}{n^2}}-\beta\sqrt{\frac{\beta\epsilon_{\chi_1}\epsilon_{\chi_2}}{\epsilon_{\chi_3}}}\sum_{n=1}^{\infty}\frac{a_2(n)}{n^2}e^{\frac{-\beta^2}{n^2}}$$

$$= -\frac{\sqrt{\epsilon_{\chi_1}\epsilon_{\chi_2}}}{2\sqrt{\beta\epsilon_{\chi_3}}}\left[\sum_{\rho}\frac{\Gamma\left(\frac{2-\rho}{2}\right)L(\rho,\chi_3)}{L'(\rho,\chi_1)L(\rho,\chi_2)}\beta^{\rho}+\sum_{\rho'}\frac{\Gamma\left(\frac{2-\rho'}{2}\right)L(\rho',\chi_3)}{L(\rho',\chi_1)L'(\rho',\chi_2)}\beta^{\rho'}\right]. \tag{1.10}$$

Here $\rho = \tau + i\nu$ and $\rho' = \tau' + i\nu'$ run through the non-trivial zeros of $L(s,\chi_1)$ and $L(s,\chi_2)$ respectively, and the sums over ρ and ρ' involve bracketing the terms so that the terms for which

$$|\nu_1 - \nu_2| < \exp(-A_1\nu_1/\log\nu_1) + \exp(-A_1\nu_2/\log\nu_2)$$

and

$$|\nu_1' - \nu_2'| < \exp(-A_2\nu_1'/\log\nu_1') + \exp(-A_2\nu_2'/\log|\nu_2'),$$

where A_1 and A_2 are suitable positive constants, are included in the same bracket.

2. Preliminaries

The Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1-p^{-s})^{-1}, \text{ for } \operatorname{Re} s > 1, \tag{2.1}$$

has an analytic continuation to the entire complex plane except for a simple pole at $s = 1$, and satisfies the functional equation

$$\xi(s) := \frac{s(s-1)}{2}\pi^{\frac{-s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \xi(1-s). \tag{2.2}$$

For a primitive Dirichlet character χ mod q, the Dirichlet L-function

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s} = \prod_p(1-\chi(p)p^{-s})^{-1}, \text{ for } \operatorname{Re} s > 1, \tag{2.3}$$

has an analytic continuation to the entire complex plane and satisfies the functional equation

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-\frac{s+a}{2}}\Gamma(\frac{s+a}{2})L(s,\chi) = i^{-a}q^{-\frac{1}{2}}\epsilon_{\chi}\xi(1-s,\bar{\chi}), \tag{2.4}$$

where a and ϵ_{χ} are as in the Introduction (see [6]).

Let us consider the functions

$$F(s,\chi) := \frac{\zeta^2(s)}{L(s,\chi)} \tag{2.5}$$

and

$$G(s,\chi_1,\chi_2,\chi_3) := \frac{L(s,\chi_1)L(s,\chi_2)}{L(s,\chi_3)}. \tag{2.6}$$

The Dirichlet series associated to F and G converge absolutely for $\operatorname{Re} s > 1$. From (2.2) and (2.4) we see that for an even primitive character χ mod q, the function $F(\chi, s)$ satisfies the functional equation

$$\frac{\Gamma\left(\frac{1-s}{2}\right)}{F(s,\chi)} = \sqrt{\pi}\epsilon_\chi(\pi q)^{-s}\frac{\Gamma\left(\frac{s}{2}\right)}{F(1-s,\bar{\chi})}. \tag{2.7}$$

Similarly we obtain a functional equation for $G(s,\chi_1,\chi_2,\chi_3)$. For primitive characters χ_1, χ_2 and χ_3 of the same parity and conductors q_1, q_2 and q_3 respectively, we have

$$\frac{\Gamma\left(\frac{s+a}{2}\right)}{G(1-s,\bar{\chi_1},\bar{\chi_2},\bar{\chi_3})} = i^{-a}\pi^{-\frac{1}{2}}\frac{\epsilon_{\chi_1}\epsilon_{\chi_2}}{\epsilon_{\chi_3}}\left(\frac{\pi q_3}{q_1 q_2}\right)^s \frac{\Gamma\left(\frac{1-s+a}{2}\right)}{G(s,\chi_1,\chi_2,\chi_3)}. \tag{2.8}$$

Using (1.1), (2.1), and (2.3) one sees that

$$\frac{1}{F(s,\chi)} = \sum_{n=1}^{\infty}\frac{a_1(n)}{n^s}$$

for $\operatorname{Re} s > 1$. Similarly using (1.2) and (2.3) one has

$$\frac{1}{G(s,\chi_1,\chi_2,\chi_3)} = \sum_{n=1}^{\infty}\frac{a_2(n)}{n^s}$$

for $\operatorname{Re} s > 1$.

We will also need the following result, which is Lemma 3.1 from [1].

Lemma 1. *Let χ be a primitive character of conductor N, and let $k \geq 2$ be an integer such that $\chi(-1) = (-1)^k$. Then*

$$\frac{(k-2)!N^{k-2}\epsilon_\chi}{2^{k-1}\pi^{k-2}i^{k-2}}L(k-1,\bar{\chi}) = L^{'}(2-k,\chi). \tag{2.9}$$

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Throughout this proof we assume that χ is an even primitive character mod q. We make use of the inverse Mellin transform for the Γ-function,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}\,ds = \begin{cases} e^{-x} & \text{if } c > 0, \\ e^{-x}-1 & \text{if } -1 < c < 0. \end{cases} \tag{3.1}$$

One has

$$\sum_{n=1}^{\infty}\frac{\tilde{a}_1(n)}{n} = \sum_{n=1}^{\infty}\frac{(\bar{\chi}*\mu*\mu)(n)}{n} = 0, \tag{3.2}$$

as a consequence of the prime number theorem (consistent with the fact that $1/F(s,\bar{\chi})$ vanishes at $s=1$). Using (3.1) and (3.2) for $-1<$ Re $s=c<0$, we have

$$\sum_{n=1}^{\infty}\frac{\tilde{a}_1(n)}{n}e^{\frac{-\alpha^2}{n^2}}=\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\tilde{a}_1(n)}{n}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\left(\frac{\alpha^2}{n^2}\right)^{-s}\,ds. \tag{3.3}$$

By an application of Stirling's formula one sees that the right-hand side of (3.3) converges uniformly, and the summation and integration in (3.3) can be interchanged. For $-1<c<0$ we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\tilde{a}_1(n)}{n}e^{\frac{-\alpha^2}{n^2}}&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\sum_{n=1}^{\infty}\frac{\tilde{a}_1(n)}{n^{1-2s}}\alpha^{-2s}\,ds\\&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{F(1-2s,\bar{\chi})}\alpha^{-2s}\,ds.\end{aligned} \tag{3.4}$$

Using the functional equation (2.7) in (3.4) we find that

$$\sum_{n=1}^{\infty}\frac{\tilde{a}_1(n)}{n}e^{\frac{-\alpha^2}{n^2}}=\frac{1}{\sqrt{\pi}\epsilon_\chi 2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\pi^{2s}q^{2s}\alpha^{-2s}\,ds,$$

for $-1<c<0$. For a large positive real number T, consider the contour $\mathcal{C}$ determined by the line segments $[c-iT,c+iT]$, $[c+iT,\lambda+iT]$, $[\lambda+iT,\lambda-iT]$, and $[\lambda-iT,c-iT]$ in order, where $\frac{1}{2}<\lambda<\frac{3}{2}$. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\pi^{2s}q^{2s}\alpha^{-2s}\,ds=-\sum_{-T<\text{ Im }(\rho/2)<T}\text{res}_{\rho/2}, \tag{3.5}$$

where the sum is over nontrivial zeros ρ of the Riemann zeta-function. Note that by our assumptions, every nontrivial zero ρ of the Riemann zeta-function is a zero of $F(s,\chi)$, and hence $\rho/2$ is a pole for the integrand on the left-hand side above. Therefore each such ρ which lies inside the above rectangle needs to be counted on the right hand side above. Letting m_ρ denote the multiplicity of ρ, the residue $\text{res}_{\rho/2}$ is given by

$$\text{res}_{\rho/2}=\frac{1}{(m_\rho-1)!}\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho/2)^{m_\rho}\,\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\left(\frac{\pi q}{\alpha}\right)^{2s}\Big|_{s=\rho/2}.$$

Moreover, by our assumptions, all the zeros of the Riemann zeta-function are simple, and all the above poles are double poles. Therefore

$$\text{res}_{\rho/2}=\frac{d}{ds}(s-\rho/2)^2\,\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\left(\frac{\pi q}{\alpha}\right)^{2s}\Big|_{s=\rho/2}.$$

Using the equality $\alpha\beta = q\pi$ we write

$$\begin{aligned}\mathrm{res}_{\rho/2} &= \lim_{s\to\rho/2}\frac{d}{ds}\left[(s-\rho/2)^2\,\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\beta^{2s}\right] \\ &= \frac{\beta^{\rho}L(\rho,\chi)\Gamma\left(\frac{1-\rho}{2}\right)}{2\left(\zeta'(\rho)\right)^2}\left(\log\beta-\frac{\zeta''(\rho)}{\zeta'(\rho)}+\frac{L'(\rho,\chi)}{L(\rho,\chi)}-\frac{1}{2}\psi\left(\frac{1-\rho}{2}\right)\right),\end{aligned} \tag{3.6}$$

where $\psi(z) = \frac{\Gamma'}{\Gamma}(z)$ is the digamma function. By (3.5), we may write

(3.7)
$$\frac{1}{2\pi i}\int\limits_{\lambda-iT}^{\lambda+iT}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\left(\frac{\pi q}{\alpha}\right)^{2s}\,ds-\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\left(\frac{\pi q}{\alpha}\right)^{2s}\,ds=\sum_{-T<\ \mathrm{Im}\ (\rho/2)<T}\mathrm{res}_{\rho/2}+I_2+I_3,$$

where

$$I_2 = \frac{1}{2\pi i}\int\limits_{c+iT}^{\lambda+iT}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\left(\frac{\pi q}{\alpha}\right)^{2s}\,ds,$$

and

$$I_3 = \frac{1}{2\pi i}\int\limits_{\lambda-iT}^{c-iT}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\left(\frac{\pi q}{\alpha}\right)^{2s}\,ds.$$

We now proceed to show that $I_2 \to 0$ and $I_3 \to 0$ as $T \to \infty$ along a suitable sequence of values. We have

$$\begin{aligned}I_2 &= \frac{1}{2\pi i}\int\limits_{c+iT}^{\lambda+iT}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)}\left(\frac{\pi q}{\alpha}\right)^{2s}\,ds \\ &= \frac{1}{2\pi i}\int\limits_{c+iT}^{\lambda+iT}\frac{\Gamma\left(\frac{1-2s}{2}\right)L(2s,\chi)}{\zeta^2(2s)}\left(\frac{\pi q}{\alpha}\right)^{2s}\,ds \\ &= \frac{1}{2\pi i}\int\limits_{c}^{\lambda}\frac{\Gamma\left(\frac{1}{2}(1-2x-2iT)\right)L(2x+2iT,\chi)}{\zeta^2(2x+2iT)}\left(\frac{\pi q}{\alpha}\right)^{2x+2iT}\,dx.\end{aligned}$$

Thus

$$|I_2| \ll \int\limits_{c}^{\lambda}\frac{\left|\Gamma\left(\frac{1-2x}{2}-iT\right)\right|\,|L(2x+2iT,\chi)|}{|\zeta^2(2x+2iT)|}\left(\frac{\pi q}{\alpha}\right)^{2x}\,dx.$$

Substituting $x = \sigma/2$, we have

$$|I_2| \ll \int_{2c}^{2\lambda} \frac{\left|\Gamma\left(\frac{1-\sigma}{2} - iT\right)\right| \, |L(\sigma+2iT,\chi)|}{|\zeta^2(\sigma+2iT)|} \left(\frac{\pi q}{\alpha}\right)^{\sigma} \, d\sigma. \tag{3.8}$$

By Stirling's formula for $\Gamma(s)$, $s = \sigma + it$ in a vertical strip $\alpha \le \sigma \le \beta$,

$$|\Gamma(s)| = |(2\pi)^{\frac{1}{2}} |t|^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \left(1 + O\left(\frac{1}{|t|}\right)\right),$$

as $|t| \to \infty$, so we get

$$\left|\Gamma\left(\frac{1-\sigma}{2} - iT\right)\right| = (2\pi)^{\frac{1}{2}} |T|^{-\frac{1}{2}\sigma} e^{-\frac{1}{2}\pi|T|} \left(1 + O\left(\frac{1}{|T|}\right)\right). \tag{3.9}$$

From [16, p. 218, Equation(9.7.3)], we find that

$$\log|\zeta(\sigma+i2T)| \ge \sum_{|2T-\gamma|\le 1} \log|2T-\gamma| + O(\log T).$$

Let $N(T)$, $T > 0$, denote the number of zeros of $\zeta(s)$ in the region $0 < \sigma < 1,\ 0 \le t \le T$, where $s = \sigma + it$. Then,

$$N(T+1) - N(T) = O(\log T) \text{ as } T \to \infty$$

(see [16, p. 211, Equation(9.2.1)]). So, we may find T with arbitrarily large absolute value such that for any ordinate γ of a zero of $\zeta(s)$,

$$|2T-\gamma| \gg \frac{1}{\log T}.$$

In what follows, we only require that T is such that for any ordinate γ of a zero of $\zeta(s)$,

$$|2T-\gamma| > e^{-\frac{A_1\gamma}{\log\gamma}},$$

where A_1 is some suitable positive constant. Let us remark that this vastly increases the set of admissible values of T. At the same time let us also note that, while the assumptions from the statement of the theorem force zeros to be distinct, this does not prohibit two consecutive zeros from being extremely close to each other. In that case we may not be able to find a T which is admissible in the above sense, and such that $2T$ lies between the ordinates of these two zeros. This remark explains the idea, alluded to in the Introduction, of bracketing such zeros together. Returning to the proof of the theorem, for T admissible in the above sense, and large, we have

$$\begin{aligned}
\log|\zeta(\sigma+i2T)| &\ge -\sum_{|2T-\gamma|\le 1} \frac{A_1\gamma}{\log\gamma} + O(\log T) \\
&> -\frac{4A_1T}{\log 2T} \sum_{|2T-\gamma|\le 1} 1 \ + \ O(\log T) \\
&> -A_2T,
\end{aligned}$$

where $A_2 < \frac{\pi}{4}$ if A_1 is small enough. It follows that

$$\frac{1}{|\zeta(\sigma + i2T)|} < e^{A_2 T}. \tag{3.10}$$

From [6, p. 82, Equation(14)] we know that

$$|L(s,\chi)| \leq 2q|s| \text{ for } \operatorname{Re} s \geq \frac{1}{2}.$$

Applying the functional equation of $L(s,\chi)$, we see that

$$|L(\sigma + 2iT, \chi)| \ll |T|^{A_0} \text{ for all } \sigma \in [2c, 2\lambda], \tag{3.11}$$

where A_0 is a positive constant. Using the relations (3.9), (3.10), and (3.11) in (3.8), we find that

$$|I_2| \ll T^A \exp\left(-\frac{1}{2}\pi|T| + 2A_2 T\right)$$

We conclude that $I_2 \to 0$ as $T \to \infty$ through the above values. One finds similarly that $I_3 \to 0$ as $T \to \infty$ through these values.

Therefore, using $\alpha\beta = \pi q$, (3.4) and (3.7) we find that

$$\sum_{n=1}^{\infty} \frac{\tilde{a}_1(n)}{n} e^{\frac{-\alpha^2}{n^2}} = \frac{1}{\sqrt{\pi}\epsilon_\chi 2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{\Gamma\left(\frac{1-2s}{2}\right)}{F(2s,\chi)} \beta^{2s}\, ds - \frac{1}{\sqrt{\pi}\epsilon_\chi} \sum_{\rho} \operatorname{res}_{\rho/2}, \tag{3.12}$$

for $\frac{1}{2} < \lambda < \frac{3}{2}$. Taking $w = \frac{1-2s}{2}$, we may rewrite the integral on the right-hand side of (3.12) as

$$\frac{1}{\sqrt{\pi}\epsilon_\chi 2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{\Gamma(w)}{F(1-2w,\chi)} \beta^{1-2w}\, dw = \frac{\beta}{\sqrt{\pi}\epsilon_\chi} \sum_{n=1}^{\infty} \frac{a_1(n)}{n} e^{\frac{-\beta^2}{n^2}}, \tag{3.13}$$

for $-1 < \delta < 0$. The above identity is obtained by proceeding similarly as in (3.4). Denoting $\operatorname{res}_{\rho/2}$ by $S_\rho/2$ and combining (3.6), (3.12) and (3.13) one finds that

$$\sum_{n=1}^{\infty} \frac{\tilde{a}_1(n)}{n} e^{\frac{-\alpha^2}{n^2}} - \frac{\beta}{\sqrt{\pi}\epsilon_\chi} \sum_{n=1}^{\infty} \frac{a_1(n)}{n} e^{\frac{-\beta^2}{n^2}} = -\frac{1}{2\sqrt{\pi}\epsilon_\chi} \sum_{\rho} S_\rho,$$

from which one obtains the desired formula

$$\sqrt{\frac{\alpha}{\epsilon_{\bar{\chi}}}} \sum_{n=1}^{\infty} \frac{\tilde{a}_1(n)}{n} e^{\frac{-\alpha^2}{n^2}} - \sqrt{\frac{\beta}{\epsilon_\chi}} \sum_{n=1}^{\infty} \frac{a_1(n)}{n} e^{\frac{-\beta^2}{n^2}} = -\frac{1}{2\sqrt{\beta\epsilon_\chi}} \sum_{\rho} S_\rho,$$

which completes the proof of the theorem. $\square$

Proof of Theorem 2. For the sake of completeness we will give a brief proof of part a) and we will skip the proof of part b), which is similar.

Using (3.1) for $-1 <$ Re $s = c < 0$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\tilde{a}_2(n)}{n}\left(e^{\frac{-\alpha^2}{n^2}}-1\right) &= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\tilde{a}_2(n)}{n}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(s)\left(\frac{\alpha^2}{n^2}\right)^{-s}\, ds \\
&= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(s)\sum_{n=1}^{\infty}\frac{\tilde{a}_2(n)}{n^{1-2s}}\alpha^{-2s}\, ds \\
&= \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)}{G(1-2s,\bar{\chi_1},\bar{\chi_2},\bar{\chi_3})}\alpha^{-2s}\, ds, \qquad (3.14)
\end{aligned}$$

where in the penultimate step one uses Stirling's formula to justify the interchange of summation and integration. Employing the functional equation (2.8) in (3.14) for $a = 0$ we find that

$$\sum_{n=1}^{\infty} \frac{\tilde{a}_2(n)}{n}\left(e^{\frac{-\alpha^2}{n^2}}-1\right) = \frac{\epsilon_{\chi_1}\epsilon_{\chi_2}}{\sqrt{\pi}\epsilon_{\chi_3}}\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{G(2s,\chi_1,\chi_2,\chi_3)}\left(\frac{\pi q_3}{q_1q_2\alpha}\right)^{2s}\, ds. \qquad (3.15)$$

Consider the contour $\mathcal{C}$ defined by the line segments $[c+iT, c-iT]$, $[c-iT, \lambda-iT]$, $[\lambda-iT, \lambda+iT]$, and $[\lambda+iT, c+iT]$ in the counterclockwise direction, where $\frac{1}{2} < \lambda < \frac{3}{2}$. By the residue theorem,

(3.16)

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{\mathcal{C}}\frac{\Gamma\left(\frac{1-2s}{2}\right)}{G(2s,\chi_1,\chi_2,\chi_3)}\left(\frac{\pi q_3}{q_1q_2\alpha}\right)^{2s}\, ds &= \mathrm{res}_0 + \mathrm{res}_{1/2} \\
&\quad + \sum_{-T<\text{ Im }(\rho/2)<T}\mathrm{res}_{\rho/2} + \sum_{-T<\text{ Im }(\rho'/2)<T}\mathrm{res}_{\rho'/2},
\end{aligned}$$

where ρ and ρ' denote the non-trivial zeros of $L(s,\chi_1)$ and $L(s,\chi_2)$ respectively. Next, recalling (2.6),

$$\begin{aligned}
\mathrm{res}_0 &= \lim_{s\to 0}\frac{s\Gamma\left(\frac{1-2s}{2}\right)}{G(2s,\chi_1,\chi_2,\chi_3)}\left(\frac{\pi q_3}{q_1q_2\alpha}\right)^{2s} \\
&= \frac{\sqrt{\pi}L'(0,\chi_3)}{2L'(0,\chi_1)L'(0,\chi_2)} \\
&= \frac{\sqrt{\pi}\epsilon_{\chi_3}L(1,\bar{\chi}_3)}{\epsilon_{\chi_1}\epsilon_{\chi_2}L(1,\bar{\chi}_1)L(1,\bar{\chi}_2)}, \qquad (3.17)
\end{aligned}$$

where in the last step we used Lemma 1 with $k = 2$. Similarly,

$$\mathrm{res}_{1/2} = \lim_{s\to 1/2}\frac{(s-1/2)\Gamma\left(\frac{1-2s}{2}\right)}{G(2s,\chi_1,\chi_2,\chi_3)}\left(\frac{\pi q_3}{q_1q_2\alpha}\right)^{2s}$$

$$= -\frac{\beta L(1,\chi_3)}{L(1,\chi_1)L(1,\chi_2)}\,. \tag{3.18}$$

In the last step we used the fact $\alpha\beta = \frac{\pi q_3}{q_1 q_2}$. By the assumptions from the statement of the theorem it follows that ρ and ρ' run over all the zeros of $G(s,\chi_1,\chi_2,\chi_3)$ in the critical strip, and are simple. Hence

$$\begin{aligned} \mathrm{res}_{\rho/2} &= \lim_{s\to\rho/2} \frac{(s-\rho/2)\Gamma\left(\frac{1-2s}{2}\right)}{G(2s,\chi_1,\chi_2,\chi_3)}\left(\frac{\pi q_3}{q_1 q_2\alpha}\right)^{2s} \\ &= \frac{\Gamma\left(\frac{1-\rho}{2}\right)L(\rho,\chi_3)}{2L'(\rho,\chi_1)L(\rho,\chi_2)}\beta^{\rho}\,, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \mathrm{res}_{\rho'/2} &= \lim_{s\to\rho'/2} \frac{(s-\rho'/2)\Gamma\left(\frac{1-2s}{2}\right)}{G(2s,\chi_1,\chi_2,\chi_3)}\left(\frac{\pi q_3}{q_1 q_2\alpha}\right)^{2s} \\ &= \frac{\Gamma\left(\frac{1-\rho'}{2}\right)L(\rho',\chi_3)}{2L(\rho',\chi_1)L'(\rho',\chi_2)}\beta^{\rho'}. \end{aligned} \tag{3.20}$$

Arguing as with I_2 in the proof of Theorem 1, one similarly shows that the integrals along the horizontal lines on the left-hand side of (3.16) tend to 0 as T tends to infinity along a suitable sequence of values. Next, the integral along the entire vertical line on the right equals

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\Gamma\left(\frac{1-2s}{2}\right)}{G(2s,\chi_1,\chi_2,\chi_3)}\left(\frac{\pi q_3}{q_1 q_2\alpha}\right)^{2s} ds = \beta\sum_{n=1}^{\infty}\frac{a_2(n)}{n}\left(e^{\frac{-\beta^2}{n^2}}-1\right). \tag{3.21}$$

Therefore from (3.15) (3.16), (3.17), (3.18) (3.19) and (3.20) we have

(3.22)

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{\tilde{a}_2(n)}{n}\left(e^{\frac{-\alpha^2}{n^2}}-1\right) - \beta\frac{\epsilon_{\chi_1}\epsilon_{\chi_2}}{\sqrt{\pi}\epsilon_{\chi_3}}\sum_{n=1}^{\infty}\frac{a_2(n)}{n}\left(e^{\frac{-\beta^2}{n^2}}-1\right) \\ &\qquad = -\frac{L(1,\bar{\chi}_3)}{L(1,\bar{\chi}_1)L(1,\bar{\chi}_2)} + \frac{\epsilon_{\chi_1}\epsilon_{\chi_2}}{\sqrt{\pi}\epsilon_{\chi_3}}\frac{\beta L(1,\chi_3)}{L(1,\chi_1)L(1,\chi_2)} \\ &\qquad\quad - \frac{\epsilon_{\chi_1}\epsilon_{\chi_2}}{\sqrt{\pi}\epsilon_{\chi_3}}\left[\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)L(\rho,\chi_3)}{2L'(\rho,\chi_1)L(\rho,\chi_2)}\beta^{\rho} + \sum_{\rho'}\frac{\Gamma\left(\frac{1-\rho'}{2}\right)L(\rho',\chi_3)}{2L(\rho',\chi_1)L'(\rho',\chi_2)}\beta^{\rho'}\right]. \end{aligned}$$

This further gives

$$\begin{aligned} &\sqrt{\frac{\alpha\epsilon_{\bar{\chi}_1}\epsilon_{\bar{\chi}_2}}{\epsilon_{\bar{\chi}_3}}}\sum_{n=1}^{\infty}\frac{\tilde{a}_2(n)}{n}e^{\frac{-\alpha^2}{n^2}} - \sqrt{\frac{\beta\epsilon_{\chi_1}\epsilon_{\chi_2}}{\epsilon_{\chi_3}}}\sum_{n=1}^{\infty}\frac{a_2(n)}{n}e^{\frac{-\beta^2}{n^2}} \\ &\qquad = -\frac{\sqrt{\epsilon_{\chi_1}\epsilon_{\chi_2}}}{2\sqrt{\beta\epsilon_{\chi_3}}}\left[\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)L(\rho,\chi_3)}{L'(\rho,\chi_1)L(\rho,\chi_2)}\beta^{\rho} + \sum_{\rho'}\frac{\Gamma\left(\frac{1-\rho'}{2}\right)L(\rho',\chi_3)}{L(\rho',\chi_1)L'(\rho',\chi_2)}\beta^{\rho'}\right], \end{aligned}$$

which completes the proof of the theorem. □

TABLE 1. LHS and RHS of (1.8) for non-principal even χ mod 5.

β	Left Hand Side	Right Hand Side (100 zeros only)
1	0.000079785691396833	0.00007928285295117631754
e	0.000169777686441965	0.00016973590106011563813
π	−0.000012303600084319	−0.00001233307922499084479
$\sqrt{5\pi}$	0	0
4	0.000020154154038479	0.00002013709331659680732
7	0.000180011886465743	0.00018000254564760968655
10	−0.000043358710204736	−0.00004337362445789078109
20	0.000029713242212197	0.00002963758777799228780
23	−0.000308430762980821	−0.00030853785736609676219
36	−0.00037242097928460	−0.00037274868444915176415
37	−0.00031730534266211	−0.00031765627245956205028
45	0.00040595940742795	0.00040538700768650472134
50	−0.00004301685266729	−0.00004376172325350321380
68	0.00045206867413433	0.00045046206080427273606
70	0.00047588610091427	0.00047415873947210204156
100	0.00020807117871447	0.00020385783736287122257

Acknowledgements

The authors would like to express their sincere gratitude to the referee for his valuable comments on the earlier version of this paper.

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