Sample Test 3

The actual test will only have about 5 questions

1. Using the quotient and chain rules, find the derivative of $f(x) = \dfrac{\sqrt{x^2+1}}{x-2}$.
2. Find the critical values of the function $f(x) = \dfrac{x^2+3}{x+1}$.
3. Compute the first derivative of $f(x) = x^7/7 - 3x^5/5 + x^4/2$ and find the intervals on which $f(x)$ is increasing. Also find the local minima and maxima. (Hint: $x = 1$ is a critical value.)
4. Use the second derivative test to find the relative minima and maxima of $f(x) = x^3/3 + 3x^2/2 - 10x$.
5. Find the vertical asymptotes and holes of the rational function $f(x) = \dfrac{x^2-x-2}{x(x-2)(x+3)^2}$ and state the left and right limits that exist at these x-values.
6. Find the intervals, on which the function $f(x) = x^5/20 - x^4/4 + 2x^2$ is concave up. (Hint: $x = -1$ is a root of the second derivative.)
7. Find the absolute minimum and maximum values of $f(x) = (x^2 + 3 \cdot x) \cdot \sqrt{x+3}$ on the interval $[-3, 1]$.
8. Find the absolute minimum and maximum values of $f(x) = x^3/3 - x^2/2 - 2x$ on the interval $[-2, 3]$.
9. The perimeter of a Norman window is 15 feet. Write the area of the window as a function of the width x and find the value of x for which the area is maximal.
10. Find $\lim_{x\to\infty} 3-2\cdot 5^x$ and $\lim_{x\to-\infty} 3-2\cdot 5^x$. Also find the domain and the range of $f(x) = 3-2\cdot 5^x$.
11. Find the domain and range of $f(x) = 1 - \ln(x+1)$ and also the appropriate limits at -1 and ∞. State whether the function is increasing or decreasing.

Solutions:

1. $$f'(x) = \frac{\frac{1}{2\cdot\sqrt{x^2+1}} \cdot 2x \cdot (x-2) - \sqrt{x^2+1}}{(x-2)^2} = \frac{x \cdot (x-2) - (x^2+1)}{(x-2)^2 \cdot \sqrt{x^2+1}} = \frac{-2x-1}{(x-2)^2 \cdot \sqrt{x^2+1}}.$$

2. The critical values are the ones where the derivative is zero or not defined. We have
$$f'(x) = \frac{2x(x+1) - (x^2+3)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2} = \frac{(x+3)(x-1)}{(x+1)^2},$$
which is zero at $x = 1$, $x = -3$, and undefined at $x = -1$. These are the critical values.

3. The derivative is $f'(x) = x^6 - 3x^4 + 2x^3 = x^3(x^3 - 3x + 2)$. Using the hint, $f'(x)$ factors as $f'(x) = x^3(x-1)^2(x+2)$. The derivative is positive on $(-\infty, -2) \cup (0, 1) \cup (1, \infty)$, so these are the intervals on which the function is increasing. The function is decreasing on $(-2, 0)$. There is a local maximum at $x = -2$ and a local minimum at $x = 0$. (There is an inflection point at $x = 1$, but this was not asked.)

4. The first derivative is $f'(x) = x^2 + 3x - 10 = (x+5)(x-2)$, so the critical values are $x = -5$ and $x = 2$. The second derivative is $f''(x) = 2x + 3$. The second derivative satisfies $f''(-5) = -7 < 0$ and $f''(2) = 7 > 0$. Hence we have a relative maximum at $x = -5$ and a relative minimum at $x = 2$.

5. The denominator is zero at $x = 0$, $x = 2$ and $x = -3$. The numerator factors as $(x-2)(x+1)$, after simplification we get
$$f(x) = \frac{x+1}{x(x+3)^2} \quad \text{for } x \neq 2.$$
We have a hole at $x = 2$ and $\lim_{x\to 2} f(x) = \frac{2+1}{2(2+3)^2} = \frac{3}{50}$. We have a vertical asymptote at $x = 0$ and $x = -3$. The multiplicity of $x = -3$ is even, there is no sign change there, the function is positive near $x = -3$. Hence $\lim_{x\to -3} f(x) = \infty$. The multiplicity of x is odd. Using test points we get $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = \infty$.

6. The first derivative is $f'(x) = x^4/4 - x^3 + 4x$, the second derivative is $f''(x) = x^3 - 3x^2 + 4$. Using the hint, $f''(x) = (x-2)^2 \cdot (x+1)$. This function is positive on $(-1, 2) \cup (2, \infty)$.

7. The first derivative is

$$f'(x) = (2x+3) \cdot \sqrt{x+3} + (x^2+3x) \cdot \frac{1}{2\sqrt{x+3}} = (2x+3) \cdot \sqrt{x+3} + (x^2+3x) \cdot \frac{\sqrt{x+3}}{2(x+3)}$$
$$= \frac{4x+6}{2} \cdot \sqrt{x+3} + x \cdot \frac{\sqrt{x+3}}{2} = \frac{5x+6}{2} \cdot \sqrt{x+3}.$$

This is negative on $[-3, -6/5)$ and positive on $(-6/5, 1)$ The function $f(x)$ is decreasing on the first interval, and increasing on the second. The absolute minimum is at $x = -6/5$ where $f(-6/5) = -162 \cdot \sqrt{5}/125$. The absolute maximum is at one of the endpoints of the interval $[-3, 1]$. We have $f(-3) = 0$ and $f(1) = 8$. The maximum value is 8.

8. The first derivative is $f'(x) = x^2 - x - 2 = (x+1) \cdot (x-2)$. The critical values are $x = -1$ and $x = 2$. We have $f(-2) = -2/3$, $f(-1) = 7/6$, $f(2) = -10/3$ and $f(3) = -3/2$. The absolute minimum is $f(2) = -10/3$, the absolute maximum is $f(-1) = 7/6$.

9. Introducing h for the height of the rectangular part, the perimeter is $15 = 2h + x + \pi \cdot x/2$. Solving for h yields $h = \frac{15 - x - \pi \cdot x/2}{2}$. Hence the area function is

$$A(x) = x \cdot h + \frac{\pi \cdot x^2}{8} = x \cdot \frac{15 - x - \pi \cdot x/2}{2} + \frac{\pi \cdot x^2}{8} = \frac{(-\pi - 4) \cdot x^2}{8} + \frac{15 \cdot x}{2}.$$

The derivative is

$$A'(x) = \frac{(-\pi - 4) \cdot x}{4} + \frac{15}{2}.$$

This is zero, when $x = 30/(4+\pi)$, positive for $x < 30/(4+\pi)$, negative for $x > 30/(4+\pi)$. The absolute maximum is at $x = 30/(4+\pi)$.

10. The domain of $f(x)$ is $(-\infty, \infty)$. (This is the domain of all exponential functions.) The range of 5^x is $(0, \infty)$, the range of -5^x is $(-\infty, 0)$, the range of $f(x)$ is $(-\infty, 3)$. The line $y = 3$ is a horizontal asymptote. We have $\lim_{x \to \infty} 3 - 2 \cdot 5^x = -\infty$ and $\lim_{x \to -\infty} 3 - 2 \cdot 5^x = 3$.

11. The domain of $f(x)$ is $(-1, \infty)$. The function is decreasing, the range is $(-\infty, \infty)$. (This range is the same for all logarithm functions.) We have $\lim_{x \to \infty} f(x) = -\infty$ and $\lim_{x \to -1^+} f(x) = \infty$.