Assignment 5

Mandatory questions to be answered orally

1. Consider the equivalence relation on $\mathbb{P}\times\mathbb{P}$ defined by $(x_1, x_2) \sim (y_1, y_2)$ if $x_1+y_2 = y_1+x_2$, and the addition on equivalence classes defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$. Prove that $(x_1, x_2) + (z_1, z_2) \sim (y_1, y_2) + (z_1, z_2)$ implies $(x_1, x_2) \sim (y_1, y_2)$. (Cancellation law for addition.)

2. Consider the same equivalence relation as in the previous exercise and the multiplication on equivalence classes defined by $(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot y_1 + x_2 \cdot y_2, x_1 \cdot y_2 + x_2 \cdot y_1)$. State and prove the cancellation law for multiplication. Is it completely analogous to the cancellation law for addition?

3. Prove Theorem 123 in Landau's book.

4. Prove that the set $\{x \in \mathbb{Q}^+ \mid x^2 < 2\}$ is a cut. How about $\{x \in \mathbb{Q}^+ \mid x^2 \leq 2\}$?

5. Assume that ξ is a cut and n is a positive integer. Define $\xi+n$ by $\{x \in \mathbb{Q}^+ \mid \exists y \in \xi(x < y+n)\}$. Prove that $\xi + n$ is a cut. (Use only the definition of a cut, or use Theorem 129, but then explain how n corresponds to a cut.)

6. Consider positive rational numbers as equivalence classes of positive natural numbers, as defined in Landau's book. Prove that $\frac{1}{3} + \frac{1}{6} \sim \frac{1}{2}$. Does this mean that every ordered pair $\frac{p}{q}$ that is equivalent to $\frac{1}{2}$ is of the form
$$\frac{p}{q} = \frac{p_1}{q_1} + \frac{p_2}{q_2} \quad \text{for some } \frac{p_1}{q_1} \sim \frac{1}{3} \quad \text{and some} \quad \frac{p_2}{q_2} \sim \frac{1}{6}?$$
Analyze $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$ from the same point of view.

Mandatory question to be answered in writing

1. Assume that $a, b \in \mathbb{P}$ satisfy $a < b$. Prove that there is an $n \in \mathbb{P}$ such that $b < n \cdot a$. (Do not use any theorem on rational numbers from the book!)