RIESZ-TYPE CRITERIA AND THETA TRANSFORMATION ANALOGUES

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Abstract. We give character analogues of a generalization of a result due to Ramanujan, Hardy and Littlewood, and provide Riesz-type criteria for the Riemann Hypothesis for the Riemann zeta function and Dirichlet L-function. We also provide analogues of the general theta transformation formula and of recent generalizations of the transformation formulas of W.L. Ferrar and G.H. Hardy for real primitive Dirichlet characters.

1. Introduction

In 1916, Riesz [31] gave the following equivalent criterion for the Riemann Hypothesis:

Let the function $F(x)$ be defined by

$$F(x) := \sum_{n=1}^{\infty} \mu(n) \frac{x}{n^2} e^{-x/n^2}.$$

The estimate $F(x) = O_\delta\left(x^{\frac{1}{4}+\delta}\right)$ for all $\delta > 0$ is a necessary and sufficient condition for the validity of the Riemann Hypothesis.

One relevant aspect of Riesz's criterion is that it involves the values of the Riemann zeta function in the region of absolute convergence, more precisely at integers $2, 3, 4, \cdots$.

Same is the case with the following variant of the above criterion due to Hardy and Littlewood [18, p. 156, Section 2.5]:

Consider the function

$$P(y) := \sum_{k=1}^{\infty} \frac{\mu(k)}{k} e^{-y/k^2} = \sum_{m=1}^{\infty} \frac{(-y)^m}{m!\zeta(2m+1)}. \tag{1.1}$$

Then, the estimate $P(y) = O_\delta\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all positive values of δ is equivalent to the Riemann Hypothesis.

Their intuition and motivation came from a beautiful identity in Ramanujan's notebooks [29] (see also [6, p. 468, Entry 37]). Ramanujan's work [29, 5, 6, 30, 2] has always had an element of surprise in it and this identity is no exception. It gives a nice transformation between infinite series of the Möbius function. Ramanujan communicated his identity to Hardy and Littlewood during his stay in Cambridge. The corrected version of this formula

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was given by Hardy and Littlewood [18, p. 156, Equation 2.516] and is as follows:

Let α and β be two positive numbers such that $\alpha\beta = \pi$. Assume that the series

$$\sum_{\rho}\left(\Gamma\left(\frac{1-\rho}{2}\right)/\zeta'(\rho)\right)a^{\rho}$$

converges, where ρ runs through the non-trivial zeros of $\zeta(s)$ and a denotes a positive real number, and that the non-trivial zeros of $\zeta(s)$ are simple. Then

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\alpha^2/n^2}-\sqrt{\beta}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\beta^2/n^2}=-\frac{1}{2\sqrt{\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}\beta^{\rho}. \tag{1.2}$$

Various aspects of this identity have been presented by Berndt [6, p. 470], Bhaskaran [10], Paris and Kaminski [27, p. 143] and Titchmarsh [33, p. 219, Section 9.8]. The following one-variable generalization of (1.2) was recently obtained in [14] in the course of studying transformation formulas of the form $F(z,\alpha) = F(iz,\beta)$, where $\alpha\beta = 1$ and $i = \sqrt{-1}$.

Let $z \in \mathbb{C}$ and let α and β be two positive numbers such that $\alpha\beta = 1$. Let ${}_1F_1(a;c;z)$ denote the confluent hypergeometric function (see (1.8)). Assume that the series

$$\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}{}_1F_1\left(\frac{1-\rho}{2};\frac{1}{2};\frac{-z^2}{4}\right)\pi^{\frac{\rho}{2}}a^{\rho}$$

converges, where ρ runs through the non-trivial zeros of $\zeta(s)$ and a denotes a positive real number, and that the non-trivial zeros of $\zeta(s)$ are simple. Then

$$\sqrt{\alpha}e^{\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\frac{\pi\alpha^2}{n^2}}\cos\left(\frac{\sqrt{\pi}\alpha z}{n}\right)-\sqrt{\beta}e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\frac{\pi\beta^2}{n^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right)$$
$$=-\frac{e^{-\frac{z^2}{8}}}{2\sqrt{\pi\beta}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}{}_1F_1\left(\frac{1-\rho}{2};\frac{1}{2};\frac{z^2}{4}\right)\pi^{\rho/2}\beta^{\rho}. \tag{1.3}$$

In (1.2) as well as in (1.3), it is not necessary to assume convergence of the series on the right-hand side. Instead one can bracket the terms of the series as explained in [18, p. 158] and [33, p. 220].

Motivated by (1.3) and the aforementioned variant of Riesz's criterion, we establish the following theorem which gives a more general Riesz-type criterion for the Riemann zeta function. An analogue of this theorem for Dirichlet L-functions is given at the end of Section 4. Our result represents a family of criteria, parametrized by a complex variable z (although only the $z = 0$ case gives a necessary and sufficient condition, while for $z \neq 0$ and $\arg(z) \neq -\frac{\pi}{4}$, it allows for finitely many possible zeros off the critical line).

Theorem 1.1. *Fix $z \in \mathbb{C}$. Consider the function*

$$\mathcal{P}_z(y) := \sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-y/n^2}\cosh\left(\frac{\sqrt{y}z}{n}\right). \tag{1.4}$$

Then we have the following:

(1) *The Riemann Hypothesis implies $\mathcal{P}_z(y) = O_{z,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all $\delta > 0$.*

(2) (a) *If* $z = 0$, *the estimate* $\mathcal{P}_z(y) = O_{z,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ *as* $y \to \infty$ *for all* $\delta > 0$ *implies the Riemann Hypothesis.*

(b) *If* $z \neq 0$ *and* $\arg(z) \neq -\frac{\pi}{4}$, *the estimate* $\mathcal{P}_z(y) = O_{z,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ *as* $y \to \infty$ *for all* $\delta > 0$ *implies that* $\zeta(s)$ *has at most finitely many non-trivial zeros off the critical line.*

In (1.2) and (1.3), one assumes simplicity of the zeros. It is known from the work of Bui, Conrey and Young [11] that at least 40.58% non-trivial zeros of the Riemann zeta function lie on the critical line and are simple. Also, the first 1.5×10^9 non-trivial zeros of the Riemann zeta function are on the critical line and are simple (see van de Lune, te Riele and Winter [21]). By an appropriate modification of the right-hand side of (1.3), one can avoid the assumption on the simplicity of the zeros and prove an unconditional result. We do this in Theorem 1.2 below which generalizes (1.3) in the context of Dirichlet characters.

Theorem 1.2. *Let* $z \in \mathbb{C}$ *and let* α *and* β *denote two positive numbers such that* $\alpha\beta = 1$. *Let* χ *be a primitive Dirichlet character modulo* q. *Let* $G(\chi) := G(1, \chi)$ *denote the Gauss sum defined more generally by*

$$G(n, \chi) := \sum_{m=1}^{q} \chi(m) e^{2\pi i m n/q}. \tag{1.5}$$

(i) *If* χ *is even,*

$$\begin{aligned}&\sqrt{\alpha G(\chi)} e^{\frac{z^2}{8}} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} e^{-\frac{\pi\alpha^2}{qn^2}} \cos\left(\frac{\sqrt{\pi}\alpha z}{n\sqrt{q}}\right) \\ &\quad - \sqrt{\beta G(\overline{\chi})} e^{-\frac{z^2}{8}} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)\mu(n)}{n} e^{-\frac{\pi\beta^2}{qn^2}} \cosh\left(\frac{\sqrt{\pi}\beta z}{n\sqrt{q}}\right) \\ &= -\frac{e^{-\frac{z^2}{8}}\sqrt{qG(\overline{\chi})}}{2\sqrt{\pi\beta}} \sum_{\rho} \frac{1}{(m_\rho - 1)!} \frac{d^{m_\rho - 1}}{ds^{m_\rho - 1}} (s-\rho)^{m_\rho} \frac{\Gamma\left(\frac{1-s}{2}\right)}{L(s, \overline{\chi})} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right) \left(\frac{\pi}{q}\right)^{\frac{s}{2}} \beta^s \Bigg|_{s=\rho}.\end{aligned} \tag{1.6}$$

(ii) *If* χ *is odd,*

$$\begin{aligned}&\sqrt{\alpha G(\chi)} e^{\frac{z^2}{8}} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} e^{-\frac{\pi\alpha^2}{qn^2}} \sin\left(\frac{\sqrt{\pi}\alpha z}{n\sqrt{q}}\right) \\ &\quad - \sqrt{\beta G(\overline{\chi})} e^{-\frac{z^2}{8}} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)\mu(n)}{n} e^{-\frac{\pi\beta^2}{qn^2}} \sinh\left(\frac{\sqrt{\pi}\beta z}{n\sqrt{q}}\right) \\ &= -\frac{ze^{-\frac{z^2}{8}}\sqrt{qG(\overline{\chi})}}{2\sqrt{\pi\beta}} \sum_{\rho} \frac{1}{(m_\rho - 1)!} \frac{d^{m_\rho - 1}}{ds^{m_\rho - 1}} (s-\rho)^{m_\rho} \frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s, \overline{\chi})} {}_1F_1\left(\frac{2-s}{2}; \frac{3}{2}; \frac{z^2}{4}\right) \left(\frac{\pi}{q}\right)^{\frac{s}{2}} \beta^s \Bigg|_{s=\rho},\end{aligned} \tag{1.7}$$

where in (1.6) (*and analogously in* (1.7)), m_ρ *is the multiplicity of the zero* $\rho := \delta + i\gamma$ *of* $L(s, \chi)$ *and the sum over* ρ *involves bracketing the terms so that the terms for which*

$$|\gamma - \gamma'| < \exp\left(-A_1|\gamma|/\log(|\gamma| + 3)\right) + \exp\left(-A_1|\gamma'|/\log(|\gamma'| + 3)\right),$$

where A_1 *is a positive constant, are included in the same bracket.*

Theorems 1.10 and 1.9 from [13] can be recovered as special cases, by letting $z = 0$ in (1.6), and respectively, by dividing both sides of (1.7) by z and then letting $z \to 0$. We expect the pairs of zeros $\{\rho, \rho'\}$ that need to be bracketed together in (1.6) and (1.7) to occur very rarely. For various results on correlation of zeros of L-functions, the reader is referred to Montgomery [23], Rudnick and Sarnak [32], Katz and Sarnak [19], [20], Murty and Perelli [24], and Murty and one of the authors [25].

In identities (1.3), (1.6) and (1.7) appears Kummer's confluent hypergeometric function

$$ {}_1F_1(a;c;w) := \sum_{n=0}^{\infty} \frac{(a)_n w^n}{(c)_n n!}, \tag{1.8} $$

where $(a)_n$ is the rising factorial defined for $a \in \mathbb{C}$ by

$$ (a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. $$

It is the special case $p = q = 1$ of the generalized hypergeometric function given by [1, p. 62]

$$ {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; w) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{w^n}{n!}, \tag{1.9} $$

and is an entire function of w.

An analogue of (1.3) for Hecke forms is provided in [15]. The identity in (1.3) can be easily rephrased in the form $F(z, \alpha) = F(iz, \beta)$ (see [14, Equation (1.23)]). The best known example of a formula of the type $F(z, \alpha) = F(iz, \beta)$ is the general theta transformation formula (see the first equality in (1.27)). For more details, the reader is referred to [7, Equations 1.1, 1.2] and the references therein.

We now discuss the second goal of this paper. Recently, the general theta transformation formula and one-variable generalizations of the transformations of Ferrar and Hardy were established in [14] as a by-product of evaluation of integrals of the form

$$ F(z, \alpha) := \int_0^{\infty} f\left(\frac{t}{2}\right) \Xi\left(\frac{t}{2}\right) \nabla\left(\alpha, z, \frac{1+it}{2}\right) dt, \tag{1.10} $$

for specific choices of $f(t)$. Here $f(t)$ is of the form

$$ f(t) = \phi(it)\phi(-it), \tag{1.11} $$

where ϕ is analytic in t as a function of a real variable, $\Xi(t)$ is the Riemann Ξ-function defined by $\Xi(t) := \xi(\frac{1}{2} + it)$, where

$$ \xi(s) := \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{1.12} $$

and $\nabla(\alpha, z, s)$ is the function defined by

$$ \nabla(x, z, s) := \rho(x, z, s) + \rho(x, z, 1-s), \tag{1.13} $$

where

$$ \rho(x, z, s) := x^{\frac{1}{2}-s} e^{-\frac{z^2}{8}} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right). \tag{1.14} $$

In this paper, we work with two analogues of the integral in (1.10) for real primitive Dirichlet characters. Let the function $\Xi(t, \chi)$ be defined by

$$ \Xi(t, \chi) := \xi\left(\frac{1}{2} + it, \chi\right), \tag{1.15} $$

where

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-(s+b)/2}\Gamma\left(\frac{s+b}{2}\right)L(s,\chi), \tag{1.16}$$

with

$$b = \begin{cases} 0, & \chi(-1)=1, \\ 1, & \chi(-1)=-1. \end{cases} \tag{1.17}$$

It is known [12] that $\xi(s,\chi)$ satisfies the functional equation

$$\xi(1-s,\overline{\chi}) = \epsilon(\chi)\xi(s,\chi), \tag{1.18}$$

where $\epsilon(\chi) = i^b q^{1/2}/G(\chi)$, where $G(\chi)$ is defined in (1.5). Since for real primitive characters, we have

$$G(\chi) = \begin{cases} \sqrt{q}, & \text{for } \chi \text{ even}, \\ i\sqrt{q}, & \text{for } \chi \text{ odd}, \end{cases} \tag{1.19}$$

the functional equation reduces simply to

$$\xi(1-s,\chi) = \xi(s,\chi). \tag{1.20}$$

This also gives

$$\Xi(-t,\chi) = \Xi(t,\chi). \tag{1.21}$$

For even real primitive Dirichlet characters, we work with the integral

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2},\chi\right)\nabla\left(\alpha,z,\frac{1+it}{2}\right)dt, \tag{1.22}$$

for different choices of $f(t)$. Here the functions f and ∇ are defined in (1.11) and (1.13) respectively. Since $\nabla\left(\alpha,z,\frac{1+it}{2}\right) = \nabla\left(\beta,iz,\frac{1+it}{2}\right)$ (see [14, (1.12)]), this integral is invariant under the simultaneous application of the maps $\alpha\to\beta$ and $z\to iz$, and hence generates transformation formulas of the type $F(z,\alpha,\chi) = F(iz,\beta,\chi)$.

For odd real primitive Dirichlet characters, we work with the integral

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2},\chi\right)\Delta\left(\alpha,z,\frac{1+it}{2}\right)dt, \tag{1.23}$$

where

$$\Delta(x,z,s) := \omega(x,z,s) + \omega(x,z,1-s), \tag{1.24}$$

with

$$\omega(x,z,s) := x^{\frac{1}{2}-s}e^{-\frac{z^2}{8}}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right). \tag{1.25}$$

By Kummer's first transformation for ${}_1F_1(a;c;w)$ [1, p. 191, Equation (4.1.11)], [28, p. 125, Equation (2)], namely,

$${}_1F_1(a;c;w) = e^w{}_1F_1(c-a;c;-w), \tag{1.26}$$

we also have $\Delta\left(\alpha,z,\frac{1+it}{2}\right) = \Delta\left(\beta,iz,\frac{1+it}{2}\right)$, and so the integral, invariant under $\alpha\to\beta$ and $z\to iz$, gives formulas of the type $F(z,\alpha,\chi) = F(iz,\beta,\chi)$.

The difference between the forms of the functional equations for $\xi(s,\chi)$, when χ is even or odd, necessitates the use of two different analogues, namely the ones given in (1.22) and (1.23). Another reason is, we want to be able to explicitly evaluate the associated inverse Mellin transforms when we convert these integrals into equivalent complex integrals, and this requires working with the integrals in (1.22) and (1.23) as we shall see later. We now give some examples.

Consider the following extended version of the general theta transformation formula established in [14]:

Let $z \in \mathbb{C}$. If α and β are positive numbers such that $\alpha\beta = 1$, then

$$\sqrt{\alpha}\left(\frac{e^{-\frac{z^2}{8}}}{2\alpha} - e^{\frac{z^2}{8}}\sum_{n=1}^{\infty} e^{-\pi\alpha^2n^2}\cos(\sqrt{\pi}\alpha nz)\right) = \sqrt{\beta}\left(\frac{e^{\frac{z^2}{8}}}{2\beta} - e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty} e^{-\pi\beta^2n^2}\cosh(\sqrt{\pi}\beta nz)\right)$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\Xi(t/2)}{1+t^2}\nabla\left(\alpha, z, \frac{1+it}{2}\right) dt. \qquad (1.27)$$

Here, we establish the following character analogue of (1.27):

Theorem 1.3. *Let $z \in \mathbb{C}$ and let α and β denote two positive numbers such that $\alpha\beta = 1$. Let χ be a real primitive Dirichlet character modulo q.*

(i) *If χ is even,*

$$\sqrt{\alpha}e^{\frac{z^2}{8}}\sum_{n=1}^{\infty}\chi(n)e^{-\frac{\pi\alpha^2n^2}{q}}\cos\left(\frac{\sqrt{\pi}\alpha nz}{\sqrt{q}}\right) = \sqrt{\beta}e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty}\chi(n)e^{-\frac{\pi\beta^2n^2}{q}}\cosh\left(\frac{\sqrt{\pi}\beta nz}{\sqrt{q}}\right)$$
$$= \frac{1}{8\pi}\int_0^\infty \Xi\left(\frac{t}{2}, \chi\right)\nabla\left(\alpha, z, \frac{1+it}{2}\right) dt. \qquad (1.28)$$

(ii) *If χ is odd,*

$$\sqrt{\alpha}e^{\frac{z^2}{8}}\sum_{n=1}^{\infty}\chi(n)e^{-\frac{\pi\alpha^2n^2}{q}}\sin\left(\frac{\sqrt{\pi}\alpha nz}{\sqrt{q}}\right) = \sqrt{\beta}e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty}\chi(n)e^{-\frac{\pi\beta^2n^2}{q}}\sinh\left(\frac{\sqrt{\pi}\beta nz}{\sqrt{q}}\right)$$
$$= \frac{z}{8\sqrt{\pi q}}\int_0^\infty \Xi\left(\frac{t}{2}, \chi\right)\Delta\left(\alpha, z, \frac{1+it}{2}\right) dt. \qquad (1.29)$$

Note that Berndt and Schoenfeld [9, Theorem 7.1] have derived a transformation formula for a periodic theta function.

Define $\psi(a, \chi)$ by

$$\psi(a, \chi) = -\sum_{n=1}^{\infty}\frac{\chi(n)}{n+a}, \qquad (1.30)$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_{<0}$. For a real character χ, this is consistent with the character analogue of the digamma function which can be obtained by the logarithmic differentiation of the following Weierstrass product form of the character analogue of the gamma function for real characters derived by Berndt [3]:

$$\Gamma(a, \chi) = e^{-aL(1,\chi)}\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right)^{-\chi(n)}e^{a\chi(n)/n}.$$

Then we prove the following character analogue of a generalization of a formula of Hardy [14, Theorem 1.3]:

Theorem 1.4. *Let $z \in \mathbb{C}$ and let α and β denote two positive numbers such that $\alpha\beta = 1$. Let χ be a real primitive Dirichlet character modulo q and let $\psi(x, \chi)$ be defined as in (1.30).*

(i) *If χ is even,*

$$\sqrt{\alpha}e^{\frac{z^2}{8}}\int_0^\infty \psi(x,\chi)e^{-\frac{\pi\alpha^2x^2}{q}}\cos\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)dx$$
$$=\sqrt{\beta}e^{-\frac{z^2}{8}}\int_0^\infty \psi(x,\chi)e^{-\frac{\pi\beta^2x^2}{q}}\cosh\left(\frac{\sqrt{\pi}\beta xz}{\sqrt{q}}\right)dx$$
$$=-\frac{1}{8}\int_0^\infty \frac{\Xi\left(\frac{t}{2},\chi\right)\nabla\left(\alpha,z,\frac{1+it}{2}\right)}{\cosh\left(\frac{1}{2}\pi t\right)}\,dt. \tag{1.31}$$

(ii) *If χ is odd,*

$$\sqrt{\alpha}e^{\frac{z^2}{8}}\int_0^\infty \psi(x,\chi)e^{-\frac{\pi\alpha^2x^2}{q}}\sin\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)dx$$
$$=\sqrt{\beta}e^{-\frac{z^2}{8}}\int_0^\infty \psi(x,\chi)e^{-\frac{\pi\beta^2x^2}{q}}\sinh\left(\frac{\sqrt{\pi}\beta xz}{\sqrt{q}}\right)dx$$
$$=-\frac{z\sqrt{\pi}}{8\sqrt{q}}\int_0^\infty \frac{\Xi\left(\frac{t}{2},\chi\right)\Delta\left(\alpha,z,\frac{1+it}{2}\right)}{\cosh\left(\frac{1}{2}\pi t\right)}\,dt. \tag{1.32}$$

In a similar vein, we prove the following character analogues of a generalization of Ferrar's formula [14, Theorem 1.4]:

Theorem 1.5. *Let $z\in\mathbb{C}$ and let $K_\nu(z)$ denote the modified Bessel function of order ν. Let α and β are positive numbers such that $\alpha\beta=1$. Let χ be a real primitive Dirichlet character modulo q.*

(i) *If χ is even,*

$$\sqrt{\alpha}e^{\frac{z^2}{8}}\int_0^\infty e^{-\frac{\pi\alpha^2x^2}{q}}\cos\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)\sum_{n=1}^\infty \chi(n)K_0\left(\frac{2\pi nx}{q}\right)dx$$
$$=\sqrt{\beta}e^{-\frac{z^2}{8}}\int_0^\infty e^{-\frac{\pi\beta^2x^2}{q}}\cosh\left(\frac{\sqrt{\pi}\beta xz}{\sqrt{q}}\right)\sum_{n=1}^\infty \chi(n)K_0\left(\frac{2\pi nx}{q}\right)dx$$
$$=\frac{\sqrt{q}}{32\pi^{\frac{3}{2}}}\int_0^\infty \Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\Xi\left(\frac{t}{2},\chi\right)\nabla\left(\alpha,z,\frac{1+it}{2}\right)dt. \tag{1.33}$$

(ii) *If χ is odd,*

$$\sqrt{\alpha}e^{\frac{z^2}{8}}\int_0^\infty xe^{-\frac{\pi\alpha^2x^2}{q}}\sin\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)\sum_{n=1}^\infty \chi(n)nK_0\left(\frac{2\pi nx}{q}\right)dx$$
$$=\sqrt{\beta}e^{-\frac{z^2}{8}}\int_0^\infty xe^{-\frac{\pi\beta^2x^2}{q}}\sinh\left(\frac{\sqrt{\pi}\beta xz}{\sqrt{q}}\right)\sum_{n=1}^\infty \chi(n)nK_0\left(\frac{2\pi nx}{q}\right)dx$$
$$=\frac{zq}{32\pi^2}\int_0^\infty \Gamma\left(\frac{3+it}{4}\right)\Gamma\left(\frac{3-it}{4}\right)\Xi\left(\frac{t}{2},\chi\right)\Delta\left(\alpha,z,\frac{1+it}{2}\right)dt. \tag{1.34}$$

The convergence of the integrals on the extreme right-hand sides of (1.28)-(1.29) and of (1.31)-(1.34) follows from (2.6) and (2.7). When $z=0$, we have $\nabla\left(\alpha,0,\frac{1+it}{2}\right)=2\cos\left(\frac{1}{2}t\log\alpha\right)$.

This gives analogues of the theta transformation formula and formulas of Hardy and Ferrar (Equations (1.1), (1.15), (6.19) respectively in [14]) for even real primitive Dirichlet characters. Since $\Delta\left(\alpha, 0, \frac{1+it}{2}\right) = 2\cos\left(\frac{1}{2}t\log\alpha\right)$ too, we get corresponding analogues for odd real primitive characters by dividing both sides in each of the Theorems 1.3, 1.4 and 1.5 by non-zero z, letting $z \to 0$ and then by using Lebesgue's dominated convergence theorem.

This paper is organized as follows. In Section 2, we give preliminary results which are used in subsequent sections. In Section 3, a proof of Theorem 1.2 is given. Then in Section 4, we give a proof of Theorem 1.1 and give a corresponding Riesz-type criterion for the Riemann Hypothesis associated with the Dirichlet L-function. Section 5 is devoted to proving Theorems 1.3, 1.4 and 1.5, where we prove them only in the case when χ is odd and leave the even case for the reader.

2. Preliminary results

In this section, we give a result which transforms the integrals in (1.22) and (1.23) into equivalent complex integrals. The latter can then be evaluated using Cauchy's residue theorem and the theory of Mellin transforms. We omit its proof since it is similar to the analogous one for the Riemann zeta function [14, Theorem 3.1].

Theorem 2.1. *Let*
$$f(t) = \phi(it)\phi(-it),$$
where ϕ is analytic in t as a function of a real variable. Let χ be a real primitive Dirichlet character modulo q. Let $\nabla(x,z,s)$, $\rho(x,z,s)$ and $\Xi(t,\chi)$ be defined in (1.13), (1.14) *and* (1.15) *respectively. Assume that the integrals below converge. Then,*

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2},\chi\right)\nabla\left(\alpha,z,\frac{1+it}{2}\right)dt = \frac{2}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(s-\frac{1}{2}\right)\phi\left(\frac{1}{2}-s\right)\xi(s,\chi)\rho(\alpha,z,s)\,ds,$$
$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi\left(\frac{t}{2},\chi\right)\Delta\left(\alpha,z,\frac{1+it}{2}\right)dt = \frac{2}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\phi\left(s-\frac{1}{2}\right)\phi\left(\frac{1}{2}-s\right)\xi(s,\chi)\omega(\alpha,z,s)\,ds. \tag{2.1}$$

In the proofs of Theorem 1.5, we will be making use of the following special case [8, Theorem 2.1] of a general result of Berndt [4, Theorem 10.1]:

Theorem 2.2. *Let $x > 0$. If χ is even with period q and* $\operatorname{Re}\nu \geq 0$*, then*
$$\sum_{n=1}^{\infty}\chi(n)n^{\nu}K_{\nu}\left(\frac{2\pi nx}{q}\right) = \frac{\pi^{\frac{1}{2}}}{2xG(\overline{\chi})}\left(\frac{qx}{\pi}\right)^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)\sum_{n=1}^{\infty}\overline{\chi}(n)(n^2+x^2)^{-\nu-\frac{1}{2}}; \tag{2.2}$$
if χ is odd with period k and $\operatorname{Re}\nu > -1$*, then*
$$\sum_{n=1}^{\infty}\chi(n)n^{\nu+1}K_{\nu}\left(\frac{2\pi nx}{q}\right) = \frac{i\pi^{\frac{1}{2}}}{2x^2G(\overline{\chi})}\left(\frac{qx}{\pi}\right)^{\nu+2}\Gamma\left(\nu+\frac{3}{2}\right)\sum_{n=1}^{\infty}\overline{\chi}(n)n(n^2+x^2)^{-\nu-\frac{3}{2}}. \tag{2.3}$$

We will frequently use the following two lemmas in the proofs of subsequent theorems. These lemmas already exist in the literature, for example, see [26, p. 47, Formulas 5.29, 5.30] and [16, p. 318, 320, Formulas (10), (30)]. However, the first of these lemmas is incorrectly given in both these references since the first argument of the confluent hypergeometric function in the formula should be $1-\frac{s}{2}$ instead of $-\frac{s}{2}$. The correct version, which is also given in [17, p. 503, Formula (3.952.7)], is as follows.

Lemma 2.3. *For $c = Re\ s > -1$ and $Re\ a > 0$, we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{b}{2}a^{-\frac{1}{2}-\frac{s}{2}}e^{-\frac{b^2}{4a}}\Gamma\left(\frac{s+1}{2}\right){}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{b^2}{4a}\right)x^{-s}\,ds = e^{-ax^2}\sin bx. \tag{2.4}$$

Lemma 2.4. *For $c = Re\ s > 0$ and $Re\ a > 0$, we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2}a^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)e^{-\frac{b^2}{4a}}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};\frac{b^2}{4a}\right)x^{-s}\,ds = e^{-ax^2}\cos bx. \tag{2.5}$$

We note that [14, Equation (2.10)]

$${}_1F_1\left(\tfrac{1}{4}-\lambda;\tfrac{1}{2};\tfrac{z^2}{4}\right)\sim e^{z^2/8}\cos\left(\sqrt{\lambda}z\right), \tag{2.6}$$

as $|\lambda|\to\infty$ and $|\arg(\lambda z)|<2\pi$. Stirling's formula for $\Gamma(s)$, $s=\sigma+it$, in a vertical strip $\alpha\le\sigma\le\beta$ given by

$$|\Gamma(s)| = (2\pi)^{\frac{1}{2}}|t|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1+O\left(\frac{1}{|t|}\right)\right), \tag{2.7}$$

as $|t|\to\infty$. Finally we note that the Whittaker function $M_{\lambda,\mu}(z)$ is defined by [17, p. 1024, formula 9.220, no.2]

$$M_{\lambda,\mu}(z) = z^{\mu+\frac{1}{2}}e^{-z/2}{}_1F_1\left(\mu-\lambda+\tfrac{1}{2};2\mu+1;z\right). \tag{2.8}$$

3. Character analogues of the generalization of Ramanujan-Hardy-Littlewood result

We prove only Part (ii) of Theorem 1.2. First, letting $a=1$, $b=z$, $x=\sqrt{\pi}\alpha/(n\sqrt{q})$ in Lemma 2.3, we see that for Re $s>-1$,

$$e^{-\frac{\pi\alpha^2}{qn^2}}\sin\left(\frac{\sqrt{\pi}\alpha z}{n\sqrt{q}}\right) = \frac{ze^{-\frac{z^2}{4}}}{4\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{s+1}{2}\right){}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{n\sqrt{q}}\right)^{-s}ds. \tag{3.1}$$

Hence for $-1<$ Re $s<0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}e^{-\frac{\pi\alpha^2}{qn^2}}\sin\left(\frac{\sqrt{\pi}\alpha z}{n\sqrt{q}}\right)\\
&=\frac{ze^{-\frac{z^2}{4}}}{4\pi i}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{s+1}{2}\right){}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{n\sqrt{q}}\right)^{-s}ds\\
&=\frac{ze^{-\frac{z^2}{4}}}{4\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma\left(\frac{s+1}{2}\right){}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s}\left(\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}\right)ds\\
&=\frac{ze^{-\frac{z^2}{4}}}{4\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{s+1}{2}\right)}{L(1-s,\chi)}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s}ds,
\end{aligned}\tag{3.2}$$

where in the penultimate step, we interchanged the order of summation and integration, which is valid because of absolute convergence, and in the ultimate step, we used the fact that $1/L(1-s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}$ for Re $s<0$. Now for odd χ, the functional equation for $L(s,\chi)$ can be put in the form

$$\frac{\Gamma\left(\frac{s+1}{2}\right)}{L(1-s,\chi)} = \frac{\pi^{s-\frac{1}{2}}G(\overline{\chi})}{iq^s}\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}. \tag{3.3}$$

Employing (3.3) in (3.2), we have

$$\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}e^{-\frac{\pi\alpha^2}{qn^2}}\sin\left(\frac{\sqrt{\pi}\alpha z}{n\sqrt{q}}\right)$$
$$=-\frac{ze^{-\frac{z^2}{4}}G(\overline{\chi})}{4\pi^{3/2}}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}ds. \tag{3.4}$$

In order to be able to use the Dirichlet series for $1/L(s,\overline{\chi})$, we now shift the line of integration from $-1<c=\text{Re } s<0$ to $1<\lambda=\text{Re } s<2$. In this process, we encounter the non-trivial zeros of $L(s,\overline{\chi})$. Consider a positively oriented rectangular contour formed by $[c-iT,\lambda-iT],[\lambda-iT,\lambda+iT],[\lambda+iT,c+iT]$ and $[c+iT,c-iT]$, where T be a positive real number. Let $R_h(a)$ denote the residue of the function

$$h(s):=\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}$$

at $s=a$. By the residue theorem, we have

$$\left[\int_{c-iT}^{\lambda-iT}+\int_{\lambda-iT}^{\lambda+iT}+\int_{\lambda+iT}^{c+iT}+\int_{c+iT}^{c-iT}\right]\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}ds$$
$$=2\pi i\sum_{-T<\text{Im }\rho<T}R_h(\rho), \tag{3.5}$$

where

$$R_h(\rho):=\frac{1}{(m_\rho-1)!}\lim_{s\to\rho}\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho)^{m_\rho}\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}. \tag{3.6}$$

As $T\to\infty$, the integrals along the horizontal segments $[c-iT,\lambda-iT]$ and $[\lambda+iT,c+iT]$ tend to zero, which can be shown using a similar reasoning as in the proof of Theorem 1.6 in [14]. Thus, we find that

$$\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}ds$$
$$=\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}ds-2\pi i\sum_{\rho}R_h(\rho). \tag{3.7}$$

We now evaluate the integral on the right-hand side of (3.7). Let $w=1-s$ so that for $-1<\lambda'=\text{Re } w<0$, we have

$$\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}ds$$
$$=\int_{\lambda'-i\infty}^{\lambda'+i\infty}\frac{\Gamma\left(\frac{w+1}{2}\right)}{L(1-w,\overline{\chi})}{}_1F_1\left(\frac{w+1}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{w-1}dw$$
$$=\frac{\sqrt{\pi}}{\sqrt{q}\alpha}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\mu(n)}{n}\int_{\lambda'-i\infty}^{\lambda'+i\infty}\Gamma\left(\frac{w+1}{2}\right){}_1F_1\left(\frac{w+1}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}}{n\alpha\sqrt{q}}\right)^{-w}dw$$
$$=\frac{\sqrt{\pi}e^{\frac{z^2}{4}}}{\sqrt{q}\alpha}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\mu(n)}{n}\int_{\lambda'-i\infty}^{\lambda'+i\infty}\Gamma\left(\frac{w+1}{2}\right){}_1F_1\left(\frac{2-w}{2};\frac{3}{2};-\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\beta}{n\sqrt{q}}\right)^{-w}dw$$

$$=\frac{4\pi^{3/2}i\beta}{z\sqrt{q}}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\mu(n)}{n}e^{-\frac{\pi\beta^2}{qn^2}}\sinh\left(\frac{\sqrt{\pi}\beta z}{n\sqrt{q}}\right), \tag{3.8}$$

where in the penultimate step, we used (1.26), and in the last step we used (3.1) with z replaced by iz and α replaced by β. From (3.4), (3.7) and (3.8), we see that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}e^{-\frac{\pi\alpha^2}{qn^2}}\sin\left(\frac{\sqrt{\pi}\alpha z}{n\sqrt{q}}\right)\\&=-\frac{i\beta e^{-\frac{z^2}{4}}G(\overline{\chi})}{\sqrt{q}}\sum_{n=1}^{\infty}\frac{\overline{\chi}(n)\mu(n)}{n}e^{-\frac{\pi\beta^2}{qn^2}}\sinh\left(\frac{\sqrt{\pi}\beta z}{n\sqrt{q}}\right)\\&\quad+\frac{ize^{-\frac{z^2}{4}}G(\overline{\chi})}{2\sqrt{\pi}}\frac{1}{(m_\rho-1)!}\sum_{\rho}\frac{d^{m_\rho-1}}{ds^{m_\rho-1}}(s-\rho)^{m_\rho}\frac{\Gamma\left(\frac{2-s}{2}\right)}{L(s,\overline{\chi})}{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{q}\alpha}{\sqrt{\pi}}\right)^{-s}\Bigg|_{s=\rho}.\end{aligned} \tag{3.9}$$

Finally, multiplying both sides of (3.9) by $\sqrt{\alpha}\sqrt{G(\chi)}e^{\frac{z^2}{8}}$ and making use of the facts that $\alpha\beta=1$ and that $\sqrt{G(\chi)G(\overline{\chi})}=i\sqrt{q}$ for odd primitive χ, we arrive at (1.7) upon simplification.

The proof of (1.6) employs (2.5). The method involves similar steps to the one above, and the proof is left to the reader.

4. Riesz-type criteria for the Riemann Hypothesis

We prove Theorem 1.1 here. Before beginning the proof, let us give the heuristic behind why one gets the bound $P_z(y)=O_z(y^{-1/4})$. Representing $e^{-\pi y/n^2}$ and $\cosh\left(\frac{\sqrt{\pi y}z}{n}\right)$ in the definition of $\mathcal{P}_z(\pi y)$ in (1.4) by their Taylor series and interchanging the order of summation, we have

$$\begin{aligned}P_z(y):=\mathcal{P}_z(\pi y)&=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{m=0}^{\infty}\frac{(-\pi y/n^2)^m}{m!}\sum_{t=0}^{\infty}\frac{(\sqrt{\pi y}z/n)^{2t}}{(2t)!}\\&=\sum_{m=0}^{\infty}\frac{(-\pi y)^m}{m!}\sum_{t=0}^{\infty}\frac{(\sqrt{\pi y}z)^{2t}}{(2t)!}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{2m+2t+1}}\\&=\sum_{\substack{m,t=0\\(m,t)\neq(0,0)}}^{\infty}\frac{(-1)^m(\pi y)^{m+t}z^{2t}}{m!(2t)!\zeta(2m+2t+1)}.\end{aligned} \tag{4.1}$$

Thus, $P_z(y)$ is an entire function of y. From (1.3), we know that for $\alpha\beta=1$,

$$\sqrt{\alpha}e^{\frac{z^2}{8}}P_{iz}(\alpha^2)-\sqrt{\beta}e^{-\frac{z^2}{8}}P_z(\beta^2)=-\frac{e^{-\frac{z^2}{8}}}{2\sqrt{\pi}}\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}{}_1F_1\left(\frac{1-\rho}{2};\frac{1}{2};\frac{z^2}{4}\right)\pi^{\rho/2}\beta^{\rho-\frac{1}{2}}. \tag{4.2}$$

Assume the Riemann Hypothesis and the absolute convergence of

$$\sum_{\rho}\frac{\Gamma\left(\frac{1-\rho}{2}\right)}{\zeta'(\rho)}{}_1F_1\left(\frac{1-\rho}{2};\frac{1}{2};\frac{z^2}{4}\right)\pi^{\rho/2}.$$

Then the right-hand side of (4.2) is $O_z(1)$ as $\beta \to \infty$. From definition for $\alpha \to 0$, we have

$$\begin{aligned} P_{iz}(\alpha^2) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{-\pi^2\alpha^2/n^2} \cos\left(\frac{\sqrt{\pi}\alpha z}{n}\right) \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(1 + O\left(\frac{\alpha^2}{n^2}\right)\right)\left(1 + O\left(\frac{\alpha^2}{n^2}\right)\right) \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(1 + O\left(\frac{\alpha^2}{n^2}\right)\right) \\ &= O(\alpha^2), \end{aligned}$$

where in the last step, we used the prime number theorem in the form $\sum_{n=1}^{\infty} \mu(n)/n = 0$. Hence $P_{iz}(\alpha^2) \to 0$ as $\alpha \to 0$, or equivalently, as $\beta \to \infty$. Thus from (4.2), we find that

$$P_z(\beta^2) = \sum_{m=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^m (\pi\beta^2)^{m+t} z^{2t}}{m!(2t)!\zeta(2m+2t+1)} = O_z\left(\beta^{-1/2}\right), \tag{4.3}$$

which can also be rephrased as

$$P_z(y) = O_z\left(y^{-\frac{1}{4}}\right) \tag{4.4}$$

as $y \to \infty$. This completes the heuristic.

Now we begin with the actual proof of Theorem 1.1, where we first prove the necessary condition, i.e., we show that the bound $P_z(y) = O_{z,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$, as $y \to \infty$ for all positive values of δ, implies the Riemann Hypothesis when $z = 0$, and when $z \neq 0$ and $\arg(z) \neq -\frac{\pi}{4}$, it implies that all but finitely many non-trivial zeros of $\zeta(s)$ are on the critical line. We first prove the following identity involving $P_z(y)$:

Lemma 4.1. *Let $0 < \mathrm{Re}(s) < 1$. Then for any z, we have*

$$\int_0^{\infty} y^{-s-1} P_z(y)\, dy = \frac{\pi^s \Gamma(-s)\, {}_1F_1\left(-s; \frac{1}{2}; \frac{z^2}{4}\right)}{\zeta(2s+1)}. \tag{4.5}$$

Proof. Let

$$\varphi(s,z) := \int_0^{\infty} y^{-s-1} P_z(y)\, dy. \tag{4.6}$$

Employing the change of variable $y = x/n^2$, we see that

$$n^{-2s}\varphi(s,z) := \int_0^{\infty} x^{-s-1} P_z\left(\frac{x}{n^2}\right) dx. \tag{4.7}$$

Now multiply both sides of (4.7) by n^{-1} and sum over n from 1 to ∞ to obtain

$$\zeta(2s+1)\varphi(s,z) = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{n} x^{-s-1} P_z\left(\frac{x}{n^2}\right) dx. \tag{4.8}$$

It can be shown using Weierstrass M-test and the Lebesgue dominated convergence theorem that

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{n} x^{-s-1} P_z\left(\frac{x}{n^2}\right) dx = \int_0^{\infty} x^{-s-1} \sum_{n=1}^{\infty} \frac{1}{n} P_z\left(\frac{x}{n^2}\right) dx. \tag{4.9}$$

From (4.1), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}P_z\left(\frac{x}{n^2}\right)&=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\frac{(-1)^k(\pi x/n^2)^{k+t}z^{2t}}{k!(2t)!\zeta(2k+2t+1)}\\&=\sum_{\substack{k,t=0\\(k,t)\neq(0,0)}}^{\infty}\frac{(-1)^k(\pi x)^{k+t}z^{2t}}{k!(2t)!\zeta(2k+2t+1)}\sum_{n=1}^{\infty}\frac{1}{n^{2k+2t+1}}\\&=\sum_{\substack{k,t=0\\(k,t)\neq(0,0)}}^{\infty}\frac{(-1)^k(\pi x)^{k+t}z^{2t}}{k!(2t)!}\\&=\sum_{k=1}^{\infty}\frac{(-\pi x)^k}{k!}+\sum_{t=1}^{\infty}\frac{(\pi x)^tz^{2t}}{t!}+\sum_{k,t=1}^{\infty}\frac{(-1)^k(\pi x)^{k+t}z^{2t}}{k!(2t)!}\\&=(e^{-\pi x}-1)+(\cosh(\sqrt{\pi x}z)-1)+(e^{-\pi x}-1)(\cosh(\sqrt{\pi x}z)-1)\\&=e^{-\pi x}\cosh(\sqrt{\pi x}z)-1.\end{aligned}\tag{4.10}$$

□

Substituting this in (4.8), we have

$$\zeta(2s+1)\varphi(s,z)=\int_0^{\infty}x^{-s-1}\left(e^{-\pi x}\cosh(\sqrt{\pi x}z)-1\right)\,dx.\tag{4.11}$$

Integrating by parts, for $0<\text{Re }s<1$, we have

$$\begin{aligned}&\int_0^{\infty}x^{-s-1}\left(e^{-\pi x}\cosh(\sqrt{\pi x}z)-1\right)\,dx\\&=\left[\frac{x^{-s}\left(e^{-\pi x}\cosh(\sqrt{\pi x}z)-1\right)}{-s}\right]_0^{\infty}+\frac{1}{s}\int_0^{\infty}x^{-s}\frac{d}{dx}\left(e^{-\pi x}\cosh(\sqrt{\pi x}z)-1\right)\,dx\\&=\frac{1}{s}\int_0^{\infty}x^{-s}\frac{d}{dx}\left(e^{-\pi x}\cosh(\sqrt{\pi x}z)\right)\,dx,\end{aligned}\tag{4.12}$$

since $\lim_{x\to\infty}e^{-\pi x}\cosh(\sqrt{\pi x}z)=0$. Next, observe that

$$\begin{aligned}&\int_0^{\infty}x^{-s}\frac{d}{dx}\left(e^{-\pi x}\cosh(\sqrt{\pi x}z)\right)\,dx\\&=\pi\int_0^{\infty}x^{-s}e^{-\pi x}\left(\frac{z}{2\sqrt{\pi x}}\sinh(\sqrt{\pi x}z)-\cosh(\sqrt{\pi x}z)\right)\,dx\\&=\pi\left(\frac{z^2}{2}\sum_{n=0}^{\infty}\frac{(\sqrt{\pi}z)^{2n}}{(2n+1)!}\int_0^{\infty}x^{n-s}e^{-\pi x}\,dx-\sum_{n=0}^{\infty}\frac{(\sqrt{\pi}z)^{2n}}{(2n)!}\int_0^{\infty}x^{n-s}e^{-\pi x}\,dx\right)\\&=\pi^s\left(\frac{z^2}{2}\sum_{n=0}^{\infty}\frac{z^{2n}\Gamma(1+n-s)}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{z^{2n}\Gamma(1+n-s)}{(2n)!}\right)\\&=\pi^s\Gamma(1-s)\left(\frac{z^2}{2}\sum_{n=0}^{\infty}\frac{z^{2n}(1-s)_n}{(2n+1)!}-\sum_{n=0}^{\infty}\frac{z^{2n}(1-s)_n}{(2n)!}\right),\end{aligned}$$

where in the penultimate step, we have utilized the fact that Re $s<1$. Combining the two infinite series in the last line in the above calculation, we have

$$\begin{aligned}
&\int_0^\infty x^{-s}\frac{d}{dx}\left(e^{-\pi x}\cosh(\sqrt{\pi x}z)\right)\,dx\\
&=-\pi^s\Gamma(1-s)\left(1+\sum_{n=0}^\infty\left(\frac{(1-s)_{n+1}z^{2n+2}}{(2n+2)!}-\frac{(1-s)_nz^{2n+2}}{2(2n+1)!}\right)\right)\\
&=-\pi^s\Gamma(1-s)\left(1+\sum_{n=0}^\infty\frac{(1-s)_nz^{2n+2}}{(2n+1)!}\left(\frac{1-s+n}{2n+2}-\frac{1}{2}\right)\right)\\
&=-\pi^s\Gamma(1-s)\left(1+\sum_{n=0}^\infty\frac{(-s)_{n+1}(z)^{2n+2}}{(2n+2)!}\right)\\
&=-\pi^s\Gamma(1-s)\left(1+\sum_{n=0}^\infty\frac{(-s)_{n+1}(z^2/4)^{n+1}}{(n+1)!\left(\frac{1}{2}\right)_{(n+1)}}\right)\\
&=-\pi^s\Gamma(1-s){}_1F_1\left(-s;\frac{1}{2};\frac{z^2}{4}\right). \qquad (4.13)
\end{aligned}$$

From (4.11), (4.12) and (4.13), we arrive at

$$\varphi(s,z)=\frac{\pi^s\Gamma(-s){}_1F_1\left(-s;\frac{1}{2};\frac{z^2}{4}\right)}{\zeta(2s+1)}. \qquad (4.14)$$

Remark. Letting $z=0$ in (4.5) gives the following result of Hardy and Littlewood [18, Equation (2.544)]:

$$\int_0^\infty y^{-s-1}P(y)\,dy=\frac{\pi^s\Gamma(-s)}{\zeta(2s+1)}, \qquad (4.15)$$

where $P(y)$ is defined in (1.1). Multiplying both sides of (4.5) by $s\zeta(2s+1)$, we have

$$s\zeta(2s+1)\int_0^\infty y^{-s-1}P_z(y)\,dy=\pi^s\Gamma(1-s){}_1F_1\left(-s;\frac{1}{2};\frac{z^2}{4}\right). \qquad (4.16)$$

We now show that the bound $P_z(y)=O_{z,\delta}(y^{-\frac{1}{4}+\delta})$ for any $\delta>0$ as $y\to\infty$ implies that (4.16) holds for $-\frac{1}{4}<\text{Re } s\leq 0$ as well. Splitting the integral on the left-hand side into two integrals, one from 0 to 1 and another from 1 to ∞, and applying the bound for $P_z(y)$ for the latter integral, one can see that the integral is analytic on $-\frac{1}{4}<\text{Re } s\leq 0$. Since the pole of $\zeta(2s+1)$ at $s=0$ is annihilated by the zero of s at $s=0$, we see that the left-hand side of (4.16) is analytic. Since $\Gamma(1-s)$ does not have any poles in $-\frac{1}{4}<\text{Re } s\leq 0$ and ${}_1F_1\left(-s;\frac{1}{2};\frac{z^2}{4}\right)$ is an entire function of s, the right-hand side of (4.16) is also analytic on $-\frac{1}{4}<\text{Re } s\leq 0$. Since (4.16) holds for $0<\text{Re } s<1$, by the principle of analytic continuation, it holds on $-\frac{1}{4}<\text{Re } s\leq 0$ as well. Since the Gamma function does not have any zeros, the zeros of the right-hand side of (4.16) are the zeros of ${}_1F_1\left(-s;\frac{1}{2};\frac{z^2}{4}\right)$. Now if $z=0$, the fact that ${}_1F_1\left(-s;\frac{1}{2};\frac{z^2}{4}\right)=1$ implies that the left-hand side of (4.16) is non-zero in $-\frac{1}{4}<$ Re $s<0$. Since the integral on the left is analytic in this interval, this implies that $\zeta(2s+1)$ does not have any zeros in $-\frac{1}{4}<\text{Re } s<0$. This implies the Riemann Hypothesis, thereby proving part (2)(a) of Theorem 1.1.

Let us now consider the case when $z \neq 0$. For $|\lambda| \to \infty$ and $|\arg(\sqrt{\lambda z})| < \pi$, the following estimate for $M_{\lambda,\mu}(z)$, where $M_{\lambda,\mu}(z)$ is defined in (2.8), is well-known [22, p. 318]:

$$M_{\lambda,\mu}(z) = \pi^{-1/2} z^{1/4} \lambda^{-\mu-1/4} \Gamma(2\mu+1) \cos\left(2\sqrt{\lambda z} - \frac{\pi}{4} - \mu\pi\right) + O\left(|\lambda|^{-\mu-3/4}\right). \tag{4.17}$$

From (2.8) and (4.17), as $|\lambda| \to \infty$,

$$\begin{aligned} {}_1F_1\left(\mu - \lambda + \frac{1}{2}; 2\mu+1; z\right) &= \pi^{-1/2} (\lambda z)^{-\mu-1/4} e^{z/2} \Gamma(2\mu+1) \cos\left(2\sqrt{\lambda z} - \frac{\pi}{4} - \mu\pi\right) \\ &\quad + O_{z,\mu}\left(|\lambda|^{-\mu-3/4}\right). \end{aligned} \tag{4.18}$$

This gives for $|s| \to \infty$,

$${}_1F_1\left(-s; \frac{1}{2}; \frac{z^2}{4}\right) = e^{\frac{z^2}{8}} \cos\left(z\sqrt{\left(s+\frac{1}{4}\right)}\right) + O_z\left(\left|s+\frac{1}{4}\right|^{-\frac{1}{2}}\right). \tag{4.19}$$

For $s = \sigma + it$, since $-\frac{1}{4} < \sigma < 1$ and $|s| \to \infty$, we have $|t| \to \infty$. Since $z \neq 0$ and $\arg(z) \neq -\frac{\pi}{4}$, this implies that the main term on the right-hand side of (4.19) tends to ∞ in absolute value as $|s| \to \infty$, implying that for t large enough, we have $\left|{}_1F_1\left(-s; \frac{1}{2}; \frac{z^2}{4}\right)\right| > 0$, i.e., for a fixed non-zero z, there exists a number T_z such that for $t > T_z$, we have ${}_1F_1\left(-s; \frac{1}{2}; \frac{z^2}{4}\right) \neq 0$. Hence, the left-hand side of (4.16) has zeros at most up to a fixed height T_z depending on z. But $\zeta(s)$ has only finitely many zeros up to any fixed height T. This proves part (2)(b) of Theorem 1.1.

Remark. Note that if $\arg(z) = -\frac{\pi}{4}$, then $z\sqrt{s+1/4}$ is almost real, and so $\cos\left(z\sqrt{s+1/4}\right)$ will be essentially bounded, thereby making the main term in (4.19) bounded. Hence we need the condition $\arg(z) \neq -\frac{\pi}{4}$.

Now we prove part (1) of Theorem 1.1, i.e., the Riemann Hypothesis implies the bound $P_z(y) = O_{z,\delta}(y^{-\frac{1}{4}+\delta})$ for any $\delta > 0$ as $y \to \infty$. It is known that the Riemann Hypothesis implies $M(x) := \sum_{n \le x} \mu(n) = O_\epsilon(x^{\frac{1}{2}+\epsilon})$ for all $\epsilon > 0$. By partial summation, it is easy to see that

$$M(\nu, n) := \sum_{m=\nu}^{n} \frac{\mu(m)}{m} = O_\epsilon(\nu^{-\frac{1}{2}+\epsilon}) \tag{4.20}$$

uniformly in n. From (1.4),

$$\begin{aligned} P_z(\beta^2) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{\frac{-\pi\beta^2}{n^2}} \cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right) \\ &= \left[\sum_{n=1}^{\nu-1} + \sum_{n=\nu}^{\infty}\right] \frac{\mu(n)}{n} e^{\frac{-\pi\beta^2}{n^2}} \cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right) \\ &= P_1 + P_2, \end{aligned} \tag{4.21}$$

say, where $\nu = [\beta^{1-\epsilon}]$. Here

$$P_2 = \sum_{n=\nu}^{\infty} \frac{\mu(n)}{n} e^{\frac{-\pi\beta^2}{n^2}} \cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right). \tag{4.22}$$

For any integer $N > \nu$, we have

$$\begin{aligned}\sum_{n=\nu}^{N}\frac{\mu(n)}{n}e^{\frac{-\pi\beta^2}{n^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right) &= \frac{\mu(\nu)}{\nu}e^{\frac{-\pi\beta^2}{\nu^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{\nu}\right)\\ &\quad+\sum_{\nu<n\leq N}(M(\nu,n)-M(\nu,n-1))e^{\frac{-\pi\beta^2}{n^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right)\\ &= \frac{\mu(\nu)}{\nu}e^{\frac{-\pi\beta^2}{(\nu+1)^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{\nu+1}\right)+M(\nu,N)e^{\frac{-\pi\beta^2}{N^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{N}\right)\\ &\quad+\sum_{\nu\leq n<N-1}M(\nu,n)\left(e^{\frac{-\pi\beta^2}{n^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right)-e^{\frac{-\pi\beta^2}{(n+1)^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n+1}\right)\right)\\ &= \sum_{\nu\leq n<N-1}M(\nu,n)\left(e^{\frac{-\pi\beta^2}{n^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right)-e^{\frac{-\pi\beta^2}{(n+1)^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n+1}\right)\right)\\ &\quad+O_\epsilon(\nu^{-\frac{1}{2}+\epsilon})+\nu^{-\frac{1}{2}+\epsilon}O(N^{-2}). \qquad (4.23)\end{aligned}$$

Letting N tend to infinity, from (4.22) and (4.23) we derive

$$P_2=\sum_{n=\nu}^{\infty}M(\nu,n)e^{\frac{-\pi\beta^2}{\lambda_n^2}}\left(\frac{2\pi\beta^2}{\lambda_n^3}\cosh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)-\frac{\sqrt{\pi}\beta z}{\lambda_n^2}\sinh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)\right)+O_\epsilon(\nu^{-\frac{1}{2}+\epsilon}), \qquad (4.24)$$

where the mean value theorem with $n < \lambda_n < n+1$ is used in the last step. Using (4.20), we see that

$$\begin{aligned}P_2 &= O_\epsilon\left(\nu^{-\frac{1}{2}+\epsilon}\sum_{n=\nu}^{\infty}e^{\frac{-\pi\beta^2}{\lambda_n^2}}\left|\frac{2\pi\beta^2}{\lambda_n^3}\cosh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)-\frac{\sqrt{\pi}\beta z}{\lambda_n^2}\sinh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)\right|\right)\\ &= O_\epsilon\left(\nu^{-\frac{1}{2}+\epsilon}\left(\sum_{n=\nu}^{[C\beta]}+\sum_{n=[C\beta]+1}^{\infty}\right)e^{\frac{-\pi\beta^2}{\lambda_n^2}}\left|\frac{2\pi\beta^2}{\lambda_n^3}\cosh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)-\frac{\sqrt{\pi}\beta z}{\lambda_n^2}\sinh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)\right|\right)\\ &= O_\epsilon\left(\nu^{-\frac{1}{2}+\epsilon}(P_3+P_4)\right), \qquad (4.25)\end{aligned}$$

say, where the constant C will be chosen later. Note that for $n\geq C\beta$,

$$\begin{aligned}\frac{2\pi\beta^2}{\lambda_n^3}\cosh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)-\frac{\sqrt{\pi}\beta z}{\lambda_n^2}\sinh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right) &= \left(\frac{\beta^2}{n^3}O_{C,z}(1)+\frac{\beta}{n^2}O_{C,z}\left(\frac{\beta}{n}\right)\right)\\ &= O_{C,z}\left(\frac{\beta^2}{n^3}\right). \qquad (4.26)\end{aligned}$$

Hence,

$$\begin{aligned}P_4 &= \sum_{n=[C\beta]}^{\infty}e^{\frac{-\pi\beta^2}{\lambda_n^2}}\left|\frac{2\pi\beta^2}{\lambda_n^3}\cosh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)-\frac{\sqrt{\pi}\beta z}{\lambda_n^2}\sinh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)\right|\\ &= O_{C,z}\left(\sum_{n=[C\beta]}^{\infty}\frac{\beta^2}{n^3}\right)\\ &= O_{C,z}(1). \qquad (4.27)\end{aligned}$$

For $\nu \le n \le C\beta$,

$$\frac{2\pi\beta^2}{\lambda_n^3}\cosh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)-\frac{\sqrt{\pi}\beta z}{\lambda_n^2}\sinh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)=O_{C,z}\left(\frac{\beta^2}{n^3}e^{\frac{\sqrt{\pi}\beta|\mathrm{Re}z|}{n}}\right). \tag{4.28}$$

Hence,

$$\begin{aligned}P_3&=\sum_{n=\nu}^{[C\beta]}e^{\frac{-\pi\beta^2}{\lambda_n^2}}\left|\frac{2\pi\beta^2}{\lambda_n^3}\cosh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)-\frac{\sqrt{\pi}\beta z}{\lambda_n^2}\sinh\left(\frac{\sqrt{\pi}\beta z}{\lambda_n}\right)\right|\\&=O_{C,z}\left(\sum_{n=\nu}^{[C\beta]}\frac{\beta^2}{n^3}e^{\frac{-\pi\beta^2}{\lambda_n^2}+\frac{\sqrt{\pi}\beta|\mathrm{Re}z|}{n}}\right).\end{aligned} \tag{4.29}$$

Choose $C=\frac{\sqrt{\pi}}{1+4|\mathrm{Re}z|}$. Then one sees that for each n in the interval $[\nu,[C\beta]]$ the corresponding exponent on the right-hand side of (4.29) is negative, so

$$\left|e^{\frac{-\pi\beta^2}{\lambda_n^2}+\frac{\sqrt{\pi}\beta|\mathrm{Re}z|}{n}}\right|\le 1.$$

Hence

$$P_3=O_z\left(\sum_{n=\nu}^{[C\beta]}\frac{\beta^2}{n^3}\right)=O_z\left(\frac{\beta^2}{\nu^2}\right)=O_z(\beta^{2\epsilon}). \tag{4.30}$$

Therefore, $P_2=O_z\left(\nu^{-\frac{1}{2}+\epsilon}\beta^{2\epsilon}\right)$. Now for $\beta^{2\epsilon}>\frac{\sqrt{\pi}|\mathrm{Re}z|}{\pi-1}$,

$$\begin{aligned}P_1&=\sum_{n=1}^{\nu-1}\frac{\mu(n)}{n}e^{\frac{-\pi\beta^2}{n^2}}\cosh\left(\frac{\sqrt{\pi}\beta z}{n}\right)\\&\ll\sum_{n=1}^{\nu-1}e^{\frac{-\pi\beta^2}{n^2}+\frac{\sqrt{\pi}\beta|\mathrm{Re}z|}{n}}\\&\ll\nu e^{-\frac{\beta^2}{\nu^2}}\\&\ll\nu e^{-\beta^{2\epsilon}}.\end{aligned} \tag{4.31}$$

Combining the bounds for P_1 and P_2 above, one obtains the desired bound for $P_z(\beta^2)$. Replacing β^2 by y proves part (i) of Theorem 1.1. This completes the proof of Theorem 1.1.

Now in the case of a primitive Dirichlet character, the associated Dirichlet L-function does not have a pole at $s=1$, or in other words,

$$\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\neq 0.$$

Thus, in order to obtain an analogue of Theorem 1.1 for Dirichlet L-functions, it seems appropriate to work with the derivative of the analogue of $P_z(y)$ rather than the function itself. This analogue of $P_z(y)$ is defined by

$$P_z(y,\chi):=\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\left(e^{-\frac{\pi y}{qn^2}}\cosh\left(\frac{\sqrt{\pi y}z}{n\sqrt{q}}\right)-1\right) \tag{4.32}$$

for χ even, and by

$$P_z(y,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \left(\frac{e^{-\frac{\pi y}{qn^2}}}{\frac{\sqrt{\pi y}z}{n\sqrt{q}}} \sinh\left(\frac{\sqrt{\pi y}z}{n\sqrt{q}}\right) - 1 \right) \tag{4.33}$$

for χ odd.

Theorem 4.2. *Let χ denote a primitive Dirichlet character modulo q, where $q \geq 3$. Fix $z \in \mathbb{C}$. Let b be defined in* (1.17). *Consider the function*

$$Q_z(y,\chi) := \sum_{m,k=0}^{\infty} \frac{(m+k)(-1)^m(\pi/q)^{m+k}y^{m+k}z^{2k}}{m!(2k+b)!L(2m+2k+1,\chi)}, \tag{4.34}$$

Then we have the following:

(1) *The Riemann Hypothesis for $L(s,\chi)$ implies $Q_z(y,\chi) = O_{z,\chi,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all positive values of δ.*

(2) (a) *If $z = 0$, the estimate $Q_z(y,\chi) = O_{z,\chi,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all positive values of δ implies the Riemann Hypothesis for $L(s,\chi)$.*

(b) *If $z \neq 0$ and* $\arg(z) \neq -\frac{\pi}{4}$, *the estimate $Q_z(y,\chi) = O_{z,\chi,\delta}\left(y^{-\frac{1}{4}+\delta}\right)$ as $y \to \infty$ for all positive values of δ implies that $L(s,\chi)$ has at most finitely many non-trivial zeros off the critical line.*

Note that upon writing the right-hand sides of (4.32) and (4.33) in terms of the Taylor series of $e^{-\frac{\pi y}{qn^2}} \cosh\left(\frac{\sqrt{\pi y}z}{n\sqrt{q}}\right)$ and $e^{-\frac{\pi y}{qn^2}} \left(\sinh\left(\frac{\sqrt{\pi y}z}{n\sqrt{q}}\right) / \left(\frac{\sqrt{\pi y}z}{n\sqrt{q}}\right)\right)$, changing the order of summation and then simplifying, it is seen in both the cases that $\frac{\partial}{\partial y}P_z(y,\chi) = Q_z(y,\chi)/y$. We refrain from giving the details of the proof of the above theorem since the reasoning is similar to that in the proof of Theorem 1.1. However, we note that the following identity, which is interesting in its own right, plays a crucial role in the proof.

Lemma 4.3. *Let $Q_z(y,\chi)$ and the number b be defined as in* (4.34) *and* (1.17) *respectively. Let* $0 < \mathrm{Re}(s) < 1$. *Then for any z, we have*

$$\int_0^{\infty} y^{-s-1} Q_z(y,\chi)\, dy = \frac{-(\pi/q)^s \Gamma(1-s) {}_1F_1\left(-s; \frac{1}{2}+b; \frac{z^2}{4}\right)}{L(2s+1,\chi)}. \tag{4.35}$$

5. Proofs of the character analogues of some well-known transformation formulas

In this section, we prove Theorems 1.3, 1.4, 1.5 for odd real χ. The case when χ is even and real can be similarly proved.

Proof of Theorem 1.3. Let χ be odd and real, $z \in \mathbb{C}\backslash\{0\}$, and let $\phi(s) \equiv 1$. Using the second equation in (2.1), we have

$$\int_0^{\infty} \Xi\left(\frac{t}{2},\chi\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right) dt$$
$$= \frac{2\sqrt{\alpha q}e^{-\frac{z^2}{8}}}{i\sqrt{\pi}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) {}_1F_1\left(1-\frac{s}{2}; \frac{3}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s} ds$$

$$= \frac{2\sqrt{\alpha q}e^{-\frac{z^2}{8}}}{i\sqrt{\pi}} \sum_{n=1}^{\infty} \chi(n) \int_{1+\delta-i\infty}^{1+\delta+i\infty} \Gamma\left(\frac{s+1}{2}\right) {}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha n}{\sqrt{q}}\right)^{-s} ds, \qquad (5.1)$$

where $0 < \delta < 1$. The last expression comes from the residue theorem since one does not encounter any pole when one moves the line of integration from Re $s = 1/2$ to Re $s = 1+\delta$, and also from the fact that $\sum_{n=1}^{\infty} \chi(n) n^{-s}$ converges absolutely for Re $s > 1$ and is equal to $L(s, \chi)$. Now letting $b = z \neq 0$, $a = 1$ in Lemma 2.3 and simplifying, for $c = \text{Re } s > -1$, we obtain

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s+1}{2}\right) {}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right) x^{-s}\, ds = \frac{2}{z} e^{-x^2+\frac{z^2}{4}} \sin xz. \qquad (5.2)$$

Use (5.2) in (5.1) to get

$$\frac{1}{8\sqrt{\pi q}} \int_0^{\infty} \Xi\left(\frac{t}{2}, \chi\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right) dt = \frac{\sqrt{\alpha} e^{\frac{z^2}{8}}}{z} \sum_{n=1}^{\infty} \chi(n) e^{-\frac{\pi\alpha^2 n^2}{q}} \sin\left(\frac{\sqrt{\pi}\alpha n z}{\sqrt{q}}\right). \qquad (5.3)$$

The proof is complete once we observe the fact that the left-hand side of (5.3) is invariant under the simultaneous replacement of α by β and of z by iz. $\square$

Proof of Theorem 1.4. Again, assume that χ is odd and real. Let $\phi(s) = \frac{s+\frac{1}{2}}{2\sqrt{2\pi}} \Gamma\left(\frac{1}{4}+\frac{s}{2}\right)\Gamma\left(-\frac{1}{4}+\frac{s}{2}\right)$ so that

$$f(t) = \phi(it)\phi(-it) = \frac{\frac{1}{4}+t^2}{8\pi^2} \Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}-\frac{it}{2}\right).$$

Using twice the reflection formula $\Gamma(z)\Gamma(-z) = -\pi/(z \sin \pi z)$, $z \notin \mathbb{Z}$, we find that $f\left(\frac{t}{2}\right) = 1/\cosh\left(\frac{1}{2}\pi t\right)$. Thus, using the second equation in (2.1) and the fact that $z\Gamma(z) = \Gamma(z+1)$, we get

$$\int_0^{\infty} \frac{\Xi\left(\frac{t}{2}, \chi\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right)}{\cosh \frac{1}{2}\pi t} dt$$

$$= \frac{\sqrt{\alpha q}e^{-\frac{z^2}{8}}}{\pi^{\frac{5}{2}} i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{s}{2}\right)\Gamma^2\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right) L(s,\chi) {}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s} ds$$

$$= \frac{2\sqrt{\alpha q}e^{-\frac{z^2}{8}}}{i\sqrt{\pi}} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{s+1}{2}\right) \frac{L(s,\chi)}{\sin \pi s} {}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s} ds, \qquad (5.4)$$

where in the last step we have used the duplication formula $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ twice as well as the reflection formula.

Now in order to use the Dirichlet series representation for $L(s, \chi)$, we shift the line of integration from Re $s = \frac{1}{2}$ to Re $s = 1+\delta$, where $0 < \delta < 1$. Consider a positively oriented rectangular contour with sides $[\frac{1}{2}+iT, \frac{1}{2}-iT], [\frac{1}{2}-iT, 1+\delta-iT], [1+\delta-iT, 1+\delta+iT]$ and $[1+\delta+iT, \frac{1}{2}+iT]$, where T is any positive real number. While shifting, we have to take care of the pole of order 1 of the integrand (due to $\sin \pi s$). Thus using the residue theorem and noting that by (2.7) the integrals along the horizontal line segments tend to zero as $T \to \infty$, and then interchanging the order of summation and integration while evaluating the integral on the line Re $s = 1+\delta$, which is valid because of absolute convergence, we have

$$\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma\left(\frac{s+1}{2}\right) \frac{L(s,\chi)}{\sin \pi s} {}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s} ds$$

$$= \sum_{n=1}^{\infty} \chi(n) G(z,\alpha,n) - 2\pi i \lim_{s\to1} \frac{(s-1)}{\sin \pi s} \Gamma\left(\frac{s+1}{2}\right) L(s,\chi) {}_1F_1\left(1-\frac{s}{2}, \frac{3}{2}, \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s}, \tag{5.5}$$

where

$$G(z,\alpha,n) = \int_{1+\delta-i\infty}^{1+\delta+i\infty} \Gamma\left(\frac{s+1}{2}\right) \frac{{}_1F_1\left(1-\frac{s}{2}; \frac{3}{2}; \frac{z^2}{4}\right)}{\sin \pi s} \left(\frac{\sqrt{\pi}\alpha n}{\sqrt{q}}\right)^{-s} ds. \tag{5.6}$$

Using the residue theorem once again, we have for $0 < c = \text{Re } s < 1$,

$$\begin{aligned} G(z,\alpha,n) = &\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s+1}{2}\right) \frac{{}_1F_1\left(1-\frac{s}{2}; \frac{3}{2}; \frac{z^2}{4}\right)}{\sin \pi s} \left(\frac{\sqrt{\pi}\alpha n}{\sqrt{q}}\right)^{-s} ds \\ &+ 2\pi i \lim_{s\to1} \frac{(s-1)}{\sin \pi s} \Gamma\left(\frac{s+1}{2}\right) {}_1F_1\left(1-\frac{s}{2}; \frac{3}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\alpha n}{\sqrt{q}}\right)^{-s}. \end{aligned} \tag{5.7}$$

Since for $0 < c < 1$, we have [27, p. 91, Equation (3.3.10)]

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{\sin \pi s} ds = \frac{1}{\pi(1+x)}, \tag{5.8}$$

and since [27, p.83, Equation (3.1.13)],

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) G(s) w^{-s} ds = \int_0^{\infty} f(x) g\left(\frac{w}{x}\right) \frac{dx}{x}, \tag{5.9}$$

where $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively, we deduce using (5.2) that

$$G(z,\alpha,n) = 2\pi i \left(\frac{2e^{\frac{z^2}{4}}}{\pi z} \int_0^{\infty} \frac{e^{-x^2} \sin xz}{x + \frac{\sqrt{\pi}\alpha n}{\sqrt{q}}} dx - \frac{\sqrt{q}}{\alpha n \pi^{\frac{3}{2}}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{z^2}{4}\right) \right), \tag{5.10}$$

which after a simple change of variable gives

$$G(z,\alpha,n) = \frac{4ie^{\frac{z^2}{4}}}{z} \int_0^{\infty} \frac{e^{-\frac{\pi\alpha^2x^2}{q}} \sin\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)}{x+n} dx - \frac{2i\sqrt{q}}{\alpha n\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \frac{z^2}{4}\right). \tag{5.11}$$

Substituting (5.11) in (5.5), simplifying and then combining with (5.4), we see that

$$\begin{aligned} -\frac{\sqrt{\pi}}{8\sqrt{q}} \int_0^{\infty} \frac{\Xi\left(\frac{t}{2}, \chi\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right)}{\cosh \frac{1}{2}\pi t} dt &= -\frac{\sqrt{\alpha} e^{\frac{z^2}{8}}}{z} \sum_{n=1}^{\infty} \chi(n) \int_0^{\infty} \frac{e^{-\frac{\pi\alpha^2x^2}{q}} \sin\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)}{x+n} dx \\ &= \frac{\sqrt{\alpha} e^{\frac{z^2}{8}}}{z} \int_0^{\infty} \psi(x,\chi) e^{-\frac{\pi\alpha^2x^2}{q}} \sin\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right) dx, \end{aligned} \tag{5.12}$$

where $\psi(x,\chi)$ is defined in (1.30) and the interchange of summation and integration is justified because of absolute convergence. Finally we obtain (1.32) from the above equation by simultaneously replacing α by β and z by iz and noting that the integral on the left-hand side is invariant under these replacements. □

Proof of Theorem 1.5. Let $\phi(s) = \Gamma\left(\frac{3}{4}+\frac{s}{2}\right)$ and let

$$I(z,\alpha,\chi) := \int_0^\infty \Gamma\left(\frac{3+it}{4}\right)\Gamma\left(\frac{3-it}{4}\right)\Xi\left(\frac{t}{2},\chi\right)\Delta\left(\alpha,z,\frac{1+it}{2}\right)dt. \tag{5.13}$$

Using (2.1), we have

$$I(z,\alpha,\chi) = \frac{\sqrt{\alpha q}e^{-\frac{z^2}{8}}}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} B\left(\frac{1+s}{2},1-\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)L(s,\chi)\,{}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha}{\sqrt{q}}\right)^{-s}ds, \tag{5.14}$$

where $B(s,z-s)$ is the Euler beta function given by

$$B(s,z-s) = \int_0^\infty \frac{x^{s-1}}{(1+x)^z}\,dx = \frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)},\ \ 0 < \text{ Re } s < \text{ Re } z. \tag{5.15}$$

Now shift the line of integration from Re $s=\frac{1}{2}$ to Re $s = 1+\delta$, $0<\delta<2$, and observe that this does not introduce any poles, so that by the residue theorem and replacing $L(s,\chi)$ by its Dirichlet series, we get

$$I(z,\alpha,\chi) = \frac{\sqrt{\alpha q}e^{-\frac{z^2}{8}}}{i}\sum_{n=1}^\infty \chi(n)\int_{1+\delta-i\infty}^{1+\delta+i\infty} B\left(\frac{1+s}{2},1-\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right){}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)\left(\frac{\sqrt{\pi}\alpha n}{\sqrt{q}}\right)^{-s}ds, \tag{5.16}$$

where the interchange of summation and integration is justified by absolute convergence. Using (5.15), we have for $-1 < c = \text{Re } s < 2$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B\left(\frac{1+s}{2},1-\frac{s}{2}\right)x^{-s}\,ds = \frac{2x}{(1+x^2)^{\frac{3}{2}}}. \tag{5.17}$$

Thus, from (5.2), (5.9) and (5.17), we obtain upon simplification

$$I(z,\alpha,\chi) = \frac{4\pi\sqrt{\alpha q}e^{\frac{z^2}{8}}}{z}\sum_{n=1}^\infty \chi(n)\int_0^\infty 2\frac{\left(\sqrt{\pi}\alpha n/\sqrt{q}\right)}{x}\left(1+\left(\frac{\sqrt{\pi}\alpha n/\sqrt{q}}{x}\right)^2\right)^{\frac{-3}{2}}e^{-x^2}\sin xz\,dx. \tag{5.18}$$

Making a change of variable $x \to \frac{\sqrt{\pi}\alpha x}{\sqrt{q}}$ and interchanging the order of summation and integration, we have

$$I(z,\alpha,\chi) = \frac{8\pi\sqrt{\alpha q}e^{\frac{z^2}{8}}}{z}\int_0^\infty xe^{-\frac{\pi\alpha^2x^2}{q}}\sin\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)\sum_{n=1}^\infty \frac{n\chi(n)}{(n^2+x^2)^{\frac{3}{2}}}\,dx. \tag{5.19}$$

Now use (2.3) with χ real and $\nu = 0$, use (1.19) and simplify to obtain

$$\sum_{n=1}^\infty \frac{n\chi(n)}{(n^2+x^2)^{\frac{3}{2}}} = \frac{4\pi}{q^{\frac{3}{2}}}\sum_{n=1}^\infty \chi(n)nK_0\left(\frac{2\pi nx}{q}\right). \tag{5.20}$$

Substitute (5.20) in (5.19) to obtain

$$\frac{q}{32\pi^2}I(z,\alpha,\chi) = \frac{\sqrt{\alpha}e^{\frac{z^2}{8}}}{z}\int_0^\infty xe^{-\frac{\pi\alpha^2x^2}{q}}\sin\left(\frac{\sqrt{\pi}\alpha xz}{\sqrt{q}}\right)\sum_{n=1}^\infty \chi(n)nK_0\left(\frac{2\pi nx}{q}\right)dx. \tag{5.21}$$

Now note that this proves (1.34) completely since $I(z,\alpha,\chi)$ is invariant under the simultaneous application of the maps $\alpha \to \beta$ and $z \to iz$. □

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