The ideal Poisson-Voronoi tessellation and some applications

Amanda Wilkens
Carnegie Mellon University

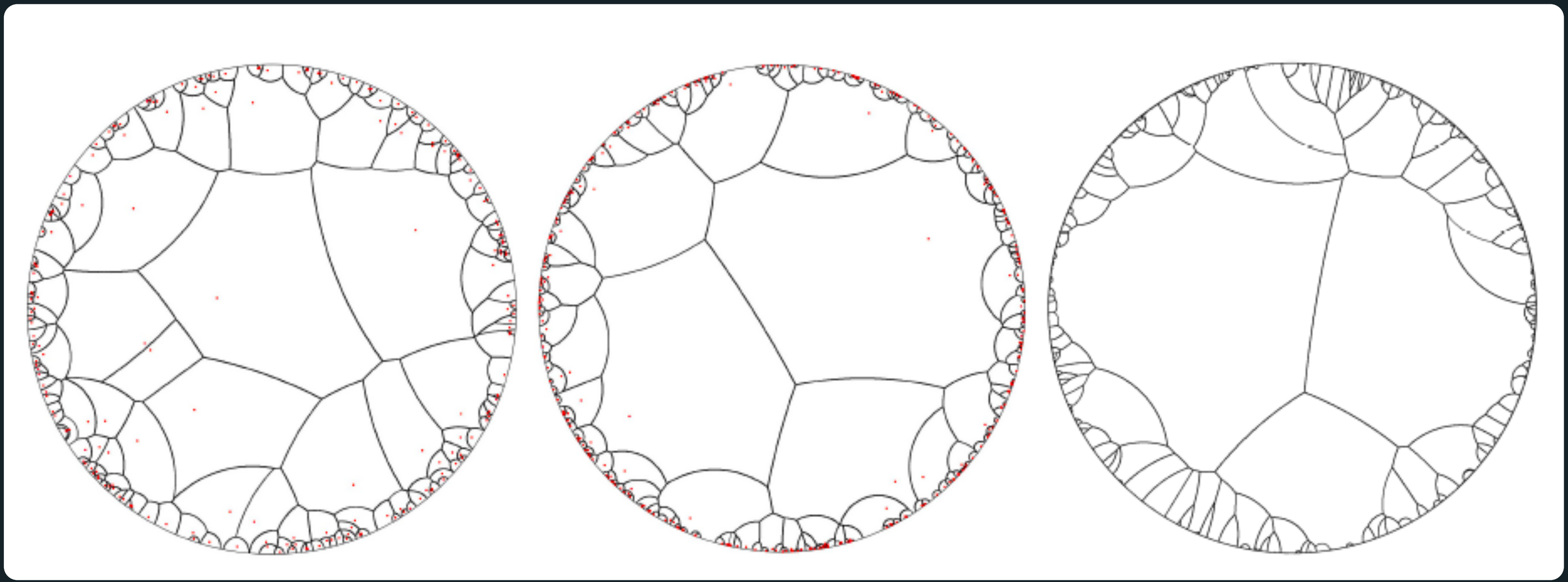
Outline

- The ideal Poisson-Voronoi tessellation: what is it?
- Proof sketch of key IPVT behavior
- Application to percolation
  - joint work with D'Achille, Grebík, Khezeli, and Recke
- Application to geometric topology
  - joint work with Frączyk and Mellick

IPVT history

First appeared in the thesis of Bhupatiraju '19
and on the arxiv in Budzinski, Curien, Petri '25 ('22)

Then constructed in Frączyk, Mellick, W. '26 ('23) and
D'Achille, Curien, Enriquez, Lyons, Ünel '26 ('23)



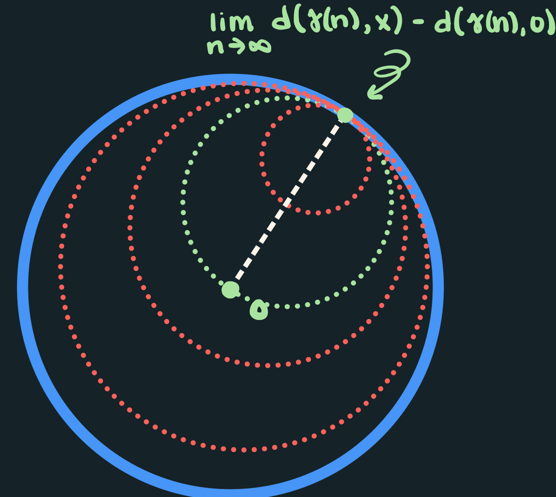
Boundary space construction

$X = \mathbb{H}^2$, $G = SL_2\mathbb{R}$, $G \curvearrowright X$

X has a G-invariant volume measure

Natural embedding

$$\iota : X \hookrightarrow \mathcal{C}(X)$$
$$x \mapsto d_x : X \to \mathbb{R}$$
$$y \mapsto d(x,y) - \boxed{d(x,0)}$$



Consider instead, for $t \in \mathbb{R}$

$$\iota_t : X \hookrightarrow \mathcal{C}(X)$$
$$x \mapsto d_x : X \to \mathbb{R}$$
$$y \mapsto d(x,y) - \boxed{t}$$

$\rightsquigarrow$ normalized push forward measures

$$\boxed{\frac{\iota_{t*}(\mathrm{vol})}{\mathrm{vol}\, B(t)} =: \mu_t}$$

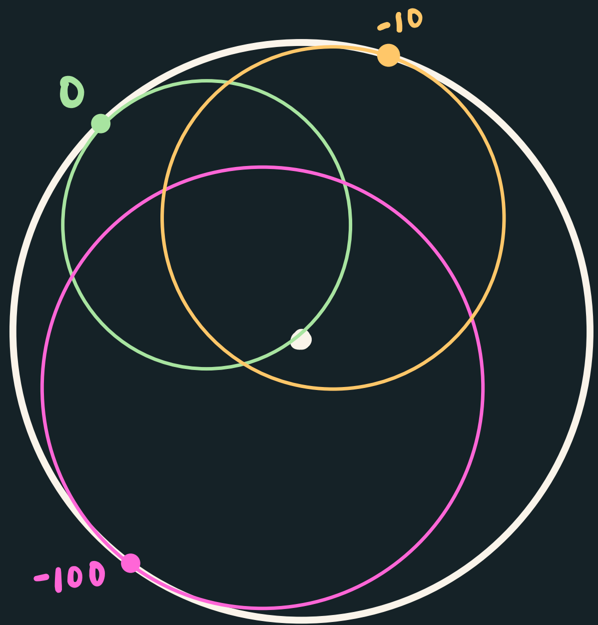
Toy example

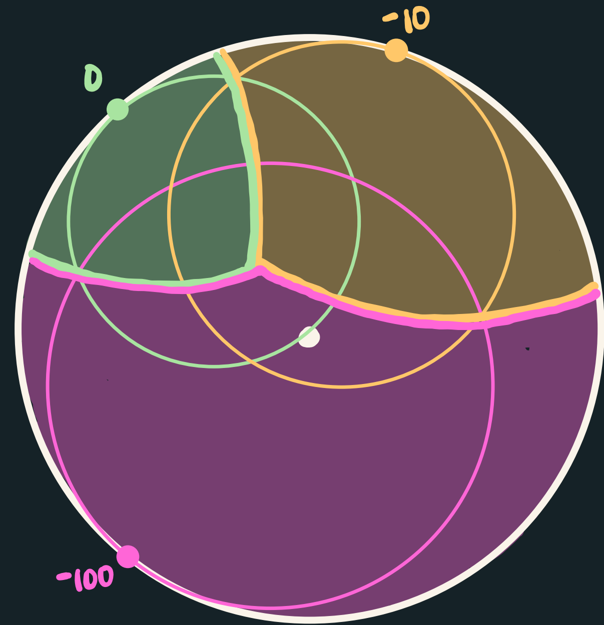
$$X = \mathbb{H}^2, \quad \partial X \times \mathbb{R} = G/P \times \mathbb{R} = G/U$$

$SL_2\mathbb{R}$

upper triangular matrices

matrices in P with 1s on diagonal





The Poisson point process

locally compact second countable space
with Haar measure vol

The PPP with intensity $\lambda > 0$ on X, Π, is a random variable taking values on

$$\mathbb{M}(X) = \left\{ \text{locally finite } \sum_{i \in \mathbb{N}} \delta(x_i) \;\middle|\; x_i \in X \right\}$$

characterized by:

1) $\forall$ measurable $A \subseteq X$, $\Pi(A)$ is a Poisson random variable with mean $\lambda \cdot \mathrm{vol}(A)$

2) $\forall$ measurable, disjoint $A, B \subseteq X$, $\Pi(A)$ and $\Pi(B)$ are independent

$\Rightarrow$ points in A are uniformly distributed in A

… as a dynamical system

$$G \curvearrowright \mathbb{M}(X): \quad g\pi(A) = \pi(g^{-1}A)$$
$$\forall g \in G, \text{ measurable } A \subseteq X$$

W'23: Any two Poisson point processes on a lcsc group are isomorphic as measure preserving dynamical systems.

(in contrast to Bernoulli shifts)

Extends results from Ornstein, Weiss 1987 and
Soo, W. 2019

Detecting geometry

The IPVT on $\mathbb{R}^n$ behaves fundamentally differently than the IPVT on $\mathbb{H}^n$.

rank 1

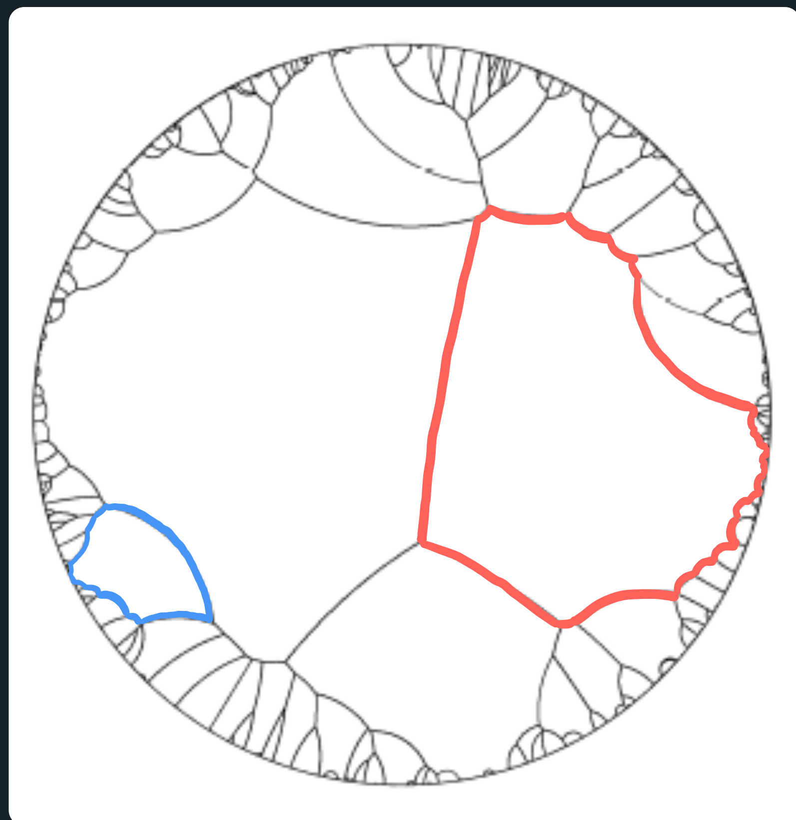
Even more surprisingly, the IPVT on $\mathbb{H}^n$ behaves fundamentally differently than the IPVT on $\mathbb{H}^n \times \mathbb{H}^n$, or the IPVT on the symmetric space for $SL_n\mathbb{R}$, $n \geq 3$.

higher rank (≥ 2)

rank n: contains isometric copy of $\mathbb{R}^n$ but not $\mathbb{R}^{n+1}$

Rank 1 vs higher rank

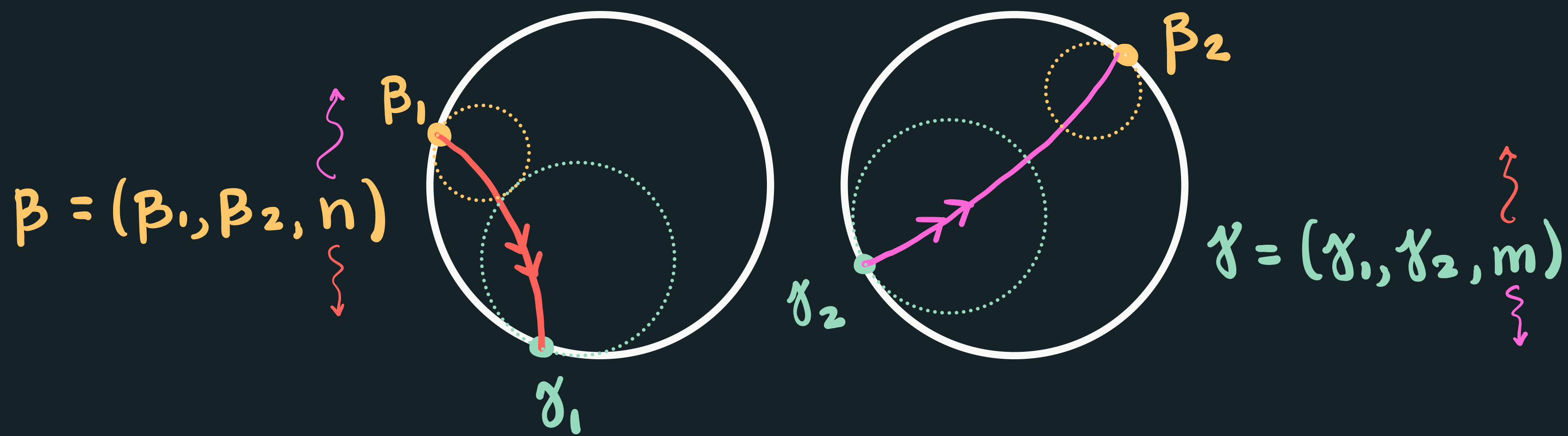
Frączyk, Mellick, W. '26: Let G be $SL_n\mathbb{R}$, $n \geq 3$, or $SL_n\mathbb{R} \times SL_n\mathbb{R}$, $n \geq 2$. Every pair of cells in the IPVT on G shares an unbounded border almost surely.



False for $SL_2\mathbb{R}$!

Proof sketch of unbounded borders

$X = \mathbb{H}^2 \times \mathbb{H}^2$, boundary $D = \partial X \times \mathbb{R}$



$S = \mathrm{stab}\{\beta\} \cap \mathrm{stab}\{\gamma\}$ is non-compact.
$x \in X$ equidistant from β, γ. $A_x = \{ \alpha \in D \mid \alpha(x) < \beta(x) \}$
has finite measure in D, so $\mathbb{P}(\Pi(A_x) = 0) = e^{-\mu(A_x)} > 0$.
Apply Borel-Cantelli.

Observation

In higher rank, graphs can be constructed from the IPVT which are HIGHLY connected across cells.

This is the key idea behind the main results in FWW and DGKRW.

Abért, Mellick '22: The P_{pp} on G has max cost out of all $G \underset{\text{mpef}}{\curvearrowright} (X, \mu)$ sps

If $\text{cost}(P_{pp}) = 1$ for G, then G has fixed price 1.

Poisson-Voronoi percolation

$X = \mathbb{H}^2$ or any noncompact metric space equipped with a Radon measure

1) Place a Poisson point process with intensity $\lambda > 0$ on X, along with its Voronoi tessellation.

2) Fix $p \in [0,1]$ and color each cell black with probability p and white with probability $1-p$.

The union of black cells is denoted $\omega(\lambda, p)$.

Uniqueness threshold

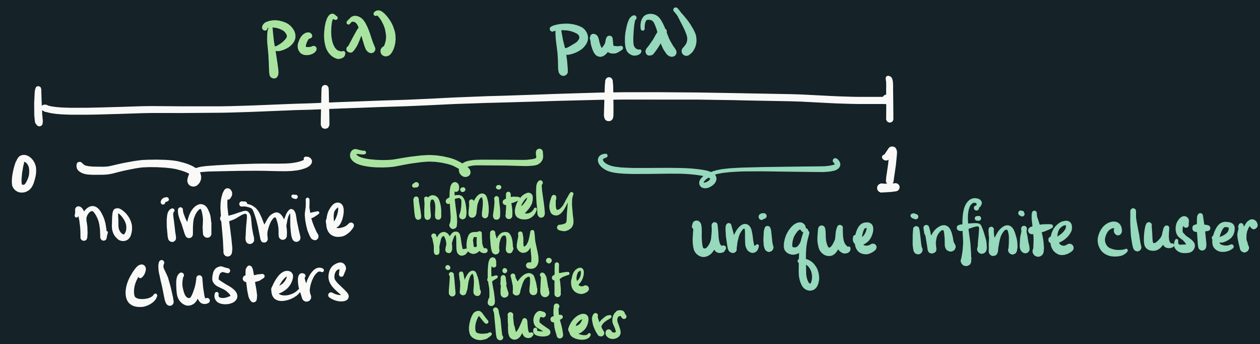
A path connected component of $\omega(\lambda, p)$ is called a cluster.

critical probability

$$p_c(\lambda) = \inf_p \{ \omega(\lambda, p) \text{ has an infinite cluster with positive probability} \}$$

uniqueness threshold

$$p_u(\lambda) = \inf_p \{ \omega(\lambda, p) \text{ has a unique infinite cluster with positive probability} \}$$



Vanishing uniqueness threshold

Benjamini, Schramm 2000: For Poisson-Voronoi percolation on $\mathbb{H}^2$,

$$\lim_{\lambda \to 0} p_u(\lambda) = 1.$$

Grebik, Recke '25: For P-V percolation on $SL_n\mathbb{R}$, $n \geq 3$,

$$\lim_{\lambda \to 0} p_u(\lambda) = 0.$$

D'Achille, Grebik, Khezeli, Recke, W. '25:
For P-V percolation on $\mathbb{H}^2 \times \mathbb{H}^2$,

$$\lim_{\lambda \to 0} p_u(\lambda) = 0.$$

We also prove the same for Bernoulli-Voronoi percolation on $T_d \times T_d$, $d \geq 3$.

Vanishing rank gradient

FMW '26: $G = SL_n\mathbb{R}$, $n \geq 3$, or $SL_n\mathbb{R} \times SL_n\mathbb{R}$, $n \geq 2$ with symmetric space X.

$\{\Gamma_i\}_{i \in \mathbb{N}}$ sequence of torsion-free lattices in G such that $\Gamma_i \backslash X \xrightarrow[i\to\infty]{BS} X$.

Then $\lim\limits_{i\to\infty} \dfrac{(\text{min \# of generators of } \Gamma_i) - 1}{\mathrm{vol}(\Gamma_i \backslash X)} = 0.$

"rank gradient"

manifold

its fundamental group

This follows as a corollary to fixed price 1 due to Abért, Mellick '22 and Carderi '23, and was conjectured in Abért, Gelander, Nikolov '17.

Mod-p homology corollary

FMW '26: $G = SL_n\mathbb{R}$, $n \geq 3$, or $SL_n\mathbb{R} \times SL_n\mathbb{R}$, $n \geq 2$ with symmetric space X.

$\{\Gamma_i\}_{i \in \mathbb{N}}$ sequence of torsion-free lattices in G such that $\Gamma_i \backslash X \xrightarrow[i\to\infty]{BS} X$.

p prime

first mod-p Betti number

first mod-p homology group

Then $$\lim_{i \to \infty} \frac{\dim_{\mathbb{F}_p} H_1(\Gamma_i, \mathbb{F}_p)}{\mathrm{vol}(\Gamma_i \backslash X)} = 0.$$

The case $p = 2$ is proved in Frączyk '22 via different methods.