TWISTED SECOND MOMENTS OF THE RIEMANN ZETA-FUNCTION AND APPLICATIONS

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Abstract. In order to compute a twisted second moment of the Riemann zeta-function, two different mollifiers, each being a combinations of two different Dirichlet polynomials were introduced separately by Bui, Conrey, and Young, and by Feng. In this article we introduce a mollifier which is a combination of four Dirichlet polynomials of different shapes. We provide an asymptotic result for the twisted second moment of $\zeta(s)$ for such choice of mollifier. A small increment on the percentage of zeros of the Riemann zeta-function on the critical line is given as an application of our results.

1. Introduction

In [1], Balasubramanian, Conrey and Heath-Brown computed the twisted second moment of the Riemann zeta-function

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \overline{\psi}\psi(\tfrac{1}{2}+it)dt \tag{1.1}$$

where ψ is a Dirichlet polynomial of the type

$$\psi(s) = \sum_{n \leq T^\theta} \frac{a(n)}{n^s}, \tag{1.2}$$

and $a(n) \ll_\varepsilon n^\varepsilon$. The length T^θ of the polynomial is sensitive to the nature of the coefficients $a(n)$. They also obtained an explicit main term in their theorem for a particular choice of $\psi(s)$.

In [5], in order to obtain a higher percentage of zeros of the Riemann zeta-function on the critical line, Conrey needed to establish such type of second moment. In his result he made an ingenious choice of $a(n)$ which allowed him to push the value of θ from $1/2$ (see [10]) to $4/7$. The possibility of obtaining a mollifier by combining two Dirichlet polynomials of different shape had been considered by Lou [11]. In [2], Bui, Conrey, and Young extended (1.1) with an explicit main term for a more sophisticated choice of $a(n)$. They considered $\psi(s)$ as a convex combination of two Dirichlet polynomials of different shape. Introducing such two-piece mollifier increases the complexity and technicality of the computation of the main term. Another such two-piece mollifier was introduced by Feng [9] and the main term was computed explicitly.

Crucial ingredients to obtaining the error term in [9] were Lemmas 1 and 2. To reach $\theta_1 < 4/7-\varepsilon$ in [5], it was required that $a(n) = \mu(n)F(n)$, for a smooth function F. In [9], the coefficient $a(n)$ in the mollifier was not of the form $\mu(n)F(n)$, for some smooth function F, and it is not clear how the techniques of [5] can be directly applied to the proofs of Lemmas 1 and 2 of [9].

Independently of each other, in [2] and [9], the possibility of obtaining a $\psi(s)$ by combining these three Dirichlet polynomials of different shape was mentioned. One can obtain the main term of (1.1) for such choice of $\psi(s)$ by going over some subtle technicalities in the calculations.

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In the present paper we introduce a new mollifier $\psi(s)$ which is a convex combination of four Dirichlet polynomials of different shape. Let

$$\chi(s) := \pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right).$$

We will use the convention

$$P_i[n] := P_i\left(\frac{\log(y_i/n)}{\log y_i}\right) \quad \text{and} \quad \tilde{P}_k[n] := \tilde{P}_k\left(\frac{\log(y_4/n)}{\log y_4}\right), \tag{1.3}$$

where P's are polynomials. Recall that $\mu(n)$ denotes the Möbius function, also $\mu_2(n)$ and $\mu_3(n)$ will denote the coefficients in the Dirichlet series of $1/\zeta^2(s)$ and $1/\zeta^3(s)$, respectively, for $\mathrm{Re}(s) > 1$. Also, let $d_k(n)$ denote the number of ways an integer n can be written as a product of $k \geq 2$ fixed factors. Note that $d_1(n) = 1$ and that $d_2(n) = d(n)$ is the number of divisors of n. With this in mind, we define

$$\psi(s) := \psi_1(s) + \psi_2(s) + \psi_3(s) + \psi_4(s), \tag{1.4}$$

where

$$\psi_1(s) = \sum_{n \leq y_1} \frac{\mu(n) n^{\sigma_0 - 1/2}}{n^s} P_1[n] \tag{1.5}$$

introduced in [5],

$$\psi_2(s) = \chi(s + \tfrac{1}{2} - \sigma_0) \sum_{hk \leq y_2} \frac{\mu_2(h) h^{\sigma_0 - 1/2} k^{1/2-\sigma_0}}{h^s k^{1-s}} P_2[hk] \tag{1.6}$$

introduced in [2],

$$\psi_3(s) = \chi^2(s + \tfrac{1}{2} - \sigma_0) \sum_{hk \leq y_3} \frac{\mu_3(h) d(k) h^{\sigma_0 - 1/2} k^{1/2-\sigma_0}}{h^s k^{1-s}} P_3[hk] \tag{1.7}$$

introduced in the present paper, and

$$\psi_4(s) = \sum_{n \leq y_4} \frac{\mu(n) n^{\sigma_0 - 1/2}}{n^s} \sum_{k=2}^{K} \sum_{p_1 \ldots p_k | n} \frac{\log p_1 \ldots \log p_k}{\log^k y_4} \widetilde{P}_k[n], \tag{1.8}$$

introduced in [9]. Here $K \geq 2$ is a positive integer of our choice and $p_1, \ldots, p_k$ are distinct primes. Also we need $P_1(0) = 0$, $P_1(1) = 1$, $P_2(0) = P_2'(0) = P_2''(0) = 0$, $P_3(0) = P_3'(0) = \cdots = P_3^{(6)}(0) = 0$, and $\tilde{P}_k(0) = 0$, for $k = 2, \ldots, K$. We use the conventions $y_i = T^{\theta_i}$ and $\sigma_0 = 1/2 - R/\log T$.

The reasoning behind introducing the new piece ψ_3 is that it approximates $1/\zeta(s)$ in some region of the complex plane. We now state our main theorem.

Theorem 1.1. *Let $\alpha, \beta \ll \frac{1}{\log T}$, $\sigma_0 = \frac{1}{2} - \frac{R}{\log T}$ and $R \ll 1$. Then for $\theta_1 < 4/7 - \varepsilon$, $\theta_2 < 1/2 - \varepsilon$, $\theta_3 < 3/7 - \varepsilon$, and $\theta_4 < 3/7 - \varepsilon$ we have*

$$I(\alpha, \beta) := \int_1^T \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi}\psi(\sigma_0 + it) dt = CT + O_\varepsilon(T(\log T)^{-1+\varepsilon}), \tag{1.9}$$

where C is an explicit constant that depends on $\alpha, \beta, Q, P_1, P_2, P_3, R, \theta_1, \theta_2, \theta_3, \theta_4$ and $\tilde{P}_k$ for $k = 2, 3, \ldots, K$.

In this manuscript we will obtain an explicit formula for the constant C and more specifically we show that

$$C = c_{11}(\alpha,\beta) + 2c_{14}(\alpha,\beta) + c_{44}(\alpha,\beta) + c_{22}(\alpha,\beta) + c_{33}(\alpha,\beta) + 2c_{12}(\alpha,\beta) + 2c_{23}(\alpha,\beta) + 2c_{24}(\alpha,\beta),$$

where the values of $c_{ij}(\alpha,\beta)$ are given in the next section. At the end of this article we will provide an application and show that optimizing the numerical value of certain derivatives of C with respect to α and β for specific values of α and β, will give an improved result towards the percentage of zeros of the Riemann zeta-function on the critical line.

2. Intermediate Results

From now on we will denote $L=\log T$. Suppose that $w(t)$ is a smooth function with the following properties:

(1) $0\le w(t)\le 1$ for all $t\in\mathbb{R}$,
(2) $w(t)$ has compact support in $[T/4,2T]$,
(3) $w^{(j)}(t)\ll_j \Delta^{-j}$ for each $j=0,1,2,\cdots$, where $\Delta=\frac{T}{5L}$.

The Fourier transform of $w(t)$ is denoted by $\widehat{w}(s)$. For $j,k\in\{1,2,3,4\}$ and $(j,k)\notin\{(1,1),(1,4),(4,1),(4,4)\}$ we define

$$I_{jk}(\alpha,\beta)=\int_{-\infty}^{\infty} w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\overline{\psi_j}\psi_k(\sigma_0+it)dt. \tag{2.1}$$

For $(j,k)\in\{(1,1),(1,4),(4,1),(4,4)\}$ we define

$$I_{jk}(\alpha,\beta,w)=\frac{1}{\sqrt{\pi}\Delta}\int_{-\infty}^{\infty} e^{-(t-w)^2\Delta^{-2}}\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\overline{\psi_j}\psi_k(\sigma_0+it)dt. \tag{2.2}$$

The following two propositions were proved in [2, Theorem 3.2 and Theorem 3.3].

Proposition 2.1. *Let $\theta_1<4/7-\varepsilon$ and $\theta_2<1/2-\varepsilon$. One has that*

$$I_{12}(\alpha,\beta)=c_{12}(\alpha,\beta)\widehat{w}(0)+O(TL^{-1}), \tag{2.3}$$

uniformly for $\alpha,\beta\ll L^{-1}$. Here $c_{12}(\alpha,\beta)$ is given in the main term of [2, Theorem 3.2].

Proposition 2.2. *Let $\theta_2<1/2-\varepsilon$. One has that*

$$I_{22}(\alpha,\beta)=c_2(\alpha,\beta)\widehat{w}(0)+O(TL^{-1+\varepsilon}), \tag{2.4}$$

where $c_2(\alpha,\beta)$ is given in the main term of [2, Theorem 3.3].

We will prove the following propositions as intermediate results.

Proposition 2.3. *Let $\theta_2<1/2-\varepsilon$ and $\theta_3<1/2-\varepsilon$. Then we have*

$$I_{23}(\alpha,\beta)=c_{23}(\alpha,\beta)\widehat{w}(0)+O(T/L),$$

uniformly for $\alpha,\beta\ll L^{-1}$, where

$$c_{23}(\alpha,\beta)=\frac{2^8}{7!}\left(\frac{\theta_3}{\theta_2}\right)^6\frac{d^4}{dx^2dy^2}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 u^4(1-u)^7\left(y_2^{-x}y_3^{au}\right)^{-\alpha}\left(y_2^{y}y_3^{-ub}T\right)^{-\beta}$$
$$\times P_2''\left(x+y+1-(1-u)\frac{\theta_3}{\theta_2}\right)abP_3^{(6)}((1-a-b)u)dudadb\Bigg]_{x=y=0}.$$

Also $I_{32}(\alpha,\beta)$ is asymptotic to $I_{23}(\alpha,\beta)$.

Proposition 2.4. *Let $\theta_3<1/2-\varepsilon$. Then we have*

$$I_{33}(\alpha,\beta)=c_{33}(\alpha,\beta)\widehat{w}(0)+O(TL^{-1+\varepsilon}),$$

uniformly for $\alpha,\beta\ll L^{-1}$, where

$$I_{33}(\alpha,\beta)=\frac{2^{12}}{12!}\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1\int_0^1(1-r)^{12}y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))}$$

$$\times\left(\frac{1}{\theta_3}+(x+y-v(y+r)-u(x+r))\right)(x+r)^2(y+r)^2$$
$$\times P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))(Ty_3^{x+y-v(y+r)-u(x+r)})^{-t(\alpha+\beta)}dtdrdudv\Bigg)\Bigg|_{x=y=0}.$$

Proposition 2.5. *Let K be an integer greater or equal to 2 , $\theta_2<1/2-\varepsilon$ and $\theta_4<1/2-\varepsilon$. Then we have*
$$I_{42}(\alpha,\beta)=c_{42}(\alpha,\beta,K)\widehat{w}(0)+O(T/L),$$
uniformly for $\alpha,\beta\ll L^{-1}$, where
$$c_{42}(\alpha,\beta,K)=\sum_{k=2}^{K}(c_{42}^{(0,0)}(\alpha,\beta)+c_{42}^{(1,0)}(\alpha,\beta)+c_{42}^{(0,1)}(\alpha,\beta)+c_{42}^{(1,1)}(\alpha,\beta)$$
$$+c_{42}^{(1,\geq 2)}(\alpha,\beta)+c_{42}^{(\geq 2,1)}(\alpha,\beta)+c_{42}^{(\geq 2,0)}(\alpha,\beta)+c_{42}^{(0,\geq 2)}(\alpha,\beta)+c_{42}^{(\geq 2,\geq 2)}(\alpha,\beta)).$$
Here we have
$$c_{42}^{(0,0)}(\alpha,\beta)=4\frac{2^k}{(k+1)!}\frac{d^2}{dxdy}\Bigg[\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1(1-u)^{1+k}(y_2^{a-x}y_4^{a(u-1)})^{-\alpha}$$
$$\times(y_2^{y-b}y_4^{b(-u+1)}T)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times\widetilde{P}_k(x+y+u)P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)dudadb\Bigg]_{x=y=0},$$
$$c_{42}^{(1,0)}(\alpha,\beta)=-4\frac{2^{k-1}}{(k-1)!}\frac{d}{dy}\Bigg[\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1(1-u)^{k}(y_4^{-(1-u)a}y_2^{a})^{-\alpha}$$
$$\times(y_4^{b(1-u)+y}y_2^{-b}T)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(y+u)dudadb\Bigg]_{y=0},$$
$$c_{42}^{(0,1)}(\alpha,\beta)=-4\frac{2^{k-1}}{(k-1)!}\frac{d}{dx}\Bigg[\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1(1-u)^{k}(y_4^{-(1-u)a-x}y_2^{a})^{-\alpha}$$
$$\times(y_4^{b(1-u)}y_2^{-b}T)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(x+u)dudadb\Bigg]_{x=0},$$
$$c_{42}^{(1,1)}(\alpha,\beta)=4\frac{2^{k-2}k}{(k-2)!}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2(1-u)^{k-1}$$
$$\times(y_4^{-(1-u)a}y_2^{a})^{-\alpha}(y_4^{b(1-u)}y_2^{-b}T)^{-\beta}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(u)dudadb,$$
$$c_{42}^{(1,\geq 2)}(\alpha,\beta)=-4k!\sum_{l_1+1+l_3=k}\frac{2^{l_1}(-1)^{l_3-2}}{l_1!l_3!(1+l_1)!(l_3-2)!}$$

$$\times \iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 \int_0^1 \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times (1-u)^{1+l_1} (y_4^{-a(1-u)} y_2^a)^{-\alpha} \left(y_4^{b(1-u)-uc} y_2^{-b} T\right)^{-\beta}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right) \widetilde{P}_k((1-c)u) u^{l_3-1} c^{l_3-2} du dc da db,$$

with $l_3 \ge 2$,

$$c_{42}^{(\ge 2,1)}(\alpha,\beta) = -4k! \sum_{l_1+l_2+1=k} \frac{2^{l_1}(-1)^{l_2-2}}{l_1! l_2! (1+l_1)! (l_2-2)!}$$
$$\times \iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 \int_0^1 \left(1-(1-u)\frac{\theta_4}{\theta_4}\right)^2 (1-u)^{1+l_1}$$
$$\times (y_4^{-a(1-u)+uc} y_2^a)^{-\alpha} (y_4^{b(1-u)} y_2^{-b} T)^{-\beta}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_4}\right)\right) \widetilde{P}_k((1-c)u) u^{l_2-1} c^{l_2-2} du dc da db,$$

with $l_2 \ge 2$,

$$c_{42}^{(\ge 2,0)}(\alpha,\beta) = 4k! \sum_{l_1+l_2=k} \frac{2^{l_1}(-1)^{l_2}}{l_1! l_2! (l_2-2)! (1+l_1)!} \frac{d}{dx}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 \int_0^1 (1-u)^{1+l_1} (x+u)^{l_2-1}$$
$$\times (y_4^{c(u+x)-(1-u)a} y_2^a)^{-\alpha} (y_4^{x+(1-u)b} y_2^{-b} T)^{-\beta} \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_2-2}$$
$$\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right) \tilde{P}_k((1-c)(x+u)) da db dc du\Bigg]_{x=0}$$

with $l_2 \ge 2$,

$$c_{42}^{(0,\ge 2)}(\alpha,\beta) = 4k! \sum_{l_1+l_3=k} \frac{2^{l_1}(-1)^{l_3}}{l_1! l_3! (l_3-2)! (1+l_1)!} \frac{d}{dy}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 \int_0^1 (1-u)^{1+l_1} (y+u)^{l_3-1}$$
$$\times (y_4^{-y-a(1-u)} y_2^a)^{-\alpha} (y_4^{-c(u+y)+b(1-u)} y_2^{-b} T)^{-\beta} \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_3-2}$$
$$\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right) \tilde{P}_k((1-c)(y+u)) da db dc du\Bigg]_{y=0}$$

with $l_3 \ge 2$, *and*

$$c_{42}^{(l_2,l_3)}(\alpha,\beta) = 4k! \sum_{l_1+l_2+l_3=k} \frac{2^{l_1}(-1)^{l_2+l_3}}{l_1! l_2! l_3! (1+l_1)! (l_2-2)! (l_3-2)!}$$
$$\times \iiiint_{\substack{0\le a+b\le 1\\ 0\le g+h\le 1\\ a,b,g,h\ge 0}} \int_0^1 (1-u)^{k+l-1} \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times (y_4^{au+gu-a} y_2^a)^{-\alpha} (y_4^{-bu-hu+b} y_2^{-b} T)^{-\beta} P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)$$
$$\times \widetilde{P}_k((1-g-h)u) u^{l_2+l_3-2} g^{l_2-2} h^{l_3-2} du da db dg dh,$$

with $l_2 \ge 2$ *and* $l_3 \ge 2$.

Also note that $I_{24}(\alpha,\beta)$ is asymptotic to $I_{42}(\alpha,\beta)$.

Proposition 2.6. *Let $\theta_1 < 4/7 - \varepsilon$, $\theta_4 < 3/7 - \varepsilon$ and $T/2 \le w \le T$. One has that*

$$I_{11}(\alpha,\beta,w) + 2I_{14}(\alpha,\beta,w) + I_{44}(\alpha,\beta,w) = c_{11}(\alpha,\beta) + 2c_{14}(\alpha,\beta) + c_{44}(\alpha,\beta) + O_\varepsilon(L^{-1+\varepsilon}), \tag{2.5}$$

uniformly for $\alpha, \beta \ll L^{-1}$. Here $c_{11}(\alpha,\beta) + 2c_{14}(\alpha,\beta) + c_{44}(\alpha,\beta)$ is given in the main term of [9, Eq. (5.1)]. *Note that the right-hand side is independent of w.*

Proposition 2.7. *Let $\theta_1 < 4/7 - \varepsilon$ and $\theta_3 < 3/7 - \varepsilon$. One has that*

$$I_{13}(\alpha,\beta) = O(TL^{-1+\varepsilon}), \tag{2.6}$$

uniformly for $\alpha, \beta \ll L^{-1}$.

Proposition 2.8. *Let $\theta_1 < 4/7 - \varepsilon$ and $\theta_4 < 3/7 - \varepsilon$. One has that*

$$I_{34}(\alpha,\beta) = O(TL^{-1+\varepsilon}), \tag{2.7}$$

uniformly for $\alpha, \beta \ll L^{-1}$.

Now we choose a $w(t)$ that satisfies (1)-(3), an upper bound (or lower bound) for the characteristic function in the interval $[T/2, T]$, and with support in $[T/2 - \Delta, T + \Delta]$. We note that in this case $\widehat{w}(0) = T/2 + O(T/L)$. Therefore one can see that

$$\int_{T/2}^{T} \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi_j}\psi_k(\sigma_0 + it)dt \tag{2.8}$$

can be bounded by

$$\int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi}_j\psi_k(\sigma_0 + it)dt$$

for above choice of w and $(j,k) \notin \{(1,1),(1,4),(4,1),(4,4)\}$. Using Propositions 2.3, 2.4, and 2.5 we can see that (2.8) can be bounded by $c_{jk}(\alpha,\beta)T/2 + O(T/L)$. Now summing over dyadic segments gives the required asymptotic for (2.8) with the limits of integration replaced by 1 to T. Let $T/4 \le T_1 < T_2 < 2T$ and we define

$$w(t, T_1, T_2) = \frac{1}{\sqrt{\pi}\Delta}\int_{T_1}^{T_2} e^{-(t-w)^2\Delta^{-2}}dt. \tag{2.9}$$

Then clearly

(a) $0 \le w(t, T_1, T_2) \le 1$
(b) $w(t, T_1, T_2) = O(\exp(-\log^2 T))$ when $t \notin [T_1 - \Delta\log T, T_2 + \Delta\log T]$
(c) $w(t, T_1, T_2) = 1 + O(\exp(-\log^2 T))$ when $t \in [T_1 - \Delta\log T, T_2 + \Delta\log T]$.

Now we can select two such $w(t, T_1, T_2)$'s, specifically $w(t, T/2 - \Delta\log T, T + \Delta\log T)$ and $w(t, T/2 + \Delta\log T, T - \Delta\log T)$. Then from the above facts, Proposition 2.6, and (2.9) we bound

$$\sum_{(j,k)\in\{(1,1),(1,4),(4,1),(4,4)\}} \int_{T/2}^{T} \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi_j}\psi_k(\sigma_0 + it)dt \tag{2.10}$$

by $(c_{11}(\alpha,\beta) + 2c_{14}(\alpha,\beta) + c_{44}(\alpha,\beta))T/2 + O_\varepsilon(TL^{-1+\varepsilon})$. Now summing over dyadic segments gives the required asymptotic for (2.9) with the limits of integration replaced by 1 to T.

Since $I(\alpha,\beta)$ is the sum of the terms of the form given in (2.8) and (2.9) with the limits of integration replaced by 1 to T, the equality

$$C = c_{11}(\alpha,\beta) + 2c_{14}(\alpha,\beta) + c_{44}(\alpha,\beta) + c_{22}(\alpha,\beta) + c_{33}(\alpha,\beta) + 2c_{12}(\alpha,\beta) + 2c_{23}(\alpha,\beta) + 2c_{24}(\alpha,\beta),$$

holds.

3. Auxiliary lemmas

In this section we collect all the tools, new and old, that will be needed for the forthcoming computations. Throughout this paper, the notation $\int_{(c)}$ will signify $\int_{c-i\infty}^{c+i\infty}$. The following results were proved in [2].

Lemma 3.1. *Let $\sigma_{\alpha,-\beta}(l)=\sum_{ab=l}a^{-\alpha}b^{\beta}$. For $L^2\le|t|\le 2T$ and uniformly for $\alpha,\beta\ll L^{-1}$,*

$$\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12-\beta+it)=\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3}+O(T^{-1+\varepsilon}).$$

Lemma 3.2. *Suppose $w(t)$ satisfies (1)-(3), and a and b are positive integers with $ab\le T^{1-\varepsilon}$. Then, uniformly for $\alpha,\beta\ll L^{-1}$, we have*

$$\int_{-\infty}^{\infty}\left(\frac{a}{b}\right)^{-it}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)dt=\sum_{am=bn}\frac{1}{m^{1/2+\alpha}n^{1/2+\beta}}\int_{-\infty}^{\infty}V_t(mn)w(t)dt \tag{3.1}$$

$$+\sum_{am=bn}\frac{1}{m^{1/2-\beta}n^{1/2-\alpha}}\int_{-\infty}^{\infty}V_t(mn)\left(\frac{t}{2\pi}\right)^{-\alpha-\beta}w(t)dt+O(T^{-1/2}).$$

Here $V_t(x)$ is given by

$$V_t(x)=\frac{1}{2\pi i}\int_{(1)}\left(\frac{t}{2\pi x}\right)^{z}\frac{G(z)}{z}dz,$$

where

$$G(z)=e^{z^2}p(z)\quad\text{and}\quad p(z)=\frac{(\alpha+\beta)^2-(2z)^2}{(\alpha+\beta)^2}.$$

Lemma 3.3. *Suppose that $z\le x$, $|s|\le\frac{1}{\log x}$, k is a positive integer, and let F and H be smooth in an interval containing $[0,1]$. Then*

$$\sum_{n\le z}\frac{d_k(n)}{n^{1+s}}F\left(\frac{\log x/n}{\log x}\right)H\left(\frac{\log z/n}{\log z}\right)$$

$$=\frac{(\log z)^k}{(k-1)!z^s}\int_0^1(1-u)^{k-1}F\left(1-(1-u)\frac{\log z}{\log x}\right)H(u)z^{us}du+O((\log 3z)^{k-1}).$$

Lemma 3.4. *Suppose that $-1\le\sigma\le 0$. Then*

$$\sum_{n\le x}\frac{d_k(n)}{n}\left(\frac{x}{n}\right)^{\sigma}\ll_k(\log 3x)^{k-1}\min(|\sigma|^{-1},\log 3x).$$

As an extension to the above lemma and following a similar argument to that of [2, Lemma 4.6] we have following:

Lemma 3.5. *Suppose that $-1\le\sigma\le 0$. Then*

$$\sum_{n\le x}\frac{(d_k*\Lambda*\cdots*\Lambda)(n)}{n}\left(\frac{x}{n}\right)^{\sigma}\ll_{k,l}(\log 3x)^{k+l-1}\min(|\sigma|^{-1},\log 3x),$$

where the convolution of Λ is taken l times.

We also need the following lemma which is an extension of Lemma 3.3.

Lemma 3.6. *Under the conditions of Lemma 3.3, one has*

$$S_{k,l}=\sum_{n\le z}\frac{(d_k*\Lambda*\cdots*\Lambda)(n)}{n^{1+s}}F\left(\frac{\log x/n}{\log x}\right)H\left(\frac{\log z/n}{\log z}\right)$$

$$= \frac{(\log z)^{k+l}}{(k+l-1)! z^s} \int_0^1 (1-u)^{k+l-1} F\left(1-(1-u)\frac{\log z}{\log x}\right) H(u) z^{us} du + O((\log 3z)^{k+l-1}),$$

where the convolution of Λ *is taken* l *times.*

Proof. For $l=1$ we have

$$\begin{aligned} S_{k,1} &= \sum_{n \le z} \frac{(d_k * \Lambda)(n)}{n^{1+s}} F\left(\frac{\log x/n}{\log x}\right) H\left(\frac{\log z/n}{\log z}\right) \\ &= \sum_{m \le z} \frac{\Lambda(m)}{m^{1+s}} \sum_{q \le z/m} \frac{d_k(q)}{q^{1+s}} F\left(\frac{\log \frac{x}{qm}}{\log x}\right) H\left(\frac{\log \frac{z}{qm}}{\log z}\right) \end{aligned}$$

By Lemma 3.3, we then have

$$\begin{aligned} S_{k,1} &= \sum_{m \le z} \frac{\Lambda(m)}{m^{1+s}} \bigg[\frac{(\log \frac{z}{m})^k}{(k-1)!(\frac{z}{m})^s} \\ &\quad \times \int_0^1 (1-u)^{k-1} \left(\frac{z}{m}\right)^{us} F\left(\frac{\log \frac{x}{m}}{\log x}(1-(1-u))\frac{\log \frac{z}{m}}{\log \frac{x}{m}}\right) H\left(u \frac{\log \frac{z}{m}}{\log z}\right) du \bigg] \\ &\quad + O((\log 3z)^{k-1}) \\ &= \sum_{m \le z} \frac{\Lambda(m)}{m^{1+s}} \bigg[\frac{(\log \frac{z}{m})^k}{(k-1)!(\frac{z}{m})^s} \\ &\quad \times \int_0^1 (1-u)^{k-1} \left(\frac{z}{m}\right)^{us} F\left((1-u)\left(1-\frac{\log z}{\log x}\right) + u\frac{\log \frac{x}{m}}{\log x}\right) H\left(u \frac{\log \frac{z}{m}}{\log z}\right) du \bigg] \\ &\quad + O((\log 3z)^{k-1}) \\ &= \frac{(\log z)^k}{(k-1)! z^s} \int_0^1 (1-u)^{k-1} z^{us} \sum_{m \le z} \frac{\Lambda(m)}{m^{1+us}} \\ &\quad \times F\left((1-u)\left(1-\frac{\log z}{\log x}\right) + u\frac{\log \frac{x}{m}}{\log x}\right) H\left(u \frac{\log \frac{z}{m}}{\log z}\right) \left(\frac{\log \frac{z}{m}}{\log z}\right)^k du \\ &\quad + O((\log 3z)^{k-1}) \\ &= \frac{(\log z)^k}{(k-1)! z^s} \int_0^1 (1-u)^{k-1} \log z \int_0^1 F\left((1-u)\left(1-\frac{\log z}{\log x}\right) + u\left(1-(1-b)\frac{\log z}{\log x}\right)\right) \\ &\quad \times H(ub) b^k z^{ubs} db du + O((\log 3z)^{k-1}). \end{aligned}$$

Hence, we have

$$\begin{aligned} S_{k,1} &= \frac{(\log z)^{k+1}}{(k-1)! z^s} \int_0^1 \int_0^1 b^k (1-u)^{k-1} F\left(1-(1-ub)\frac{\log z}{\log x}\right) H(ub) z^{ubs} db du \\ &\quad + O((\log 3z)^{k-1}). \end{aligned} \tag{3.2}$$

We perform three changes of variables. First, $u = 1-v$ so that

$$\begin{aligned} S_{k,1} &= \frac{(\log z)^{k+1}}{(k-1)! z^s} \int_0^1 \int_0^1 b^k v^{k-1} F\left(1-(1-b(1-v))\frac{\log z}{\log x}\right) H(b(1-v)) z^{b(1-v)s} db dv \\ &\quad + O((\log 3z)^{k-1}). \end{aligned} \tag{3.3}$$

Second, we set $v = \frac{a}{b}$ so that

$$S_{k,1} = \frac{(\log z)^{k+1}}{(k-1)!z^s} \int_0^1 \int_0^b a^{k-1} F\left(1-(1-b(1-\tfrac{a}{b}))\frac{\log z}{\log x}\right) H(b(1-\tfrac{a}{b}))z^{b(1-\frac{a}{b})s} dadb$$

$$+ O((\log 3z)^{k-1}). \tag{3.4}$$

Finally, we set $b = u + a$ and we obtain

$$S_{k,1} = \frac{(\log z)^{k+1}}{(k-1)!z^s} \iint\limits_{\substack{u+a\le 1\\ a,u\ge 0}} a^{k-1} F\left(1-(1-u)\frac{\log z}{\log x}\right) H(u) z^{us} dadu + O((\log 3z)^{k-1})$$

$$= \frac{(\log z)^{k+1}}{(k-1)!z^s} \int_0^1 (1-u)^{k-1} F\left(1-(1-u)\frac{\log z}{\log x}\right) H(u) z^{us} du + O((\log 3z)^{k}).$$

Hence, by induction on l, we obtain

$$S_{k,l} = \frac{(\log z)^{k+l}}{(k+l-1)!z^s} \int_0^1 (1-u)^{k+l-1} F\left(1-(1-u)\frac{\log z}{\log x}\right) H(u) z^{us} du + O((\log 3z)^{k+l-1}),$$

as it was to be shown. □

Also we need the following Mellin inversion formula. For $n \le y$ one has

$$P[n] = \sum_i \frac{a_i}{(\log y)^i} (\log(y/n))^i = \sum_i \frac{a_i i!}{(\log y)^i} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y}{n}\right)^s \frac{ds}{s^{i+1}}. \tag{3.5}$$

Note that if $n > y$, then the right hand side vanishes. From the inverse Mellin transform of the gamma function we have

$$e^{-l/T^3} = \frac{1}{2\pi i} \int_{(1)} T^{3z} \Gamma(z) l^{-z} dz. \tag{3.6}$$

4. Proof of Proposition 2.3

First we keep in mind that

$$\overline{\psi_2}(\sigma_0 + it) = \chi(\tfrac{1}{2} - it) \sum_{hk \le y_2} \frac{\mu_2(h)}{h^{1/2-it} k^{1/2+it}} P_2[hk],$$

as well as

$$\psi_3(\sigma_0 + it) = \chi^2(\tfrac{1}{2} + it) \sum_{mn \le y_3} \frac{\mu_3(m) d(n)}{m^{1/2+it} n^{1/2-it}} P_3[mn].$$

Inserting this in the integral yields

$$I_{23}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \zeta(\tfrac{1}{2}+\alpha+it) \zeta(\tfrac{1}{2}+\beta-it) \chi(\tfrac{1}{2}-it)$$

$$\times \sum_{hk \le y_2} \frac{\mu_2(h)}{h^{1/2-it} k^{1/2+it}} P_2[hk] \chi^2(\tfrac{1}{2}+it) \sum_{mn \le y_3} \frac{\mu_3(m) d(n)}{m^{1/2+it} n^{1/2-it}} P_3[mn] dt.$$

Recalling that $\chi(\frac{1}{2}+it)\chi(\frac{1}{2}-it) = 1$, and pulling out the sums we obtain

$$I_{23}(\alpha,\beta) = \sum_{hk \le y_2} \sum_{mn \le y_3} \frac{\mu_2(h)\mu_3(m)d(n)}{(hkmn)^{1/2}} P_2[hk] P_3[mn] J_{23}$$

where

$$J_{23} = \int_{-\infty}^{\infty} w(t) \left(\frac{km}{hn}\right)^{-it} \chi(\tfrac{1}{2}+it) \zeta(\tfrac{1}{2}+\alpha+it) \zeta(\tfrac{1}{2}+\beta-it) dt.$$

We then use the Stirling formula

$$\chi(\tfrac{1}{2}+\beta-it)\chi(\tfrac{1}{2}+it) = \left(\frac{t}{2\pi}\right)^{-\beta}(1+O(t^{-1})),$$

for $t>0$, as well as the functional equation $\zeta(\frac{1}{2}+\beta-it) = \chi(\frac{1}{2}+\beta-it)\zeta(\frac{1}{2}-\beta+it)$, which allows us to rewrite J_{23} with the $-\beta$ inside the ζ function, i.e.

$$\begin{aligned} J_{23} &= \int_{-\infty}^{\infty} w(t)\left(\frac{km}{hn}\right)^{-it}\chi(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+\alpha+it)\chi(\tfrac{1}{2}+\beta-it)\zeta(\tfrac{1}{2}-\beta+it)dt \\ &= \int_{-\infty}^{\infty} w(t)\left(\frac{km}{hn}\right)^{-it}\left(\frac{t}{2\pi}\right)^{-\beta}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it)dt + O(T^{\varepsilon}). \end{aligned}$$

We use Lemma 3.1 so that

$$\begin{aligned} J_{23} &= \int_{-\infty}^{\infty} w(t)\left(\frac{km}{hn}\right)^{-it}\left(\frac{t}{2\pi}\right)^{-\beta}\left(\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3}dt + O(T^{-1+\varepsilon})\right) + O(T^{\varepsilon}) \\ &= \sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2}}e^{-l/T^3}\int_{-\infty}^{\infty} w(t)\left(\frac{kml}{hn}\right)^{-it}\left(\frac{t}{2\pi}\right)^{-\beta}dt + O(T^{\varepsilon}). \end{aligned}$$

Define

$$w_0(t) = w(t)\left(\frac{t}{2\pi}\right)^{-\beta} \quad\text{and}\quad \widehat{w_0}\left(\frac{1}{2\pi}\log\frac{kml}{hn}\right) = \int_{-\infty}^{\infty} w_0(t)\left(\frac{kml}{hn}\right)^{-it}dt. \tag{4.1}$$

Therefore

$$\begin{aligned} I_{23}(\alpha,\beta) &= \sum_{hk\le y_2}\sum_{mn\le y_3}\frac{\mu_2(h)\mu_3(m)d(n)}{(hkmn)^{1/2}}P_2[hk]P_3[mn]\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2}}e^{-l/T^3}\widehat{w_0}\left(\frac{1}{2\pi}\log\frac{kml}{hn}\right) \\ &\quad + O_{\varepsilon}(T^{\varepsilon}(y_2y_3)^{1/2}). \end{aligned}$$

We can bound the off diagonal terms i.e. those where $kml \neq hn$ in a similar fashion as in the proof of Proposition 2.5.

4.1. **Main term ($kml = hn$):** From (3.5) and (3.6)

$$\begin{aligned} I_{23}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{i,j}\frac{a_i i! b_j j!}{\log^i y_2 \log^j y_3}\left(\frac{1}{2\pi i}\right)^3\int_{(1)}\int_{(1)}\int_{(1)} T^{3z}\Gamma(z)y_2^s y_3^u \\ &\quad \times \sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{h^{1/2+s}k^{1/2+s}m^{1/2+u}n^{1/2+u}l^{1/2+z}}\frac{dzdsdu}{s^{i+1}u^{j+1}} + O(T^{1-\varepsilon}). \end{aligned}$$

Let

$$S := \sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{h^{1/2+s}k^{1/2+s}m^{1/2+u}n^{1/2+u}l^{1/2+z}} \tag{4.2}$$

Since the functions in (4.2) are completely multiplicative, a p-adic analysis shows that

$$S = \frac{\zeta^8(1+s+u)\zeta^2(1+\alpha+u+z)\zeta^2(1-\beta+u+z)}{\zeta^2(1+2s)\zeta^6(1+2u)\zeta^2(1+\alpha+s+z)\zeta^2(1-\beta+s+z)}A(s,u,z). \tag{4.3}$$

A detailed argument to obtain (4.3) is given in the proof of (6.3). Here $A(s,u,z)$ is a certain arithmetical factor that is given by an Euler product that is absolutely and uniformly convergent in some product of fixed half-planes containing the origin. In particular when $s=u=z$, one has

$$A(s,s,s) = \sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{(kmlhn)^{1/2+s}} = \sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{(kml)^{1+2s}}$$

$$= \sum_{j=1}^{\infty} \sum_{kml=j} \frac{\mu_3(m)\sigma_{\alpha,-\beta}(l)}{(kml)^{1+2s}} \sum_{hn|j} \mu_2(h)d(n) = 1,$$

since $\sum_{hn|j} \mu_2(h)d(n) = 1$ when $j = 1$ and vanishes when $j > 1$. Hence,

$$I_{23}(\alpha,\beta) = \widehat{w_0}(0) \sum_{i,j} \frac{a_i i! b_j j!}{\log^i y_2 \log^j y_3} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)}\int_{(1)}\int_{(1)} T^{3z}\Gamma(z) y_2^s y_3^u$$
$$\times \frac{\zeta^8(1+s+u)\zeta^2(1+\alpha+u+z)\zeta^2(1-\beta+u+z)}{\zeta^2(1+2s)\zeta^6(1+2u)\zeta^2(1+\alpha+s+z)\zeta^2(1-\beta+s+z)} A(s,u,z) \frac{dz ds du}{s^{i+1}u^{j+1}}$$
$$+ O(T^{1-\varepsilon}). \tag{4.4}$$

The next step is to deform the s- and u- contours to $\mathrm{Re}(s) = \mathrm{Re}(u) = \delta$, and then deform the z-contour to $-2\delta/3$, where $\delta > 0$ is some fixed constant such that the arithmetical factor converges absolutely. This implies that we pick up a pole at $z = 0$ coming from $\Gamma(z)$. The bound for the integral on the new lines of integration is

$$|\widehat{w_0}(0)| \left(\frac{y_2 y_3}{T^2}\right)^{\delta} \ll T^{1-\varepsilon}.$$

Consequently, we are left with

$$I_{23}(\alpha,\beta) = \widehat{w_0}(0) \sum_{i,j} \frac{a_i i! b_j j!}{\log^i y_2 \log^j y_3} K_{23} + O(T^{1-\varepsilon}), \tag{4.5}$$

where

$$K_{23} = \left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)}\int_{(\delta)} y_2^s y_3^u \frac{\zeta^8(1+s+u)\zeta^2(1+\alpha+u)\zeta^2(1-\beta+u)}{\zeta^2(1+2s)\zeta^6(1+2u)\zeta^2(1+\alpha+s)\zeta^2(1-\beta+s)}$$
$$\times A(s,u,0) \frac{ds du}{s^{i+1}u^{j+1}}.$$

Let K'_{23} be the same integral as K_{23} but with $A(s,u,0)$ replaced by $A(0,0,0)$. Since $A(s,u,0) = 1 + O(|s|) + O(|u|)$, then $K_{23} = K'_{23} + O(L^{i+j-1})$. The variables s and u are coupled together in the term $\zeta^8(1+s+u)$, so let us replace this by its Dirichlet series and reverse the order of summation and integration. Hence, we get

$$K'_{23} = \sum_{n \le \min(y_2,y_3)} \frac{d_8(n)}{n} K_2 K_3,$$

where

$$K_2 = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_2}{n}\right)^s \frac{1}{\zeta^2(1+2s)\zeta^2(1+\alpha+s)\zeta^2(1-\beta+s)} \frac{ds}{s^{i+1}},$$

and

$$K_3 = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_3}{n}\right)^u \frac{\zeta^2(1+\alpha+u)\zeta^2(1-\beta+u)}{\zeta^6(1+2u)} \frac{du}{u^{j+1}}.$$

The truncation of n is at $\min(y_2,y_3) = y_3$ since $\theta_3 < \theta_2$ and this is accomplished by moving the u-integral to the far right. Let us now compute each integral separately.

Lemma 4.1. *Suppose $i \ge 3$ and $j \ge 7$. Then*

$$K_2 = \frac{4}{(i-2)!} \frac{d^4}{dx^2 dy^2} e^{\alpha x - \beta y} \left(x + y + \log\frac{y_2}{n}\right)^{i-2} \Bigg|_{x=y=0} + O(L^{i-7}), \tag{4.6}$$

as well as

$$K_3 = \frac{64(\log y_3/n)^{j-2}}{(j-6)!} \iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} (1-a-b)^{j-6} ab \Big(\frac{y_3}{n}\Big)^{-a\alpha+b\beta} da db + O(L^{j-3}). \tag{4.7}$$

Proof. First we examine K_2. An argument along the lines of the prime number theorem indicates that the integral K_2 is captured by the residue at $s=0$, with an error of size $(\log y_2/n)^{-A}$ for arbitrarily large A. But since $n \le y_3$ we have that $\log(y_2/n) \ge \log(y_2/y_3) = (\theta_2-\theta_3)L$ and hence this error is as desired. Using

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_n (-1)^n \gamma_n (s-1)^n, \tag{4.8}$$

where γ_n are the Stieltjes' constants, indicates that

$$K_2 = 4\frac{1}{2\pi i}\oint \Big(\frac{y_2}{n}\Big)^s (\alpha+s)^2(-\beta+s)^2 \frac{ds}{s^{i-1}} + O(L^{i-7}),$$

where the contour is a small circle enclosing 0. Hence

$$\begin{aligned} K_2 &= 4\frac{d^4}{dx^2dy^2} e^{\alpha x-\beta y} \frac{1}{2\pi i}\oint \Big(\frac{y_2}{n}e^{x+y}\Big)^s \frac{ds}{s^{i-1}}\Big|_{x=y=0} + O(L^{i-7}) \\ &= \frac{4}{(i-2)!}\frac{d^4}{dx^2dy^2} e^{\alpha x-\beta y}\Big(x+y+\log\frac{y_2}{n}\Big)^{i-2}\Big|_{x=y=0} + O(L^{i-7}). \end{aligned}$$

Let us now move on to K_3. As we reasoned previously, the prime number theorem shows that we can replace the contour by a small circle around the origin with radius $\asymp L^{-1}$, with error $O(1)$. On this contour and by the use of (4.8) we obtain

$$K_3 = 64\frac{1}{2\pi i}\oint \Big(\frac{y_3}{n}\Big)^u \frac{1}{(\alpha+u)^2(-\beta+u)^2}\frac{du}{u^{j-5}} + O(L^{j-3}).$$

Note the identity

$$\int_{1/q}^1 r^{\alpha+u-1}\log^\tau r dr = \frac{(-1)^\tau \tau!}{(\alpha+u)^{\tau+1}} - \frac{q^{-\alpha-u}}{(\alpha+u)^{\tau+1}}P(u,\alpha,\log q) \tag{4.9}$$

where P is a polynomial in $\log q$ of degree $\tau-1$. Set $q = y_3/n$. Only the first term of the right-hand side above contributes when we insert this expression into K_3. This is because the contribution from the second term is

$$64q^{-\alpha}\log q\frac{1}{2\pi i}\oint \frac{1+(u+\alpha)}{(\alpha+u)^2(-\beta+u)^2}du,$$

which vanishes by taking the contour to be arbitrary large. Then K_3 becomes

$$\begin{aligned} K_3 &= 64\int_{1/q}^1\int_{1/(qr)}^1 r^{\alpha-1}t^{-\beta-1}\log r\log t\frac{1}{2\pi i}\oint (qrt)^u\frac{du}{u^{j-5}}dtdr + O(L^{j-3}) \\ &= \frac{64}{(j-6)!}\int_{1/q}^1\int_{1/(qr)}^1 r^{\alpha-1}t^{-\beta-1}(\log r\log t)\Big(\log\frac{y_3}{n}rt\Big)^{j-6}dtdr + O(L^{j-3}). \end{aligned}$$

Finally, make the change of variables $r = q^{-a}$ and $t = q^{-b}$ so that after simplifications, we get

$$K_3 = \frac{64(\log y_3/n)^{j-2}}{(j-6)!} \iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} (1-a-b)^{j-6} ab \Big(\frac{y_3}{n}\Big)^{-a\alpha+b\beta} da db + O(L^{j-3}).$$

This proves both statements of the lemma. □

The sum over i becomes

$$\sum_i \frac{a_i i!}{(\log y_2)^i} K_2 = \frac{4}{(\log y_2)^2} \frac{d^4}{dx^2 dy^2} e^{\alpha x - \beta y} \sum_i a_i i(i-1) \left(\frac{x+y}{\log y_2} + \frac{\log y_2/n}{\log y_2} \right)^{i-2} \Bigg|_{x=y=0} + O(L^{-7})$$

$$= \frac{4}{(\log y_2)^2} \frac{d^4}{dx^2 dy^2} e^{\alpha x - \beta y} P_2'' \left(\frac{x+y}{\log y_2} + \frac{\log(y_2/n)}{\log y_2} \right) \Bigg|_{x=y=0} + O(L^{-7}).$$

It is more convenient to write this as

$$\sum_i \frac{a_i i!}{(\log y_2)^i} K_2 = \frac{4}{(\log y_2)^6} \frac{d^4}{dx^2 dy^2} \left[y_2^{\alpha x - \beta y} P_2'' \left(x + y + \frac{\log(y_2/n)}{\log y_2} \right) \right]_{x=y=0} + O(L^{-7}).$$

For the sum over j we get

$$\sum_j \frac{b_j j!}{(\log y_3)^j} K_3 = \sum_j \frac{64 b_j j!}{(\log y_3)^j} \frac{(\log y_3/n)^{j-2}}{(j-6)!} \iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} (1-a-b)^{j-6} ab \Big(\log \frac{y_3}{n} \Big)^{-a\alpha + b\beta} da db + O(L^{-3})$$

$$= \frac{64 (\log y_3/n)^4}{(\log y_3)^6} \sum_j b_j j(j-1)(j-2)(j-3)(j-4)(j-5) \times \iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} \left((1-a-b) \frac{\log y_3/n}{\log y_3} \right)^{j-6} ab \Big(\frac{y_3}{n} \Big)^{-a\alpha + b\beta} da db + O(L^{-3})$$

$$= \frac{64 (\log y_3/n)^4}{(\log y_3)^6} \iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} \left(\frac{y_3}{n} \right)^{-a\alpha + b\beta} ab P_3^{(6)} \left((1-a-b) \frac{\log y_3/n}{\log y_3} \right) da db + O(L^{-3}).$$

Next, we recall that

$$\widehat{w}_0(0) = T^{-\beta} \widehat{w}(0)(1 + O(L^{-1})),$$

and therefore

$$I_{23}(\alpha, \beta) = T^{-\beta} \widehat{w}(0) \sum_{n \le y_3} \frac{d_8(n)}{n} \frac{4}{(\log y_2)^6} \frac{d^4}{dx^2 dy^2} \left[y_2^{\alpha x - \beta y} P_2'' \left(x + y + \frac{\log(y_2/n)}{\log y_2} \right) \right]_{x=y=0}$$
$$\times \frac{64 (\log y_3/n)^4}{(\log y_3)^6} \iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} \left(\frac{y_3}{n} \right)^{-a\alpha + b\beta} ab P_3^{(6)} \left((1-a-b) \frac{\log y_3/n}{\log y_3} \right) da db + O(T/L)$$

$$= \frac{256 T^{-\beta} \widehat{w}(0)}{(\log y_2)^6 (\log y_3)^2} \frac{d^4}{dx^2 dy^2} \Bigg(y_2^{\alpha x - \beta y} \iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} \sum_{n \le y_3} \frac{d_8(n)}{n} \frac{(\log y_3/n)^4}{(\log y_3)^4} \times \left(\frac{y_3}{n} \right)^{-a\alpha + b\beta} ab P_2'' \left(x + y + \frac{\log(y_2/n)}{\log y_2} \right)$$
$$\times P_3^{(6)} \left((1-a-b) \frac{\log(y_3/n)}{\log y_3} \right) da db \Bigg) \Bigg|_{x=y=0} + O(T/L).$$

The last step is to apply Lemma 3.3. We choose $k=8$, $x=y_2$, $z=y_3$, $F(u)=P_2''(x+y+u)$, $H(u)=u^4P_3^{(6)}((1-a-b)u)$. These substitutions give

$$\sum_{n\le y_3}\frac{d_8(n)}{n^{1-a\alpha+b\beta}}\frac{(\log(y_3/n))^4}{(\log y_3)^4}P_2''\left(x+y+\frac{\log(x/n)}{\log x}\right)P_3^{(6)}\left((1-a-b)\frac{\log(y_3/n)}{\log y_3}\right)$$
$$=\frac{(\log y_3)^8(y_3)^{a\alpha-b\beta}}{7!}\int_0^1(1-u)^7P_2''\left(x+y+1-(1-u)\frac{\log y_3}{\log x}\right)$$
$$\times u^4P_3^{(6)}((1-a-b)u)(y_3)^{u(-a\alpha+b\beta)}du+O(\log^7 y_3).$$

Inserting $y_2=T^{\theta_2}$ and $y_3=T^{\theta_3}$ we obtain that

$$c_{23}(\alpha,\beta)=\frac{2^8}{7!}\left(\frac{\theta_3}{\theta_2}\right)^6\frac{d^4}{dx^2dy^2}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 u^4(1-u)^7(y_2^{-x}y_3^{au})^{-\alpha}(y_2^yy_3^{-ub}T)^{-\beta}$$
$$\times P_2''\left(x+y+1-(1-u)\frac{\theta_3}{\theta_2}\right)abP_3^{(6)}((1-a-b)u)dudadb\Bigg]_{x=y=0}.$$

which is precisely the term appearing in Proposition 2.3.

5. Proof of Proposition 2.4

One has

$$\overline{\psi_3}(\sigma_0+it)=\chi^2(\tfrac12-it)\sum_{h_1k_1\le y_3}\frac{\mu_3(h_1)d(k_1)}{h_1^{1/2-it}k_1^{1/2+it}}P_3[h_1k_1],$$

as well as

$$\psi_3(\sigma_0+it)=\chi^2(\tfrac12+it)\sum_{h_2k_2\le y_3}\frac{\mu_3(h_2)d(k_2)}{h_2^{1/2+it}k_2^{1/2-it}}P_3[h_2k_2].$$

Inserting these in the integral and pulling out the sums, we obtain

$$I_{33}(\alpha,\beta)=\int_{-\infty}^{\infty}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\chi^2(\tfrac12-it)$$
$$\times\sum_{h_1k_1\le y_3}\frac{\mu_3(h_1)d(k_1)}{h_1^{1/2-it}k_1^{1/2+it}}P_3[h_1k_1]\chi^2(\tfrac12+it)\sum_{h_2k_2\le y_3}\frac{\mu_3(h_2)d(k_2)}{h_2^{1/2+it}k_2^{1/2-it}}P_3[h_2k_2]dt$$
$$=\sum_{h_1,k_1,h_2,k_2}\frac{\mu_3(h_1)\mu_3(h_2)d(k_1)d(k_2)}{h_1^{1/2}k_1^{1/2}h_2^{1/2}k_2^{1/2}}P_3[h_1k_1]P_3[h_2k_2]$$
$$\times\int_{-\infty}^{\infty}\left(\frac{k_1h_2}{h_1k_2}\right)^{-it}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)dt.$$

We now apply Lemma 3.2. Thus $I_{33}(\alpha,\beta)=I_{33}'(\alpha,\beta)+I_{33}''(\alpha,\beta)$, where I_{33}'' can be obtained from I_{33}' by switching α by $-\beta$ and multiplying by

$$\left(\frac{t}{2\pi}\right)^{-\alpha-\beta}=T^{-\alpha-\beta}+O(L^{-1}),$$

for $t\asymp T$. From (3.5) we have

$$I_{33}{}'(\alpha,\beta)=\int_{-\infty}^{\infty}w(t)\sum_{i,j}\frac{b_ib_ji!j!}{(\log y_3)^{i+j}}\sum_{h_1k_2m=k_1h_2n}\frac{\mu_3(h_1)\mu_3(h_2)d(k_1)d(k_2)}{(h_1k_1h_2k_2)^{1/2}m^{1/2+\alpha}n^{1/2+\beta}}$$
$$\times\left(\frac{1}{2\pi i}\right)^3\int_{(1)}\int_{(1)}\int_{(1)}\left(\frac{y_3}{h_1k_1}\right)^s\left(\frac{y_3}{h_2k_2}\right)^u\left(\frac{t}{2\pi mn}\right)^z\frac{G(z)}{z}dz\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}dt.$$

Let

$$S := \sum_{h_1k_2m=k_1h_2n} \frac{\mu_3(h_1)\mu_3(h_2)d(k_1)d(k_2)}{(h_1k_1)^{1/2+s}(h_2k_2)^{1/2+u}m^{1/2+\alpha+z}n^{1/2+\beta+z}}$$

(5.1)

Evaluating this p-adically (for details see the argument of the proof of (6.3)) one gets

$$S = \frac{\zeta^{13}(1+s+u)\zeta^2(1+\beta+u+z)\zeta^2(1+\alpha+s+z)\zeta(1+\alpha+\beta+2z)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s+z)\zeta^3(1+\alpha+u+z)}B(s,u,z).$$

Again $B(s,u,z)$ is an arithmetical factor converging absolutely and uniformly in a product of half-planes containing the origin. As in the proof of Proposition 2.3, one can show that $B(s,s,s)=1$. This leaves us with

$$\begin{aligned} I_{33}'(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{b_ib_ji!j!}{(\log y_3)^{i+j}} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)}\int_{(1)}\int_{(1)} \\ \times \frac{\zeta^{13}(1+s+u)\zeta^2(1+\beta+u+z)\zeta^2(1+\alpha+s+z)\zeta(1+\alpha+\beta+2z)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s+z)\zeta^3(1+\alpha+u+z)} \\ \times B(s,u,z)y_3^{s+u}\left(\frac{t}{2\pi}\right)^z \frac{G(z)}{z}dz\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}dt. \end{aligned}$$

As in the previous computation, the next step is to move contours around carefully and wisely. We take the s-, u- and z- contours of integration to $\delta>0$ small and then deform z to $-\delta+\varepsilon$ crossing the simple pole of $1/z$ at $z=0$ only. Recall that $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$. The new path of integration gives a contribution of

$$T^{1+\varepsilon}\left(\frac{y_3^2}{T}\right)^\delta \ll T^{1-\varepsilon}.$$

We end up with

$$I_{33}'(\alpha,\beta) = I_{330}'(\alpha,\beta) + O(T^{1-\varepsilon}),$$

where $I_{330}'(\alpha,\beta)$ corresponds to the residue at $z=0$, i.e.

$$I_{330}'(\alpha,\beta) = \widehat{w}(0)\zeta(1+\alpha+\beta)\sum_{i,j}\frac{b_ib_ji!j!}{(\log y_3)^{i+j}}J_3,$$

where

$$\begin{aligned} J_3 = \left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)}\int_{(\delta)} \frac{\zeta^{13}(1+s+u)\zeta^2(1+\beta+u)\zeta^2(1+\alpha+s)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s)\zeta^3(1+\alpha+u)} \\ \times y_3^{s+u}B(s,u,0)\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}. \end{aligned}$$

Since we want to decouple the function where s and u are present, we use Dirichlet series for $\zeta^{(13)}(1+s+u)$ and then reverse order of integration and summation to obtain

$$\begin{aligned} J_3 = \sum_{m\le y_3}\frac{d_{13}(m)}{m}\left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)}\int_{(\delta)} B_{\alpha,\beta}(s,u,0)\left(\frac{y_3}{m}\right)^{s+u} \\ \times \frac{\zeta^2(1+\alpha+s)\zeta^2(1+\beta+u)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s)\zeta^3(1+\alpha+u)}\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}. \end{aligned}$$

Let us now take $\delta \asymp L^{-1}$. We can trivially bound the integrals to show that

$$J_{33} = \sum_{n\le y_3}\frac{d_{13}(n)}{n}\left(\frac{1}{2\pi i}\right)^2 L_1L_2 + O(\log^{i+j-2}T) \ll \log^{i+j-1}T.$$

In particular, this means that we can use a Taylor series so that $B(s,u,0) = B(0,0,0) + O(|s|+|u|)$ and this allows us to write $J_3 = {J_3}' + O(L^{i+j-2})$, say. This process decouples the variables s and u so that

$$ {J_3}' = \sum_{m \leq y_3} \frac{d_{13}(m)}{m} L_1 L_2, \tag{5.2} $$

where

$$ L_1 = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_3}{m}\right)^s \frac{\zeta^2(1+\alpha+s)}{\zeta^6(1+2s)\zeta^3(1+\beta+s)} \frac{ds}{s^{i+1}}, \tag{5.3} $$

and

$$ L_2 = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_3}{m}\right)^u \frac{\zeta^2(1+\beta+u)}{\zeta^6(1+2u)\zeta^3(1+\alpha+u)} \frac{du}{u^{j+1}}. $$

We observe that L_2 is the same as L_1 but with i replaced by j and α and β switched. The result we will need is encapsulated below, its proof follows the proof of Lemma 6.1 of [2].

Lemma 5.1. *With L_1 defined as in (5.3) and for some $\nu \asymp (\log\log y_3)^{-1}$ we have*

$$ L_1 = 64 \frac{1}{2\pi i} \oint \left(\frac{y_3}{m}\right)^s \frac{(\beta+s)^3}{(\alpha+s)^2} \frac{ds}{s^{i-5}} + O(L^{i-8}) + O\left(\left(\frac{y_3}{m}\right)^{-\nu} L^{\varepsilon}\right), $$

where the contour is a circle of radius $\asymp L^{-1}$ around the origin.

Let us now compute this integral. The result appears below.

Lemma 5.2. *For $i \geq 6$ we have*

$$ \frac{1}{2\pi i} \oint \left(\frac{y_3}{m}\right)^s \frac{(\beta+s)^3}{(\alpha+s)^2} \frac{ds}{s^{i-5}} = \frac{1}{(i-6)!} \frac{d^3}{dx^3} \left(x + \log\frac{y_3}{m}\right)^{i-4} e^{\beta x} \int_0^1 c(1-c)^6 \left(\frac{y_3}{m}\right)^{-\alpha c} e^{-\alpha c x} dc \Big|_{x=0}. \tag{5.4} $$

Proof. Using simple derivatives one can write

$$ I := \frac{1}{2\pi i} \oint \left(\frac{y_3}{m}\right)^s \frac{(\beta+s)^3}{(\alpha+s)^2} \frac{ds}{s^{i-5}} = \frac{d^3}{dx^3} e^{\beta x} \oint \left(e^x \frac{y_3}{m}\right)^s \frac{1}{(\alpha+s)^2} \frac{ds}{s^{i-5}} \Big|_{x=0}. $$

Let us set $q = e^x y_3/m$, so that

$$ I = -\frac{d^3}{dx^3} e^{\beta x} \int_{1/q}^1 r^{\alpha-1} \log r \left(\frac{1}{2\pi i} \oint (rq)^s \frac{ds}{s^{i-5}}\right) dr \Big|_{x=0}. \tag{5.5} $$

The second term of (4.9) yields an error which vanishes by taking the contour to be arbitrarily large. Then, by Cauchy's integral formula one has

$$ \begin{aligned} I &= \frac{d^3}{dx^3} e^{\beta x} \int_{1/q}^1 r^{\alpha-1} \log r \frac{1}{(i-6)!} (\log rq)^{i-6} dr \Big|_{x=0} \\ &= \frac{1}{(i-6)!} \frac{d^3}{dx^3} \left(x + \log\frac{y_3}{m}\right)^{i-4} e^{\beta x} \int_0^1 c(1-c)^6 \left(\frac{y_3}{m}\right)^{-\alpha c} e^{-\alpha c x} dc \Big|_{x=0}, \end{aligned} \tag{5.6} $$

by the change of variable $r = q^{-c}$. □

Applying Lemmas 5.1 and 5.2 to equation (5.2) yields

$$ \begin{aligned} J_3' &= \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} L_1 L_2 \\ &= \frac{2^{12}}{(i-6)!(j-6)!} \frac{d^6}{dx^3 dy^3} e^{x\beta + \alpha y} \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} \left(x + \log\frac{y_3}{m}\right)^{i-4} \left(y + \log\frac{y_3}{m}\right)^{j-4} \end{aligned} $$

$$\times \int_0^1 \int_0^1 u(1-u)^{i-6} v(1-v)^{j-6} e^{-x\alpha u - y\beta v} \left(\frac{y_3}{m}\right)^{-\alpha u-\beta v} dudv \bigg|_{x=y=0}$$
$$+ O(L^{i+j-2}),$$

where we used Lemma 3.4 to obtain the error term. Hence, telescoping all the way back to I'_{33} and using the Dirichlet series for $\zeta(1+\alpha+\beta)$ gives us

$$I'_{33}(\alpha,\beta) = \frac{\hat{w}(0)}{\alpha+\beta} \sum_{i,j} \frac{b_i b_j i! j!}{(\log y_3)^{i+j}} \frac{2^{12}}{(i-6)!(j-6)!} \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} \frac{d^6}{dx^3 dy^3} e^{x\beta+\alpha y}$$
$$\times \left(x + \log \frac{y_3}{m}\right)^{i-4} \left(y + \log \frac{y_3}{m}\right)^{j-4} \int_0^1 \int_0^1 u(1-u)^{i-6} v(1-v)^{j-6}$$
$$\times e^{-x\alpha u - y\beta v} \left(\frac{y_3}{m}\right)^{-\alpha u - \beta v} dudv \bigg|_{x=y=0} + O(TL^{\varepsilon-1})$$
$$= \frac{2^{12}\hat{w}(0)}{\alpha+\beta} \frac{d^6}{dx^3dy^3} \Big(\int_0^1 \int_0^1 e^{x(\beta-\alpha u)+y(\alpha-\beta v)} \sum_{m\leqslant y_3} \frac{d_{13}(m)}{m} \frac{(x+\log\frac{y_3}{m})^2 (y+\log\frac{y_3}{m})^2}{(\log y_3)^{12}}$$
$$\times P_3^{(6)}\left((1-u)\frac{x+\log\frac{y_3}{m}}{\log y_3}\right) P_3^{(6)}\left((1-v)\frac{y+\log\frac{y_3}{m}}{\log y_3}\right) \left(\frac{y_3}{m}\right)^{-\alpha u-\beta v} dudv\Big)\bigg|_{x=y=0} + O(TL^{\varepsilon-1})$$

A more convenient way to write this is as:

$$I'_{33}(\alpha,\beta) = \frac{2^{12}\hat{w}(0)}{(\alpha+\beta)\log^{14} y_3} \frac{d^6}{dx^3dy^3} \bigg(\int_0^1 \int_0^1 y_3^{x(\beta-\alpha u)+y(\alpha-\beta v))} \sum_{m\leqslant y_3} \frac{d_{13}(m)}{m} \left(\frac{y_3}{m}\right)^{-\alpha u-\beta v}$$
$$\times \left(x + \frac{\log\frac{y_3}{m}}{\log y_3}\right)^2 \left(y + \frac{\log\frac{y_3}{m}}{\log y_3}\right)^2$$
$$\times P_3^{(6)}\left((1-u)\left(x+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right) P_3^{(6)}\left((1-v)\left(y+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right) dudv\bigg)\bigg|_{x=y=0}$$
$$+ O(TL^{\varepsilon-1}).$$

Using Lemma 3.4 with $k=13$, $s=-\alpha u-\beta v$, $x=z=y_3$, $F(r)=(x+r)^2P_3^{(6)}((1-u)(x+r))$ as well as $H(r)=(y+r)^2P_3^{(6)}((1-v)(y+r))$, we then obtain

$$\sum_{m\leqslant y_3} \frac{d_{13}(m)}{m^{1-\alpha u-\beta v}} \left(x+\frac{\log\frac{y_3}{m}}{\log y_3}\right)^2$$
$$\times P_3^{(6)}\left((1-u)\left(x+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right) \left(y+\frac{\log\frac{y_3}{m}}{\log y_3}\right)^2 P_3^{(6)}\left((1-v)\left(y+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right)$$
$$= \frac{(\log y_3)^{13}}{12! y_3^{-\alpha u-\beta v}} \int_0^1 (1-r)^{12}(x+r)^2 P_3^{(6)}((1-u)(x+r))(y+r)^2 P_3^{(6)}((1-v)(y+r)) z^{r(-\alpha u-\beta v)} dr.$$

Putting this into $I'_3(\alpha,\beta)$ we obtain

$$I'_{33}(\alpha,\beta) = \frac{2^{12}\hat{w}(0)}{(\alpha+\beta)\log^{14} y_3} \frac{d^6}{dx^3dy^3} \bigg(\int_0^1 \int_0^1 \int_0^1 y_3^{x(\beta-\alpha u)+y(\alpha-\beta v)} y_3^{-\alpha u-\beta v} \frac{(\log y_3)^{13}}{(12)! y_3^{-\alpha u-\beta v}} (1-r)^{12}$$
$$\times (x+r)^2 P_3^{(6)}((1-u)(x+r))(y+r)^2 P_3^{(6)}((1-v)(y+r)) y_3^{r(-\alpha u-\beta v)} drdudv \bigg)\bigg|_{x=y=0}$$
$$= \frac{2^{12}\hat{w}(0)}{12!(\alpha+\beta)\log y_3} \frac{d^6}{dx^3dy^3} \bigg(\int_0^1 \int_0^1 \int_0^1 y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))} (1-r)^{12}(x+r)^2(y+r)^2$$

$$\times P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))drdudv\bigg)\bigg|_{x=y=0}.$$

To form the full $I_{33}(\alpha,\beta)$, recall that, as we discussed earlier, we need to add I'_{33} and I''_{33}, where I''_{33} is formed by taking I'_{33}, switching α and $-\beta$, and multiplying by $T^{-\alpha-\beta}$. Letting

$$U(\alpha,\beta)=\frac{y_3^{\beta(x-v(y+r))+\alpha(y-u(x+r))}-T^{-\alpha-\beta}y_3^{-\alpha(x-v(y+r))-\beta(y-u(x+r))}}{\alpha+\beta}$$

we then have

$$I_{33}(\alpha,\beta)=\frac{2^{12}\hat{w}(0)}{12!\log y_3}\frac{d^6}{dx^3dy^3}\bigg(\int_0^1\int_0^1\int_0^1 y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))}(1-r)^{12}(x+r)^2(y+r)^2$$
$$\times U(\alpha,\beta)P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))drdudv\bigg)\bigg|_{x=y=0}+O(TL^{-1+\varepsilon}).$$

Now write

$$U(\alpha,\beta)=y_3^{\beta(x-v(y+r))+\alpha(y-u(x+r))}\frac{1-(Ty_3^{x+y-v(y+r)-u(x+r)})^{-\alpha-\beta}}{\alpha+\beta},$$

and use the integral formula

$$\frac{1-z^{-\alpha-\beta}}{\alpha+\beta}=\log z\int_0^1 z^{-t(\alpha+\beta)}dt,$$

as well as $y_3=T^{\theta_3}$ so that

$$I_{33}(\alpha,\beta)=\frac{2^{12}}{12!}\hat{w}(0)\frac{d^6}{dx^3dy^3}\bigg(\int_0^1\int_0^1\int_0^1\int_0^1(1-r)^{12}y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))}$$
$$\times\left(\frac{1}{\theta_3}+(x+y-v(y+r)-u(x+r))\right)(x+r)^2(y+r)^2$$
$$\times P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))(Ty_3^{x+y-v(y+r)-u(x+r)})^{-t(\alpha+\beta)}dtdrdudv\bigg)\bigg|_{x=y=0}.$$

Hence this proves Lemma 2.4.

6. Proof of Proposition 2.5

Inserting the relevant definitions of the mollifiers in the mean value integral yields

$$I_{42}(\alpha,\beta)=\int_{-\infty}^{\infty}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\chi(\tfrac12+it)$$
$$\times\sum_{ab\le y_2}\frac{\mu_2(a)}{a^{1/2+it}b^{1/2-it}}P_2[ab]\sum_{c\le y_4}\frac{\mu(c)}{c^{1/2-it}}\sum_{k=2}^{K}\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c]dt$$
$$=\sum_{k=2}^{K}\sum_{ab\le y_2}\sum_{c\le y_4}\frac{\mu_2(a)\mu(c)}{(abc)^{1/2}}P_2[ab]\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c]J_{42},$$

where

$$J_{42}=\int_{-\infty}^{\infty}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\chi(\tfrac12+it)\Big(\frac{a}{bc}\Big)^{-it}dt.$$

This integral was evaluated in [2, eq. (5.7)] and once we apply Lemma 4.1 of [2] it is given by

$$J_{42}=\sum_{n=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(n)}{n^{1/2}}e^{-n/T^3}\int_{-\infty}^{\infty}w(t)\Big(\frac{t}{2\pi}\Big)^{-\beta}\Big(\frac{an}{bc}\Big)^{-it}dt+O(T^{\varepsilon}).$$

Therefore, when we insert (4.1) in I_{42} we have

$$I_{42}(\alpha,\beta)=\sum_{k=2}^{K}\sum_{n=1}^{\infty}\sum_{ab\le y_2}\sum_{c\le y_4}\frac{\mu_2(a)\mu(c)\sigma_{\alpha,-\beta}(n)}{(abcn)^{1/2}}e^{-n/T^3}P_2[ab]\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c]$$
$$\times\widehat{w_0}\left(\frac{1}{2\pi}\log\frac{an}{bc}\right)+O(T^{(\theta_2+\theta_4)/2+\varepsilon}).$$

6.1. **Off diagonal terms ($an\neq bc$):** Since $c\le y_4$, then the sum satisfied

$$\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c]\ll d(c).$$

The off-diagonal terms are given by

$$C_{42}(\alpha,\beta)=\sum_{l=1}^{\infty}\sum_{bc\le y_1}\sum_{\substack{f\le y_4\\ bl\neq cf}}\frac{\mu_2(b)\mu(f)\sigma_{\alpha,-\beta}(l)}{(bcfl)^{1/2}}e^{-l/T^3}P_2[bc]$$
$$\times\sum_{k=2}^{K}\sum_{p_1\cdots p_k|f}\frac{\log p_1\ldots\log p_k}{\log^k y_4}\widetilde{P}_k[f]\int_{-\infty}^{\infty}w_0(t)\left(\frac{bl}{cf}\right)^{-it}dt.$$

In [2] it is shown that

$$\widehat{w_0}(\tfrac{1}{2\pi}\log x)\ll_B\frac{T}{(1+\frac{1}{2\pi}\frac{T}{L}\log x)^B},\tag{6.1}$$

for any $B\ge 0$. Let us split the above into

$$C_{42}=C'_{42}+C''_{42},\quad\text{where}\quad C'_{42}=\sum_{1\le l\le T^4}\quad\text{and}\quad C''_{42}=\sum_{l>T^4}.$$

We have the following bound for C''_{42}

$$C''_{42}\ll\sum_{l>T^4}\sum_{bc\le y_2}\sum_{\substack{f\le y_4\\ bl\neq cf}}\frac{|\mu_2(b)||\mu(f)|\sigma_{\alpha,-\beta}(l)d(f)}{(bcfl)^{1/2}}e^{-l/T^3}\int_{-\infty}^{\infty}w_0(t)dt$$
$$\ll_\varepsilon\sum_{l>T^4}\frac{l^\varepsilon}{l^{1/2}}e^{-l/T^3}T^\varepsilon\left(\sum_{bc\le y_2}\frac{1}{(bc)^{1/2}}\right)\left(\sum_{f\le y_4}\frac{1}{f^{1/2}}\right)$$
$$\ll_\varepsilon T^{(\theta_2+\theta_4)/2+2\varepsilon}\sum_{l>T^4}\frac{1}{l^{1/2-\varepsilon}}e^{-l/T^3}$$
$$\ll_\varepsilon T^{\frac{3}{2}+\frac{1}{2}(\theta_2+\theta_4)+5\varepsilon}e^{-T}.$$

We now assume that $\theta_2+\theta_4<1$. Fix δ_0 such that $0<\delta_0<1-\theta_2-\theta_4$. For any $1\le l\le T^4$, $1\le f\le T^4$, $1\le f\le y_4$ and any $b,c\ge 1$ such that $bc\ge y_2$ for which $cl\neq bf$ we have

$$\left|\frac{bl}{cf}-1\right|\ge\frac{1}{cf}\ge\frac{1}{y_2y_4}=\frac{1}{T^{\theta_2+\theta_4}}>\frac{1}{T^{1-\delta}}.$$

Therefore, we can write

$$\left|\log\frac{bl}{cf}\right|\ge\frac{1}{2T^{1-\delta_0}}.$$

Then by (6.1) with $B = \frac{2014}{\delta_0}$ we have, uniformly for all b, c, l and f in the above ranges, that

$$\int_{-\infty}^{\infty} w_0(t) \left(\frac{bl}{cf} \right)^{-it} dt \ll_{\delta_0} \frac{T}{(1 + \frac{1}{2\pi} \frac{T}{L} \log x)^{2014/\delta_0}} \ll_{\delta_0} \frac{(T \log T)^{2014/\delta_0}}{T^{2014/\delta_0}} \ll_{\delta_0} \frac{1}{T^{2012}}.$$

We now use this to bound C'_{42} as follows

$$C'_{42} \ll \sum_{1 \leq l \leq T^4} \sum_{bc \leq y_2} \sum_{\substack{f \leq y_4 \\ bl \neq cf}} \frac{|\mu_2(b)||\mu(f)|\sigma_{\alpha,-\beta}(l) d(f)}{(bcfl)^{1/2}} e^{-l/T^3} \left| \int_{-\infty}^{\infty} w_0(t) \left(\frac{bl}{cf} \right)^{-it} dt \right|$$

$$\ll \sum_{1 \leq l \leq T^4} \sum_{bc \leq y_2} \sum_{\substack{f \leq y_4 \\ bl \neq cf}} \frac{T^\varepsilon}{(bcfl)^{1/2}} \frac{1}{T^{2012}} \ll \frac{T^{2\varepsilon + 2 + \frac{1}{2}(\theta_2 + \theta_4)}}{T^{2012}} \ll \frac{1}{T^{2009}}.$$

This shows that the off-diagonal terms get absorbed in the error term and do not contribute to our final answer.

6.2. **Main term ($an = bc$):** From (3.5) and (3.6) we have

$$I_{42}(\alpha, \beta) = \widehat{w_0}(0) \sum_{k=2}^{K} \sum_{i,j} \frac{a_i \widetilde{a}_{k,j} i! j!}{\log^i y_2 \log^{j+k} y_4} \left(\frac{1}{2\pi i} \right)^3 \int_{(1)} \int_{(1)} \int_{(1)} T^{3z} \Gamma(z) y_2^{z_1} y_4^{z_2}$$
$$\times \sum_{an=bc} \frac{\mu_2(a)\mu(c)\sigma_{\alpha,-\beta}(n)}{(ab)^{1/2+z_1} c^{1/2+z_2} n^{1/2+z}} \sum_{p_1 \cdots p_k | c} \log p_1 \cdots \log p_k \frac{dz dz_1 dz_2}{z_1^{i+1} z_2^{j+1}}$$
$$+ O(T^{1-\varepsilon}).$$

(6.2)

Let us define

$$S_k = S_{k,\alpha,\beta}(z, z_1, z_2) = \sum_{an=bc} \frac{\mu_2(a)\mu(c)\sigma_{\alpha,-\beta}(n)}{(ab)^{1/2+z_1} c^{1/2+z_2} n^{1/2+z}} \sum_{p_1 \cdots p_k | c} \log p_1 \cdots \log p_k.$$

We begin by swapping the order of the sum so that

$$S_k = (-1)^k \sum_{\substack{p_i \neq p_j \\ i<j}} \log p_1 \ldots \log p_k \sum_{\substack{cl = bp_1 \ldots p_k d \\ (d, p_1 \ldots p_k) = 1}} \frac{\mu_2(b)\mu(f)\sigma_{\alpha,-\beta}(l)}{(bc)^{1/2+z_1} d^{1/2+z_2} l^{1/2+z}} \frac{1}{(p_1 \ldots p_k)^{1/2+z_2}}$$
$$= (-1)^k \sum_{\substack{p_i \neq p_j \\ i<j}} \frac{\log p_1 \ldots \log p_k}{(p_1 \ldots p_k)^{1/2+z_2}} \sum_{\substack{b,c,d,f,l \\ (d, p_1 \ldots p_k) = 1}} \frac{\mu_2(b)\mu(f)}{(bc)^{1/2+z_1} d^{1/2+z_2} x^{1/2+\alpha+z} y^{1/2-\beta+z}}.$$

As usual, let $\nu_p(n)$ denote the number of different prime factors of n. To simplify the expressions that will take place shortly, we simplify this notation to $\nu_p(n) = n'$. With this in mind, the above becomes

$$S_k = (-1)^k \sum_{\substack{p_i \neq p_j \\ i<j}} \frac{\log p_1 \ldots \log p_k}{(p_1 \ldots p_k)^{1/2+z_2}}$$
$$\times \prod_{p \in \{p_1, \cdots, p_k\}} \left(\sum_{b'+x'+y'=c'+1} \frac{\mu_2(p^{b'})}{(p^{b'} p^{c'})^{1/2+z_1} (p^{x'})^{1/2+\alpha+z} (p^{y'})^{1/2-\beta+z}} \right)$$
$$\times \prod_{p \notin \{p_1, \cdots, p_k\}} \left(\sum_{b'+x'+y'=c'+d'} \frac{\mu_2(p^{b'})\mu(p^{d'})}{(p^{b'} p^{c'})^{1/2+z_1} (p^{d'})^{1/2+z_2} (p^{x'})^{1/2+\alpha+z} (p^{y'})^{1/2-\beta+z}} \right) \tag{6.3}$$

$$= (-1)^k \sum_{\substack{p_i \neq p_j \\ i<j}} \frac{\log p_1 \dots \log p_k}{(p_1 \dots p_k)^{1/2+z_2}} \frac{\Pi_1(k,\alpha,\beta)}{\Pi_2(k,\alpha,\beta)}$$

$$\times \prod_p \Bigg(1 + \frac{1}{p^{1+\alpha+z_1+z}} + \frac{1}{p^{1-\beta+z_1+z}} - \frac{1}{p^{1+\alpha+z_2+z}} - \frac{1}{p^{1-\beta+z_2+z}} - \frac{2}{p^{1+2z_1}} + \frac{2}{p^{1+z_1+z_2}} + O(p^{-2})\Bigg),$$

where

$$\Pi_1(k,\alpha,\beta) = \prod_{p\in\{p_1,\cdots,p_k\}} \left(\frac{1}{p^{1+\alpha+z+z_2}} + \frac{1}{p^{1-\beta+z+z_2}} - \frac{2}{p^{1+z_1+z_2}} + O(p^{-2})\right),$$

and

$$\Pi_2(k,\alpha,\beta) = \prod_{p\in\{p_1,\cdots,p_k\}} \Bigg(1 + \frac{1}{p^{1+\alpha+z_1+z}} + \frac{1}{p^{1-\beta+z_1+z}} - \frac{1}{p^{1+\alpha+z_2+z}} - \frac{1}{p^{1-\beta+z_2+z}} - \frac{2}{p^{1+2z_1}} + \frac{2}{p^{1+z_1+z_2}} + O(p^{-2})\Bigg).$$

This reduces the expression for S_k to the more tractable

$$S_k = \frac{\zeta(1+\alpha+z_1+z)\zeta(1-\beta+z_1+z)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2+z)\zeta(1-\beta+z_2+z)\zeta^2(1+2z_1)} A(\alpha,\beta,z,z_1,z_2) \times (-1)^k \sum_{\substack{p_i \neq p_j \\ i<j}} \log p_1 \dots \log p_k$$

$$\times \prod_{p\in\{p_1,\cdots,p_k\}} \frac{E(p)+O(p^{-2})}{1+\frac{1}{p^{1+\alpha+z_1+z}}+\frac{1}{p^{1-\beta+z_2+z}}-\frac{2}{p^{1+2z_1}}-E(p)+O(p^{-2})}, \tag{6.4}$$

where

$$E(p) = \frac{1}{p^{1+\alpha+z+z_2}} + \frac{1}{p^{1-\beta+z+z_2}} - \frac{2}{p^{1+z_1+z_2}}.$$

By comparing (6.3) and (6.4) we see that

$$\frac{\zeta(1+\alpha+z_1+z)\zeta(1-\beta+z_1+z)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2+z)\zeta(1-\beta+z_2+z)\zeta^2(1+2z_1)} A(\alpha,\beta,z,z_1,z_2)$$

$$= \prod_p \left(\sum_{b'+x'+y'=c'+d'} \frac{\mu_2(p^{b'})\mu(p^{d'})}{(p^{b'}p^{c'})^{1/2+z_1}(p^{d'})^{1/2+z_2}(p^{x'})^{1/2+\alpha+z}(p^{y'})^{1/2-\beta+z}} \right). \tag{6.5}$$

Reverting the p-adic analysis on the right-hand side of (6.5) one arrives at

$$\frac{\zeta(1+\alpha+z_1+z)\zeta(1-\beta+z_1+z)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2+z)\zeta(1-\beta+z_2+z)\zeta^2(1+2z_1)} A(\alpha,\beta,z,z_1,z_2)$$

$$= \prod_{bxy=cd} \frac{\mu_2(b)\mu(d)}{(bc)^{1/2+z_1}(d)^{1/2+z_2}(x)^{1/2+\alpha+z}(y)^{1/2-\beta+z}}$$

$$= \prod_{bl=cd} \frac{\mu_2(b)\mu(d)\sigma_{\alpha,-\beta}(l)}{(bc)^{1/2+z_1}d^{1/2+z_2}l^{1/2+z}}, \tag{6.6}$$

where in the ultimate step, we have used the definition of $\sigma_{\alpha,-\beta}(l)$. Using [2, §5.6], we can conclude that $A(\alpha,\beta,z,z,z)=1$ for all z. Let us denote the last line of (6.4) by H_k. Then

$$\begin{aligned} H_k &= (-1)^k \sum_{\substack{p_i \neq p_j \\ i<j}} \prod_{p \in \{p_1,\cdots,p_k\}} (\log p)(E(p)+O(p^{-2})) \\ &\quad \times \left(1+E(p)-\frac{1}{p^{1+\alpha+z_1+z}}-\frac{1}{p^{1-\beta+z_2+z}}+\frac{2}{p^{1+2z_1}}+O(p^{-2})\right) \\ &= (-1)^k \sum_{\substack{p_i \neq p_j \\ i<j}} \prod_{p \in \{p_1,\cdots,p_k\}} \left(E(p)\log p + O\left(\frac{\log p}{p^2}\right)\right). \end{aligned}$$

Next, we use the principle of inclusion-exclusion to write

$$H_k = (-1)^k \left(\sum_p E(p)\log p + O\left(\frac{\log p}{p}\right)\right)^k + \sum_p B(p),$$

where

$$B(p) \ll_{\alpha,\beta,z,z_1,z_2} \frac{1}{p^2}.$$

The final step is to identify sums over p containing $\log p$ with their analytic counterparts in terms of logarithmic derivatives of the zeta function by the use of

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = -\sum_p \frac{\log p}{p^s}\left(1-\frac{1}{p^s}\right)^{-1},$$

to see that

$$\begin{aligned} H_k &= \left(-\frac{\zeta'}{\zeta}(1+\alpha+z+z_2)-\frac{\zeta'}{\zeta}(1-\beta+z_2+z)+2\frac{\zeta'}{\zeta}(1+z+z_2)+O(\alpha,\beta,z,z_1,z_2)\right)^k \\ &\quad + D(\alpha,\beta,z,z_1,z_2) \\ &= U^k + \sum_{m=0}^{k-1} U^m B_m(\alpha,\beta,z,z_1,z_2), \end{aligned} \tag{6.7}$$

where

$$U := 2\frac{\zeta'}{\zeta}(1+z_1+z_2)-\frac{\zeta'}{\zeta}(1+\alpha+z+z_2)-\frac{\zeta'}{\zeta}(1-\beta+z+z_2)$$

and

$$B_m(\alpha,\beta,z,z_1,z_2) \ll_{\alpha,\beta,z,z_1,z_2} \sum_p \frac{\log p}{p^2}.$$

All of these terms are analytic in a larger region, thus we need only be concerned with U^k. Next, we move the lines of integration to $\mathrm{Re}(z)=-\delta+\varepsilon$ and $\mathrm{Re}(z_1)=\mathrm{Re}(z_2)=\delta$. By deforming the contours like this, we cross the simple pole at $z=0$ of $\Gamma(z)$. The integral on $\mathrm{Re}(z_1)=\mathrm{Re}(z_2)=\delta$, and $\mathrm{Re}(z)=-\delta+\varepsilon$ can be bounded by

$$\left|\widehat{w_0}(0)\frac{y_2^{\delta_0}y_4^{\delta_0}}{T^{3\delta_0}}\right| T^{3\varepsilon} \ll T^{1-\varepsilon}.$$

Hence

$$I_{42}(\alpha,\beta) = \widehat{w_0}(0)\sum_{k=2}^{K}\sum_{i,j}\frac{a_i\widetilde{a}_{k,j}i!j!}{\log^i y_2 \log^{j+k} y_4}K_{42} + O(T^{1-\varepsilon}),$$

where

$$K_{42} = \left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)}\int_{(\delta)} y_2^{z_1} y_4^{z_2} \frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)\zeta^2(1+2z_1)} A(\alpha,\beta,0,z_1,z_2)$$
$$\times \left(2\frac{\zeta'}{\zeta}(1+z_1+z_2) - \frac{\zeta'}{\zeta}(1+\alpha+z_2) - \frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^k \frac{dz_1 dz_2}{z_1^{i+1} z_2^{j+1}}.$$

Let K'_{42} be the same integral as K_{42} but with $A(\alpha,\beta,z,z_1,z_2)$ replaced by $A(\alpha,\beta,0,0,0) = (-1)^k$. Then, just as before, $K'_{42} = K_{42} + O(L^{i+j-1})$. We wish to separate the variables z_1 and z_2 by the use of a suitable Dirichlet series. Let us define the term involving ζ's in the integrand of K_{42} by Π_{42}. Using the multinomial theorem we have

$$\begin{aligned}\Pi_{42} = &\frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)\zeta^2(1+2z_1)} \\ &\times \left(2\frac{\zeta'}{\zeta}(1+z_1+z_2) - \frac{\zeta'}{\zeta}(1+\alpha+z_2) - \frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^k \\ = &(-1)^k k! \sum_{l_1+l_2+l_3=k} \frac{2^{l_1}}{l_1! l_2! l_3!} \sum_{n=1}^{\infty} \frac{(d*\Lambda^{*l_1})(n)}{n^{1+z_1+z_2}} \frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)}{\zeta^2(1+2z_1)} \\ &\times \frac{1}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)} \left(\frac{\zeta'}{\zeta}(1+\alpha+z_2)\right)^{l_2} \left(\frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^{l_3},\end{aligned}$$

where we have used the Dirichlet convolution of

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad \text{and} \quad -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

for $\mathrm{Re}(s) > 1$ and where Λ^{*l_1} stands for convolving $\Lambda * \cdots * \Lambda$ exactly l_1 times. Hence, we get the splitting

$$K'_{42} = \frac{k!}{\log^k y_4} \sum_{l_1+l_2+l_3=k} \frac{2^{l_1}}{l_1! l_2! l_3!} \sum_{n \le \min(y_2,y_4)} \frac{(d*\Lambda^{l_1})(n)}{n} K_1 K_2(l_2,l_3) + O(L^{i+j-1}),$$

where

$$K_1 = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_2}{n}\right)^{z_1} \frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)}{\zeta^2(1+2z_1)} \frac{dz_1}{z_1^{i+1}},$$

and

$$K_2(l_2,l_3) = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_4}{n}\right)^{z_2} \frac{1}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)}$$
$$\times \left(\frac{\zeta'}{\zeta}(1+\alpha+z_2)\right)^{l_2} \left(\frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^{l_3} \frac{dz_2}{z_2^{j+1}}. \tag{6.8}$$

From [2, eq. (5.41)] we have

$$K_1 = \frac{4(\log(y_2/n))^i}{(i-2)!} \iint_{a+b\le 1} (1-a-b)^{i-2} \left(\frac{y_2}{n}\right)^{-a\alpha+b\beta} da\, db + O(L^{i-1}).$$

By the Laurent series expansion around $s=1$ of the logarithmic derivative of $\zeta(s)$ we have

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \gamma + O(|s-1|). \tag{6.9}$$

Now we will compute the following contour integrations for different choices of l_2 and l_3.

(1) If $l_2 = l_3 = 0$,

$$\frac{1}{2\pi i}\oint q^{z_2}(\alpha+z_2)(-\beta+z_2)\frac{dz_2}{z_2^{j+1}} = \frac{d^2}{dxdy}e^{\alpha x-\beta y}\frac{1}{2\pi i}\oint (qe^{x+y})^{z_2}\frac{dz_2}{z_2^{j+1}}\bigg|_{x=y=0}$$

$$= \frac{1}{j!}\frac{d^2}{dxdy}\left[e^{\alpha x-\beta y}\left(x+y+\log\frac{y_4}{n}\right)^j\right]_{x=y=0}. \tag{6.10}$$

(2) If $l_2 = 1$ and $l_3 = 0$,

$$-\frac{1}{2\pi i}\oint q^{z_2}(-\beta+z_2)\frac{dz_2}{z_2^{j+1}} = -\frac{d}{dy}e^{-\beta y}\frac{1}{2\pi i}\oint (qe^{y})^{z_2}\frac{dz_2}{z_2^{j+1}}\bigg|_{y=0}$$

$$= -\frac{1}{j!}\frac{d}{dy}\left[e^{-\beta y}\Big(y+\log\frac{y_4}{n}\Big)^j\right]_{y=0}. \tag{6.11}$$

(3) By symmetry, if $l_2 = 0$ and $l_3 = 1$, then

$$-\frac{1}{2\pi i}\oint q^{z_2}(\alpha+z_2)\frac{dz_2}{z_2^{j+1}} = -\frac{1}{j!}\frac{d}{dy}\left[e^{\alpha y}\Big(y+\log\frac{y_4}{n}\Big)^j\right]_{y=0}. \tag{6.12}$$

(4) If $l_2 = l_3 = 1$,

$$\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{dz_2}{z_2^{j+1}} = \frac{1}{j!}\log^j\frac{y_4}{n}.$$

(5) If $l_2 = 1$ and $l_3 \geq 2$,

$$(-1)^{1+l_3}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{1}{(-\beta+z_2)^{l_3-1}}\frac{dz_2}{z_2^{j+1}}$$

$$= -\frac{1}{(l_3-2)!}\int_{1/q}^{1} t^{-\beta-1}\log^{l_3-2}t\frac{1}{2\pi i}\oint (qt)^{z_2}\frac{dz_2}{z_2^{j+1}}dt$$

$$= -\frac{1}{j!(l_3-2)!}\int_{n/y_4}^{1} t^{-\beta-1}\log^j\left(\frac{y_4}{n}t\right)\log^{l_3-2}t\,dt$$

$$= -\frac{(-1)^{l_3-2}(\log(y_4/n))^{j+l_3-1}}{j!(l_3-2)!}\int_0^1 (1-b)^j\left(\frac{y_4}{n}\right)^{b\beta}b^{l_3-2}db. \tag{6.13}$$

(6) Again, by symmetry, if $l_2 \geq 2$ and $l_3 = 1$, then

$$(-1)^{1+l_2}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{1}{(\alpha+z_2)^{l_2-1}}\frac{dz_2}{z_2^{j+1}}$$

$$= -\frac{(-1)^{l_2-2}(\log(y_4/n))^{j+l_2-1}}{j!(l_2-2)!}\int_0^1 (1-b)^j\left(\frac{y_4}{n}\right)^{-b\alpha}b^{l_2-2}db. \tag{6.14}$$

(7) If $l_2 = 0$ and $l_3 \geq 2$,

$$(-1)^{l_3}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{\alpha+z_2}{(-\beta+z_2)^{l_3-1}}\frac{dz_2}{z_2^{j+1}}$$

$$= \frac{(-1)^{l_3}}{j!(l_3-2)!}\frac{d}{dx}\left(x+\log\frac{y_4}{m}\right)^{j+l_3-1}e^{\alpha x}\int_0^1 c^{l_3-2}(1-c)^j\left(\frac{y_4}{m}\right)^{\beta c}e^{\beta cx}dc\bigg|_{x=0}. \tag{6.15}$$

(8) If $l_2 \geq 2$ and $l_3 = 0$,

$$(-1)^{l_2}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{-\beta+z_2}{(\alpha+z_2)^{l_2-1}}\frac{dz_2}{z_2^{j+1}} + O(L^{j-3-l_2})$$

$$= \frac{(-1)^{l_2}}{j!(l_2-2)!}\frac{d}{dx}\left(x+\log\frac{y_4}{n}\right)^{j+l_2-1}e^{-\beta x}\int_0^1 c^{l_2-2}(1-c)^j\left(\frac{y_4}{n}\right)^{-\alpha c}e^{-\alpha cx}dc\bigg|_{x=0}. \tag{6.16}$$

(9) Finally, if $l_2 \geq 2$ and $l_3 \geq 2$,

$$\begin{aligned}
&(-1)^{l_2+l_3}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{1}{(\alpha+z_2)^{l_2-1}}\frac{1}{(-\beta+z_2)^{l_3-1}}\frac{dz_2}{z_2^{j+1}}\\
&=(-1)^{l_2+l_3}\frac{(-1)^{2-l_2}}{(l_2-2)!}\frac{(-1)^{2-l_3}}{(l_3-2)!}\int_{1/q}^{1}\int_{1/(qr)}^{1} r^{\alpha-1}t^{-\beta-1}\log^{l_2-2}r\log^{l_3-2}t\\
&\qquad\times\frac{1}{2\pi i}\oint (qrt)^{z_2}\frac{dz_2}{z_2^{j+1}}dtdr\\
&=\frac{1}{j!(l_2-2)!(l_3-2)!}\int_{1/q}^{1}\int_{1/(qr)}^{1} r^{\alpha-1}t^{-\beta-1}\log^{l_2-2}r\log^{l_3-2}t\log\left(rt\frac{y_4}{n}\right)^{j}dtdr\\
&=\frac{(-1)^{l_2+l_3}(\log(y_4/n))^{j+l_2+l_3-2}}{j!(l_2-2)!(l_3-2)!}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}(1-a-b)^j\left(\frac{y_4}{n}\right)^{-a\alpha+b\beta}a^{l_2-2}b^{l_3-2}dadb. \qquad (6.17)
\end{aligned}$$

In the last step we used the substitutions $r=q^{-a}$ and $t=q^{-b}$.

Going back to $I_{42}(\alpha,\beta)$, we now have to perform the sums over i and j and then insert them back into

$$\begin{aligned}
I_{42}(\alpha,\beta)=\widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+l_2+l_3=k}\frac{2^{l_1}}{l_1!l_2!l_3!}\sum_{n\leq\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\\
\times\sum_i K_1\frac{a_i i!}{\log^i y_2}\sum_j\frac{\widetilde{a}_{k,j}j!}{\log^j y_4}K_2(l_2,l_3)+O(TL^{-1+\varepsilon}). \qquad (6.18)
\end{aligned}$$

Since $\theta_4<\theta_2$, we will now use $\min(y_2,y_4)=y_4$. From [2, §5.5] we find

$$\sum_i\frac{a_i i!}{\log^i y_2}K_1=4\frac{(\log(y_2/n))^2}{(\log y_2)^2}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\left(\frac{y_2}{n}\right)^{-a\alpha+b\beta}P_2''\left((1-a-b)\frac{\log(y_2/n)}{\log y_2}\right)dadb+O(L^{-1}).$$

For the j-sum, we need to consider each case separately.

6.2.1. *The case $l_2=l_3=0$.* In this case, from (4.8), (6.9), (6.10), and a similar argument to that of of Lemma 4.1 we have

$$\begin{aligned}
\sum_j{}^{(0,0)}&=\sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(0,0)=\frac{d^2}{dxdy}e^{\alpha x-\beta y}\sum_j\widetilde{a}_{j,k}\left(\frac{x+y}{\log y_4}+\frac{\log(y_4/n)}{\log y_4}\right)^j\Bigg|_{x=y=0}+O(L^{-3})\\
&=\frac{1}{(\log y_4)^2}\frac{d^2}{dxdy}y_4^{\alpha x-\beta y}\widetilde{P}_k\left(x+y+\frac{\log(y_4/n)}{\log y_4}\right)\Bigg|_{x=y=0}+O(L^{-3}).
\end{aligned}$$

Inserting this expression in (6.18) yields

$$\begin{aligned}
I_{42}^{(0,0)}(\alpha,\beta)&=\widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1=k}\frac{2^{l_1}}{l_1!}\sum_{n\leq\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(0,0)}+O(TL^{-1+\varepsilon})\\
&=\widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\frac{2^k}{k!}\sum_{n\leq y_4}\frac{(d*\Lambda^{*k})(n)}{n}\sum_i\sum_j{}^{(0,0)}+O(TL^{-1+\varepsilon})\\
&=\frac{4\widehat{w_0}(0)}{(\log y_4)^2}\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\frac{2^k}{k!}\frac{d^2}{dxdy}\Bigg[y_2^{\alpha x-\beta y}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\sum_{n\leq y_4}\frac{(d*\Lambda^{*k})(n)}{n^{1-a\alpha+b\beta}}y_2^{-a\alpha+b\beta}\\
&\quad\times P_2''\left((1-a-b)\frac{\log(y_2/n)}{\log y_2}\right)\left(\frac{\log(y_2/n)}{\log y_2}\right)^2\widetilde{P}_k\left(x+y+\frac{\log(y_4/n)}{\log y_4}\right)dadb\Bigg]_{x=y=0}
\end{aligned}$$

$$
\begin{aligned}
&+O(TL^{-1+\varepsilon})\\
&= 4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{2^k}{(1+k)!}\frac{d^2}{dxdy}\bigg[y_2^{\alpha x-\beta y}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2(1-u)^{1+k}\\
&\times\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(x+y+u)dudadb\bigg]_{x=y=0}\\
&+O(TL^{-1+\varepsilon}),
\end{aligned}
$$

where we have applied Lemma 3.6 with $k=2$, $l=k$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(x+y+u)$.

6.2.2. *The case $l_2=1$, $l_3=0$.* In this case, from (4.8), (6.9), (6.11), and a similar argument to that of Lemma 4.1 we have

$$
\begin{aligned}
\sum_j{}^{(1,0)} &= \sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(1,0) = -\frac{d}{dy}e^{-\beta y}\sum_j\widetilde{a}_{j,k}\left(\frac{y}{\log y_4}+\frac{\log(y_4/n)}{\log y_4}\right)^j\bigg|_{y=0}+O(L^{-4})\\
&= -\frac{1}{\log y_4}\frac{d}{dy}y_4^{-\beta y}\widetilde{P}_k\left(y+\frac{\log(y_4/n)}{\log y_4}\right)\bigg|_{y=0}+O(L^{-4}).
\end{aligned}
$$

By an analogue argument as in the previous case

$$
\begin{aligned}
I_{42}^{(1,0)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+1=k}\frac{2^{l_1}}{l_1!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(1,0)}+O(TL^{-1+\varepsilon})\\
&= -4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{2^{k-1}}{(k-1)!}\frac{d}{dy}\bigg[y_4^{-\beta y}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2(1-u)^k\\
&\times\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(y+u)dudadb\bigg]_{y=0}+O(TL^{-1+\varepsilon}),
\end{aligned}
$$

where we have used $k=2$, $l=k-1$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(y+u)$ in Lemma 3.6.

6.2.3. *The case $l_2=0$, $l_3=1$.* In this case, from (4.8), (6.9), (6.12), and a similar argument to that of Lemma 4.1 we have

$$
\begin{aligned}
\sum_j{}^{(0,1)} &= \sum_j\frac{\tilde{a}_{k,j}j!}{\log^j y_4}K_2(0,1) = -\frac{d}{dx}e^{\alpha x}\sum_j\tilde{a}_{k,j}\left(\frac{x}{\log y_4}+\frac{\log(y_4/n)}{\log y_4}\right)^j\bigg|_{x=0}\\
&= -\frac{1}{\log y_4}\frac{d}{dx}y_4^{\alpha x}\tilde{P}_k\left(x+\frac{\log(y_4/n)}{\log y_4}\right)\bigg|_{x=0}+O(L^{-4}).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
I_{42}^{(0,1)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+1=k}\frac{2^{l_1}}{l_1!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(0,1)}+O(TL^{-1+\varepsilon})\\
&= -4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{2^{k-1}}{(k-1)!}\frac{d}{dx}\bigg[y_4^{\alpha x}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2(1-u)^k\\
&\times\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(x+u)dudadb\bigg]_{x=0}+O(TL^{-1+\varepsilon}),
\end{aligned}
$$

by setting $k=2$, $l=k-1$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(x+u)$ in Lemma 3.6.

6.2.4. *The case $l_2=1$, $l_3=1$.* In this case, from (4.8), (6.9), and Cauchy's theorem we have

$$\sum_j{}^{(1,1)}=\sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(1,1)=\sum_j\widetilde{a}_{j,k}\left(\frac{\log(y_4/n)}{\log y_4}\right)^j+O(L^{-5})=\widetilde{P}_k\left(\frac{\log(y_4/n)}{\log y_4}\right)+O(L^{-5}).$$

Hence

$$\begin{aligned}I_{42}^{(1,1)}(\alpha,\beta)&=\widehat{w_0}(0)\sum_{k=2}^K\frac{k!}{\log^k y_4}\sum_{l_1+2=k}\frac{2^{l_1}}{l_1!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(1,1)}+O(TL^{-1+\varepsilon})\\&=4\widehat{w_0}(0)\sum_{k=2}^K\frac{2^{k-2}k}{(k-2)!}\iint_{\substack{0\le a+b\le1\\a,b\ge0}}\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2(1-u)^{k-1}\\&\times\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(u)dudadb+O(TL^{-1+\varepsilon}),\end{aligned}$$

by setting $k=2$, $l=k-2$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(u)$ in Lemma 3.6.

6.2.5. *The case $l_2=1$, $l_3\ge2$.* In this case, from (4.8), (6.9), (6.13), and a similar argument to that of Lemma 4.1 we have

$$\begin{aligned}\sum_j{}^{(1,l_3)}&=\sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(1,l_3)\\&=-\frac{(-1)^{l_3-2}(\log(y_4/n))^{l_3-1}}{(l_3-2)!}\int_0^1\sum_j\widetilde{a}_{j,k}\left((1-b)\frac{(\log(y_4/n))}{\log y_4}\right)^j\left(\frac{y_4}{n}\right)^{b\beta}b^{l_3-2}db\\&+O(L^{-4-l_3})\\&=-\frac{(-1)^{l_3-2}(\log(y_4/n))^{l_3-1}}{(l_3-2)!}\int_0^1\widetilde{P}_k\left((1-c)\frac{\log(y_4/n)}{\log y_4}\right)\left(\frac{y_4}{n}\right)^{c\beta}c^{l_3-2}dc\\&+O(L^{-4-l_3}).\end{aligned}$$

Therefore

$$\begin{aligned}I_{42}^{(1,\ge2)}(\alpha,\beta)&=\widehat{w_0}(0)\sum_{k=2}^K\frac{k!}{\log^k y_4}\sum_{l_1+1+l_3=k}\frac{2^{l_1}}{l_1!l_3!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(1,l_3)}+O(TL^{-1+\varepsilon})\\&=-4\widehat{w_0}(0)\sum_{k=2}^K k!\sum_{l_1+1+l_3=k}\frac{2^{l_1}(-1)^{l_3-2}}{l_1!l_3!(1+l_1)!(l_3-2)!}\\&\times\iint_{\substack{0\le a+b\le1\\a,b\ge0}}\int_0^1\int_0^1\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2(1-u)^{1+l_1}u^{l_3-1}c^{l_3-2}\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}\\&\times y_4^{uc\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)\right)\widetilde{P}_k((1-c)u)dudcdadb+O(TL^{-1+\varepsilon}),\end{aligned}$$

by setting $k=2$, $l=l_1$, $z=y_4$, $x=y_2$, $s=-a\alpha+b\beta+c\beta$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=u^{l_3-1}\widetilde{P}_k((1-c)u)$ in Lemma 3.6.

6.2.6. *The case $l_2 \geq 2$, $l_3 = 1$.* In this case, from (4.8), (6.9), (6.14), and a similar argument to that of Lemma 4.1 we have

$$\begin{aligned}\sum_j{}^{(l_2,1)} &= \sum_j \frac{\widetilde{a}_{j,k} j!}{\log^j y_4} K_2(l_2,1) + O(L^{-4-l_2}) \\ &= -\frac{(-1)^{l_2-2}(\log(y_4/n))^{l_2-1}}{(l_2-2)!}\int_0^1 \widetilde{P}_k\left((1-c)\frac{\log(y_4/n)}{\log y_4}\right)\left(\frac{y_4}{n}\right)^{-c\alpha} c^{l_2-2} dc \\ &+ O(L^{-4-l_2}).\end{aligned}$$

Similarly one has

$$\begin{aligned}I_{42}^{(\geq 2,1)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^K \frac{k!}{\log^k y_4}\sum_{l_1+l_2+1=k}\frac{2^{l_1}}{l_1! l_2!}\sum_{n\leq \min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i \sum_j{}^{(l_2,1)} + O(TL^{-1+\varepsilon}) \\ &= -4\widehat{w_0}(0)\sum_{k=2}^K k! \sum_{l_1+l_2+1=k}\frac{2^{l_1}(-1)^{l_2-2}}{l_1! l_2! (1+l_1)!(l_2-2)!} \\ &\times \iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1\int_0^1 \left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2 (1-u)^{1+l_1}u^{l_2-1}c^{l_2-2}\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta} \\ &\times y_4^{-uc\alpha}P_2''\left((1-a-b)\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)\right)\widetilde{P}_k((1-c)u)\,dudcdadb + O(TL^{-1+\varepsilon}),\end{aligned}$$

by setting $k=2$, $l=l_1$, $z=y_4$, $x=y_2$, $s=-a\alpha+b\beta-c\alpha$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=u^{l_2-1}\widetilde{P}_k((1-c)u)$ in Lemma 3.6.

6.2.7. *The case $l_2 = 0$, $l_3 \geq 2$.* By a similar argument to that of Lemmas 5.1, 5.2, and using Lemma 3.5 together with equation (6.15) we have

$$\begin{aligned}\sum_j{}^{(0,l_3)} &= \sum_j \frac{\tilde{a}_{k,j} j!}{\log^j y_4}K_2(0,l_3) + O(L^{-3-l_3}) \\ &= \sum_j \frac{\tilde{a}_{k,j} j!}{\log^j y_4}\frac{(-1)^{l_3}}{j!(l_3-2)!}\frac{d}{dx}\left(x+\log\frac{y_4}{n}\right)^{l_3+j-1} e^{\alpha x}\int_0^1\left(\frac{y_4}{n}\right)^{c\beta}(1-c)^j c^{l_3-2}e^{c\beta x}dc\bigg|_{x=0} + O(L^{-3-l_3}) \\ &= \frac{(-1)^{l_3}}{(l_3-2)!}\log^{l_3-2} y_4 \frac{d}{dx}\Bigg[y_4^{\alpha x}\left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)^{l_3-1} \\ &\qquad \times \int_0^1 \tilde{P}_k\left((1-c)\left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)\right)\left(\frac{y_4}{n}\right)^{c\beta}c^{l_3-2}e^{c\beta x\log y_4}dc\Bigg]_{x=0} + O(L^{-3-l_3}),\end{aligned}$$

after the change $y = x/\log y_4$. As done previously

$$\begin{aligned}I_{42}^{(0,l_3)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^K\frac{k!}{\log^k y_4}\sum_{l_1+l_3=k}\frac{2^{l_1}}{l_1! l_3!}\sum_{n\leqslant \min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(0,l_3)} + O(TL^{-1+\varepsilon}) \\ &= 4\widehat{w_0}(0)\sum_{k=2}^K\frac{k!}{\log^k y_4}\sum_{l_1+l_3=k}\frac{2^{l_1}}{l_1! l_3!}\frac{(-1)^{l_3}\log^{l_3-2}y_4}{(l_3-2)!}\frac{d}{dx}\Bigg[y_4^{\alpha x}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1 \\ &\times y_2^{-a\alpha+b\beta}\sum_{n\leqslant y_4}\frac{(d*\Lambda^{*l_1})(n)}{n^{1-a\alpha+b\beta+c\beta}}\left(\frac{\log(y_2/n)}{\log y_2}\right)^2 P''_2\left((1-a-b)\frac{\log(y_2/n)}{\log y_2}\right) \\ &\times \left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)^{l_3-1}\tilde{P}_k\left((1-c)\left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)\right)y_4^{c\beta(1+x)}c^{l_3-2}dadbdc\Bigg]_{x=0}\end{aligned}$$

$$
\begin{aligned}
&+O(TL^{-1+\varepsilon})\\
&=4\widehat{w_0}(0)\sum_{k=2}^{K}k!\sum_{l_1+l_3=k}\frac{2^{l_1}(-1)^{l_3}}{l_1!l_3!(l_3-2)!(1+l_1)!}\frac{d}{dx}\bigg[y_4^{\alpha x}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1\\
&\times(1-u)^{1+l_1}(x+u)^{l_3-1}y_4^{c\beta(u+x)}\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2c^{l_3-2}\\
&\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(x+u))dadbdcdu\bigg]_{x=0}\\
&+O(TL^{-1+\varepsilon}),
\end{aligned}
$$

by setting $k=2,\ l=l_1,\ z=y_4,\ x=y_2,\ s=-a\alpha+b\beta+c\beta,\ F(u)=u^2P''_2((1-a-b)u)$ and $H(u)=(x+u)^{l_3-1}\tilde{P}_k((1-c)(x+u))$ in Lemma 3.6.

6.2.8. *The case $l_2\geq 2,\ l_3=0$.* Again, by a similar argument to that of Lemmas 5.1, 5.2, and using Lemma 3.5 together with equation (6.16) we have

$$
\begin{aligned}
\sum_j{}^{(l_2,0)}&=\sum_j\frac{\tilde{a}_{k,j}j!}{\log^j y_4}K_2(l_2,0)+O(L^{-3-l_2})\\
&=\sum_j\frac{\tilde{a}_{k,j}j!}{\log^j y_4}\frac{(-1)^{l_2}}{j!(l_2-2)!}\frac{d}{dx}\left(x+\log\frac{y_4}{n}\right)^{j+l_2-1}e^{-\beta x}\int_0^1\left(\frac{y_4}{n}\right)^{-\alpha c}(1-c)^jc^{l_2-2}e^{-\alpha cx}dc|_{x=0}+O(L^{-3-l_2})\\
&=\frac{(-1)^{l_2}}{(l_2-2)!}\log^{l_2-2}y_4\frac{d}{dx}\left[y_4^{-\beta x}\left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)^{l_2-1}\right.\\
&\qquad\times\left.\int_0^1\tilde{P}_k\left((1-c)\left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)\right)\left(\frac{y_4}{n}\right)^{-\alpha c}c^{l_2-2}y_4^{-\alpha cx}dc\right]_{x=0}+O(L^{-3-l_2}),
\end{aligned}
$$

after the change $y=x/\log y_4$. Likewise, one has

$$
\begin{aligned}
I_{42}^{(l_2,0)}(\alpha,\beta)&=\widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+l_2=k}\frac{2^{l_1}}{l_1!l_2!}\sum_{n\leqslant\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(l_2,0)}+O(TL^{-1+\varepsilon})\\
&=4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+l_2=k}\frac{2^{l_1}}{l_1!l_2!}\frac{(-1)^{l_2}\log^{l_2-2}y_4}{(l_2-2)!}\frac{d}{dx}\bigg[y_4^{-\beta x}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\\
&\times y_2^{-a\alpha+b\beta}\sum_{n\leqslant y_4}\frac{(d*\Lambda^{*l_1})(n)}{n^{1-a\alpha+b\beta-\alpha c}}\left(\frac{\log(y_2/n)}{\log y_2}\right)^2P''_2\left((1-a-b)\frac{\log(y_2/n)}{\log y_2}\right)\\
&\times\left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)^{l_2-1}\tilde{P}_k\left((1-c)\left(x+\frac{\log\frac{y_4}{n}}{\log y_4}\right)\right)y_4^{-\alpha c(1+x)}c^{l_2-2}dadbdc\bigg]_{x=0}\\
&O(TL^{-1+\varepsilon})\\
&=4\widehat{w_0}(0)\sum_{k=2}^{K}k!\sum_{l_1+l_2=k}\frac{2^{l_1}}{l_1!l_2!}\frac{(-1)^{l_2}}{(l_2-2)!(1+l_1)!}\frac{d}{dx}\bigg[y_4^{-\beta x}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1\\
&\times(1-u)^{1+l_1}(x+u)^{l_2-1}y_4^{-\alpha c(u+x)}\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2c^{l_2-2}\\
&\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(x+u))dadbdcdu\bigg]_{x=0}
\end{aligned}
$$

$$+ O(TL^{-1+\varepsilon}),$$

by the use of $k = 2$, $l = l_1$, $z = y_4$, $x = y_2$, $s = -a\alpha + b\beta - \alpha c$, $F(u) = u^2 P''_2((1-a-b)u)$ and $H(u) = (x+u)^{l_3-1}\tilde{P}_k((1-c)(x+u))$ in Lemma 3.6.

6.2.9. *The case $l_2 \geq 2$, $l_3 \geq 2$.* Lastly, from (4.8), (6.9), and a similar argument to that of Lemma 4.1 we have

$$\begin{aligned}\sum_j{}^{(l_2,l_3)} &= \sum_j \frac{\widetilde{a}_{k,j} j!}{\log^j y_4} K_2(l_2,l_3) + O(L^{-3-l_2-l_3}) \\ &= \frac{(-1)^{l_2+l_3}(\log(y_4/n))^{l_2+l_3-2}}{(l_2-2)!(l_3-2)!} \\ &\qquad \times \iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}} \widetilde{P}_k\left((1-a-b)\left(\frac{\log(y_4/n)}{\log y_4}\right)\right)\left(\frac{y_4}{n}\right)^{-a\alpha+b\beta} a^{l_2-l_2} b^{l_3-2}\, da db \\ &\quad + O(L^{-3-l_2-l_3}).\end{aligned}$$

Finally,

$$\begin{aligned} I_{42}^{(l_2,l_3)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+l_2+l_3=k}\frac{2^{l_1}}{l_1! l_2! l_3!}\sum_{n\leq \min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j{}^{(l_2,l_3)} + O(TL^{-1+\varepsilon}) \\ &= 4\widehat{w_0}(0)\sum_{k=2}^{K} k! \sum_{l_1+l_2+l_3=k}\frac{2^{l_1}(-1)^{l_2+l_3}}{l_1! l_2! l_3! (1+l_1)!(l_2-2)!(l_3-2)!} \\ &\times \iiiint_{\substack{0\leq a+b\leq 1\\ 0\leq g+h\leq 1\\ a,b,g,h\geq 0}}\int_0^1 (1-u)^{k+l-1}\left(\frac{y_4}{y_2}\right)^{a\alpha-b\beta}\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2 \\ &\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)\right)\widetilde{P}_k((1-g-h)u) \\ &\times y_4^{-a\alpha u - g\alpha u + b\beta u + h\beta u} u^{l_2+l_3-2} g^{l_2-2} h^{l_3-2}\, du da db dg dh + O(TL^{-1+\varepsilon}),\end{aligned}$$

by setting $k = 2$, $l = l_1$, $z = y_4$, $x = y_2$, $s = -a\alpha + b\beta - g\alpha + h\beta$, $F(u) = u^2 P_2''((1-a-b)u)$ and $H(u) = u^{l_2+l_3-2}\widetilde{P}_k((1-g-h)u)$ in Lemma 3.6.

7. Proof of Proposition 2.6

We will first focus on the error terms. From [5, p. 11, Proposition] we can obtain the right order of magnitude of the error term for $I_{11}(\alpha,\beta,w)$ when $\theta_1 < 4/7 - \varepsilon$. To see the error terms for $I_{14}(\alpha,\beta,w)$ and $I_{44}(\alpha,\beta,w)$, we will proceed as follows. First we set $\psi_1(s) = \sum_{n\leq y_1} b(n) n^{-s}$ and $\psi_4(s) = \sum_{m\leq y_4} c(n) n^{-s}$. We state our result following a similar style to that of Proposition of [5].

Proposition 7.1. *Let $\theta_1 < 4/7 - \varepsilon$, $\theta_4 < 3/7 - \varepsilon$, and $T/2 \leq w \leq T$. Then we have*

$$I_{14}(\alpha,\beta,w) = \frac{\sum(\beta,\alpha) - e^{-(\alpha+\beta)L}\sum(-\alpha,-\beta)}{\alpha+\beta} + O(L^{-1+\varepsilon}), \tag{7.1}$$

where

$$\sum(\beta,\alpha) := \sum_{\substack{n\leq y_1\\ m\leq y_4}} \frac{\mathfrak{b}(n)\mathfrak{c}(m)}{n^{1+\alpha} m^{1+\beta}}(n,m)^{1+\alpha+\beta}.$$

Proof. For the sake of brevity, we will follow the proof of Proposition of [5]. More precisely we will follow the steps starting from equation (50) and ending in equation (69). The only modification we need is that $b(h,P_1)=\mathfrak{b}(h)$ and $b(k,P_2)=\mathfrak{c}(k)$. By doing so, we arrive to the following step:

$$\mathcal{M}(\alpha,\beta,s)=\sum_{m,n} m^{\alpha+\beta-s}n^{-s}\sum_{\substack{h\le y_1\\ k\le y_4}}\frac{\mathfrak{b}(h)\mathfrak{c}(k)}{h^{1-s+\beta}k^{1-s+\alpha}}e\left(\frac{mn\bar{H}}{K}\right), \tag{7.2}$$

where $H=h/(h,k)$, $K=k/(h,k)$ and $e(x)=e^{2\pi ix}$. The fact that stops us from following the next step in [5] is that $\mathfrak{c}(k)\neq\mu(k)F(k)$, for some smooth function F. We estimate (7.2) trivially. Using the fact $\mathfrak{b}(h)\ll h^{\varepsilon}$ and $\mathfrak{c}(k)\ll k^{\varepsilon}$ we have

$$\mathcal{M}(\alpha,\beta,s)\ll y_1^{1+\varepsilon+\eta}y_4^{1+\varepsilon+\eta}$$

for $s=1+\eta+it$ and $\eta\ll 1/L$. This gives us the required error term. □

Following the same ideas, we also have

Proposition 7.2. *Let $\theta_4<1/2-\varepsilon$ and $T/2\le w\le T$. Then we have*

$$I_{44}(\alpha,\beta,w)=\frac{\sum'(\beta,\alpha)-e^{-(\alpha+\beta)L}\sum'(-\alpha,-\beta)}{\alpha+\beta}+O(L^{-1+\varepsilon}), \tag{7.3}$$

where

$$\sum{}'(\beta,\alpha)=\sum_{n,m\le y_4}\frac{\mathfrak{c}(n)\mathfrak{c}(m)}{n^{1+\alpha}m^{1+\beta}}(n,m)^{1+\alpha+\beta}. \tag{7.4}$$

Combining the main term of $I_{11}(\alpha,\beta,w)$, $I_{14}(\alpha,\beta,w)$, and $I_{44}(\alpha,\beta,w)$ yields the main term of Lemma 2 of [9] provided that $\theta_1<4/7-\varepsilon$ and $\theta_4<3/7-\varepsilon$. This completes the proof of Proposition 2.6.

8. Proof of Proposition 2.7

When we insert the definitions of the mollifiers

$$\psi_1(s)=\sum_{a\le y_1}\frac{\mu(a)}{a^{s-\sigma_0-1/2}}P_1[a],$$

and

$$\psi_3(s)=\chi^2(s+\tfrac12-\sigma_0)\sum_{bc\le y_3}\frac{\mu_3(b)d(c)}{b^{s-\sigma_0-1/2}c^{1/2-s+\sigma_0}}P_3[bc],$$

in the mean-value integral we have

$$\begin{aligned}I_{13}(\alpha,\beta)&=\int_{-\infty}^{\infty}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\overline{\psi_1}\psi_3(\sigma_0+it)dt\\&=\sum_{a\le y_1}\sum_{bc\le y_3}\frac{\mu(a)\mu_3(b)d(c)}{(abc)^{1/2}}P_1[a]P_3[bc]J_{13},\end{aligned} \tag{8.1}$$

where

$$J_{13}=\int_{-\infty}^{\infty}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\chi^2(\tfrac12+it)\left(\frac{b}{ac}\right)^{-it}dt.$$

Using the same procedure as in the previous section (i.e. approximation of $\chi(\frac12+\beta-it)\chi(\frac12+it)$, followed by the functional equation of $\zeta(\frac12+\beta-it)$), we obtain

$$J_{13}=\int_{-\infty}^{\infty}w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12-\beta+it)\chi(\tfrac12+it)dt+O(T^{\varepsilon}).$$

From the Stirling formula we have

$$\chi(\tfrac{1}{2}+it) = F(t) + E(t),$$

where

$$F(t) = e^{i\pi/4}\left(\frac{t}{2\pi e}\right)^{-it} \quad \text{and} \quad E(t) \ll \frac{1}{t}.$$

Note that

$$\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it)E(t)dt$$
$$\ll \frac{1}{T}\int_{T/4}^{2T}|\zeta(\tfrac{1}{2}+\alpha+it)||\zeta(\tfrac{1}{2}-\beta+it)|dt \ll \frac{1}{T}\log T.$$

Thus, we are left with

$$J_{13} = \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it)F(t)dt + O_\varepsilon(T^\varepsilon).$$

Now we use Lemma 3.1 to see that

$$\begin{aligned} J_{13} &= \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\left(\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3} + O(T^{-1+\varepsilon})\right)F(t)dt + O_\varepsilon(T^\varepsilon) \\ &= \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\left(\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3}\right)F(t)dt + O_\varepsilon(T^\varepsilon) \\ &= \sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3}\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{bl}{ac}\right)^{-it}F(t)dt + O_\varepsilon(T^\varepsilon). \end{aligned}$$

For all $1 \le a \le y_1$, $1 \le b \le y_3$, $1 \le c \le y_3$ and any $l \ge 1$ we have

$$\frac{tbl}{2\pi eac} \ge \frac{T}{4}\frac{1}{2\pi e y_1 y_3} = \frac{T}{8\pi e T^{\theta_1+\theta_3}} \ge T^{\varepsilon_0}, \tag{8.2}$$

provided $\theta_1 < 4/7 - \varepsilon$ and $\theta_3 < 3/7 - \varepsilon$. We also recall the fact $w^{(r)}(t) \ll (L/T)^r$. Therefore from (8.2) and by the aid of integration by parts we have

$$\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{tbl}{2\pi eac}\right)^{-it}dt \ll_{r,\varepsilon_0} \frac{1}{T^r}$$

for any fixed integer r. This leaves us with

$$J_{13} \ll_{r,\varepsilon_0} \frac{1}{T^r}\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3} + O_\varepsilon(T^\varepsilon) \ll_{\varepsilon_0,\varepsilon} T^\varepsilon.$$

Putting this back into $I_{13}(\alpha,\beta)$ we see that

$$I_{13} \ll_{\varepsilon_0,\varepsilon} T^\varepsilon \sum_{\substack{a\le y_1\\ bc\le y_3}}\frac{|\mu(a)\mu_3(b)d(c)|}{(abc)^{1/2}}|P_1[a]P_3[bc]| \ll_{\varepsilon_0,\varepsilon} T^{2\varepsilon}\sum_{\substack{a\le y_1\\ bc\le y_3}}\frac{1}{(abc)^{1/2}},$$

since P_1 and P_3 are real polynomials in logarithms. Finally, we have

$$\begin{aligned} I_{13} &\ll_{\varepsilon_0,\varepsilon} T^{2\varepsilon}\left(\sum_{a\le y_1}\frac{1}{\sqrt{a}}\right)\left(\sum_{m\le y_3}\frac{d(m)}{\sqrt{m}}\right) \ll_{\varepsilon_0,\varepsilon} T^{3\varepsilon}y_1^{1/2}y_3^{1/2} \ll_{\varepsilon_0,\varepsilon} T^{3\varepsilon+(\theta_1+\theta_3)/2} \\ &= T^{\frac{1}{2}+3\varepsilon-2\varepsilon_0}. \end{aligned} \tag{8.3}$$

This completes the proof the proposition.

9. Proof of Proposition 2.8

First we note that the extra term of the logarithms satisfies

$$\frac{\log p_1 \cdots \log p_k}{\log^k y_4} \ll 1.$$

Moreover, their sum is

$$\sum_{p_1 \cdots p_k | c} \frac{\log p_1 \cdots \log p_k}{\log^k y_4} \widetilde{P}_k[c] \ll d(c) \ll c^{\varepsilon}.$$

Hence, this proof follows the exact same procedure as when we dealt with the cross term $I_{13}(\alpha, \beta)$ in Section 8.

10. Application

Let $N(T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$ and $0 < \beta < 1$. Let $N_0(T)$ denote the number of such zeros with $\beta = \frac{1}{2}$, and let $N_0^*(T)$ denote the number of such zeros and which are simple as well. We define κ and κ^* by

$$\kappa = \liminf_{T \to \infty} \frac{N_0(T)}{N(T)}, \quad \kappa^* = \liminf_{T \to \infty} \frac{N_0^*(T)}{N(T)}.$$

In 1942 Selberg [12] proved that $\kappa > 0$; in other words, a positive proportion of the zeros of the Riemann zeta-function lies on the critical line $\sigma = \frac{1}{2}$. Since then there have been improvements on the actual value of κ. Of these results we note Levinson's 1974 [10] result that $\kappa \geq .3474$. In 1985, Conrey and Ghosh [6] simplified Levinson's method and later in 2010, Young [13] gave a much shorter proof of Levinson's result. In 1989, Conrey [5] used deep arithmetical results on Kloosterman sums due to Deshouillers and Iwaniec [7, 8] and his own analytic devices [1, 3, 4] to set the record at $\kappa \geq .4088$. In the early 2010's Bui, Conrey and Young [2], and slightly afterward Feng [9], improved this to $\kappa \geq .4105$ and $\kappa \geq .4128$, respectively. However, as mentioned in introduction the result $\kappa \geq .4128$ is not clear. In this section we provide the following application of Theorem 1.1.

Theorem 10.1. *We have*

$$\kappa \geq .410725 \quad \text{and} \quad \kappa^* \geq .405824.$$

Let $Q(x)$ be a real polynomial satisfying $Q(0) = 1$ as well as $Q(x) + Q(1-x) = \text{constant}$, and define

$$V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s).$$

Since

$$|\psi_2(s)| \ll \sqrt{t}\left(\frac{y_2}{t}\right)^{\sigma} L^2 \quad \text{and} \quad |\psi_3(s)| \ll t\left(\frac{y_3}{t^2}\right)^{\sigma} L^4, \tag{10.1}$$

then $\log \psi(s)$ is analytic. Hence $\psi(s)$ is a valid mollifier in Levinson's method (see [10]) and it satisfies the inequality

$$\kappa \geq 1 - \frac{1}{R}\log\left(\frac{1}{T}\int_1^T |V\psi(\sigma_0 + it)|^2 dt\right) + o(1),$$

where $\sigma_0 = 1/2 - R/L$, and where R is a bounded positive real number of our choice. Choosing $Q(x)$ to be a linear polynomial yields a lower bound on the percent of simple zeros κ^*. Let us denote the integral in (1.9) by $I(\alpha, \beta)$. Then we have

$$\int_1^T |V\psi(\sigma_0 + it)|^2 dt = Q\left(\frac{-1}{L}\frac{d}{d\alpha}\right) Q\left(\frac{-1}{L}\frac{d}{d\beta}\right) I(\alpha, \beta)\Big|_{\alpha=\beta=-R/L}. \tag{10.2}$$

Also one has

$$Q\left(\frac{-1}{\log T}\frac{d}{d\alpha}\right)X^{-\alpha}=Q\left(\frac{\log X}{\log T}\right)X^{-\alpha}. \tag{10.3}$$

Combining (10.2) and (10.3) we have

Theorem 10.2. *Suppose that $\theta_1 = 4/7-\varepsilon$, $\theta_2 = 1/2-\varepsilon$, $\theta_3 = 3/7-\varepsilon$ and $\theta_4 = 3/7-\varepsilon$ for $\varepsilon>0$ small. Then*

$$\int_1^T |V\psi(\sigma_0+it)|^2 dt = cT + O_\varepsilon(TL^{-1+\varepsilon}),$$

where c is an explicit constant that depends on $Q, P_1, P_2, P_3, R, \theta_1, \theta_2, \theta_3, \theta_4$ and $\tilde{P}_k$ for $k=2,3,\dots,K$.

The constant c is given by $c = c_{11}+2c_{12}+c_{22}+2c_{23}+c_{33}+2c_{14}+2c_{24}+c_{44}$. The value of $c_{11}+2c_{14}+c_{44}$ was given in the main term of [9, Eq. (5.3)]. The expressions of c_{12} and c_{22} were given in [2, Eq. (3.4) and Eq. (3.6)]. The remaining values, i.e. c_{23}, c_{33}, c_{13}, c_{24} and c_{34} are now given below. Applying (10.3) on Propositions 2.3 and 2.4 and setting $\alpha=\beta=-R/L$, we get

$$\begin{aligned} c_{23} = \frac{2^8}{7!}\left(\frac{\theta_3}{\theta_2}\right)^6 e^R \frac{d^6}{dx^3dy^3}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 & u^4(1-u)^7 e^{R[\theta_2(y-x)+u\theta_3(a-b)]}Q(-x\theta_2+au\theta_3) \\ & \times Q(1+y\theta_2-bu\theta_3)P_2''\left(x+y+1-(1-u)\frac{\theta_3}{\theta_2}\right) \\ & \times abP_3^{(6)}((1-a-b)u)dudadb\Bigg]_{x=y=0}, \end{aligned} \tag{10.4}$$

and

$$\begin{aligned} c_{33} = \frac{2^{12}}{12!}\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1\int_0^1 & \left(\frac{1}{\theta_3}+(x+y-v(y+r)-u(x+r))\right) \\ & \times (1-r)^{12}e^{-\theta_3R(x+y-v(y+r)-u(x+r))}e^{2Rt(1+\theta_3(x+y-v(y+r)-u(x+r)))} \\ & \times Q(\theta_3(-x+v(y+r))+t(1+\theta_3(x+y-v(y+r)-u(x+r)))) \\ & \times Q(\theta_3(-y+u(x+r))+t(1+\theta_3(x+y-v(y+r)-u(x+r)))) \\ & \times (x+r)^2(y+r)^2P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))dtdrdudv\Bigg)\Bigg|_{x=y=0}. \end{aligned}$$

Finally, from using (10.3) on Proposition 2.5 and setting $\alpha=\beta=-R/L$, we obtain

$$\begin{aligned} c_{24}=c_{42}=4e^R\sum_{k=2}^K(&c_{42}^{(0,0)}(k)+c_{42}^{(0,1)}(k)+c_{42}^{(1,0)}(k)+c_{42}^{(1,1)}(k)+c_{42}^{(1,\ge 2)}(k)+c_{42}^{(\ge 2,1)}(k) \\ &+c_{42}^{(0,\ge 2)}(k)+c_{42}^{(\ge 2,0)}(k)+c_{42}^{(\ge 2,\ge 2)}(k)), \end{aligned}$$

where

$$\begin{aligned} c_{42}^{(0,0)}(k) = & \frac{2^k}{(k+1)!}\frac{d^2}{dxdy}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^{1+k} \\ & \times e^{R[\theta_2(a-x)+\theta_4a(u-1)]+R[\theta_2(y-b)+\theta_4b(-u+1)]} \\ & \times \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2\widetilde{P}_k(x+y+u)P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right) \\ & \times Q(\theta_2(a-x)+\theta_4a(u-1))Q(\theta_2(y-b)+\theta_4b(-u+1)+1)dudadb\Bigg]_{x=y=0}, \end{aligned}$$

$$c_{42}^{(1,0)}(k) = -\frac{2^{k-1}}{(k-1)!}\frac{d}{dy}\bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^k e^{R[-\theta_4(1-u)a+\theta_2 a]+R[\theta_4(b(1-u)+y)-\theta_2 b]}$$
$$\times\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(y+u)$$
$$\times Q(-\theta_4(1-u)a+\theta_2 a)Q(\theta_4(b(1-u)+y)-\theta_2 b+1)dudadb\bigg]_{y=0},$$

$$c_{42}^{(0,1)}(k) = -\frac{2^{k-1}}{(k-1)!}\frac{d}{dx}\bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^k e^{R[-\theta_4((1-u)a+x)+\theta_2 a]}e^{R[\theta_4 b(1-u)-\theta_2 b]}$$
$$\times\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(x+u)$$
$$\times Q(-\theta_4((1-u)a+x)+\theta_2 a)Q(\theta_4 b(1-u)-\theta_2 b+1)dudadb\bigg]_{x=0},$$

$$c_{42}^{(1,1)}(k) = \frac{2^{k-2}k}{(k-2)!}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^{k-1} e^{R[\theta_4(-(1-u)a)+\theta_2 a]}e^{R[\theta_F b(1-u)-\theta_2 b]}$$
$$\times\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(u)$$
$$\times Q(\theta_4(-(1-u)a)+\theta_2 a)Q(\theta_F b(1-u)-\theta_2 b+1)dudadb,$$

$$c_{42}^{(1,\ge 2)}(k) = -k!\sum_{l_1+1+l_3=k}\frac{2^{l_1}(-1)^{l_3-2}}{l_1!l_3!(1+l_1)!(l_3-2)!}$$
$$\times\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2(1-u)^{1+l_1}$$
$$\times e^{R[\theta_4 a(u-1)+\theta_2 a]}e^{R[\theta_4(b(1-u)-uc)-\theta_2 b]}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k((1-c)u)u^{l_3-1}c^{l_3-2}$$
$$\times Q(\theta_4 a(u-1)+\theta_2 a)Q(\theta_4(b(1-u)-uc)-\theta_2 b+1)dudcdadb,$$

with $l_3\ge 2$,

$$c_{42}^{(\ge 2,1)} = -k!\sum_{l_1+l_2+1=k}\frac{2^{l_1}(-1)^{l_2-2}}{l_1!l_2!(1+l_1)!(l_2-2)!}$$
$$\times\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_4}\right)^2(1-u)^{1+l_1}e^{R[\theta_4(a(u-1)+uc)+\theta_2 a]}e^{R[\theta_4 b(1-u)-\theta_2 b]}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_4}\right)\right)\widetilde{P}_k((1-c)u)u^{l_2-1}c^{l_2-2}$$
$$\times Q(\theta_4(a(u-1)+uc)+\theta_2 a)Q(\theta_4 b(1-u)-\theta_2 b+1)dudcdadb,$$

with $l_2\ge 2$,

$$c_{42}^{(\ge 2,0)} = k!\sum_{l_1+l_2=k}\frac{2^{l_1}(-1)^{l_2}}{l_1!l_2!(l_2-2)!(1+l_1)!}\frac{d}{dx}\bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1$$

$$\times (1-u)^{1+l_1}(x+u)^{l_2-1}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_2-2}$$
$$\times e^{R[\theta_4(c(u+x)-(1-u)a)+\theta_2 a]+R[\theta_4(x+(1-u)b)-\theta_2 b]}$$
$$\times Q(\theta_4(c(u+x)-(1-u)a)+\theta_2 a)Q(\theta_4(x+(1-u)b)-\theta_2 b+1)$$
$$\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(x+u))dadbdcdu\Bigg]_{x=0}$$

with $l_2 \geq 2$,

$$c_{42}^{(0,\geq 2)} = k!\sum_{l_1+l_3=k}\frac{2^{l_1}(-1)^{l_3}}{l_1!l_3!(l_3-2)!(1+l_1)!}\frac{d}{dy}\Bigg[\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1\int_0^1$$
$$\times (1-u)^{1+l_1}(y+u)^{l_3-1}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_3-2}$$
$$\times e^{R[\theta_4(-y-a(1-u))+\theta_2 a]+R[\theta_4(-c(u+y)+b(1-u))-\theta_2 b]}$$
$$\times Q(\theta_4(-y-a(1-u))+\theta_2 a)Q(1-\theta_2 b+\theta_4(-c(u+y)+b(1-u)))$$
$$\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(y+u))dadbdcdu\Bigg]_{y=0}$$

with $l_3 \geq 2$,

$$c_{42}^{(\geq 2,\geq 2)}(k) = k!\sum_{l_1+l_2+l_3=k}\frac{2^{l_1}(-1)^{l_2+l_3}}{l_1!l_2!l_3!(1+l_1)!(l_2-2)!(l_3-2)!}$$
$$\times \iiiint_{\substack{0\leq a+b\leq 1\\ 0\leq g+h\leq 1\\ a,b,g,h\geq 0}}\int_0^1 (1-u)^{k+l-1}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times e^{R[\theta_4(au+gu-a)+\theta_2 a]}e^{R[\theta_4(b-bu-hu)-\theta_2 b]}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k((1-g-h)u)$$
$$\times Q(\theta_4(au+gu-a)+\theta_2 a)Q(\theta_4(b-bu-hu)-\theta_2 b+1)$$
$$\times u^{l_2+l_3-2}g^{l_2-2}h^{l_3-2}dudadbdgdh,$$

with $l_2 \geq 2$ and $l_3 \geq 2$.

Finally, we use `Mathematica` to numerically evaluate c with the following particular choices of parameters. With $R = 1.295$, $\theta_1 = 4/7$, $\theta_2 = 1/2$, $\theta_3 = 3/7$, $\theta_4 = 3/7$ and $K = 3$,

$$Q(x) = 0.492203 + 0.621972(1-2x) - 0.148163(1-2x)^3 + 0.033988(1-2x)^5$$
$$P_1(x) = x + 0.229117x(1-x) - 2.932318x(1-x^2) + 4.856163x(1-x^3) - 2.309993x(1-x^4),$$
$$\tilde{P}_2(x) = -0.072644x + 1.559440x^2,$$
$$\tilde{P}_3(x) = 0.701568x - 0.554403x^2,$$

and all the other polynomials have their coefficients temporarily set to zero, we then have $\kappa \geq .410725$.

Moreover, by setting $R = 1.1195$, $\theta_1 = 4/7$, $\theta_2 = 1/2$, $\theta_3 = 3/7$, $\theta_4 = 3/7$ and taking,

$$Q^*(x) = .483872 + .516128(1-2x),$$
$$P_1^*(x) = .827329x + .0108498x^2 + .0815758x^3 + .181027x^4 - .100781x^5,$$
$$P_2^*(x) = .0326349x^3 - .0056269x^4 + .00783646x^5,$$

and all the other polynomials have their coefficients temporarily set to zero, we get $\kappa^* \geq .405824$.

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