Sample Test II.

The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. The questions below are sample questions related to stating and proving theorems. Besides trying to answer these questions, make sure you also review all homework exercises. The test may also have questions similar to those exercises. During the test, the usage of books or notes, or communicating with other students will not be allowed.

1. Define the Cartesian product of two rings, and prove that it is a ring.
2. Prove that the even integers form a subring of the ring of integers, and explain why odd integers are not a subring.
3. Is the Cartesian product of two integral domains an integral domain? Justify your answer!
4. Let R be a ring and let a be any ring element. Prove that the solution x of the equation $a + x = 0_R$ is unique. Explain how this may be used to prove that $a + b = a + c$ implies $b = c$.
5. What is $a \cdot 0_R$ equal to in a ring? Prove your claim!
6. Prove that $-(-a) = a$ in a ring.
7. Prove that $-(a + b) = (-a) + (-b)$ in a ring.
8. Prove that a subset S is a subring if it is not empty, and it is closed under subtraction and multiplication.
9. Describe the unique solution of the equation $a + x = b$ in a ring.
10. If $ac = bc$ in a ring, does it always follow that $a = b$? When does it follow? Justify your claim with example and/or proof, as appropriate.
11. Prove that every field is an integral domain. Is the converse true?
12. Prove that every finite integral domain is a field.
13. Give an example of a zero divisor, a nilpotent element and an idempotent element.
14. Let $f : R \to S$ be a homomorphism. Prove that $f(0_R) = 0_S$, and that $f(-a) = -f(a)$ holds for all $a \in R$.
15. Let $f : R \to S$ be a surjective homomorphism and assume that R and S are rings with identity. Prove that $f(1_R) = 1_S$ and that $f(u^{-1}) = f(u)^{-1}$ holds for all units in R.
16. Let R be a ring. When is it true that $\deg(f \cdot g) = \deg(f) + \deg(g)$ holds for all nonzero polynomials $f, g \in R[x]$?

17. Let F be a field. Describe the units of $F[x]$. Justify your description.
18. State the division algorithm theorem in $F[x]$ and prove it.
19. Define the greatest common divisor of two polynomials in $F[x]$ and explain how the Euclidean algorithm may be used to find it. (You do not have to prove your claim.)
20. Prove that every irreducible polynomial is prime in $F[x]$. Explain how this statement may be used to prove unique factorization in $F[x]$. (Are we showing uniqueness or existence using this claim? How do we prove the rest?)
21. State and prove the remainder and factor theorems.
22. Up to which degree is it true that a reducible polynomial $f \in F[x]$ must have a root? Justify your claim with proofs and examples.
23. When can we say that two polynomials $f, g \in F[x]$ induce the same function from F to F if and only if they are equal? Prove your claim, and give an example for the situation when the claim is false.

Good luck. Gábor Hetyei