Inversion in the complex plane

Given a circle C centered at O with radius r, the inversion with base circle C sends P into P', where O, P and P' are on the same line, P' is between O and P and $OP \cdot OP' = r^2$.

Lemma 1 *If the base circle is centered at the origin and has radius r, then the inversion sends $z \neq 0$ into $r^2/\overline{z}$, where $\overline{z}$ is the conjugate of z.*

Proof: Let ϕ be the angle between the horizontal ray of positive real numbers and z. Then $z = |z| \cdot (\cos(\phi) + i \cdot \sin(\phi))$ and the inverse of z is $r^2/|z| \cdot (\cos(\phi) + i \cdot \sin(\phi))$. Since

$$r^2/z = r^2/|z| \cdot (\cos(-\phi) + i \cdot \sin(-\phi)) = r^2/|z| \cdot (\cos(\phi) - i \cdot \sin(\phi))$$

the conjugate of r^2/z is the inverse of z. $\diamond$

Theorem 1 *Consider an inversion with base circle centered at O of radius r. Let C_1 be a circle centered at O_1 of radius r_1. If $r_1 \neq |OO_1|$ then the inverse image of C_1 is a circle. This circle may be obtained from C_1 by a dilation centered at O by the factor of $\dfrac{r^2}{|OO_1|^2 - r_1^2}$.*

Proof: Without loss of generality we may assume that O is the origin. If we rotate the P around O by any fixed angle, its inverse gets rotated by the same angle. Hence we may assume that the ray $\overrightarrow{OO_1}$ is horizontal, pointing towards ∞. The center O_1 is then represented by the real number c_1 where $c_1 = |OO_1|$. The equation of the circle centered at O_1, of radius r_1 is

$$(z - c_1)(\overline{z} - c_1) = r_1^2. \tag{1}$$

This may be rewritten as

$$z\overline{z} - c_1(z + \overline{z}) + (c_1^2 - r_1^2) = 0.$$

Multiplying both sides by $\frac{r^2}{z\overline{z}}$ yields

$$r^2 - c_1\left(\frac{r^2}{z} + \frac{r^2}{\overline{z}}\right) + (c_1^2 - r_1^2)\frac{r^2}{z\overline{z}} = 0,$$

which may be rewritten as

$$r^2 - c_1\left(\overline{\frac{r^2}{\overline{z}}} + \frac{r^2}{\overline{z}}\right) + \frac{c_1^2 - r_1^2}{r^2} \cdot \frac{r^2}{\overline{z}} \cdot \overline{\frac{r^2}{\overline{z}}} = 0.$$

Since, by Lemma 1, the inverse of z is $r^2/\overline{z}$, the inverse of the circle C_1 is the set of points satisfying the equation

$$r^2 - c_1(z + \overline{z}) + \frac{c_1^2 - r_1^2}{r^2} z\overline{z} = 0.$$

Since we assume $c_1 \neq r_1$, we may multiply both sides by $\frac{r^2}{c_1^2 - r_1^2}$ and get

$$\frac{r^4}{c_1^2 - r_1^2} - \frac{r^2 c_1}{c_1^2 - r_1^2}(z + \overline{z}) + z\overline{z} = 0,$$

or, equivalently

$$z\overline{z} - \frac{r^2 c_1}{c_1^2 - r_1^2}(z + \overline{z}) = \frac{r^4}{r_1^2 - c_1^2}.$$

Adding $\frac{r^4 c_1^2}{(c_1^2-r_1^2)^2}$ to both sides yields

$$\left(z - \frac{r^2 c_1}{c_1^2 - r_1^2}\right)\left(\overline{z} - \frac{r^2 c_1}{c_1^2 - r_1^2}\right) = \frac{r^4(r_1^2 - c_1^2)}{(r_1^2 - c_1^2)^2} + \frac{r^4 c_1^2}{(c_1^2 - r_1^2)^2}.$$

After simplifying on the right hand side we obtain

$$\left(z - \frac{r^2 c_1}{c_1^2 - r_1^2}\right)\left(\overline{z} - \frac{r^2 c_1}{c_1^2 - r_1^2}\right) = \frac{r^4 r_1^2}{(r_1^2 - c_1^2)^2},$$

the equation of the circle centered at $\frac{r^2}{c_1^2-r_1^2} \cdot c_1$, of radius $\frac{r^2}{|r_1^2-c_1^2|} \cdot r_1$. ◇

Theorem 2 *Consider an inversion with base circle centered at O of radius r. Let C_1 be a circle centered at O_1 of radius r_1. If $r_1 = |OO_1|$ then the inverse image of C_1 is a line. This line is orthogonal to OO_1, and its distance from O is $\dfrac{r^2}{2r_1}$. (Conversely, the inverse image of any line is a circle containing the center of the base circle.)*

Proof: Just like in the previous theorem, we start by observing that we may assume that the center O_1 is a positive real number c_1. Since C_1 contains O, now we have $c_1 = r_1$, and equation (1) may be simplified to

$$z\overline{z} - r_1(z + \overline{z}) = 0.$$

Multiplying both sides by $\frac{r^2}{z\overline{z}}$ yields

$$r^2 - r_1\left(\frac{r^2}{z} + \frac{r^2}{\overline{z}}\right) = 0,$$

which may be rewritten as

$$r^2 - r_1\left(\overline{\frac{r^2}{\overline{z}}} + \frac{r^2}{\overline{z}}\right) = 0.$$

Using again Lemma 1, the inverse of the circle C_1 is the set of points satisfying the equation

$$r^2 - r_1(z + \overline{z}) = 0,$$

which may be rewritten as

$$\frac{z + \overline{z}}{2} = \frac{r^2}{2r_1}.$$

Since $\frac{z+\overline{z}}{2}$ is the real part of z, we obtained the equation of a vertical line at distance $\frac{r^2}{2r_1}$ from O. ◇