Assignment 3

Mandatory questions to be answered orally

1. Introduce the following relation on the set of ordered pairs of positive integers: $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + y_2 = x_2 + y_1$. Prove that $\sim$ is an equivalence relation.
2. Define an addition operation on ordered pairs of positive integers as follows:
$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2).$$
Prove that this addition is compatible with the equivalence relation $\sim$ of the previous question, that is, $(x_1, x_2) \sim (x_1', x_2')$ and $(y_1, y_2) \sim (y_1', y_2')$ imply $(x_1, x_2) + (y_1, y_2) \sim (x_1', x_2') + (y_1', y_2')$. (As a consequence, addition may be defined on equivalence classes.)
3. Prove that the set of ordered pairs $\{(x_1, x_2) \mid x_1 = x_2\}$ is an equivalence class in itself. Denote this equivalence class by 0. Prove that $0 + (x_1, x_2) \sim (x_1, x_2) + 0 \sim (x_1, x_2)$ holds for any ordered pair of positive integers (x_1, x_2).
4. Given a positive integer x, prove that the set of all ordered pairs $\{(x+t, t) \mid t \in \mathbb{P}\}$ is an equivalence class. (Here $\mathbb{P}$ stands for the set of positive integers.) Denote this equivalence class by A_x. Prove that the map $x \mapsto A_x$ is injective and that it satisfies $A_{x+y} = A_x + A_y$.
5. Similarly to the previous question one may prove that given a positive integer x, the set of all ordered pairs $B_x := \{(t, x+t) \mid t \in \mathbb{P}\}$ is an equivalence class. One may also prove that the map $x \mapsto B_x$ is injective and that it satisfies $B_{x+y} = B_x + B_y$. What is $A_x + B_x$ equal to? Prove your answer.
6. Prove that every equivalence class satisfies exactly one of the following: it is either 0, or of the form A_x or of the form B_x.

Mandatory questions to be answered in writing

1. Introduce multiplication on ordered pairs of positive integers as follows. Set
$$(x_1, x_2) \cdot (y_1, y_2) := (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1).$$
Prove that this multiplication operation is compatible with the equivalence relation introduced in the oral questions, that is, $(x_1, x_2) \sim (x_1', x_2')$ and $(y_1, y_2) \sim (y_1', y_2')$ imply $(x_1, x_2) \cdot (y_1, y_2) \sim (x_1', x_2') \cdot (y_1', y_2')$.
2. Consider again the equivalence relation on $\mathbb{P} \times \mathbb{P}$ introduced in the mandatory exercises. Prove that every ordered pair (x_1, x_2) satisfies $(x_1, x_2) \cdot 0 = 0$. (Here 0 is the equivalence class from oral question 3.)
3. Prove that multiplication of positive integers satisfies the Cancellation Law, i.e., $xz = yz$ implies $x = y$.

(Turn page for a Bonus question)

Bonus question

1. A *ring* is a set R with a commutative and associative addition operation $+$ and an associative multiplication operation $\cdot$ such that:

   - Multiplication is distributive over addition, that is $(x+y)\cdot z = x\cdot z + y\cdot z$ and $(x+y)\cdot z = z\cdot x + z\cdot y$ for all $x, y, z \in R$.
   - There is an additive zero element $0 \in R$ satisfying $x = 0 + x$ for all $x \in R$.
   - Every element $x \in R$ has an additive inverse $\overline{x}$ satisfying $x + \overline{x} = 0$.

   Prove that in a ring $0 \cdot x = x \cdot 0 = 0$ holds for all $x \in R$.