Fractional linear transformations preserving the Poincaré upper half plane

A fractional linear transformation $z \mapsto \dfrac{az+b}{cz+d}$ maps the Poincaré upper half plane onto itself exactly when it has following properties:

1. there is a nonzero complex number w such that aw, bw, cw and dw are real numbers, so a, b, c and d may be assumed to be real;
2. after rewriting the map with real coefficients a, b, c, d, the determinant $ad - bc$ is positive.

We leave the proof of this claims as an exercise. After dividing each of the real a, b, c, and d by $\sqrt{ad-bc}$, if necessary, from now on we may assume $ad - bc = 1$.

Corollary 1 *The group of fractional linear transformations mapping the Poincaré upper half plane onto itself is isomorphic to the group* $\mathrm{SL}_2(\mathbb{R})$.

For details, see Professor Royster's notes [1]

Theorem 1 *The group* $\mathrm{SL}_2(\mathbb{R})$ *is generated by the map*

$$\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \textit{and the maps} \quad \tau_r = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \quad \textit{for all } r \in \mathbb{R}.$$

Proof: Consider a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

Case 1: $a \neq 0$. As seen in our lecture notes [1], in this case we may look for our matrix in the form

$$M = \sigma\tau_t\sigma\tau_s\sigma\tau_r = \begin{bmatrix} -s & 1-rs \\ st-1 & rst-r-t \end{bmatrix}. \tag{1}$$

We may set $s = -a$. Solving $b = 1 - rs = 1 + ra$ yields $r = (b-1)/a$ and solving $c = st - 1 = -at - 1$ yields $t = (-c-1)/a$. Setting r and s as above implies $d = rst = r - t$, because of $ad - bc = 1$.

Case 2: $a = 0$. In this case, b and c can not be zero by $ac - bd = 1$. We will look for our matrix in the form

$$M = \tau_t\sigma\tau_s\sigma\tau_r = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -r \\ s & rs-1 \end{bmatrix} = \begin{bmatrix} -1+st & -r-t+rst \\ s & rs-1 \end{bmatrix}. \tag{2}$$

We set $c = s$ and solving $a = 0$ yields $t = 1/c$. Solving $d = rs - 1$, that is $d = rc - 1$, yields $r = (d+1)/c$. Substituting the obtained values of r, s and t gives

$$-r - t + rst = -\frac{d+1}{c} - \frac{1}{c} + \frac{d+1}{c} = -\frac{1}{c},$$

which is exactly b by $ac - bd = 1$. $\Diamond$

References

[1] D. Royster, “Non-Euclidean Geometry and a Little on How We Got There,” Lecture notes, December 11, 2011.