Sample Test I.

The real test will have approximately 5 questions plus 2 bonus questions and you will have about 50 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Prove by induction that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ holds for all $n \geq 1$.
2. State the pigeonhole principle.
3. Let $n = rm + 1$ and suppose we distribute n identical balls into m identical boxes. Prove that there is at least one box that receives at least $r + 1$ balls.
4. How many lists of length k can be formed using elements of the set $[n]$ if repetition of letters is not allowed? Justify your answer!
5. How many ways are there to line up 3 apples 4 oranges and 1 banana on a shelf? State the formula you are using and prove it!
6. How many lists of length k can be formed using elements of the set $[n]$ if repetition of letters is allowed? Justify your answer!
7. Explain how counting all subsets of $[n]$ is related to counting binary numbers.
8. State a formula for the number of k-element subsets of an n element set. Justify your answer by using your answer to question 4 and the equivalence principle.
9. Prove that the binomial coefficients satisfy $\binom{n}{k} = \binom{n}{n-k}$. Use the observation to calculate $\binom{100}{98}$.
10. State and prove Pascal's identity.
11. Express the number of positive integer solutions of $x_1 + x_2 + \cdots + x_k = n$ as a binomial coefficient. Prove your formula!
12. Express the number of nonnegative integer solutions of $x_1 + x_2 + \cdots + x_k = n$ as a binomial coefficient. Prove your formula!
13. State a formula for the number of k-element multisets taken from an n-element set. Justify your answer by reducing your formula to your answer to question 8.
14. State and prove the binomial theorem.
15. Provide an algebraic and a combinatorial proof of the formula $\sum_{k=1}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$.
16. State and prove the Chu-Vandermonde formula.

17. Use the multinomial theorem to find the coefficient of x^2y^3z in $(x+y+z)^6$. (Also state the theorem.)
18. State the general version of the binomial theorem and use it to find the coefficient of x^2 in $\sqrt[3]{1-2x}$.
19. Evaluate $\binom{-1/3}{3}$.
20. State and prove a closed form formula for $\binom{\frac{1}{2}}{n}$ in terms of semifactorials.

Good luck. Gábor Hetyei